LAMC Advanced Circle **October 9, 2016**

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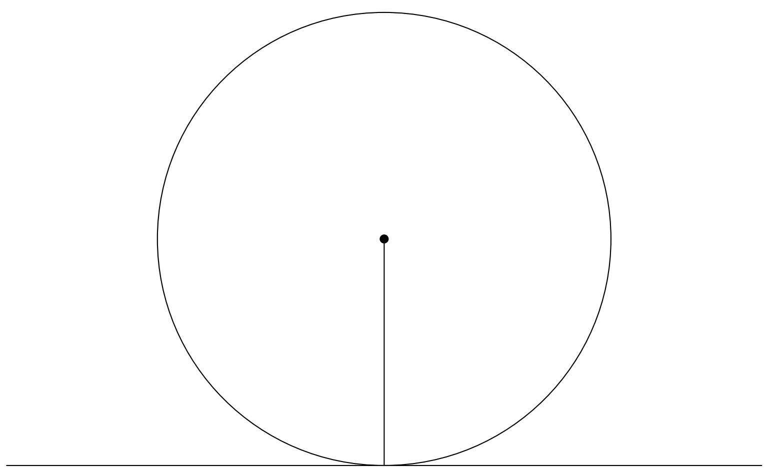
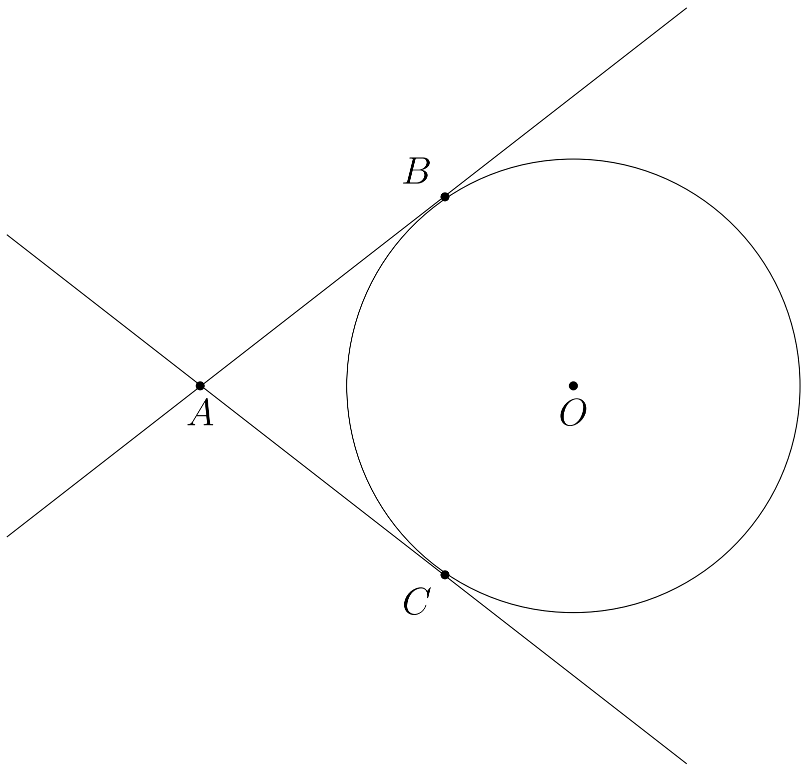
Warm-up

Problem 1 *Prove that a straight line tangent to a circle is perpendicular to the radius connecting the tangency point to the circle's center.*



Problem 2 *The straight lines AB and AC are tangent to the circle on the picture below.*



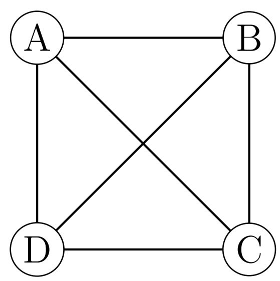
Prove that $|AB| = |AC|$.

Problem 3 *Formulate and prove the triangle inequality.*

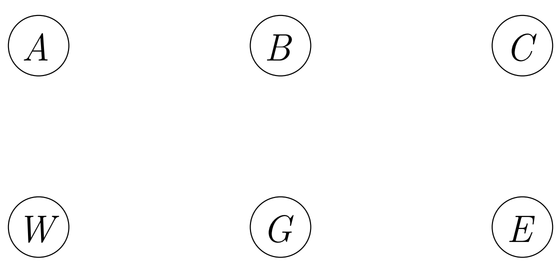
Problem 4 *Prove that given any polygon circumscribed around a circle, one can always pick three sides that can form a triangle.*

Planar graphs

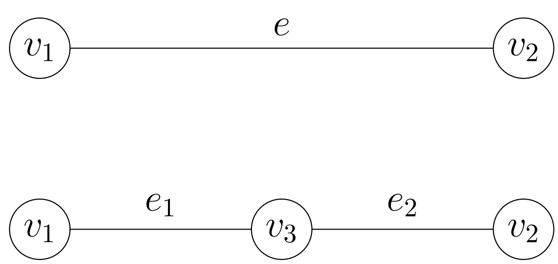
A graph is called *planar*, if it can be drawn in the plane in such a way that no edges cross one another. For example, the following graph is planar.



Problem 5 *Is it possible to connect three houses, A, B, and C, to three utility sources, water (W), gas (G), and electricity (E), without using the third dimension, either on the plane or sphere, so that the utility lines do not intersect?*



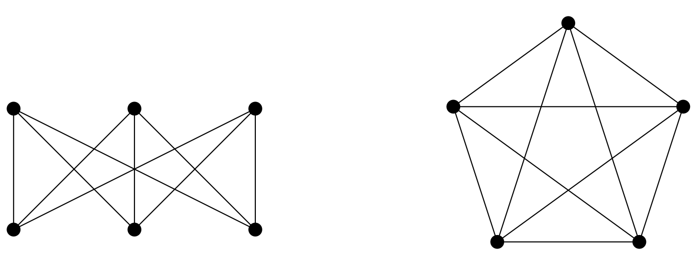
A *subdivision* of a graph G is a graph resulting from the subdivision of the edges of G. The subdivision of an edge $e = (v_1, v_2)$ is a graph containing one new vertex v_3, with the edges $e_1 = (v_1, v_3)$ and $e_2 = (v_3, v_2)$ replacing the edge e.



Problem 6 *What is the degree of a subdivision vertex?*

A graph H is called a *subgraph* of a graph G if the sets of vertices and edges of H are subsets of the sets of vertices and edges of G.

The following graphs are known as $K_{3,3}$ and K_5.



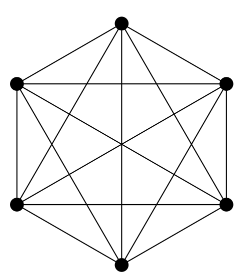
$K_{3,3}$ K_5

Let H be a graph that is a subdivision of either $K_{3,3}$ or K_5. If H is a subgraph of a graph G, then H is called a *Kuratowski subgraph*, after a famous Polish mathematician Kazimierz Kuratowski (1896-1980).

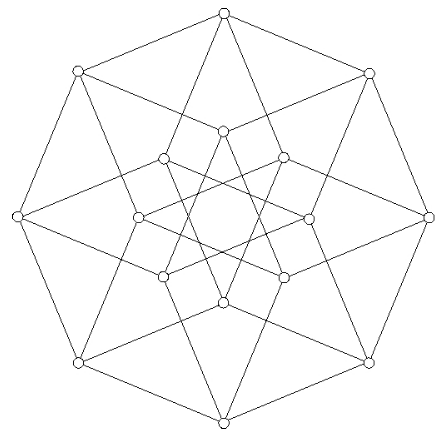
Kazimierz Kuratowski

Theorem 1 *(Kuratowski) A graph is planar if and only if it has no Kuratowski subgraph.*

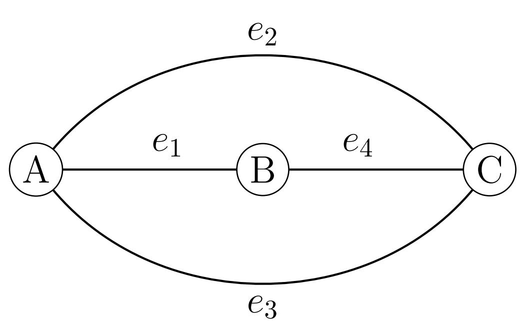
Problem 7 *Is the following graph planar? Why or why not?*



Problem 8 *Is the following graph planar? Why or why not?*



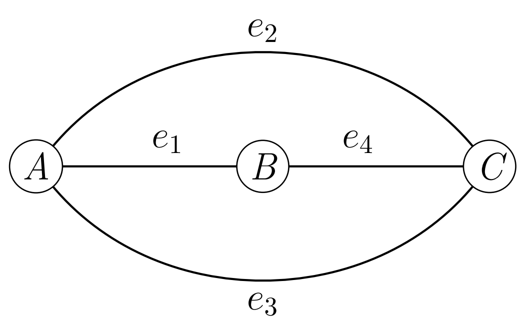
Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior (or infinite) face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1, bounded by the edges e_1, e_2, e_4; and F_2, bounded by the edges e_1, e_3, e_4. Its exterior face, F_3, is bounded by the edges e_2, e_3.



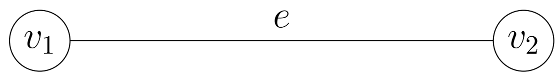
The *Euler characteristic* of a graph is the number of the graph's vertices minus the number of the edges plus the number of the faces.

$$\chi = V - E + F \tag{1}$$

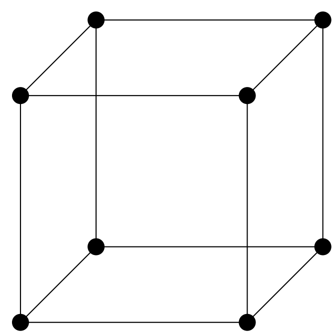
Problem 9 *Compute the Euler characteristic of the graph we have seen on the previous page, reproduced for your convenience below.*



Problem 10 *Compute the Euler characteristic of the following graph.*

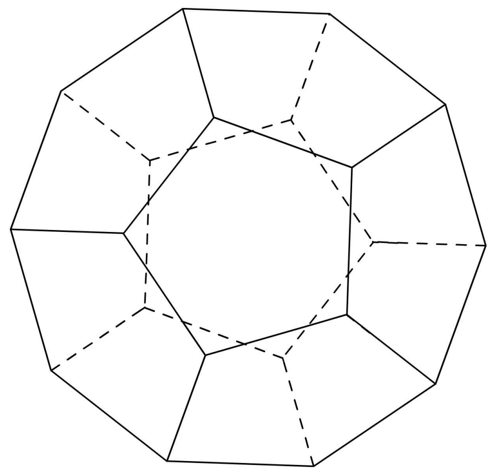


Problem 11 *Is the following graph planar? If you think it is, please draw an equivalent graph with no intersecting edges. If you think the graph is not planar, please explain why.*



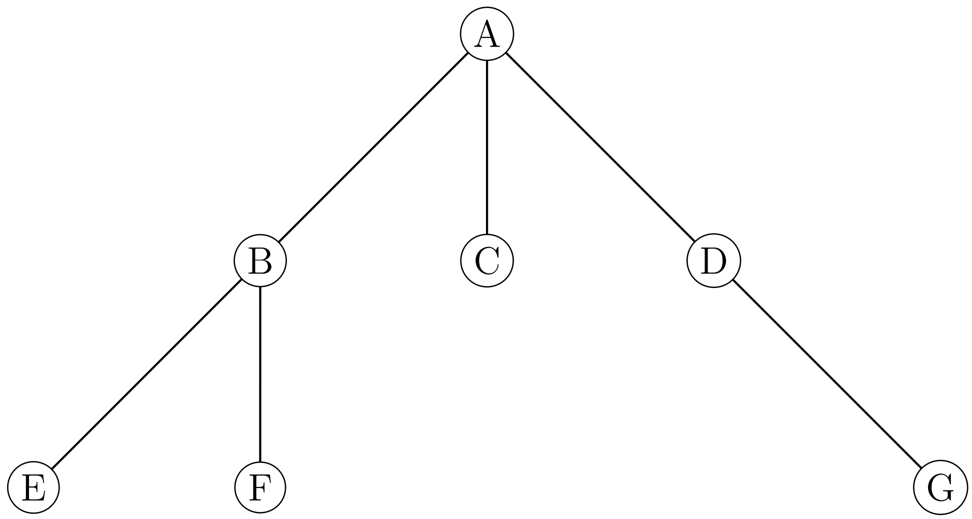
Problem 12 *Compute the Euler characteristic of the above graph.*

Problem 13 *Is the following graph planar? If you think it is, please draw an equivalent graph with no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 14 *Compute the Euler characteristic of the above graph. Can you conjecture what the Euler characteristic of every planar graph is equal to?*

A graph is called a *tree* if it is connected and has no cycles. Here is an example.

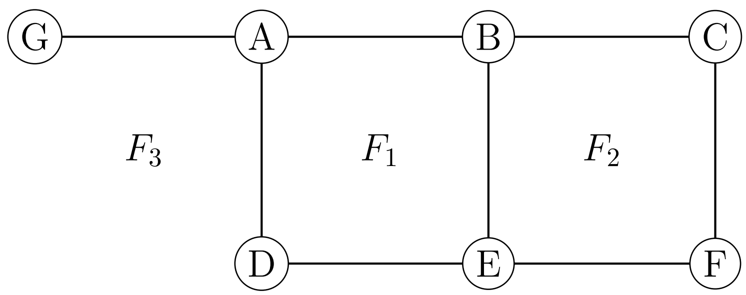


Problem 15 *What is the Euler characteristic of a finite tree?*

Theorem 2 *Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.*

Problem 16 *Use Problem 15 as a tool to complete the proof of Theorem 2.*

Let G be a planar graph with E edges. Let us call the *degree of its face*, $deg(F_i)$, the number of the edges one needs to traverse to get around the face F_i. For example, the following are the degrees of the faces of the graph below: $deg(F_1) = deg(F_2) = 4$, $deg(F_3) = 8$.



Note that in order to get around the exterior face of the graph, F_3, one has to traverse the edge (A, G) twice.

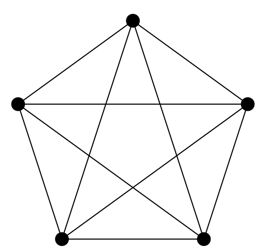
Problem 17 *Prove that* $\sum deg(F_i) = 2E$.

A graph is called *simple* if it is undirected, has no loops, and no multiple edges. The latter means that every pair of vertices connected by an edge is connected by only one edge. For example, the graph at the top of this page is simple, the graph in Problem 9 is not.

Problem 18 *Let a finite connected simple planar graph have* $E > 1$ *edges and* F *faces. Prove that then* $2E \geq 3F$.

Problem 19 *Prove that for a finite connected simple planar graph,* $E \leq 3V - 6$.

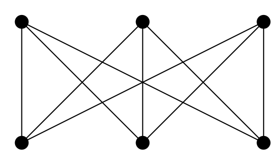
Problem 20 *Prove that the graph* K_5 *is not planar.*



Problem 21 *Let* G *a finite connected simple planar graph with* $E > 1$ *edges and no triangular faces. Prove that then* $E \geq 2F$.

Problem 22 *Let G be a finite connected simple planar graph with $E > 1$ edges and no triangular faces. Prove that then $E \leq 2V - 4$.*

Problem 23 *Prove that the graph $K_{3,3}$ is not planar.*



If you are finished doing all the above, but there still remains some time..

Problem 24 *Show that there is no prime number $p > 2$ such that $p^2 + 5$ is a perfect square.*