Number Patterns and Geometry

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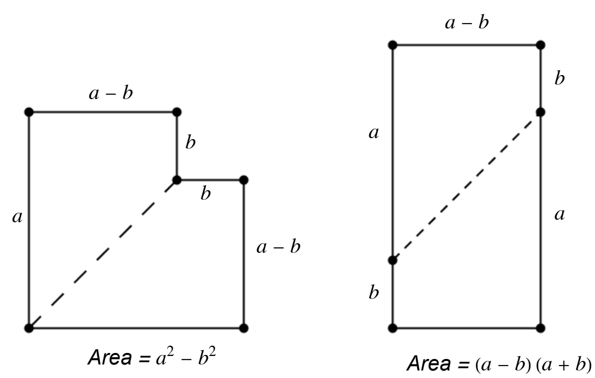
Difference of Squares



Interpreting the Picture

We start with a square of side a, and remove a square of side b (diagram on the left).

Area on the left is: $a^2 - b^2$.

Next, we identify the lengths of each segment in the picture in terms of a and b.

Finally, we cut along the dotted line and rearrange the pieces to get a the rectangle on the right.

Area on the right: $(a - b)(a + b)$.

First Identity

We reach the $\boxed{\text{Difference of Squares Identity : } a^2 - b^2 = (a - b)(a + b)}$

Examples:

$$15^2 - 12^2 = (15 - 12)(15 + 12)$$
$$5^2 - 3^2 = (5 - 3)(5 + 3)$$
$$13^2 - 5^2 = (13 - 5)(13 + 5)$$

Notice that we can compute these quantities without having to compute the values of the squares

Second Identity

We reach the $\boxed{\text{Alternate Identity : } a^2 = (a - b)(a + b) + b^2}$

Examples:

$$13^2 = (13 - 3)(13 + 3) + 3^2$$
$$17^2 = (17 + 3)(17 - 3) + 3^2$$
$$31^2 = (31 - 1)(31 + 1) + 1^3$$

- **Applying the Second Identity**

We use the identity to compute squares of numbers.

$$\begin{aligned} 14^2 - 4^2 &= 10 \cdot 18 \\ 15^2 - 5^2 &= 10 \cdot 20 \\ 16^2 - 6^2 &= 10 \cdot 22 \\ 17^2 - 7^2 &= 10 \cdot 24 \end{aligned} \quad \text{which tells us that} \quad \begin{aligned} 14^2 &= 180 + 16 \\ 15^2 &= 200 + 25 \\ 16^2 &= 220 + 36 \\ 17^2 &= 240 + 49 \end{aligned} \quad \text{and all of these are easy to compute.}$$

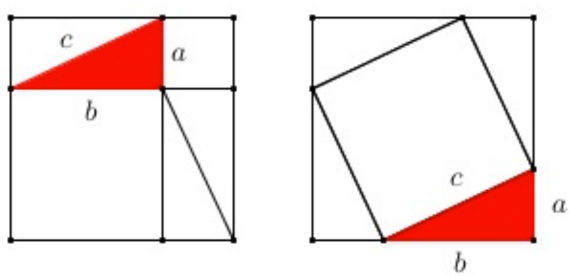
- **Exercises**

1. Compute $11^2,\ 12^2,\ 13^2,\ 14^2,\ 15^2,\ 16^2,\ 17^2,\ 18^2,\ 19^2$ using this technique.
2. Compute $28^2,\ 29^2,\ 31^2$ and 32^2 using 30 as the nearby multiple of 10.
3. Compute: $48^2, 49^2, 51^2$ and 52^2 using 50 as the nearby multiple of 10.
   (Hint: Multiplying by 50 is the same as dividing by two and multiplying by 100.)
4. Compute $97^2, 98^2,\ 99^2,\ 101^2$ and 102^2 using 100 as the nearby multiple of 10.

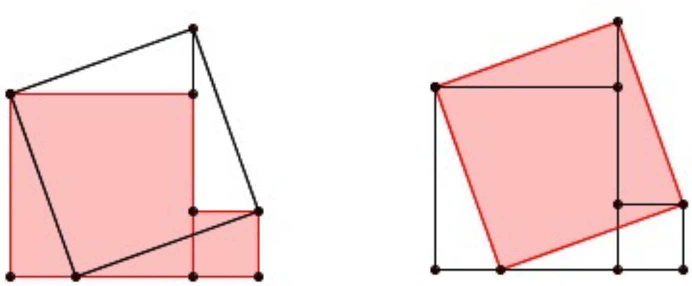
- **The Pythagorean Theorem**

In a right triangle, the areas of the squares on the sides equals the area of the square on the hypotenuse.

Of the many independent proofs of this theorem, perhaps the simplest is found by examining this image, in which a square of side $a + b$ is dissected in two ways.
If the first, when we remove the four right triangles we are left with $a^2 + b^2$.
In the second, removing four identical triangles leaves c^2.
(By some accounts, this proof was known to the Chinese before Euclid!)



Another geometric proof of the Pythagorean Theorem may be gleaned by comparing the images below.



Square Numbers and The Pythagorean Theorem

We will use the identity to verify that we have Pythagorean Triples, i.e. numbers a, b, and c such that $a^2 + b^2 = c^2$.
In these examples we *do not need to know the values of the squares*!

To verify that $20^2 + 21^2 = 29^2$. We compute $29^2 - 21^2 = 8 \cdot 50 = 16 \cdot 25 = 20^2$

This tells us that the triangle with sides 20, 21, and 29 is a right triangle!

Exercise Show that the following are Pythagorean Triples: 99-101-200, 60-61-11, 8-15-17, 16-63-65.

It can be shown that all Pythagorean triples are of the form $m^2 - n^2,\ 2\,m\,n,\ m^2 + n^2$ for some pair of integers m and n.
This means that we can use a multiplication table to find Pythagorean Triples!

Pick two entries on the diagonals; these are m^2 and n^2.

The off diagonal entries in the same row and column equal $m \cdot n$.

So the Pythagorean triple $m^2 - n^2,\ 2\,m\,n,\ m^2 + n^2$ consists of the sum and difference of the diagonal entries,
and sum of the two entries in the same row and column.

Example Look at the upper left 2×2 square

1	2
2	4

. The different and sum of the diagonal entries equal 3 and 5 respectively,

and the sum of the off diagonal entries equals 4. We have found the famous 3-4-5 right triangle!

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Can you find the entries that give you the 20-21-29 triangle?
The 5-12-13 triangle? 7-24-25? Find other right triangles.
In each case, use the difference of squares method to verify that you have found a right triangle.

Ideas What triples result from adjacent diagonal entries?
When do triples have common factors? How will you know that your triple has no common factors?

Other Patterns with Squares

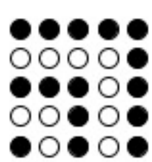
Observe that

$$\begin{aligned} 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ 1 + 3 + 5 + 7 &= 16 \end{aligned}$$

and so on.

In fact, since $2\,n - 1$ equals the n^{th} odd number, we can show that the sum $1 + 3 + 5 + \cdots + (2\,n - 1) = n^2$.

This image provides a hint to the reason for this fact:



Challenge Consider the fractions $\frac{1+3}{5+7},\ \frac{1+3+5}{7+9+11},\ \frac{1+3+5+7}{9+11+13+15},\ \frac{1+3+5+7+9}{11+13+15+17+19},\ \ldots$

In each case reduce the fraction to lowest terms. Can you find an explanation for the result?