LAMC Advanced Circle **October 2, 2016**

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Warm-up

Recall that the numbers $1, 2, 3, \ldots$ are called *natural*.

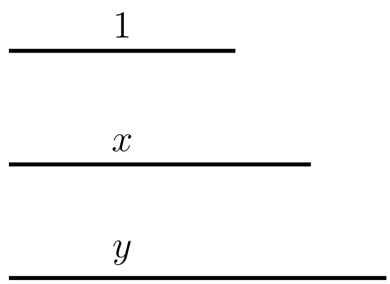
Problem 1 *Consider a natural number with all the digits distinct. If one swaps any two digits of the number different by one, the number increases. If one swaps any two digits of the number different by two, the number decreases. What is the greatest number having this property?*

Problem 2 *N is a natural number with 2016 digits such that its central digits are 2, 0, 1, and 6, in this very order. Can N be a square of an integer? If you think it can, please show an integer M such that $M^2 = N$. If you think it can't, please explain why.*

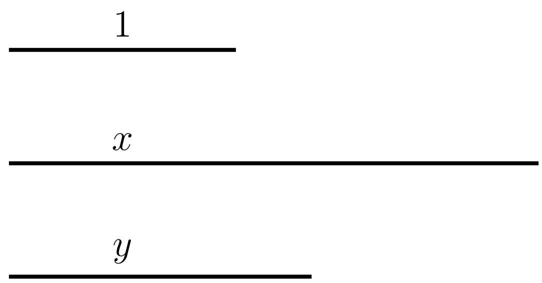
Problem 3 *Use a compass and a ruler to draw a straight line passing through the point and parallel to the straight line on the picture below.*

Geometric multiplication and division

Problem 4 *Given three segments of lengths* 1 *unit,* x *units, and* y *units, use a compass and a ruler to construct a segment of the length* $x \times y$ *units.*

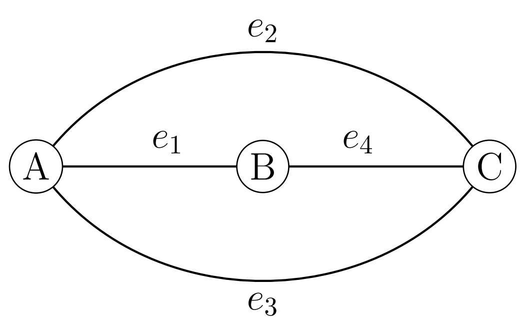


Problem 5 *Given three segments of lengths* 1 *unit,* x *units, and* y *units, use a compass and a ruler to construct a segment of the length* $x \div y$ *units.*

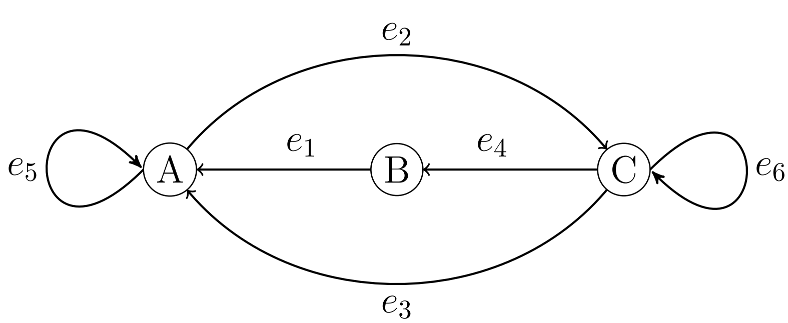


Graphs

A *graph* is a set of vertices, $V = \{v_1, v_2, \ldots\}$, connected by edges, $E = \{e_1, e_2, \ldots\}$. If an edge e connects the vertices v_i and v_j, then we write $e = (v_i, v_j)$. If the order of the vertices does not matter, the graph is called *undirected*. Typically, the word *graph* means an undirected graph. A graph is called a *directed graph*, or a *digraph*, if the order of the vertices does matter. For example, the (undirected) graph below has three vertices, A, B, and C, and four edges, $e_1 = (A, B)$, $e_2 = (A, C)$, $e_3 = (A, C)$, and $e_4 = (B, C)$.



An edge connecting a vertex to itself is called a *loop*. For example, the digraph below has two loops, $e_5 = (A, A)$ and $e_6 = (C, C)$, in addition to the edges $e_1 = (B, A)$, $e_2 = (A, C)$, $e_3 = (C, A)$, and $e_4 = (C, B)$.

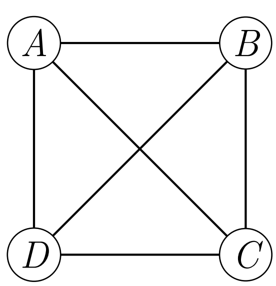


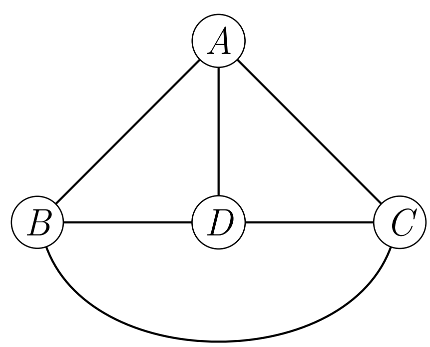
The endpoint of a directed edge e is called its *head* and denoted $h(e)$. The starting point on an edge e is called its *tail* and denoted $t(e)$. For example, $h(e_4) = B$ and $t(e_4) = C$ for the digraph above.

Problem 6 *Draw an undirected graph that has the vertices A, B, C, D, and E and the edges (A, B), (A, C), (A, D), (A, E), (B, C), (C, D), and (D, E) in the space below.*

Two different pictures of a graph can look very dissimilar.

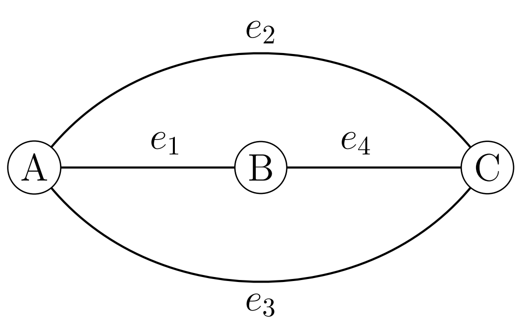
Problem 7 *Prove that the two pictures below represent the same graph by comparing the sets of their vertices and edges.*





Two vertices of a graph are called *adjacent*, if they are connected by an edge. Two edges of a graph are called *incident*, if they share a vertex. Also, a vertex and an edge are called *incident*, if the vertex is one of the two the edge connects.

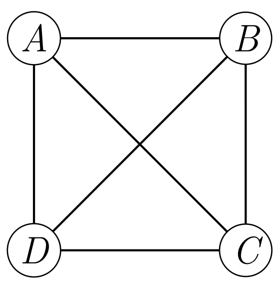
Consider a graph with m vertices, $v_1, v_2, \ldots, v_m$ and n edges, $e_1, e_2, \ldots, e_n$. The *incidence matrix* of the graph is an $m \times n$ matrix T such that $T_{ij} = 1$ in the case v_i is incident to e_j and $T_{ij} = 0$ otherwise. For example, on the right hand side below is the incidence matrix of the first graph on page 5, reproduced for your convenience on the left hand side.

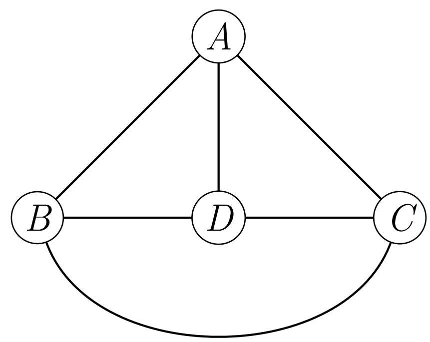


$$T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Problem 8 *Looking at the incidence matrix T above, it is not hard to notice that the sum of the entries in every column equals 2. Would it always be the case (for an undirected graph)? Why or why not?*

Problem 9 *Write down the incidence matrices T_1 and T_2 for the graphs from Problem 7, reproduced for your convenience below. Do the matrices tell you that you are looking at two different pictures of one and the same graph? Why or why not?*





The degree $d(X)$ of a vertex X of a graph is the number of the edges of the graph incident to the vertex.

Theorem 1 *For any graph, the sum of the degrees of the vertices equals twice the number of the edges.*

Problem 10 *Prove Theorem 1.*

Problem 11 *Prove the following corollary of Theorem 1. The number of vertices of odd degree in any graph is even.*

Many students in our class have seen a problem similar to the following.

One girl tells another, "There are 25 kids in my class. Isn't it funny that each of them has 5 friends in the class?" "This cannot be true," immediately replies the other girl. How did she know?

Problem 12 *Let us represent the children in the first girl's class as vertices of a graph. Let us represent the friendships as the graph's edges. What is the degree of each vertex?*

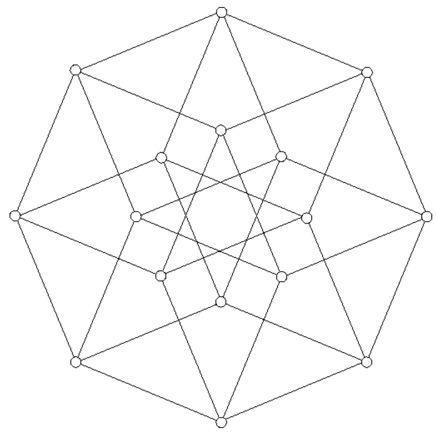
Problem 13 *So how did the second girl know right away?*

A graph is called *connected* when there is a path between every pair of its vertices. A graph is called *disconnected* otherwise.

Problem 14 *Draw a graph with four vertices, all of degree one, in the space below.*

Problem 15 *Draw a hypercube (4D cube a.k.a. tesseract) in the space below. Consider the picture as a graph and write down its incidence matrix.*

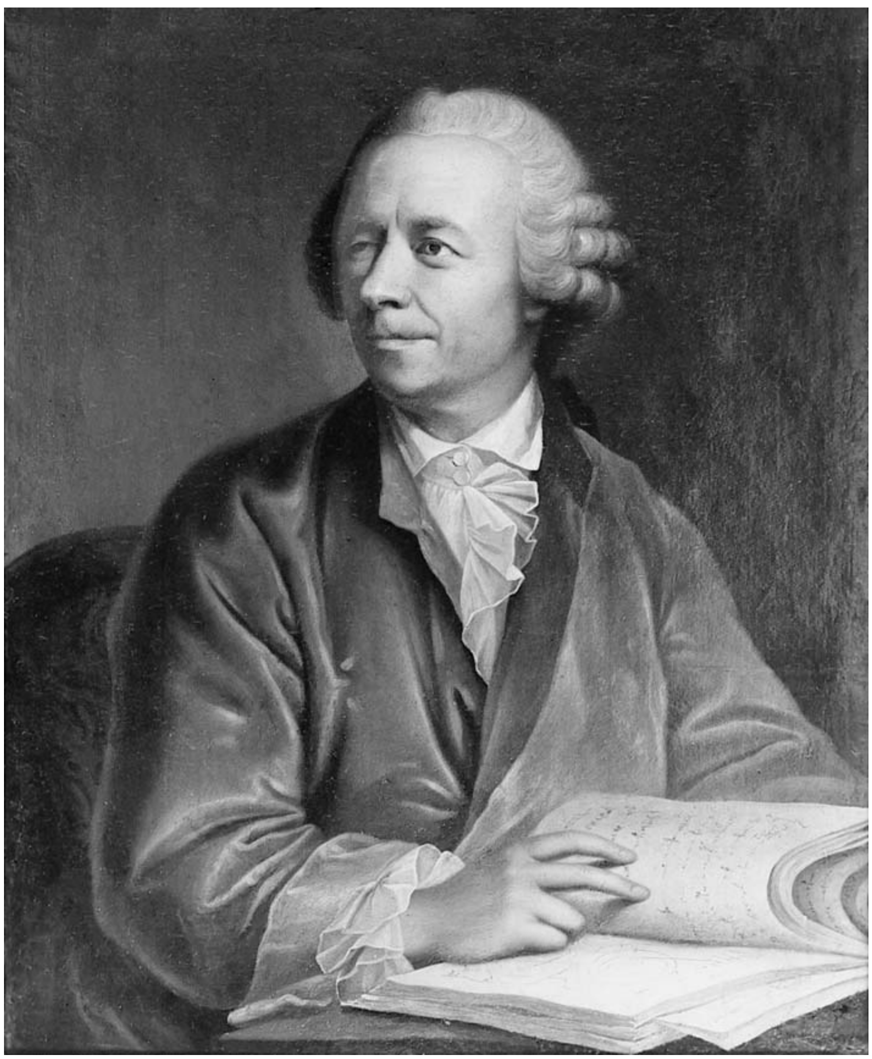
Problem 16 *Prove that the following picture*



is that of a hypercube by writing down the incidence matrix of the corresponding graph and comparing it to the matrix from Problem 15.

Eulerian paths and cycles

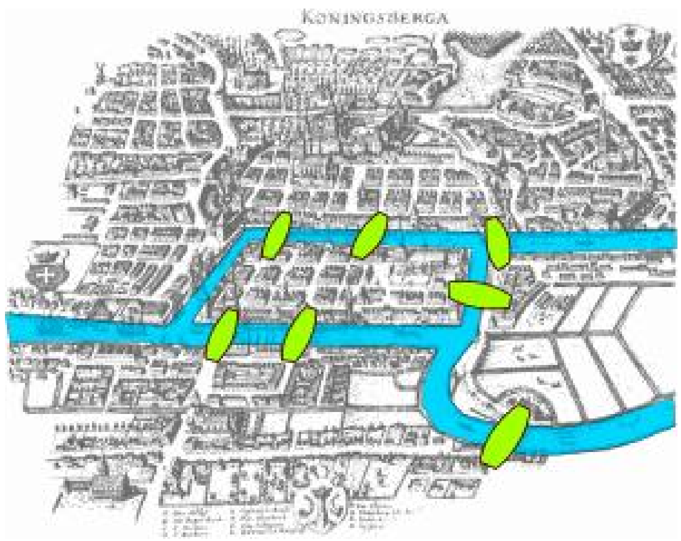
A *Eulerian path* is a path in an (undirected) graph that traverses each edge exactly once. A *Eulerian cycle* is a closed Eulerian path. They are named in honour of a great Swiss mathematician, Leonhard Euler (1707-1783), considered by many as the founding father of the graph theory.



Leonhard Euler

The seven bridges of Königsberg problem

During his stay in the city of Königsberg, then the capital of Prussia, Euler came up with, and solved, the following problem. Can one design a walk that crosses each of the Königsberg's seven bridges once and only once? The picture of Königsberg of Euler's time is provided below.

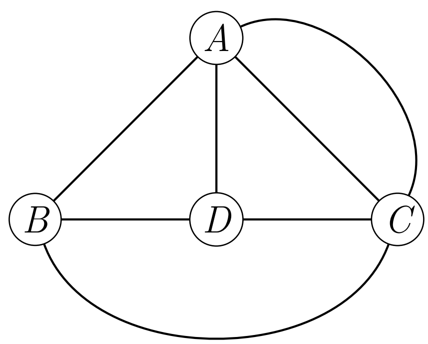


Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges.

Problem 17 *Draw a graph with the vertices corresponding to the landmasses from the picture above and with the edges corresponding to the Königsberg's seven bridges. What are the degrees of each of the graph's vertices?*

Problem 18 *Was it possible to design a Eulerian walk in the city of Königsberg at the time of Euler? Why or why not?*

Problem 19 *Find a Eulerian path in the following graph.*



Problem 20 *Does the above graph contain a Eulerian cycle? Why or why not?*

Hamiltonian paths and cycles

A *Hamiltonian path* is a path in a graph that visits each vertex exactly once. A *Hamiltonian cycle* is a closed Hamiltonian path. They are named in honour of the great Irish mathematician, physicist, and astronomer, Sir William Rowan Hamilton (1805-1865).

Sir William Rowan Hamilton

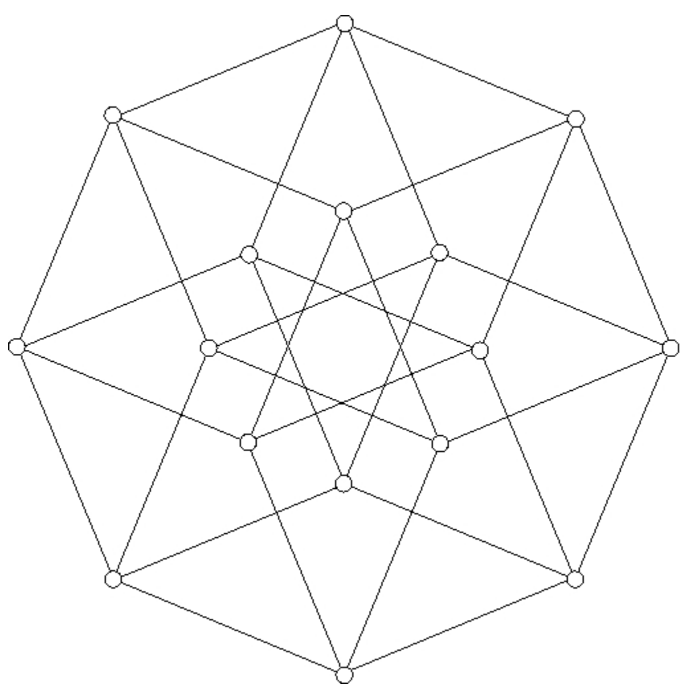
Problem 21 *A graph contains a Hamiltonian cycle. What is the minimal number of its vertices?*

Theorem 2 *(Ore) Let G be a connected graph with $n \geq 3$ vertices. If $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v, then G is Hamiltonian.*

Problem 22

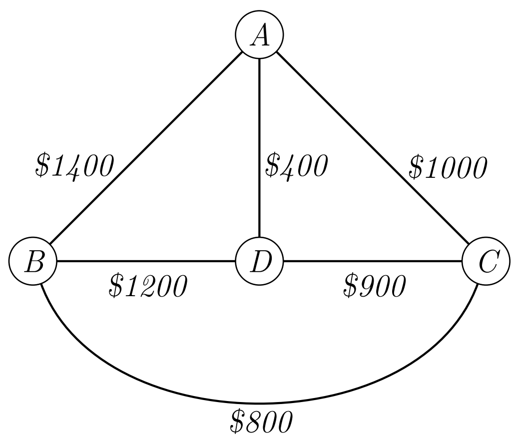
- *Does the graph of the hypercube below satisfy the conditions of Ore's theorem?*

- *Find a Hamiltonian cycle in the hypercube.*



Travelling salesman problem

Problem 23 *A salesman with the home office in Albuquerque has to fly to Boston, Chicago, and Denver, visiting each city once, and then to come back to the home office. The airfare prices, shown on the graph below, do not depend on the direction of the travel. Find the cheapest way.*



Problem 23 is a simple case of the *travelling salesman problem* (TSP). Let G be a graph, directed or undirected, with vertices $v_1, v_2, \ldots, v_n$. Its edges (v_i, v_j) are *weighted* – have numbers assigned to them. The TSP is to find a Hamiltonian cycle of lowest weight. The TSP appears in areas as different as scheduling, microchip design, DNA sequencing, and more.