Problem 1. Let $f(x) = \sqrt{x}$. Using the definition of the derivative prove that

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Solution. The function $f(x)$ is only defined when $x \geq 0$, so we will assume that $x \geq 0$ for the remainder of the solution.

By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}. \end{aligned}$$

We multiply the numerator and denominator by the conjugate of $\sqrt{x+h} - \sqrt{x}$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x}) \cdot (\sqrt{x+h} + \sqrt{x})}{h \cdot (\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}. \end{aligned}$$

Since the square root function is continuous, $\sqrt{x+h} \to \sqrt{x}$ as $h \to 0$, so that as $h \to 0$, $\sqrt{x+h} + \sqrt{x} \to 2\sqrt{x}$. Thus if $x = 0$, the limit in the definition of the derivative does not exist. If $x > 0$, the quotient rule for limits gives us that the limit in the definition of $f'(x)$ is exactly

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

$\square$

Problem 2. Given that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1,$$

prove, using the definition of the derivative, that if $f(x) = \sin x$, then $f'(x) = \cos x$.

Solution. Let $f(x) = \sin x$. By the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}.$$

Using the formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, we find

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\ &= \lim_{h\to 0} \left(\frac{\sin h}{h} \cos x + \frac{\cos h - 1}{h} \sin x \right). \end{aligned}$$

Since $\frac{\sin h}{h} \to 1$ as $h \to 0$, we get (using additivity of limits) that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin h}{h} \cos x + \lim_{h\to 0} \frac{\cos h - 1}{h} \sin x \\ &= \cos x + \lim_{h\to 0} \frac{\cos h - 1}{h} \sin x. \end{aligned}$$

Using the half-angle formula

$$\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos\alpha),$$

we can rewrite this as

$$\begin{aligned} f'(x) &= \cos x + \lim_{h\to 0} \frac{2\sin^2 \frac{h}{2}}{h} \\ &= \cos x + \lim_{h\to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \sin \frac{h}{2}. \end{aligned}$$

Since $\frac{h}{2} \to 0$ as $h \to 0$, we find that $\lim_{h\to 0} \frac{\sin(h/2)}{(h/2)} = \lim_{h\to 0} \frac{\sin h}{h} = 1$. Thus, using multiplicativity of limits, we can write

$$\begin{aligned} f'(x) &= \cos x + \left(\lim_{h\to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \cdot \left(\lim_{h\to 0} \sin \frac{h}{2} \right) \\ &= \cos x + 1 \cdot \lim_{h\to 0} \sin \frac{h}{2}. \end{aligned}$$

Since $\sin \frac{h}{2}$ is a continuous function, its limit at 0 is $\sin 0 = 0$. We therefore conclude that

$$f'(x) = \cos x.$$

$\square$

Problem 3. Find $\frac{dy}{dx}$ given that: (a) $y = 4x^2$; (b) $y = \sin(x) + \sin(2x^2) \cdot \cos(3x^3)$; (c) $y = \frac{\sin(x)+1}{\cos(x)+2}$; (d) $y = \sec(\tan(x+y))$.

Solution. (a)

$$\frac{d(4x^2)}{dx} = \frac{4d(x^2)}{dx} = 4 \cdot 2x = 8x.$$

(b)

$$\begin{aligned}
\frac{d(\sin x + (\sin 2x^2)\cdot(\cos(3x^3))}{dx} &= \frac{d\sin x}{dx} + \frac{d((\sin 2x^2)(\cos(3x^3))}{dx} \\
&= \cos x + \frac{d(\sin 2x^2)}{dx}\cos 3x^3 + \left(\sin 2x^2\right)\frac{d(\cos 3x^3)}{dx} \\
&= \cos x + \cos(2x^2)\frac{d(2x^2)}{dx}\cos 3x^3 \\
&\quad + \left(\sin 2x^2\right)(-\sin(3x^3))\frac{d(3x^3)}{dx} \\
&= \cos x + \cos(2x^2)4x(\cos 3x^3) - \sin(2x^2)\sin(3x^3)9x^2 \\
&= \cos x + 4x\cos(2x^2)\cos(3x^3) - 9x^2\sin(2x^2)\sin(3x^3).
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d(\sin x+1)}{dx}(\cos x+2) - \frac{d(\cos x+2)}{dx}(\sin x+1)}{(\cos x+2)^2} \\
&= \frac{(\cos x)(\cos x+2) - (-\sin x)(\sin x+1)}{(\cos x+2)^2} \\
&= \frac{\cos^2 x + 2\cos x + \sin^2 x + \sin x}{(\cos x+2)^2} \\
&= \frac{1+2\cos x+\sin x}{(\cos x+2)^2}.
\end{aligned}$$

(d) We use implicit differentiation:

$$\begin{aligned}
\frac{dy}{dx} &= \sec'(\tan(x+y))\cdot\tan'(x+y)\cdot\frac{d(x+y)}{dx} \\
&= \sec(\tan(x+y))\tan(\tan(x+y))\cdot\sec^2(x+y)\cdot(1+\frac{dy}{dx}) \\
&= \sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y) \\
&\quad +\frac{dy}{dx}(\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)).
\end{aligned}$$

Thus, by moving the terms involving dy/dx to the left side of the equation, we get

$$\frac{dy}{dx}(1-\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)) = \sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y).$$

Solving this we get

$$\frac{dy}{dx} = \frac{\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)}{1-\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)}.$$

$\square$

Problem 4. Find a point where the curve $y = x^3 + 3x^2 + 3x + 5$ has a horizontal tangent.

Solution. The curve has a horizontal tangent at the points where $\frac{dy}{dx}$ is zero.

We find

$$\frac{dy}{dx} = 3x^2 + 6x + 3 = 3(x^2 + 2x + 1) = 3(x+1)^2.$$

This expression is zero when $x = -1$. The corresponding value of y is $-1+3-3+5 = 4$. Therefore, the curve has a horizontal tanget at the point $(-1, 4)$. □

Problem 5. A trough is 10 feet long and its ends have the shapes of isosceles triangles that are 3ft across at the top and have a height of 1 foot. If the trough is filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 9 inches deep.

Solution. Let $V(t)$ be the volume of the water in the trough at a given time t, and let $h(t)$ be the height of the water level at the same time. Then $V'(t)$ is the rate at which the tank is filled, and $h'(t)$ is the rate at which the water level is rising. Thus we are given $V'(t)$ and we need to find $h'(t)$.

If the water level is $h(t)$, then the amount of water the trough holds is given by its length (10ft) times the area of the part of the end which is covered with water. The shape of this part is an isosceles triangle with height h, which is similar to the given triangle (whose height is 1ft and width is 3ft). Thus the desired area is $h(t)^2$ times the area of the end of the trough, which is $\frac{1}{2}1 \cdot 3 = 3/2$. Thus

$$V(t) = 10 \cdot \frac{3}{2} \cdot h^2(t) = 15h^2(t).$$

Differentiating this equation implicitely in t we get

$$V'(t) = 15\frac{dh^2(t)}{dt} = 15 \cdot 2h(t) \cdot h'(t) = 30h(t)h'(t).$$

Solving for $h'(t)$ gives

$$h'(t) = \frac{V'(t)}{30h(t)}.$$

We are given that at the time we are interested in, $V'(t) = 12$ and $h(t)$ is 9 inches (i.e., $9/12 = 3/4$ of a foot). Thus

$$h'(t) = \frac{12}{30\frac{3}{4}} = \frac{8}{15}.$$

□

Problem 6. Let C be the curve defined by $x^3 + xy + y^3 = 3$ and which goes through the point $(1,1)$. What is the slope fo the tangent line to C at $(1,1)$?

Solution. We implicitely differentiate the quation for C:

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

Thus

$$\frac{dy}{dx}(x + 3y^2) = -3x^2 - y.$$

Thus

$$\frac{dy}{dx} = \frac{-3x^2 - y}{x + 3y^2}.$$

At the given point $x = 1$ and $y = 1$. Substituting these in we get

$$\frac{dy}{dx} = -1.$$

Thus the slope is -1. □

Problem 7. Give an example of a function that is continuous on $I = [-1, 1]$ but not differentiable at at least two points in $(-1, 1)$.

Solution. Let $f(x) = |x + \frac{1}{2}||x - \frac{1}{2}|$. We claim that the function is not differenentiable at $-\frac{1}{2}$ and $\frac{1}{2}$, but is continuous.

The function $f(x)$ is a continuous function, because the functions $|x + \frac{1}{2}|$ and $|x - \frac{1}{2}|$ are continuous, and therefore so is their product, $f(x)$.

To see that $f(x)$ is not differentiable at $\frac{1}{2}$, we compute its derivative at that point using the definition of the derivative:

$$\begin{aligned} f'(\frac{1}{2}) &= \lim_{h\to 0} \frac{|(\frac{1}{2} + h) + \frac{1}{2}||(\frac{1}{2} + h) - \frac{1}{2}| - |\frac{1}{2} + \frac{1}{2}||\frac{1}{2} - \frac{1}{2}|}{h} \\ &= \lim_{h\to 0} \frac{|h+1||h|}{h}. \end{aligned}$$

Since $|x|$ is a continuous function, $\lim_{h\to 0} |h + 1| = |1| = 1$. Using the product rule for limits, we therefore conclude that

$$f'(\frac{1}{2}) = \lim_{h\to 0} |h+1| \cdot \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

But this limit does not exist. Indeed, if $h < 0$, then $|h| = -h$ and so $\frac{|h|}{h} = -1$. If $h > 0$, $\frac{|h|}{h} = 1$. Thus

$$\lim_{h\to 0^-} \frac{|h|}{h} = -1, \qquad \lim_{h\to 0^+} \frac{|h|}{h} = 1.$$

Since the one-sided limits are different, the limit does not exist. Thus $f'(\frac{1}{2})$ does not exist.

Note that

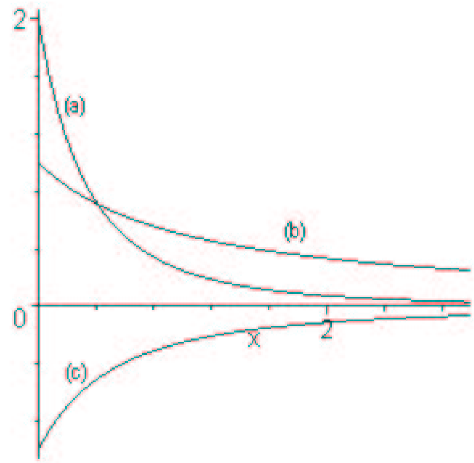
$$f(-x) = |-x - \frac{1}{2}||-x + \frac{1}{2}| = |-(x + \frac{1}{2})||-(x - \frac{1}{2})| = f(x).$$

Moreover,

$$\frac{d f(-x)}{dx} = -f'(x),$$

by the chain rule. Thus if $f'(x)$ exists, then so must $f'(-x)$. We conclude that since $f'(\frac{1}{2})$ does not exist, then $f'(-\frac{1}{2})$ cannot exist, either. Thus $f(x)$ is not differentiable at either of the two points $\pm\frac{1}{2}$. □

Problem 8. A function $f(x)$ and its first two derivatives are graphed below. Label which one is which.



Solution. Let us label the graphs as in the picture and let us call the function $f(x)$.

Suppose that the graph of $f(x)$ is given by (c). Since $f(x)$ is increasing, its derivative is positive, and so its graph could be either (a) or (b). Suppose that (a) is the graph of $f'(x)$, so that the graph of $f''(x)$ is the remaining choice, (b). Since the function with graph (a) is decreasing, its derivative $f''(x)$ must be negative. But this contradicts the assumption that the graph of $f''(x)$ is (b). Similarly, if we assume that the graph of $f'(x)$ is (b), so that the graph of $f''(x)$ is (a), we again arrive at a contradiction, since $f'(x)$ is decresing, so that $f''(x) < 0$, which is not the case with (a).

Suppose next that the graph of $f(x)$ is given by (a). Since $f(x)$ is decreasing, its derivative must be negative, so that the graph of $f'(x)$ is (c). The slope of the graph (a) at $x \approx 0.4$ appears to be -1, since the line with this slope is tangent to (a). Thus $f'(0.4) \approx -1$. However, we see that (c) is approximately -0.4 at $x \approx 0.4$, which is a contradiction.

Suppose lastly that the graph of $f(x)$ is given by (b). Since $f(x)$ is decreasing, its derivative must be (c) and its second derivative must be (a). This is the only remaining choice and must therefore be the answer. $\square$