Problem 1. Approximate $\sqrt{\dfrac{5}{2}}$ using the linear approximation of the function $f(x) = \sqrt{1+3x}$ at $a = 0$ and $a = 1$. Given that $\sqrt{\dfrac{5}{2}} \approx 1.58$, which of the two approximations is better?

SOLUTION: First, notice that $\sqrt{\frac{5}{2}} = f(\frac{1}{2})$. Compute the derivative:

$$f'(x) = \frac{3}{2}\frac{1}{\sqrt{1+3x}}$$

Since $f(0) = 1$, $f'(0) = \frac{3}{2}$, the linear approximation at 0 is

$$L_0(x) = 1 + \frac{3}{2}x,$$

which for $x = \frac{1}{2}$ gives

$$L_0(\frac{1}{2}) = 1\frac{3}{4} = 1.75.$$

Similarly, since $f(1) = 2$, $f'(1) = \frac{3}{4}$, we have

$$L_1(x) = 2 + \frac{3}{4}(x-1),$$

which for $x = \frac{1}{2}$ gives

$$L_1(\frac{1}{2}) = 1\frac{5}{8} = 1.625$$

Since $L_0(1/2) - 1.58 = 0.17$ is bigger then $L_1(1/2) - 1.58$, the approximation at 1 gives better result.

Problem 2. True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive 4 points for a correct answer, 0 points if there is no answer, and -4 points if the answer is wrong.

1. If $f(x)$ is continuous on $[a, b]$, it is differentiable on (a, b)	False
2. If $f''(a) = 0$, then $f(x)$ can not have a local minimum at a	False
3. The limit $\lim_{x\to\infty} \frac{\sin^2 x + 3}{\sqrt{x} + 3}$ does not exist	False
4. The graph of $y = \tan x$ is concave down on $(\pi/2, \pi)$	True
5. If $f'(x)$ is increasing, then $f(x)$ is concave up	True

Problem 3. Find the minimal and maximal values of the function $f(x) = \cos 2x - 2\cos x$ on the interval $[\pi/4,\ 3\pi/4]$.

SOLUTION: Since $f(x)$ is continuous, it attains its maximum and minimum on the given interval.

$$f'(x) = -2\sin 2x + 2\sin x = -2\cdot 2\sin x\cos x + 2\sin x = -2\sin x(2\cos x - 1)$$

$f'(x) = 0 \Leftrightarrow \quad \sin x = 0 \text{ or } \cos x = 1/2 \quad \Rightarrow \quad$ on $[\pi/4,\ 3\pi/4$, the only solution is $x = \pi/3$.

Since at $x = \pi/3$ the derivative changes sign from negative to positive, $\pi/3$ is a local minimum.

$$f(\pi/3) = \cos 2\pi/3 - 2\cos\pi/3 = -\frac{3}{2}$$

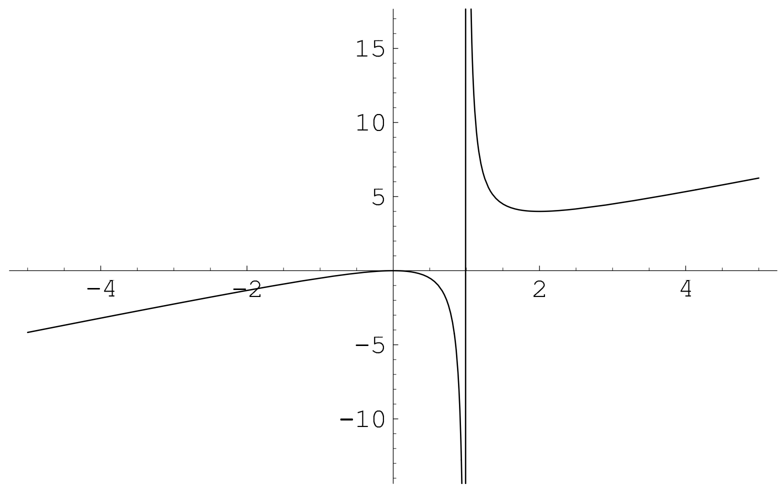
Compute the values at the end points of the interval:

$$f(\pi/4) = \cos\pi/2 - 2\cos\pi/4 = 0 - 2\cdot\sqrt{2}/2 = -\sqrt{2}$$
$$f(3\pi/4) = \cos 6\pi/4 - 2\cos 3\pi/4 = 0 - 2\cdot(-\sqrt{2}/2) = \sqrt{2}$$

Comparing the values at $\pi/3$, $\pi/4$ and $3\pi/4$, we obtain that the minimal value is $f(\pi/3) = -3/2$ and the maximal value is $f(3\pi/4) = \sqrt{2}$.

Problem 4. Let $f(x) = \dfrac{x^2}{x-1}$. Compute the first and second derivatives of f and sketch the graph of f, indicating all properties of the function, such as asymptotes, minima, maxima, convexity, points of inflection and intercepts.

Solution:



$f(x)$ is defined on the following domain: $(-\infty, 1) \cup (1, \infty)$. Since $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} = -\infty$, the line $x = 1$ is a vertical asymptote. Since

$$f(x) = \frac{x^2 - 1 + 1}{x - 1} = (x - 1) + \frac{1}{x - 1},$$

and, therefore, the limit $\lim_{x\to\pm\infty}(f(x) - (x - 1)) = \lim_{x\to\pm\infty} \frac{1}{x-1}$ is zero, the line $y = x - 1$ is a slant asymptote.

$$f'(x) = 1 - \frac{1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$f'(x) = 0 \quad \Leftrightarrow \quad x = 0, x = 2$$

Since at $x = 0$ the derivative changes sign from $-$ to $+$, $x(0) = 0$ is a local maximum. Since at $x = 2$ the derivative changes sign from $+$ to $-$. $x(2) = 4$ is a local minimum.

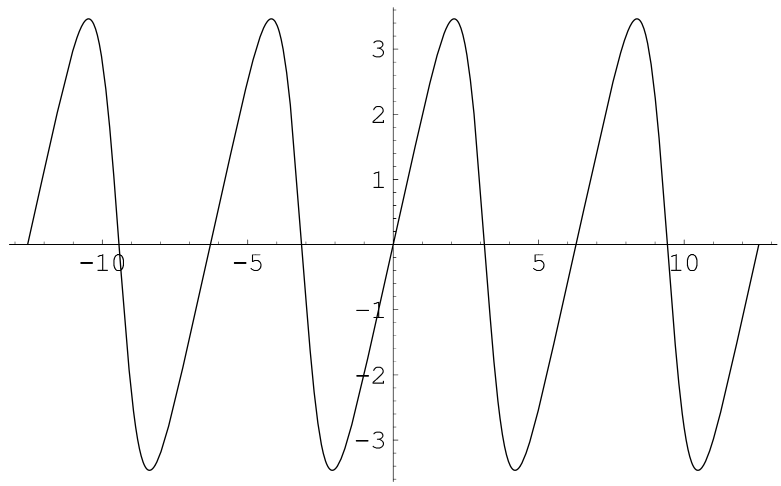
$$f''(x) = \frac{2}{(x-1)^3}$$

Since on $(-\infty, 1)$ $f''(x) < 0$, on this interval the curve $f(x)$ is concave down. Since on $(1, \infty)$ $f''(x) > 0$, on this interval the curve $f(x)$ is concave up.

We can now sketch the graph of $f(x)$:

Problem 5. Let $f(x) = \dfrac{6\sin x}{2+\cos x}$. Sketch the graph of this function, indicating all properties of the function, such as asymptotes, minima, maxima, convexity, points of inflection and intercepts.

SOLUTION:



Since $\sin x$ and $\cos x$ are periodic functions with a period 2π, and 2 is a constant, $f(x)$ is periodic with a period 2π. We will consider $f(x)$ on the interval $[-\pi, \pi]$ and sketch its graph on this interval, and then extend it periodically.

$f(0) = 0$, $f(-\pi) = 0 = f(\pi)$.

$$f'(x) = \frac{6\cos x(2+\cos x) + \sin x \cdot 6\sin x}{(2+\cos x)^2} = \frac{12\cos x + 6}{(2+\cos x)^2} = 6\frac{2\cos x + 1}{(2+\cos x)^2}$$

$$f'(x) = 0 \;\Leftrightarrow\; \cos x = -\frac{1}{2} \;\Leftrightarrow\; x = 2\pi/3 \,\text{and}\, x = -2\pi/3$$

Since at $x = 2\pi/3$ the derivative changes the sign from positive to negative, this is a point of local maximum, $f(2\pi/3) = \frac{6\cdot\sqrt{3}/2}{2-1/2} = 2\sqrt{3}$. Since at $x = -2\pi/3$ the sign of the derivative changes from negative to positive, this is a point of local minimum, $f(-2\pi/3) = -2\sqrt{3}$.

$$f''(x) = 6\frac{-2\sin x(2+\cos x)^2 + 2(2+\cos x)\sin x(2\cos x + 1)}{(2+\cos x)2^4} = -\frac{12\sin x(1-\cos x)}{(2+\cos x)^3}$$

$$f''(x) = 0 \;\Leftrightarrow\; \sin x = 0 \,\text{or}\, \cos x = 1 \;\Leftrightarrow\; x = -\pi,\ 0,\ \pi$$

Since the second derivative changes sign from positive to negative at $x = 0$, $f(x)$ is concave up on $(-\pi, 0)$ and concave down on $(0, \pi)$.

We can now sketch the graph of $f(x)$ on $[-\pi, \pi]$ and then periodically extend it:

Problem 6. Prove that the equation $x^5 + 20x + \cos x = 0$ has exactly one solution.

SOLUTION: Let $f(x) = x^5 + 20x + \cos x$. This function is continuous and differentiable everywhere.

Consider $f(x)$ on the interval $[-\pi/2, 0]$. Since $f(-\frac{\pi}{2}) = -\frac{\pi^5}{64} - 10\pi + 0 = -\pi \cdot (\frac{\pi^4}{64} + 10) < 0$ and $f(0) = 1 > 0$, by Intermediate Value Theorem, there is a point $a \in [-\pi/2, 0]$ such that $f(a) = 0$. Therefore, there is at least one root.

Suppose that there are two roots, a and b. By the Mean Value Theorem applied to the interval $[a, b]$, there is a point c on the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. However,

$$f'(x) = 5x^4 + 20 - \sin x$$

and since $5x^4 > 0$ and $20 - \sin x > 0$ (since $|\sin x| \leq 1$), $f'(x) > 0$. We obtained a contradiction. Therefore, there can't be two roots of this equation.