MATH 31A 2ND PRACTICE MIDTERM

Problem 1. For each of the following functions, compute its derivative and then use linear approximation around the given point a to estimate the value of the function.
(a) $f(x) = \sin x$, $a = \pi$, $f(\frac{5\pi}{4})$;
(b) $f(x) = \sqrt{1+\sqrt{1+x}}$, $a = 0$, $f(0.1)$;

SOLUTION. (a) $f'(x) = \cos x$. Since $f(\pi) = 0$ and $f'(\pi) = \cos(\pi) = -1$, the linear approximation at $a = \pi$ is given by $L(x) = -1(x-\pi)$. Therefore, $f(5\pi/4) \simeq L(5\pi/4) = -\pi/4$;
(b) We have

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{1+\sqrt{1+x}}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1+x}} = \frac{1}{4} \cdot \frac{1}{\sqrt{1+\sqrt{1+x}} \cdot \sqrt{1+x}};$$

Since $f(0) = \sqrt{2}$ and $f'(0) = \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{4\sqrt{2}}$, the linear approximation at $a = 0$ is given by $L(x) = \sqrt{2} + \frac{1}{4\sqrt{2}}x$. Therefore, $f(0.1) \simeq L(0.1) = \sqrt{2} + \frac{0.1}{4\sqrt{2}} = \frac{81}{80}\sqrt{2}$;

Problem 2. True or False. For each of the following statements, indicate if it is true or false (you may assume that f is everywhere differentiable for all of the questions below):

1. If $f'(x) > 0$ for all x, then $f(x)$ is increasing	True
2. If $f'(a) = 0$, then f attains either a maximum or a minimum at a	False
3. If f is concave up, then f' is increasing	True
4. The function $\sqrt{x^2+1}$ is concave down	False
5. The function $\sin x$ has an inflection point at π.	True

Comments:

1. The statement is true by the first derivative test.
2. If $f'(a) = 0$, $f(x)$ can have an inflection point at a, which is neither a local minimum nor a local maximum. For example, take $f(x) = x^3$ and $a = 0$.
3. If f is concave up (on some interval), then $f'' > 0$ on this interval. But $f''(x) = (f'(x))'$. Therefore, $(f'(x))' > 0$, therefore, $f'(x)$ is increasing (by the first derivative test applied to $f'(x)$).

4. Compute the derivatives: $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$;

$$f''(x) = \frac{\sqrt{x^2+1} - \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}x}{x^2+1} = \frac{(x^2+1) - x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0 \qquad \text{for all } x$$

Therefore, the function is concave up.

5. Either recall the graph of $\sin(x)$, or compute the second derivative: $(\sin(x))' = \cos x$, $(\sin x)'' = -\sin x$. Since $\sin\pi = 0$, π is an inflection point.

Problem 3. Let $f(x) = x^3 - 3x + 7$. Find the minimum and maximum values of f in the interval $[0, 2]$.

Solution: Compute the derivative: $f'(x) = 3x^2 - 3$. $f'(x) = 0$ for $x = -1$ and $x = 1$. Only $x = 1$ is on the interval $[0, 2]$. Since at $x = 1$ the derivative changes sign from negative to positive, $x = 1$ is a local minimum. The value of the function at 1 is $f(1) = 5$.

Compute the values of $f(x)$ at the end points of $[0, 2]$. We have $f(0) = 7$ and $f(2) = 9$.

Comparing all of the values we computed, we obtain that on the interval $[0, 2]$, $f(1) = 5$ is the minimum of $f(x)$ and $f(2) = 9$ is its maximum.

Problem 4. Let $f(x) = \frac{1-2x^2}{1-x^2}$. Sketch the graph of f, indicating all properties of the function, such as asymptotes, extreme points, minima, maxima, convexity, points of inflection and intersepts.

Solution: The domain of $f(x)$ is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

1). The intercept points:

- $f(0) = 1$;
- $f(x) = 0$ when $x = \pm\frac{1}{\sqrt{2}}$.

2). First, notice that

$$f(x) = \frac{1-2x^2}{1-x^2} = \frac{1-x^2-x^2}{1-x^2} = 1 - \frac{x^2}{1-x^2};$$

Compute the derivative:

$$f'(x) = -\frac{2x(1-x^2) - (-2x)x^2}{(1-x^2)^2} = -\frac{2x}{(1-x^2)^2};$$

Since $f'(x) > 0$ for $x < 0$, on this interval the function is increasing. Since $f'(x) < 0$ for $x > 0$, on this interval the function is decreasing. Since $f'(0) = 0$ and the derivative changes sign from positive to negative, $f(x)$ has a local maximum at $x = 0$. The value at local maximum is 1.

3). Compute the second derivative:

$$f''(x) = -2\frac{(1-x^2)^2 - 2(1-x^2)\cdot(-2x)\cdot x}{(1-x^2)^4}$$
$$= -2\frac{(1-x^2)(1-x^2+4x^2)}{(1-x^2)^4} = -2\frac{(1+3x^2)}{(1-x^2)^3}$$

For $x < -1$ and $x > 1$ $f''(x) < 0$ and the graph of the function is concave up. For $x \in (-1, 1)$ $f''(x) < 0$ and the graph of the function is concave down. There are no points of inflection.

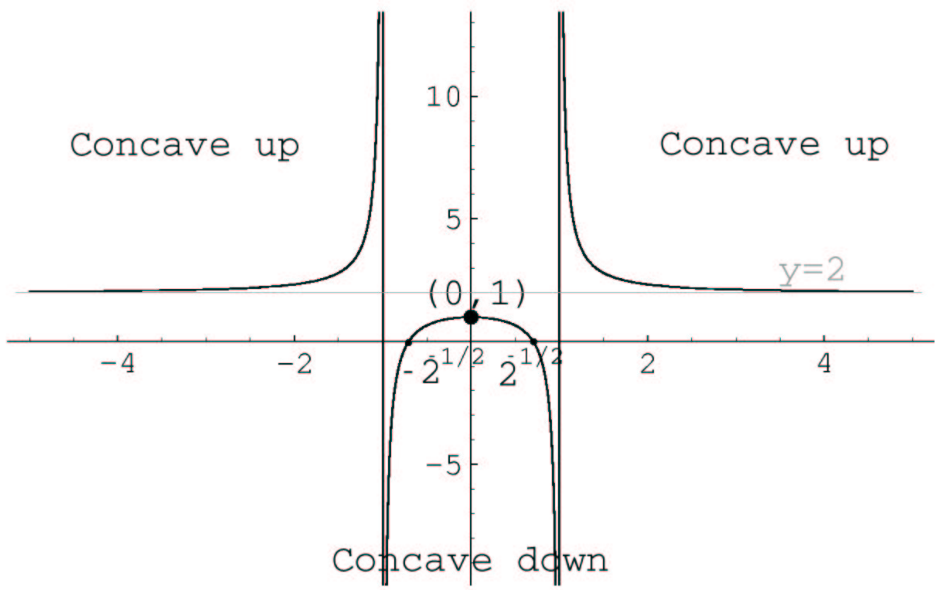
4). Since $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$, $x = 1$ is a vertical asymptote. Since $\lim_{x\to 1^-} f(x) = +\infty$ and $\lim_{x\to -1^+} f(x) = -\infty$, $x = -1$ is a vertical asymptote.

5). To find a horizontal asymptote, compute the limit

$$\lim_{x\to\infty}\frac{1-2x^2}{1-x^2}=\lim_{x\to\infty}\frac{\frac{1}{x^2}-2}{\frac{1}{x^2}-1}=1$$

Therefore, $y=1$ is the horizontal asymptote.

6). We can now draw the graph of $f(x)$:



Problem 5. Let $y=\sin 2x-2\sin x$. Sketch the graph of this function, indicating all properties of the function.

SOLUTION: 1). Since the period of the function $\sin 2x$ is π and the period of the function $\sin x$ is 2π, the function $y=\sin 2x-2\sin x$ is periodic with period 2π. It is also convenient to notice that $y=2\sin 2x-2\sin x=2\sin x\cos x-2\sin x=2\sin x(\cos x-1)$.

2). The intercept points:

- $y(0)=0$.
- $y(x)=0$ is equivalent to $\sin x(\cos x-1)=0$, i.e., either $\sin x=0$ or $\cos x=1$. The first equation has the solutions $x=\pi n$ for any integer n, and the second has the solution $x=2\pi n$ for all n. Therefore, $y(x)=0$ at the points $y=\pi n$ for all n.

3). $y'=2\cos 2x-2\cos x$. We have $y'=0$ iff $\cos 2x=\cos x$. Since $\cos 2x=2\cos^2 x-1$, $\cos 2x=\cos x$ is equivalent to $4\cos^2 x-2\cos x-2=0$. This is a quadratic equation with respect to $\cos x$. Solving it, we obtain $\cos x=\frac{1\pm\sqrt{9}}{4}$, that is $\cos x=-\frac{1}{2}$ or $\cos x=1$. The first of these has the following solutions:

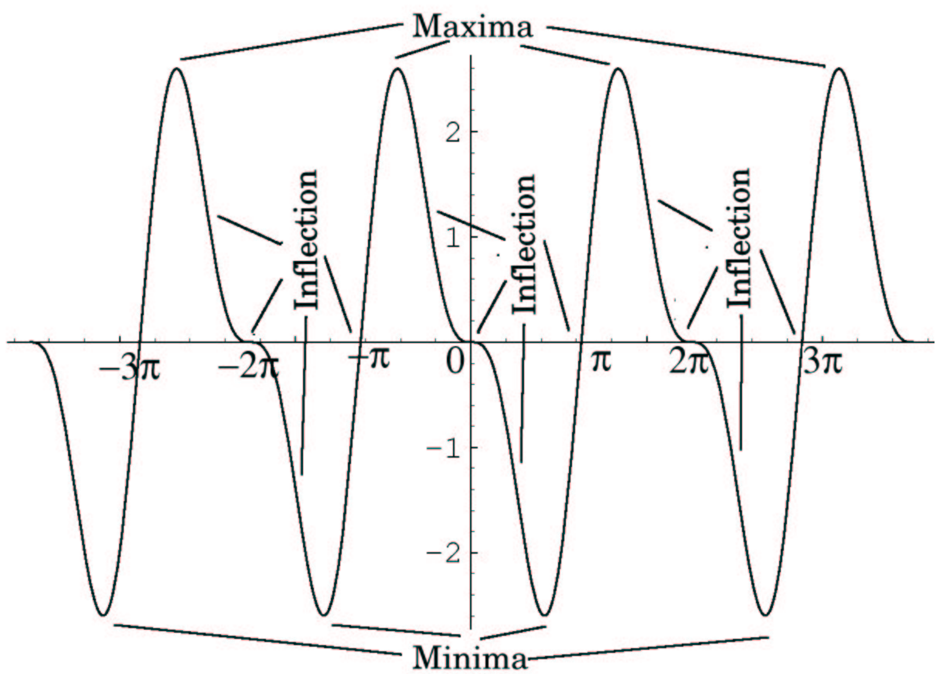
$x_1 = 2\pi/3 + 2\pi n$ and $x_2 = 4\pi/3 + 2\pi n$, where n is any integer. The second has the solutions: $x_3 = 2\pi n$ where n is any integer.

Since at $x_1 = 2\pi/3 + 2\pi n$ the derivative changes sign from $-$ to $+$, these are the points of local minima. At $x_2 = 4\pi/3 + 2\pi n$ the derivative changes sign from $+$ to -; therefore, x_2 are the points of local maxima. The values of the function at these points are $y(x_1) = -\frac{3\sqrt{3}}{2}$ and $y(x_2) = \frac{3\sqrt{3}}{2}$.

Near $x_3 = 2\pi n$ the derivative is negative. Since it does not change sign, these are not extremal points.

4). $y'' = -4\sin 2x + 2\sin x = -8\sin x\cos x + 2\sin x = 2\sin x(-4\cos x + 1)$. We have $y'' = 0$ iff $\sin x = 0$ or $\cos x = \frac{1}{4}$. The first equation has the solutions $x_5 = \pi n$ and the second one has the solutions $x_{6,7} = \pm\arccos(1/4) + 2\pi n$ for all integers n. These are the points of inflection. Since at $(x_5)_1 = 2\pi n$ the second derivative changes sign from $+$ to $-$, the concavity changes from upward to downward. Since at $(x_5)_2 = \pi + 2\pi n$ the second derivative changes the sign from $+$ to $-$, the concavity changes from upward to downward. At $x_6 = \arccos(1/4) + 2\pi n$ the second derivative changes the sign from $-$ to $+$, therefore, the concavity changes from down to up. At $x_7 = -\arccos(1/4) + 2\pi n$ the second derivative changes sign from $-$ to $+$, which implies that the function changes concavity from down to up.

5). We can now draw the graph of this function.



Problem 6. Let f be a differentiable function. Prove that if the equation $f(x) = x$ has more than one solution, then there must be a point c at which $f'(c) = 1$.

PROOF: By assumption, the equation $f(x) = x$ has at least two distinct solutions. Suppose that a and b are two solutions, $a \neq b$. Then $f(a) = a$ and $f(b) = b$. By the Mean Value Theorem applied on the interval $[a, b]$, there exists a point

$c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b} = \frac{a - b}{a - b} = 1.$$

Problem 7. Compute the limits:

1. $\lim_{x \to \infty} \frac{x^4 + 9x^3 + \pi x^2 - 17x + 106}{3x^4 - 16x^3 - 149} = \frac{1}{3}$ by dividing both the numerator and denominator by x^4.
2. $\lim_{x \to \infty} \frac{5x^3 + 55x^2 + 555x + 5555}{x^8 - .003} = 0$ since the degree of the numerator is less then the degree of the denominator.
3. $\lim_{x \to \infty} \sqrt{x^4 + 2x + 4} - \sqrt{x^4 - 2x - 4}$:

$$\lim_{x \to \infty} \sqrt{x^4 + 2x + 4} - \sqrt{x^4 - 2x - 4} =$$

$$\lim_{x \to \infty} \frac{(\sqrt{x^4 + 2x + 4} - \sqrt{x^4 - 2x - 4})(\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4})}{\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4}} =$$

$$\lim_{x \to \infty} \frac{x^4 + 2x + 4 - x^4 + 2x + 4}{\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4}} = \lim_{x \to \infty} \frac{4(x + 2)}{\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4}} =$$

$$\lim_{x \to \infty} \frac{4(\frac{1}{x} + \frac{2}{x^2})}{\sqrt{1 + \frac{2}{x^3} + \frac{4}{x^4}} + \sqrt{1 - \frac{2}{x^3} - \frac{4}{x^4}}} = 0$$

4. $\lim_{x \to \infty} \frac{\sin^2 x}{x^2}$: First, notice that $0 \leq \sin^2 x \leq 1$. Therefore, $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$. Since $\lim_{x \to \infty} \frac{1}{x^2}$, by the Squeeze Thereom, it follows that $\lim_{x \to \infty} \frac{\sin x^2}{x^2} = 0$.