

1. Let $f(x) = x^4$. Using the definition of the derivative, prove that $f'(x) = 4x^3$.

Proof: Let a be a fixed number. Then

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} =$$

$$\lim_{x \rightarrow a} \frac{x^4 - a^4}{x - a} =$$

$$\lim_{x \rightarrow a} \frac{(x - a)(x^3 + ax^2 + a^2x + a^3)}{x - a} =$$

$$\lim_{x \rightarrow a} (x^3 + ax^2 + a^2x + a^3) =$$

$$a^3 + aa^2 + a^2a + a^3 =$$

$$4a^3$$

2. Let $f(x) = \cos(x)$. You are given that

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1.$$

(a) (4 points) Prove that

$$\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = 0.$$

Proof: By the given limit

$$\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = \lim_{h \rightarrow 0} \frac{(1 - \cos(h))(1 + \cos(h))}{h(1 + \cos(h))} =$$

$$\lim_{h \rightarrow 0} \frac{1 - \cos^2(h)}{h} = \lim_{h \rightarrow 0} \frac{\sin^2(h)}{h} = \lim_{h \rightarrow 0} \sin(h) * \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 0 * 1 = 0$$

(b) (16 points) Using the definition of the derivative, prove that if $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$.

Proof: Using the given limit, the identity

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B),$$

and part (a) we have

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{-\sin(x)\sin(h) - \cos(x)(1 - \cos(h))}{h} =$$

$$-\sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} - \cos(x) \lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = -\sin(x) - \cos(x) * 0 =$$

$$-\sin(x)$$

3. Using the usual rules of differentiation find dy/dx for

a. $y = \cos(x^2)$

$$\frac{dy}{dx} = -\sin(x^2)2x$$

b. $y = (\sin(x) + \cos(2x))^3$

$$\frac{dy}{dx} = 3(\sin(x) + \cos(2x))^2(\cos(x) - 2\sin(2x))$$

c. $y = \sec(\cos(\sin(x)))$

$$\frac{dy}{dx} = \sec(\cos(\sin(x))) \tan(\cos(\sin(x)))(-1)\sin(\sin(x))(\cos(x))$$

d. $y^3 + xy^2 + \cos(y) = \sin(x)$

$$3y^2y'(x) + y^2 + x2yy'(x) - \sin(y)y'(x) = \cos(x)$$

\Rightarrow

$$(3y^2 + 2xy - \sin(y))y'(x) = \cos(x) - y^2$$

\Rightarrow

$$y'(x) = \frac{\cos(x) - y^2}{3y^2 + 2xy - \sin(y)}$$

4. What is the smallest positive value of x for which $f(x) = \cos(2x) - 2\cos(x)$ has a horizontal tangent line?

Solution: We are looking for the smallest positive x for which $f'(x) = 0$. Now

$$f(x) = \cos(2x) - 2\cos(x) \Rightarrow f'(x) = -2\sin(2x) + 2\sin(x) = 0 \Rightarrow$$

$$-\sin(2x) + \sin(x) = 0 \Rightarrow -2\sin(x)\cos(x) + \sin(x) = 0 \Rightarrow$$

$$\sin(x)(-2\cos(x) + 1) = 0 \Rightarrow \sin(x) = 0 \text{ or } -2\cos(x) + 1 = 0$$

Then

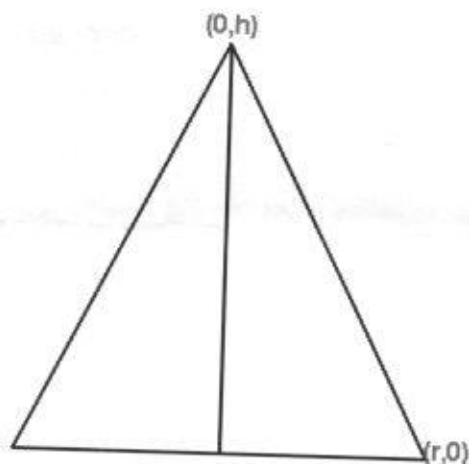
$$\sin(x) = 0 \Rightarrow x = 0, \pi$$

$$-2\cos(x) + 1 = 0 \Rightarrow \cos(x) = 1/2 \Rightarrow x = \pi/3,$$

Consequently, the smallest positive x such that $f'(x) = 0$ is $x = \pi/3$.

5. Gravel is being pumped from a conveyor belt at the rate of $30 \text{ ft}^3/\text{min}$, and it forms a pile in the form of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 feet high?

Solution: A cross-section of the conical pile is sketched below:



If V is the volume of the cone then $V = (\pi/3)r^2h$. In addition we are given that $r = h/2$ so

$$V(t) = (\pi/3)(h(t)/2)^2h(t) = (\pi/12)h^3(t)$$

We are also given that $V'(t) = 30$, so

$$V'(t) = 3(\pi/12)h^2(t)h'(t) = 30.$$

Thus, when $h(t) = 10$

$$(\pi/4)(10)^2 h'(t) = 30$$

That is, the height is increasing at a rate given by

$$h'(t) = (1.2/\pi) \text{ ft/min.}$$

6. Let y be the function of x defined by the curve

$$x^4 - xy^3 + y^4 = 9.$$

The point $(1,2)$ is on this curve. What is the area of the triangle formed by the tangent line to the curve at $(1,2)$, the x -axis, and the y -axis?

Solution: Differentiating implicitly, we get

$$(4y^3 - 3xy^2) (dy/dx) = y^3 - 4x^3.$$

Setting $x = 1$, $y = 2$ gives

$$(32-12) (dy/dx) = (8-4).$$

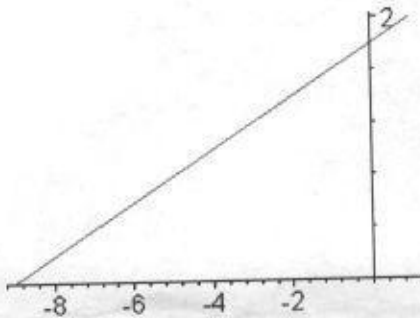
That is $dy/dx = 1/5$, so the equation of the tangent line is

$$(y-2)/(x-1) = 1/5$$

or

$$y = 2 + (1/5)(x-1)$$

This line intersects the y -axis when $x = 0$, $y = 9/5$. It intersects the x -axis when $y = 0$, $x = -9$:



Consequently, the area bounded by the tangent line, the x -axis, and the y -axis is equal to $(1/2) * 9 * (9/5) = 8.1$