1. Let $f(x) = x^4$. Using the definition of the derivative, prove that $f'(x) = 4x^3$.

Proof: Let a be a fixed number. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} =$$

$$\lim_{x \to a} \frac{x^4 - a^4}{x - a} =$$

$$\lim_{x \to a} \frac{(x - a)(x^3 + ax^2 + a^2x + a^3)}{x - a} =$$

$$\lim_{x \to a}(x^3 + ax^2 + a^2x + a^3) =$$

$$a^3 + aa^2 + a^2a + a^3 =$$

$$4a^3$$

2. Let f(x) = cos(x). You are given that

$$\lim_{h\to 0}\frac{\sin(h)}{h}=1.$$

(a) (4 points) Prove that

$$\lim_{h\to 0}\frac{1-\cos(h)}{h}=0.$$

Proof: By the given limit

$$\lim_{h\to 0}\frac{1-\cos(h)}{h}= \lim_{h\to 0}\frac{(1-\cos(h))}{h}\frac{(1+\cos(h))}{(1+\cos(h))}=$$

$$\lim_{h\to 0}\frac{1-\cos^2(h)}{h}=\lim_{h\to 0}\frac{\sin^2(h)}{h}=\lim_{h\to 0}\sin(h)*\lim_{h\to 0}\frac{\sin(h)}{h}=0*1=0$$

(b) (16 points) Using the definition of the derivative, prove that if f(x) = cos(x) then f'(x) = -sin(x).

Proof: Using the given limit, the identity

cos(A+B) = cos(A)cos(B)-sin(A)sin(B),

and part (a) we have

$$f'(x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}=$$

$$\lim_{h\to 0}\frac{-\sin(x)\sin(h)-\cos(x)(1-\cos(h))}{h}=$$

$$-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}-\cos(x)\lim_{h\to 0}\frac{1-\cos(h)}{h}=-\sin(x)-\cos(x)*0=$$

$$-\sin(x)$$

3. Using the usual rules of differentiation find dy/dx for

a. y = cos(x^2)

$$\frac{dy}{dx} = -\sin(x^2)2x$$

b. y = (sin(x) + cos(2x))^3

$$\frac{dy}{dx} = 3(\sin(x)+\cos(2x))^2(\cos(x)-2\sin(2x))$$

c. y = sec(cos(sin(x)))

$$\frac{dy}{dx} = \sec(\cos(\sin(x))\tan(\cos(\sin(x))(-1)\sin(\sin(x))(\cos(x))$$

d. $y^3 + xy^2 + \cos(y) = \sin(x)$

$$3y^2y'(x) + y^2 + x2yy'(x) - \sin(y)y'(x) = \cos(x)$$

$$\Rightarrow$$

$$(3y^2 + 2xy - \sin(y))y'(x) = \cos(x) - y^2$$

$$\Rightarrow$$

$$y'(x) = \frac{\cos(x) - y^2}{3y^2 + 2xy - \sin(y)}$$

4. What is the smallest positive value of x for which

f(x) = cos(2x) - 2cos(x) has a horizontal tangent line?

Solution: We are looking for the smallest positive x for which f'(x) = 0. Now

$$f(x) = \cos(2x) - 2\cos(x) \Rightarrow f'(x) = -2\sin(2x) + 2\sin(x) = 0 \Rightarrow$$

$$-\sin(2x) + \sin(x) = 0 \Rightarrow -2\sin(x)\cos(x) + \sin(x) = 0 \Rightarrow$$

$$\sin(x)(-2\cos(x) + 1) = 0 \Rightarrow \sin(x) = 0 \text{ or } -2\cos(x) + 1 = 0$$

Then

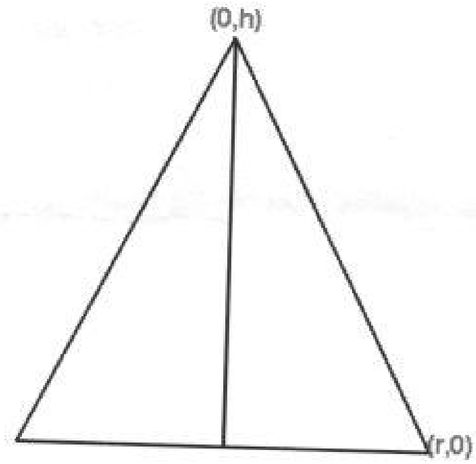
$$\sin(x) = 0 \Rightarrow x = 0,\ \pi$$

$$-2\cos(x) + 1 = 0 \Rightarrow \cos(x) = 1/2 \Rightarrow x = \pi/3,$$

Consequently, the smallest positive x such that f'(x) = 0 is x = π /3.

5. Gravel is being pumped from a conveyor belt at the rate of 30 ft^3/min, and it forms a pile in the form of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 feet high?

Solution: A cross-section of the conical pile is sketched below:



If V is the volume of the cone then $V = (\pi/3)r^2h$. In addition we are given that $r = h/2$ so

$$V(t) = (\pi/3)(h(t)/2)^2 h(t) = (\pi/12)h^3(t)$$

We are also given that $V'(t) = 30$, so

$$V'(t) = 3(\pi/12)h^2(t)h'(t) = 30.$$

Thus, when $h(t) = 10$

$$(\pi/4)(10)^2\ h'(t) = 30$$

That is, the height is increasing at a rate given by

$$h'(t) = (1.2/\pi)\ \text{ft/min}.$$

6. Let y be the function of x defined by the curve

$$x^4 - xy^3 + y^4 = 9.$$

The point (1,2) is on this curve. What is the area of the triangle formed by the tangent line to the curve at (1,2), the x-axis, and the y-axis?

Solution: Differentiating implicitly, we get

$$(4y^3 - 3xy^2)(dy/dx) = y^3 - 4x^3.$$

Setting x = 1, y = 2 gives

$$(32-12)(dy/dx) = (8-4).$$

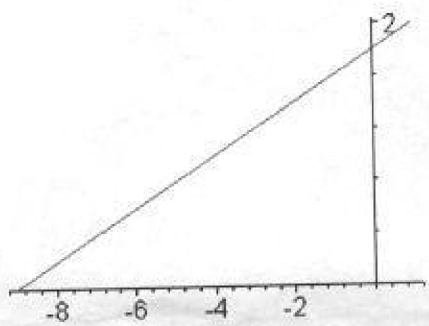
That is dy/dx = 1/5, so the equation of the tangent line is

$$(y-2)/(x-1) = 1/5$$

or

$$y = 2 + (1/5)(x-1)$$

This line intersects the y-axis when x = 0, y = 9/5. It intersects the x-axis when y = 0, x = -9:



Consequently, the area bounded by the tangent line, the x-axis, and the y-axis is equal to (1/2) * 9 * (9/5) = 8.1