The Knot Quandle

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Abstract

A quandle is a set with two operations that satisfy three conditions. For example, there is a quandle naturally associated to any group. It turns out that one can associate a quandle to any knot. The knot quandle is invariant under Reidemeister moves (and is thus an invariant of ambient isotopy). However, if fails to distinguish some non-isotopic knots, and is therefore not a complete invariant. The knot quandle allows to distinguish some knots that we could not distinguish using the 3-coloring invariant.

1. Introduction

The quandle is an algebraic object which was first introduced by Joyce in [1]. To each knot or link Joyce associated a quandle in such a way that it is an invariant of ambient isotopy. We will discuss the original presentation in section 2, and relations of the quandle to the knot group in section 3.

2. The Knot Quandle

Def. A <u>quandle</u> is a set Q with two binary operations, called conjugations and denoted by $>: Q\times Q\to Q$ and $<: Q\times Q\to Q$. These two operations satisfy the following three conditions.

<u>Q1</u> $x>x=x$.

<u>Q2</u> $(x>y)<y=x=(x<y)>y$.

Q3 $(x>y)>z=(x>z)>(y>z)$.

Equivalently, a quandle can be defined as a set Q with an operation $>: Q\times Q\to Q$ satisfying the following conditions.

Q1 $x>x=x$.

Q2' for every $x,y\in Q$ there exists a unique $z\in Q$ with $x=z>y$.

Q3 $(x>y)>z=(x>z)>(y>z)$.

The existence and uniqueness requirements of Q2' imply that each quandle comes with a second operation $<$ satisfying $(x<y)>y=x$. Specifically, Q2' is equivalent to the statement that for all y, the map $f_y: Q\to Q$ given by $f_y(x)=x>y$ is a bijection. We may define the second quandle operation by $x<y=f_y^{-1}(x)$. Then we have $(x>y)<y=f_y^{-1}(f_y(x))=x$ and $(x<y)>y=f_y(f_y^{-1}(x))=x$.

Ex. The easiest example of a quandle is the quandle associated to a group. Let G be a group. The underlying set of the associated quandle Q(G) is the same as the underlying set of the group. The quandle operations are defined by:

$$x \triangleright y = y^{-1}xy, \qquad x \triangleleft y = yxy^{-1}.$$

To show that this indeed defines a quandle, we must check that the two conjugation operations satisfy the three axioms of a quandle.
For Q1 we have:

$$x \triangleright x = x^{-1}xx = x.$$

Q2: $(x \triangleright y) \triangleleft y = (y^{-1}xy) \triangleleft y = yy^{-1}xyy^{-1} = x.$

Q3: $(x \triangleright y) \triangleright z = (y^{-1}xy) \triangleright z = z^{-1}y^{-1}xyz.$

$$(x \triangleright z) \triangleright (y \triangleright z) = (z^{-1}xz) \triangleright (z^{-1}yz) = z^{-1}y^{-1}zz^{-1}xzz^{-1}yz = z^{-1}y^{-1}xyz.$$

Note that the quandle operations of Q(G) satisfy $x \triangleright y = x \triangleleft y^{-1}$ for all x,y.

It turns out that to each quandle Q one can associate a group G(Q) in the following way. Elements of the group are equivalence classes of the elements of the quandle. Let $\bar{x} \in G(Q)$ be the equivalence class of an element $x \in Q$. The multiplication of equivalence classes satisfies the relation

$$\overline{x \triangleright y} = \bar{y}^{-1}\bar{x}\bar{y}.$$

Explicitly the group defined in this way will be:

G(Q)={ $\bar{x}$, for $x \in Q$ | $\overline{x \triangleright y} = \bar{y}^{-1}\bar{x}\bar{y}$ for $x, y \in Q$}

Thus quandles and groups are closely related:

Theorem. Let G be a group and $Q = Q(G)$ be the associated quandle. Let $G(Q) = G(Q(G))$ be the group associated to Q. Then $G(Q) \cong G$. Similarly, Let Q be a quandle and G(Q) be the associated group. If Q(G(Q)) is the quandle associated to G (Q), then $Q(G(Q)) \cong Q$.

The proof follows from the definitions.

To an oriented knot diagram one can associate a quandle in the following way. Label the arcs, and let the elements of the quandle be the labels of the arcs. Then relate the elements of the quandle using the crossings in the knot diagram and the following definition of the quandle operations.

$$c > b = a$$
$$a < b = c$$

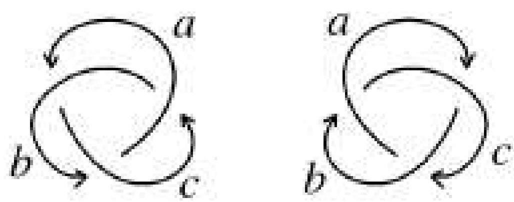
Now, to show that the knot quandle we have defined is a knot invariant, we must show that it is an invariant under Reidemeister moves.
For the first Reidemeister move we get:

For R2 we have:

And for R3:

Notice that the conditions initially set on the quandle make it an invariant under the three Reidemeister moves.
So if two knot quandles are isomorphic then the unoriented knots are equivalent.
However, as the following example shows, the quandle is not a complete invariant of knot type.

Ex. Consider the left and the right trefoil:



$$Q = \langle a, b, c \mid a \sim b \triangleright c, b \sim c \triangleright a, c \sim a \triangleright b \rangle$$

This gives an example of two knots which are not equivalent, but which have isomorphic quandles. This example is an illustration of the fact that for two knots which are mirror images of each other, their quandles will be isomorphic. This stems from the equivalence of the operations $x > y = z$ and $z < y = x$, and their association to mirror image crossing diagrams.

From here we can compute some simple knot quandles. We have already done the trefoil, but we can do 4-1.

Ex.
{a,b,c,d | $a < c = b$, $b > d = c$, $c < a = c$, $d > c = c$}

Knots that the quandle does allow us to distinguish are, for example 5-1 and the unknot, and 6-3 and 5-1. We couldn't distinguish these knots using the 3-coloring invariant.

Def. The 3-coloring invariant is the number of ways to color a knot diagram with three colors. To three color a diagram, each arc must be assigned a color, and the colors must satisfy the rule that at each crossing, either only one color occurs on all arcs, or all three colors occur on the intersecting arcs.

The knot quandle is a generalization of the 3-coloring invariant.
From the 3-coloring invariant, replace the colors by arbitrary labels for each of the arcs in the diagram. Replace the coloring rule by a method for combining these new labels.

Ex.

From 5-1 we get the quandle {a,b,c,d,e | $a \triangleright b = e$, $b \triangleright c = c$, $c \triangleright d = e$, $d \triangleright e = c$, $e \triangleright a = b$} which differs from the quandle of the unknot {a} which has only one element.

{a,b,c,d,e,f | $a \triangleleft c = b$, $b \triangleleft f = c$, $c \triangleright e = c$, $d \triangleright a = e$, $e \triangleleft b = f$, $f \triangleright d = c$} {a,b,c,d,e | $a \triangleright b = e$, $b \triangleright c = c$, $c \triangleright d = e$, $d \triangleright e = c$, $e \triangleright a = b$}

The quandles for 6-3 and 5-1 have different numbers of elements, and different explicit relations, thus they cannot correspond to equivalent knots.

3. The Knot Group and Knot Quandle

The knot group of a knot has many different presentations. The presentation most closely associated with quandles is called the Wirtinger presentation.

Def. To each arc of the knot diagram, assign a distinct generator. To each crossing of arcs associate the relation CB=BA, where B is the generator for the overcrossing arc, and C and A are the generators for the undercrossing arcs.

Ex. For the trefoil:
we have three distinct generators, a, b, and c. From the diagram, we can relate them in the following way:

$$G = \{a, b, c \mid ca = ab, ab = bc\}.$$

Since G is a group,

i) For $g,h \in G$, $gh \in G$.

ii) $(gh)k = g(hk)$ for $g,h,k \in G$.

iii) The is an identity element, $e \in G$ such that $ge = eg = g$ for all $g \in G$.

iv) Every element in G has an inverse, such that $gg^{-1} = e = g^{-1}g$ for all $g \in G$.

For G we know that it cannot be abelian. If G were abelian, then $ab = ba$. Then the relation $ca = ab$ implies $ca = ba$. By multiplying by a^{-1} on the right on both sides, we get $b = c$, which contradicts the fact that b and c are distinct generators.
Also, we know that there must be a distinct identity element e. If a,b, or c were the identity, then the rules for G would yield equality between two of the elements that are distinct.
Since we know that groups with four and five elements are abelian, then we know that G must have at least six elements. The only nonabelian group with six elements is S_3, the symmetric group on three letters. The multiplication table in S_3 can be written as:

Thus we know for the knot quandle, each relation $a \triangleright b = c$ tells us that b is the overcrossing arc, and a and c are the undercrossing arcs. So we can relate a quandle operation $a \triangleright b = c$ to a relation of group elements $ab = bc$.

For the quandle for the trefoil we have

$$Q = \{a, b, c \mid a \triangleright b = c, b \triangleright c = a, c \triangleright a = b\}.$$

The associated group is:

$$G = \{a, b, c \mid ca = ab, ab = bc, bc = ca\}.$$

This is the same as the knot group of the trefoil in Wirtinger presentation. We can see from the multiplication table for G that $ca = ab$ and $bc = ca$. Also, if we use the original construction of a group from a quandle presented earlier, we get:

$$G' = \{a, b, c \mid c = b^{-1}ab, a = c^{-1}bc, b = a^{-1}ca\}$$

If we multiply by the appropriate element on the right side of each of these equations we can get the same relations in G' as we did for G.

Proposition. The knot quandle isomorphic to the obtained from the knot group in Wirtinger presentation. The opposite is also true, the Wirtinger presentation of the knot group can be obtained as the group associated to the knot quandle.

4. Conclusion

The knot quandle is a useful invariant of knots, and is closely tied to the knot group. The main weakness of both of these invariants is their inability to distinguish knots that are mirror images of each other. The knot quandle is also closely tied to coloring invariants, and can be used to compute the Alexander Matrix of a knot, which can then be used to compute the Alexander polynomial of a knot.

References

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