Braids, the Artin Group, and the Jones Polynomial

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Abstract

This paper is about Braids and the Artin braid group B_n. After some initial definitions and examples, I proceed to show how the Jones polynomial can be derived through a representation of the braid group by the Temperley-Lieb Algebra, an approach similar to Jones' original construction.

1 Braids, An Introduction

Perhaps the most obvious place to start is with the definition [3] of a braid:

Definition 1 *Consider two parallel planes A and B in $\mathbb{R}^3$, each containing n distinct points $\{a_i\}$ and $\{b_i\}$ respectively. Then an n-strand braid is a collection of n curves $\{x_i\}$ such that:*

1. *Each x_i has one endpoint at an a_i and an endpoint at a b_i.*
2. *All the x_i are pairwise disjoint.*
3. *Every plane parallel to A and B intersects each of the x_i at one point or not at all.*

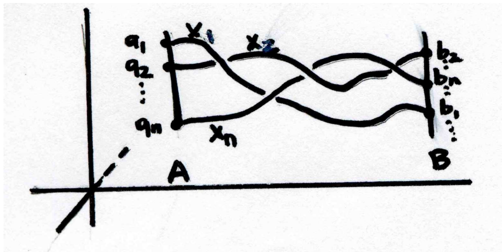


Figure 1: A braid in $\mathbb{R}^3$

There are several ways to think of braids. Perhaps the most intuitive is by imagining a number of strings attached at even intervals to a pole, then pulled out in a given direction and woven about one another. This follows our definition closely (and should seem natural to anyone who has braided long hair). Alternatively, one can imagine the paths traversed by n particles in a plane. To be more specific, suppose n particles are initially positioned at the points $(0,1,0),(0,2,0),\ldots,(0,n,0)$ in $\mathbb{R}^3$, and let them move around along the trajectories

$$a_1(t),\ a_2(t),\ \ldots,\ a_n(t),\ a_i(t)\in\mathbb{R}^3$$

A braid then is the trace of these trajectories $a=(a_1(t),a_2(t),\ldots,a_n(t))$, $0\leq t\leq 1$ with the conditions that the particles do not collide (ie $a_i(t_1)\neq a_j(t_2)$ if $i\neq j$ for any t_1,t_2), that they end at the points $(1,1,0),(1,2,0),\ldots,(1,n,0)$, and they do not move in the negative direction along the x-axis. For simplicity we can assume that they end at $x=1$ with the y coordinates possibly permuted, so that we have the following:

$$\begin{aligned} a_i(0)=(0,i),\ a_i(1)=(1,j(i)) \text{ where } j(i)\in\{1,2,\ldots,n\} \\ \text{and } j(i)\neq j(i') \text{ for } i\neq i'. \end{aligned}$$

See Figure 2a for an example of such particle trajectories.

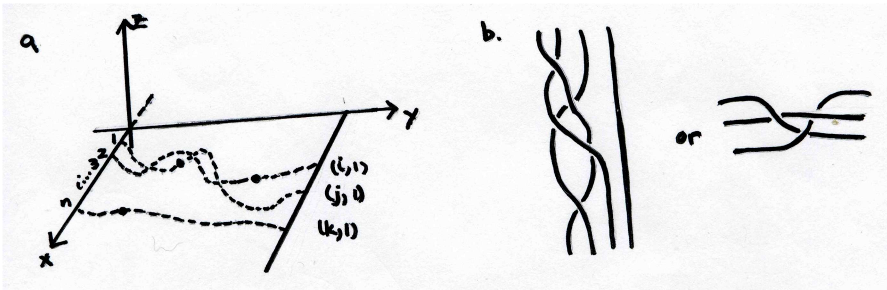


Figure 2: a. A braid as trajectories, b. Typical braid diagrams

We will make use of both of the above approaches to braids. In general we will just consider simple diagrams like the one in figure 2b in our study of braids. Now that we have a definition of our basic object of study, we will discuss a few basic properties of braids, such as what it means for two braids to be equivalent. We will then show that braids form a group under the operation of concatenation. This group is called the Artin braid group. It can be defined using simple generators and relations and has many interesting algebraic properties. Once we have described the braid group we will construct a correspondence between braids and knots and demonstrate how the Jones polynomial for knots can be derived from a representation of the corresponding braid group.

1.1 The Artin Braid Group

As with knots, we say that two braids, b and b' are *equivalent* if they are ambient isotopic. For braids, this means that if we keep the end-points of the braid b fixed we can continuously deform b into b' without breaking any of the strands. Again this is similar to the case of knots and the deformation can be described using the Reidemeister moves (see figure 3a) and planar isotopies of braid diagrams. However, for braids we have no use for R1 type moves, since we don't allow our strands to loop backwards. See Figure 3b for an example of two equivalent braids.

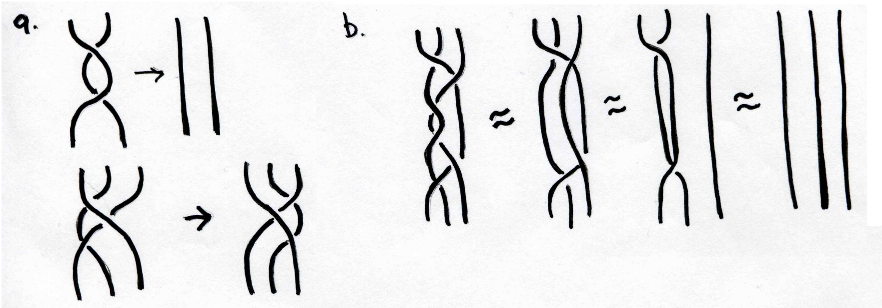


Figure 3: a. Reidemester moves for braids, b. Equivalent braids

As was mentioned, if we consider the operation of concatenation of braids, we obtain a group structure. The elements of the group are equivalence classes of braids under the equivalence defined above. Given two n-strand braids a and b we form their product ab by joining b to the end of a (see Figure 4a). Clearly ab again forms an n-strand braid. The identity with respect to this operation is the unbraid, represented by n parallel strands (Figure 4b). This operation is associative. Finally, braids, unlike knots, have inverses. The easiest way to see this is with a picture (Figure 4c). Essentially to get the inverse, given a diagram, you take the mirror image of the braid with respect to the axis formed by drawing a line intersecting each strand at its endpoint as shown in the drawing. Thus equivalence classes of n-strand braids with the operation of concatenation satisfy all the axioms of a group. We call the group formed by n-strand braids the *Artin Braid Group*, denoted B_n.

In order to make our studies easier and to save a lot of picture drawing, it is convenient to notice the fact that every braid can be written as a product of the following generators. Let σ_i be the n-braid in which the i^{th} strand passes

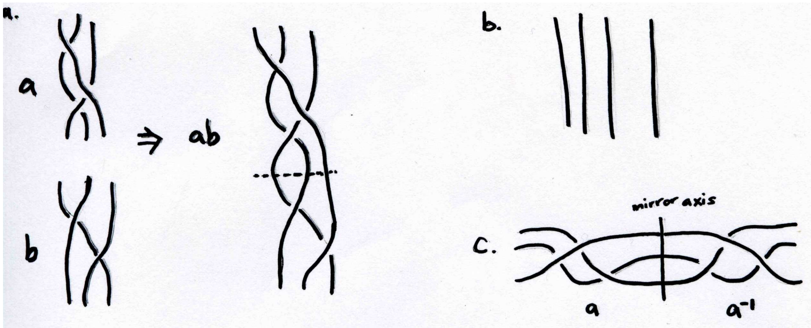


Figure 4: a. Concatenation of braids, b. The identity element i_n, c. Inverse braid

under the $(i+1)^{th}$ strand and let σ_i^{-1} be the n-braid in which the i^{th} strand passes over the $(i+1)^{th}$ strand. Thus we get the generators σ_1, $\sigma_2, \ldots,$ σ_{n-1} and their inverses $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$. This gives a convenient way to describe the weaving patterns of braids, see Figure 5 below for a couple of examples.

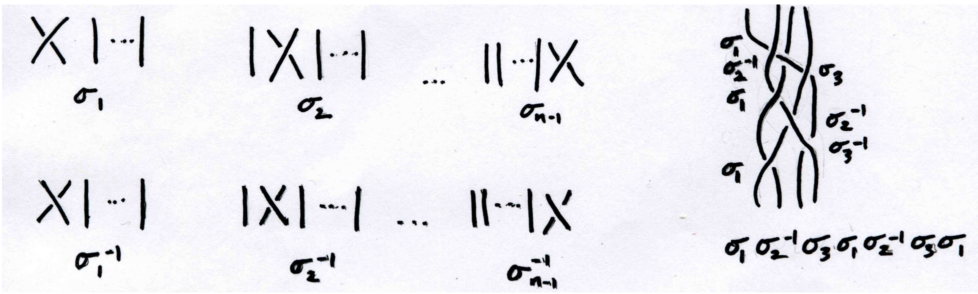


Figure 5: The generators of B_n and an example braid.

With these generators it then becomes easier to determine when braids are equivalent. The braid group B_n is described by the above generators and the following relations:

$$\sigma_i \sigma_i^{-1} = 1, \qquad i = 1, \ldots, n-1. \tag{1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-2. \tag{2}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| > 1. \tag{3}$$

It is easy to see that these relations describe equivalences between braids. The first relation is just a version of a type II Reidemeister move (Figure 6a). The second relation is a type III Reidemeister move (Figure 6b). The third is easily understood with a diagram (Figure 6c). Essentially, what it means is that if two consecutive crossings occur on two disjoint sets of strands, the the first crossing can be slid further down along the braid than the second. With these relations we are ready to construct the Jones polynomial for braids.

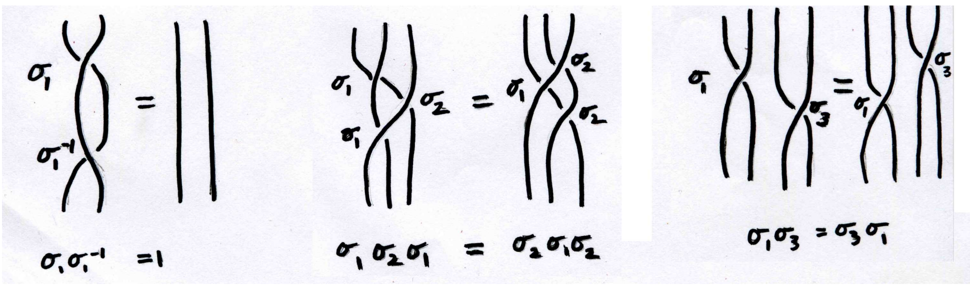


Figure 6: The relations of B_n

1.2 The Braid-Knot Connection

Before we construct the Jones Polynomial we need to make a few connections between knots and braids. Constructing a link from a braid is an easy task:

Definition 2 *Let b be a braid, the closure, $\bar{b}$, of b is formed by connecting the starting points with its endpoints by parallel, non-weaving lines (Figure 7).*

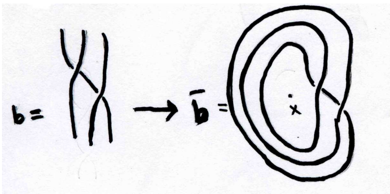


Figure 7: The closure of a braid

Clearly the closure of a braid is a knot or a link. It will be convenient to notice that if you orient the strands of your braid so that they all point in the same direction, when you make the closure, the link then circles in one direction around the axis of the closure (in figure 7, the axis is the line through x perpendicular to the plane of the picture). On the other hand, every link can be obtained from a braid by taking its closure:

Theorem 1 ***(Alexander's Theorem)*** *Every knot or link in* $\mathbb{R}^3$ *is ambient isotopic to a closed braid.*

The proof of this theorem is quite involved (see [1]), however it is a very algorithmic process and we can get an idea of how it is done through an example. The first step is to put an orientation on your knot, then choose a proposed braid axis. Once you have chosen your axis, pick a point on the knot and follow the orientation around the axis (see Figure 8 step i). The idea is to have the knot to wind around the axis in one direction (clockwise or counterclockwise). If the knot begins to circle the axis incorrectly (as in step ii of Figure 8) you throw the strand over the axis so that it is winding in the right direction (step iii of Figure 8). Repeating this process as you traverse the knot we will eventually arrive in a position so that all the strands are winding around the axis in the same direction. It is easy to see in this situation that, after some planar isotopy, we have a closed braid (step iiii).

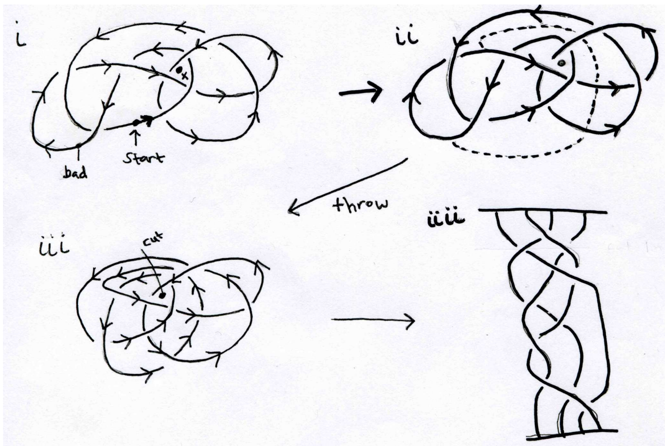
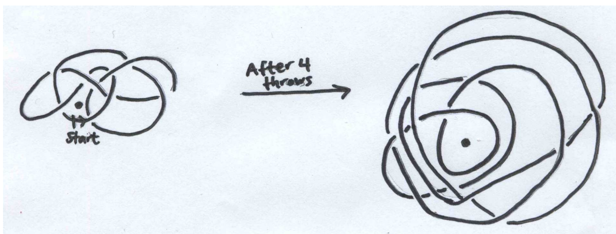


Figure 8: An example of Alexander's Theorem

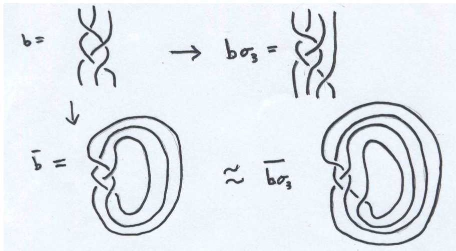
With Alexander's Theorem, we can move easily between knots and braids. However, we see from the following diagram, taking the same knot as above, that if we choose a different axis, the construction of a closed braid from a knot does not produce a unique closed braid.



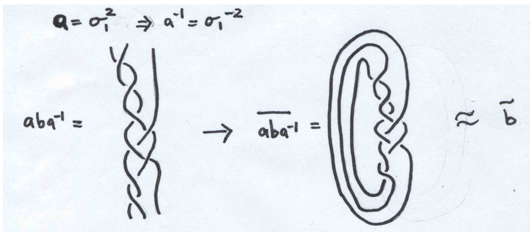
In order to effectively study knots using the braid group we need to understand how many different closed braids can represent a given knot. Furthermore, we would like to understand how we can modify our braids to produce ambient isotopic knots.

There are two basic moves that can be used to modify braids so that their closures produce ambient isotopic knots:

1. **Markov Move:** If b is a braid in B_n and if we add a strand to b to make it an $(n+1)$-strand braid, the braids $b\sigma_n$ and $b\sigma_n^{-1}$ in B_{n+1} have closures ambient isotopic to $\bar{b}$. Essentially, this tells us how we affect the braid b when we do a Reidemeister move one on the closed braid.



2. **Conjugate Braid:** Let a and b be braids in B_n. If we take the conjugate braid aba^{-1} then it can happen that when we take the closure $\overline{aba^{-1}}$, the terms a and a^{-1} cancel each other out through the closure strands, thus producing a link which is ambient isotopic to b.



With these two moves we have the following theorem which relates equivalence in braids with ambient isotopy in links or knots.

Theorem 2 ***(Markov's Theorem)*** *Let $\beta_n \in B_n$ and $\beta'_m \in B_m$ be two braids in the braid groups B_n and B_m respectively. Then the links (closures of the braids β_n, β'_m) $L = \overline{\beta_n}$ and $L' = \overline{\beta'_m}$ are ambient isotopic if and only if β'_m can be obtained from β_n by a series of*

1. *Equivalences in a given braid group.*

2. *Conjugation in a given braid group.*

3. *Markov Moves: replacing $\beta \in B_n$ by $\beta\sigma_n^{\pm 1} \in B_{n+1}$ or the inverse of this operation, replacing $\beta\sigma_n^{\pm 1} \in B_{n+1}$ by $\beta \in B_n$ if β has no occurrences of σ_n.*

For a proof of this theorem see [1]. To understand it intuitively, we recall that two knots, L and L' are ambient isotopic if we can derive the diagram of L from the diagram of L' by a sequence of Reidemeister moves. Markov's Theorem tells us how the three Reidemeister moves affect the corresponding braids: we have equivalency in the braid group for the R2 and R3 moves, and Markov moves for the R1 move. Braid conjugation is just one extra thing we need to be aware of that can give us seemingly different braid representations.

2 The Jones Polynomial

The Markov Theorem gives us a direct relationship between knots/links and braids. With this theorem we can now use the braid group to better understand knots and links. In particular, we can now derive knot invariants such as the Jones Polynomial using the braid group. In order to derive the Jones polynomial, we will construct a representation (see section 2.2) from the braid group B_n into the *Temperley-Lieb Algebra* (see section 2.1) and show that this gives us a version of the bracket polynomial for braids, which we can extend to the original bracket polynomial for knots. Markov's theorem will then guarantee that this is, in fact, an invariant for knots.

2.1 The Temperley-Lieb Algebra

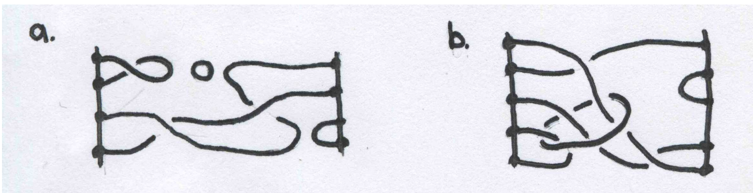
As mentioned above, we will construct a representation of the braid group into the Temperley-Lieb Algebra.

Recall that an *algebra* over a commutative ring R, is a ring A such that:

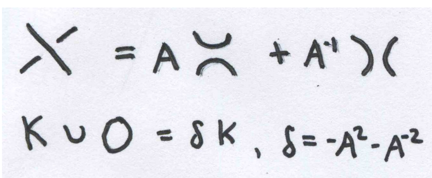
1. $(A, +)$ is an R-Module, and
2. $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in A$.

In other words, an algebra is essentially a module (or a vector space if R is a field) with multiplication of the module elements, satisfying condition 2 above. For example, $\mathbb{C}$ is an algebra over the field $\mathbb{R}$.

The Temperley-Lieb (TL) Algebra A_n is an algebra over the ring $\mathbb{Z}[A, A^{-1}]$, of polynomials in A and A^{-1} with integer coefficients. To understand the TL algebra we will construct it in much the same way we did with the braid group. We consider elements of A_n to be diagrams similar to braids. An element of A_n can be represented as follows: take n points each on a left plane and a right plane in $\mathbb{R}^3$ with strings connected to them. However, unlike in the case of braids the strings are allowed to loop backwards and don't necessarily have to move from one side to the other. As usual, this is much easier to understand with a diagram. The following are some typical elements of the Temperley-Lieb Algebra:



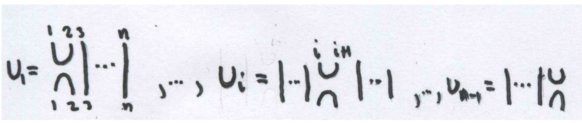
We can think of these elements as "tangle" diagrams. Now we define A_n to be the set of all tangles under the following equivalence relations:



Where K is an element of A_n. The first relation allows us to replace a tangle with a crossing by a linear combination of two tangles with the crossing resolved. The second relation allows us to remove any unconnected loop in the diagram, multiplying the resulting tangle by $\delta = -A^2 - A^{-2}$. Using these relations, it is possible to write all tangles in A_n as linear combinations of tangles with no crossings and no closed curves.

Just as we did with braids, we can multiply elements in A_n by concatenation. The identity element $1_n \in A_n$ is identical to the identity of B_n. It is easy to check that this construction gives us an algebra over $\mathbb{Z}[A, A^{-1}]$.

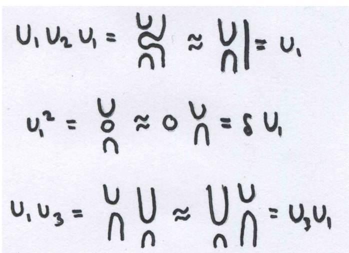
As we did with the braid group, we can describe the TL algebra with generators and relations. We have the following generators $U_1, U_2, \ldots, U_{n-1}$ given by



The relations in A_n are as follows:

$$\begin{aligned} U_i U_{i\pm1} U_i &= U_i, \\ U_i^2 &= \delta U_i, \text{ where } \delta = -A^2 - A^{-2}, \\ U_i U_j &= U_j U_i, \text{ if } |i-j| > 1. \end{aligned}$$

Here are several examples of diagrams representing these relations:



Finally, we will be interested in the closure of tangles. If we reduce our tangle to have no crossings and no additional loops and take its closure (by a process similar to the closure of braids), then we will be have a diagram of an unlink. We define $\|U\|$ to be the number of components in the unlink minus 1. For example:

2.2 Representations

Recall that a *linear representation* of a group G is a homomorphism $\phi : G \to GL_n(\mathbb{R})$, where $GL_n(\mathbb{R})$ is the group of invertible $n \times n$ matrices over the real numbers. For example, the map $\phi : \mathbb{Z}/2\mathbb{Z} \to GL_1(\mathbb{R})$ given by $\phi(0)v = v$ and $\phi(1)v = -v$, for $v \in \mathbb{R}$ is a representation of $\mathbb{Z}/2\mathbb{Z}$ in $GL_1(\mathbb{R})$. The study of group representations is a very big subject which we won't go into very deeply here. More generally one can consider a representation of a group to be a homomorphism of the group into some other algebraic object such as another group, a ring, a module or an algebra.

We are now in a position to give our representation of the braid group by the Termperley-Lieb algebra. We define the following homomorphism ρ_n:

$$\rho_n : B_n \to A_n$$

by the formulas:

$$\begin{aligned} \rho_n(\sigma_i) &= A(1_n) + A^{-1}U_i, \\ \rho_n(\sigma_i^{-1}) &= A^{-1}(1_n) + AU_i. \end{aligned}$$

Note: we will generally omit the 1_n.

Proposition 1 *$\rho_n : B_n \to A_n$ is a representation of the Artin Braid Group.*

Proof: We must verify that ρ_n preserves the relations on the generators of the braid group. In other words we must show that $\rho_n(\sigma_i)\rho_n(\sigma_i^{-1}) = 1$, $\rho_n(\sigma_i\sigma_{i+1}\sigma_i) = \rho_n(\sigma_{i+1}\sigma_i\sigma_{i+1})$, and that $\rho_n(\sigma_i\sigma_j) = \rho_n(\sigma_j\sigma_i)$ for $|i-j| > 1$.
For $\rho_n(\sigma_i)\rho_n(\sigma_i^{-1}) = 1$:

$$\begin{aligned}
\rho_n(\sigma_i)\rho_n(\sigma_i^{-1}) &= (A + A^{-1}U_i)(A^{-1} + AU_i) \\
&= 1 + (A^{-2} + A^2)U_i + U_i^2 \\
&= 1 + (A^{-2} + A^2)U_i + \delta U_i \\
&= 1 + (A^{-2} + A^2)U_i + (-A^{-2} + -A^2)U_i \\
&= 1
\end{aligned}$$

For $\rho_n(\sigma_i\sigma_{i+1}\sigma_i) = \rho_n(\sigma_{i+1}\sigma_i\sigma_{i+1})$:

$$\begin{aligned}
\rho_n(\sigma_i\sigma_{i+1}\sigma_i) &= (A + A^{-1}U_i)(A + A^{-1}U_{i+1})(A + A^{-1}U_i) \\
&= (A^2 + U_{i+1} + U_i + A^{-2}U_iU_{i+1})(A + A^{-1}U_i) \\
&= A^3 + AU_{i+1} + AU_i + A^{-1}U_iU_{i+1} + A^{-1}U_i^2 + AU_i \\
&\quad +A^{-1}U_{i+1}U_i + A^{-3}U_iU_{i+1}U_i \\
&= A^3 + AU_{i+1} + (A^{-1}\delta + 2A)U_i + A^{-1}(U_iU_{i+1} + U_{i+1}U_i) \\
&\quad +A^{-3}U_i \\
&= A^3 + AU_{i+1} + (A^{-1}(-A^2 - A^{-2}) + 2A + A^{-3})U_i \\
&\quad +A^{-1}(U_iU_{i+1} + U_{i+1}U_i) \\
&= A^3 + A(U_{i+1} + U_i) + A^{-1}(U_iU_{i+1} + U_{i+1}U_i)
\end{aligned}$$

And it is clear that this final equation is symmetric in i and $i+1$ which implies that $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho_n(\sigma_{i+1}\sigma_i\sigma_{i+1})$.
Finally, given that $|i-j| > 1$, we need $\rho_n(\sigma_i\sigma_j) = \rho_n(\sigma_j\sigma_i)$:

$$\begin{aligned}
\rho_n(\sigma_i\sigma_j) &= \rho_n(\sigma_i)\rho(\sigma_j) \\
&= (A + A^{-1}U_i)(A + A^{-1}U_j) \\
&= (A + A^{-1}U_j)(A + A^{-1}U_i)\ [U_iU_j = U_jU_i \text{ if } |i-j| > 1] \\
&= \rho_n(\sigma_j\sigma_i).
\end{aligned}$$

Since all the relations of the group generators are satisfied, ρ_n is a representation of the braid group. $\Box$

The best way to understand ρ_n is to think of it as a function that resolves the crossings of the braid. Each σ_i in the braid group represents a crossing of the strands in a braid. When we take ρ_n of it, we get the sum of the two possible resolutions of the crossing with the coefficients of A or A^{-1} depending on which type of crossing we are resolving. Understood this way, the representation ρ_n is clearly suggestive of the bracket polynomial for knots.

2.3 The Bracket Polynomial

Recall that the bracket polynomial for knots was defined as a sum over all states of a knot with the crossings resolved. We have seen that the bracket polynomial is closely related to the Jones Polynomial. We shall now see that we can easily derive the bracket polynomial from our representation ρ_n. Given a braid $\beta = \sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_s}^{a_s} \in B_n$ we plug it into ρ and get a product in A_n:

$$\rho_n(\beta) = (A + A^{-1}U_{i_1})^{a_1} \cdots (A + A^{-1}U_{i_s})^{a_s}.$$

We define $\langle\beta|t\rangle$ to be the coefficient of the U_t in the expanded expression of $\rho_n(\beta)$. Thus we see that:

$$\rho_n(\beta) = \sum_t \langle\beta|t\rangle U_t,$$

where t indexes all the terms in the sum. As we discussed above, $\rho_n(\beta)$ essentially resolves all the crossings of β. Furthermore, the U_t represent the resolved states of the braids. We define the bracket for the resolved states by

$$\langle U_t\rangle = \langle\overline{U_t}\rangle = \delta^{\|U\|}$$

and we can conclude that the bracket for the closed braid $\overline{\beta}$ is

$$\langle\overline{\beta}\rangle = \sum_t \langle\beta|t\rangle\delta^{\|U\|}.$$

We will see a few calculations of these techniques in section 2.4. However, we must first fill in some gaps in our construction.

We have shown how we can use a representation of the braid group to construct the bracket polynomial for knots. However our construction is not quite complete. Recall that in order for our bracket polynomial to be an invariant for knots, Markov's theorem tells us that it must be invariant under the three moves listed in Markov's theorem. It is easily checked that the above bracket polynomial fails to be invariant under Markov moves. To correct this we introduce a new factor. We define the **writhe of a braid**, $w(\beta)$ to be the sum of exponents of the generators in the braid:

$$w(\beta) = \sum_{t=1}^{s} a_t \text{ for a braid } \beta = \sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \cdots \sigma_{i_s}^{a_s} \in B_n$$

And finally, we define the **normalized bracket** for a closed braid $\overline{\beta}$ as follows:

$$\langle \overline{\beta} \rangle = (-A^3)^{-w(\beta)} \sum_t \langle \beta | t \rangle \delta^{\|U\|}.$$

Proposition 2 *The normalized bracket $\langle K \rangle$ is an invariant of ambient isotopy for knots. That is, if $K \approx K'$, then $\langle K \rangle = \langle K' \rangle$ for two knots K and K'*

Proof: Suppose, by Alexander's Theorem, that $K \approx \overline{\beta}$ and $K' \approx \overline{\beta'}$ for $\beta, \beta' \in B_n$. Since K and K' are ambient isotopic, Markov's theorem tells us that that $\overline{\beta}$ and $\overline{\beta'}$ are ambient isotopic and thus $\overline{\beta'}$ can be obtained from $\overline{\beta}$ by a sequence of Markov moves of type 1, 2, and 3 (from the theorem). So it suffices to show that the bracket is invariant under these three moves.

1. The first possibility is equivalences in a given braid group. The invariance under these equivalencies follows directly from our construction of ρ_n and the fact that it was a representation of the braid group, thus invariant under equivalences.

2. The second move is conjugation in a given braid group. So we must show that $\langle\overline{\beta}\rangle = \langle\overline{\alpha\beta\alpha^{-1}}\rangle$ for some $\alpha \in B_n$. It will be simpler and sufficient to show that $\rho(\beta) = \rho(\alpha\beta\alpha^{-1})$. Since ρ is a homomorphism, it is sufficient to show that $\rho(\beta) = \rho(\sigma_i\beta\sigma_i^{-1})$:

$$\begin{aligned}
\rho(\sigma_i\beta\sigma_i^{-1}) &= (A + A^{-1}U_i)\rho(\beta)(A^{-1} + AU_i) \\
&= (A\rho(\beta) + A^{-1}U_i\rho(\beta))(A^{-1} + AU_i) \\
&= \rho(\beta) + (A^{-2} + A^2)U_i\rho(\beta) + U_i^2\rho(\beta) \\
&= \rho(\beta) + (A^{-2} + A^2)U_i\rho(\beta) + (-A^{-2} - A^2)U_i\rho(\beta) \\
&= \rho(\beta)
\end{aligned}$$

3. Finally, we must check that the bracket is invariant under Markov moves. We must show $\langle\overline{\beta\sigma_n}\rangle = \langle\overline{\beta}\rangle$. First note that $w(\beta\sigma_n) = w(\beta)+1$. Thus:

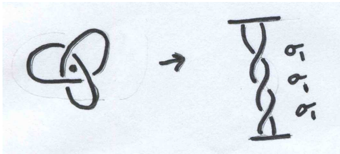
$$\begin{aligned}
\langle\overline{\beta\sigma_n}\rangle &= (-A^3)^{w(\beta)+1}(\sum_t \langle\beta|t\rangle\delta^{\|U\|})(A + A^{-1}\langle U_i\rangle) \\
&= \langle\overline{\beta}\rangle(-A^3(A + A^{-1}\langle U_n\rangle) \\
&= \langle\overline{\beta}\rangle(-A^4 - A^2\langle U_n\rangle) \\
&= \langle\overline{\beta}\rangle(-A^4 - A^2(-A^2 - A^{-2})) \\
&= \langle\overline{\beta}\rangle(-A^4 + 1 + A^4) \\
&= \langle\overline{\beta}\rangle
\end{aligned}$$

And we conclude that the normalized bracket is an invariant of ambient isotopy for knots. □

2.4 Examples

Let's see how we can compute brackets with the braid group:

1. **Trefoil** First we need the closed representation braid of the trefoil:



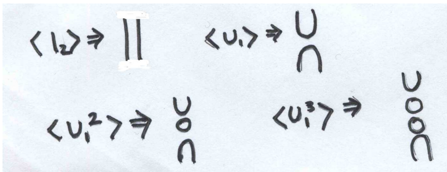
And we see that the trefoil is ambient isotopic to $\overline{\sigma_1^3}$. Now let's compute the bracket polynomial. First we resolve crossings using ρ:

$$\begin{aligned}\rho(\sigma_1^3) &= (A + A^{-1}U_i)^3 \\ &= A^3 + 3AU_1 + 3A^{-1}U_1^2 + A^{-3}U_1^3\end{aligned}$$

We note that writhe $w(\sigma_1^3) = 3$. Now we can compute the bracket:

$$\langle\overline{\sigma_1^3}\rangle = (-A^3)^{-3}(A^3\langle 1_2\rangle + 3A\langle U_1\rangle + 3A^{-1}\langle U_1^2\rangle + A^{-3}\langle U_1^3\rangle).$$

The following diagram gives us our loops counts:



We see that:

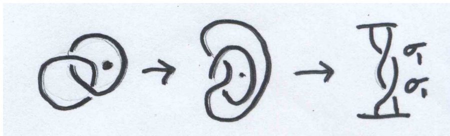
$$\begin{aligned}\langle 1_2\rangle &= \delta, \\ \langle U_1\rangle &= 1, \\ \langle U_1^2\rangle &= \delta, \\ \langle U_1^3\rangle &= \delta^2.\end{aligned}$$

Substituting for $\langle U\rangle$ we get:

$$\begin{aligned}\langle\overline{\sigma_1^3}\rangle &= (-A^3)^{-3}(A^3\delta + 3A + 3A^{-1}\delta + A^{-3}\delta^2) \\ &= (-A^3)^{-3}(A^3(-A^2 - A^{-2}) + 3A + 3A^{-1}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2) \\ &= (-A^3)^{-3}(A^{-7} - A^{-3} - A^5) \\ &= A^{-4} + A^{-10} - A^{-16}\end{aligned}$$

Which is exactly the polynomial which we have calculated for the standard bracket of the trefoil.

2. **Hopf Link** Again we start by finding the braid corresponding to the Hopf Link:



So the Hopf link is ambient isotopic to $\overline{\sigma_1^2}$. Now let's compute the bracket polynomial. First we resolve crossings using ρ:

$$\begin{aligned} \rho(\sigma_1^3) &= (A + A^{-1}U_i)^2 \\ &= A^2 + 2U_1 + A^{-2}U_1^2. \end{aligned}$$

The writhe for the Hopf Link is $w(\sigma_1^2) = 2$, and we have the loop counts from the diagram above, so putting it all together we get:

$$\begin{aligned} \langle\overline{\sigma_1^2}\rangle &= (-A^3)^{-2}(A^2\langle 1_2\rangle + 2\langle U_1\rangle + A^{-2}\langle U_1^2\rangle) \\ &= (-A^3)^{-2}(A^2\delta + 2 + A^{-2}\delta) \\ &= (-A^3)^{-2}(A^2(-A^2 - A^{-2}) + 2 + A^{-2}(-A^2 - A^{-2})) \\ &= (-A^3)^{-2}(-A^4 - 1 + 2 - 1 - A^{-4}) \\ &= -A^{-2} - A^{-10}. \end{aligned}$$

And again we verify that we get the same bracket polynomial we calculated directly with the state expansions.

As we have seen, braids can be very useful in the study of knots and links. Besides the representation of the braid group by the Termperley-Lieb algebra which we've seen here, there are several other representations, such as the Burau representation, of the braid group which lead to other polynomials and invariants of knots. There are, however, some limitations to this approach. For example, the representations we have seen here are not faithful for all braid groups B_n, in other words the maps are not injective for all n. This means that unequivalent braids can be mapped to the same element in our algebra, and thus the representations do not provide complete invariants for knots. Despite this deficiency, braid group representations have spurred much activity in the study of knots and provide a great deal of insight.

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