Generalized Knot Polynomials and Some Applications

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Generalized Polynomials

HOMFLY

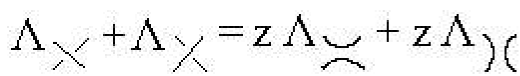
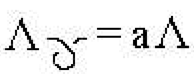
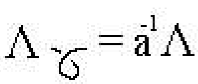
1. 
2. the HOMFLY polynomial of the unknot is 1
3. the HOMFLY polynomial is an invariant of ambient isotopy

Here are a few examples of HOMFLY polynomials:

1. P(unknot) = 1
2. P(n-component unlink)= $[(a+a^{-1})/z]^{n-1}$
3. P(Hopf link) = $a z^{-1} – a^3 z^{-1} + a z$
4. P(trefoil) = $2a^2 – a^4 + a^2 z^2$

By definition we must have $P(L)(t, t^{½} + t^{-½}) = V(L)(t)$ for all links L.

Pre-Kauffman

1. 
2. 
3. 
4. Λ (unknot) = 1
5. Λ is invariant under regular isotopy (R2 and R3 moves)

Kauffman

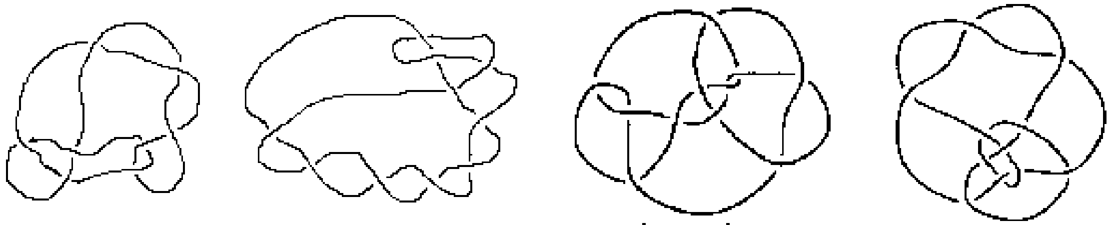
$F(L)(a,z) = a^{w(L)}\ \Lambda(L)(a,z)$ (definition)

By construction, $F(K)(-t^{¾}, t^{-¼} + t^{-¼}) = V(K)(t)$.

Here are some examples of Kauffman polynomials:

1. F(unknot) = 1
2. F(n-component unlink) = $[(a + a^{-1})/z - 1]^{n-1}$
3. F(Hopf link) = $(-a^3 - a^{-1})\, z^{-1} – a^{-2} + (a^3 + a^{-1})\, z$
4. F(trefoil) = $-2a^{-2} - a^{-4} + (a^{-3} + a^{-5})\, z + (a^{-2} + a^{-4})\, z^2$

Distinct Invariants



8-8 10-129 11-alternating-79 11-alternating-255

Knots 8-8 and 10-129 have the same HOMFLY polynomial but distinct Kauffman polynomials. Knots 11-alternating-79 and 11-alternating-255 have distinct HOMFLY polynomials but identical Kauffman polynomials.

F(8-8) = $2a^{-5}z - 3a^{-5}z^3 + a^{-5}z^5 - a^{-4} + 4a^{-4}z^2 - 6a^{-4}z^4 + 2a^{-4}z^6 + 3a^{-3}z - 5a^{-3}z^3 + a^{-3}z^7 - a^{-2} + 5a^{-2}z^2 - 9a^{-2}z^4 + 4a^{-2}z^6 + a^{-1}z - 3a^{-1}z^3 + a^{-1}z^5 + a^{-1}z^7 + 2 - z^2 - z^4 + 2z^6 - az + 2az^5 + a^2 - 2a^2z^2 + 2a^2z^4 - a^3z + a^3z^3$

P(8-8) = $- a^{-4} - a^{-4}z^2 + a^{-2} + 2a^{-2}z^2 + a^{-2}z^4 + 2 + 2z^2 + z^4 - a^2 - a^2z^2$

F(10-129) = $2a^{-5}z - 3a^{-5}z^3 + a^{-5}z^5 - a^{-4} + 4a^{-4}z^2 - 6a^{-4}z^4 + 2a^{-4}z^6 + 3a^{-3}z - 5a^{-3}z^3 + a^{-3}z^7 - a^{-2} + 5a^{-2}z^2 - 9a^{-2}z^4 + 4a^{-2}z^6 + a^{-1}z - 3a^{-1}z^3 + a^{-1}z^5 + a^{-1}z^7 + 2 - z^2 - z^4 + 2z^6 - az + 2az^5 + a^2 - 2a^2z^2 + 2a^2z^4 - a^3z + a^3z^3$

P(10-129) = $-a^{-2} - a^{-2}z^2 + 2 + 2z^2 + z^4 + a^2 + 2a^2z^2 + a^2z^4 - a^4 - a^4z^2$

F(11-alternating-79) = $a^{-2}z^2 - 2a^{-2}z^4 + a^{-2}z^6 - a^{-1}z + 5a^{-1}z^3 - 9a^{-1}z^5 + 4a^{-1}z^7 + 2 - 5z^2 + 11z^4 - 16z^6 + 7z^8 - 3az + 10az^3 - 9az^5 - 6az^7 + 6az^9 + 3a^2 - 19a^2z^2 + 44a^2z^4 - 45a^2z^6 + 14a^2z^8 + 2a^2z^{10} - 5a^3z + 14a^3z^3 - a^3z^5 - 20a^3z^7 + 13a^3z^9 + 3a^4 - 19a^4z^2 + 47a^4z^4 - 48a^4z^6 + 17a^4z^8 + 2a^4z^{10} - 5a^5z + 16a^5z^3 - 14a^5z^5 - 2a^5z^7 + 7a^5z^9 + a^6 - 5a^6z^2 + 11a^6z^4 - 16a^6z^6 + 10a^6z^8 - 2a^7z + 6a^7z^3 - 12a^7z^5 + 8a^7z^7 + a^8z^2 - 5a^8z^4 + 4a^8z^6 - a^9z^3 + a^9z^5$

P(11-alternating-79) = $2 + 3z^2 + 3z^4 + z^6 - 3a^2 - 9a^2z^2 - 10a^2z^4 - 5a^2z^6 - a^2z^8 + 3a^4 + 8a^4z^2 + 7a^4z^4 + 2a^4z^6 - a^6 - 2a^6z^2 - a^6z^4$

F(11-alternating-255) = $- 2a^{-2}z^4 + a^{-2}z^6 + 3a^{-1}z^3 - 9a^{-1}z^5 + 4a^{-1}z^7 + 2 - 8z^2 + 19z^4 - 22z^6 + 8z^8 + az - 6az^3 + 15az^5 - 19az^7 + 8az^9 + 3a^2 - 22a^2z^2 + 50a^2z^4 - 42a^2z^6 + 9a^2z^8 + 3a^2z^{10} + a^3z - 13a^3z^3 + 38a^3z^5 - 40a^3z^7 + 16a^3z^9 + 3a^4 - 20a^4z^2 + 44a^4z^4 - 39a^4z^6 + 11a^4z^8 + 3a^4z^{10} - a^5z + 2a^5z^3 + a^5z^5 - 9a^5z^7 + 8a^5z^9 + a^6 - 5a^6z^2 + 10a^6z^4 - 16a^6z^6 + 10a^6z^8 - a^7z + 5a^7z^3 - 12a^7z^5 + 8a^7z^7 + a^8z^2 - 5a^8z^4 + 4a^8z^6 - a^9z^3 + a^9z^5$

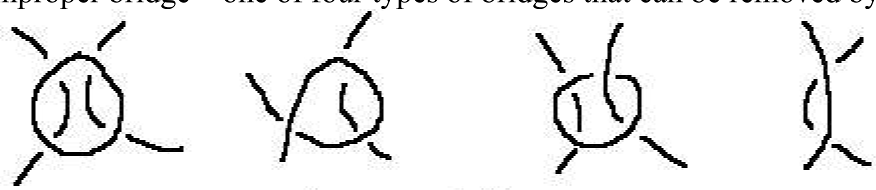
P(11-alternating-255) = $2 + 3z^2 + 3z^4 + z^6 - 3a^2 - 9a^2z^2 - 10a^2z^4 - 5a^2z^6 - a^2z^8 + 3a^4 + 8a^4z^2 + 7a^4z^4 + 2a^4z^6 - a^6 - 2a^6z^2 - a^6z^4$

This shows that the two invariants are distinct, and give more information than the Jones polynomial. Since F(8-8) = F(10-129) we must have V(8-8) = V(10-129), so the HOMFLY is able to distinguish knots that the Jones cannot, and similarly with the Kauffman polynomial.

Applications

Theorem 1: The writhe of a reduced alternating diagram of an alternating link is invariant.

Definitions:
prime – a link not representable as the connected sum of two or more non-trivial links
bridge – an arc in a planar link diagram that forms an overcrossing
length of a bridge – the number of overcrossings formed by a bridge
bridge length – a number assigned to any link diagram equal to length of the longest bridge (Note: a link diagram is alternating iff the bridge length is one)
improper bridge – one of four types of bridges that can be removed by ambient isotopy



improper bridges

centered polynomial - a Laurent polynomial whose highest x degree and highest x^{-1} degree are equal; e.g. $x^{-4} + 6x^{-1} + 5x^{3} + 2x^{4}$

Main Ideas:

- If an unoriented link diagram is prime, connected and alternating, then both of the link diagrams formed by the splicings of that diagram at any crossing are connected and alternating, and at least one of the spliced diagrams is prime.

- There are 4 types of improper bridges. If a bridge is not improper, then it does not terminate by going under itself.

- The z degree of F(L) and Λ(L) is less than or equal to the number of crossings N minus the bridge length b. In symbols, $\deg_z F(L) = \deg_z \Lambda(L) \leq N - b$

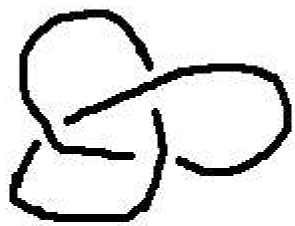
- If a link projection is prime, connected and alternating then the leading z coefficient of Λ(L) is a centered polynomial in a with positive first and last coefficients, i.e. of the form $c_{-m}\, a^{-m} + c_{-m+1}\, a^{-m+1} + \ldots + c_0 + \ldots + c_{m-1}\, a^{m-1} + c_m\, a^m$ where $c_{-m} > 0$, $c_m > 0$.

- The Kauffman polynomial of any two reduced, prime, alternating projections of a knot are equal, so their pre-Kauffman polynomials differ by a factor of a^{w1-w2} where w_1 and w_2 are the writhes of each projection. Since the leading coefficient of both pre-Kauffman polynomials is a centered polynomial in a, then we must have $w_1 - w_2 = 0$

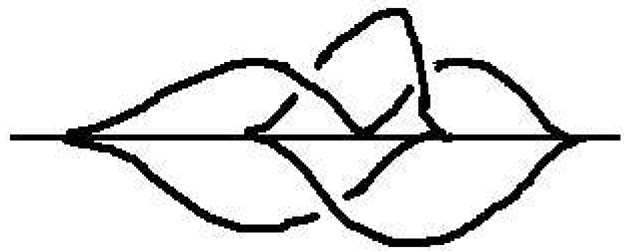
- The general case follows from the prime case since the writhe is additive and knots are uniquely decomposable into primes.

Arc Presentations

An arc presentation of a link L is a drawing of L on a finite number of half-planes in 3 space with the following conditions. All the half-planes share a single boundary line called the spine. On each plane is one arc of the link which does not self intersect and whose endpoints are distinct points the spine.



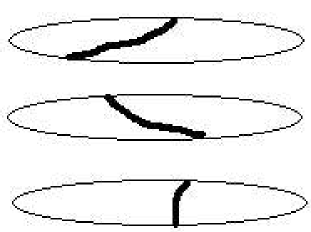
planar diagram



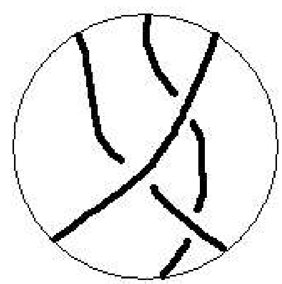
arc presentation

Stacked Tangle Links

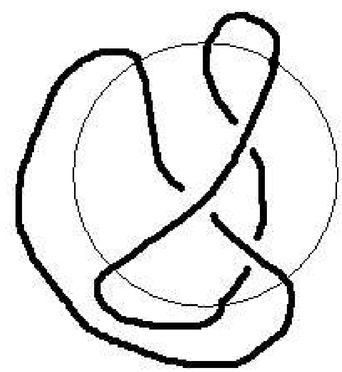
A stacked tangle is an ordering of arcs on disks as below. Naturally there is a projection onto the plane and we use the standard crossing scheme to draw the projection of arcs that overlap. A stacked tangle link projection is a link projection made by connecting the boundries of the disks by arcs in such a way that no crossings occur outside of the disk and we get a continuous link.



stacked tangles

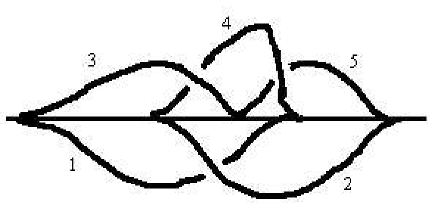


stacked tangle projection

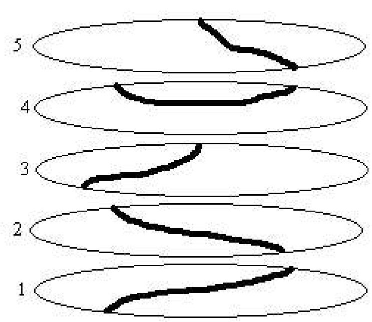


stacked tangle link projection

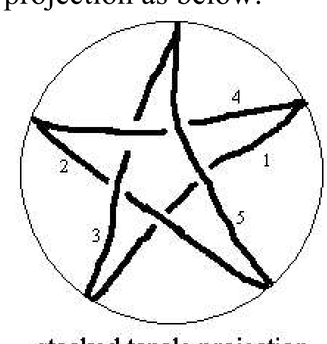
We can turn any arc presentation of a link into a stacked tangle projection as below:



arc presentation



stacked tangles



stacked tangle projection (already linked)

Definitions:

arc index – invariant of links; the minimal number of arcs in any arc presentation of a given link spread of a polynomial – the difference between the highest and lowest degrees of a polynomial; for a Laurent polynomial in x, the difference between the highest x power and the highest x^{-1} power

cap – an arc of a stacked tangle link diagram that lies outside the tangle disk and connects two adjacent arcs of the tangle (in the picture above labeled "**stacked tangle link projection**" there are two caps)

positive/negative curls – the part of a diagram removed or added by a Reiedemeister 1 move; positive curls are ones which factor out of Λ with an a; negative curls factor out of Λ with an a^{-1}

Theorem 2: If L is a link and A is the arc index of L, then $A \geq \text{spr}_z F(L) + 2$.

Main ideas:

- We look again at the pre-Kauffman polynomial, this time focusing on the highest M and lowest m degrees in a. For any stacked tangle link projection made of d tangles, $M \leq d\text{-}1$ and $m \geq 1\text{-}d$.

- A stacked tangle link projection formed from an arc presentation has only caps outside of the tangle disk.

- For each cap, we can do a Reidemeister 1 move to form a crossing in which the higher tangle arc goes over the lower, i.e. preserves the ordering of the tangles. Some will give positive curls and some will give negative curls.

- If we call the alternating d-stacked tangle link projection L, let L(pos) be the same projection with all p positive cap curls added. Let L(neg) be L with all (d-p) negative cap curls added. By the axioms of the pre-Kauffman polynomial, $\Lambda(L(pos))= a^p \Lambda(L)$ and $\Lambda(L(neg))= a^{(p-d)} \Lambda(L)$. Applying our degree bounds then yields the theorem.