PROBLEM # 27 (p. 188)

Let $||\cdot||$ be a norm on a real vector space V which satisfies the parallelogramm law:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \tag{1}$$

The problem asks to prove that the following:

$$\langle x, y\rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right) \tag{2}$$

defines an inner product product on V, and check that with this definition, $\langle x, x\rangle = ||x||^2$.

To prove this, one has to verify all of the properties given in the definition of an inner product.

The third and forth properties are easily proved, and I leave them to you as an exercise.

To prove the first two properties (linearity in the first argument), we need to check that

(a) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

and

(b) $\langle cx, y\rangle = c\langle x, y\rangle$;

To prove (a), consider the function

$$\Phi(x,y,z) = 4\cdot(\langle x+y, z\rangle - \langle x, z\rangle - \langle y, z\rangle) \tag{3}$$

(a) is equivalent to $\Phi(x,y,z) \equiv 0$ for all $x, y, z \in V$. Compute Φ using the definition of $\langle\cdot,\cdot\rangle$:

$$\begin{aligned} \Phi(x,y,z) &= ||x+y+z||^2 - ||x+y-z||^2 - \quad (4)\\ &\quad -||x+z||^2 + ||x-z||^2 - ||y+z||^2 + ||y-z||^2; \quad (5) \end{aligned}$$

Using the parallelogramm law, we obtain:

$$||x+y\pm z||^2 = 2||x\pm z||^2 + 2||y||^2 - ||x\pm z-y||^2 \tag{6}$$

Substitutin this into (5), we obtain

$$\begin{aligned} \Phi(x,y,z) &= \frac{1}{2}(||y+z+x||^2 + ||y+z-x||^2) - \quad (7)\\ &\quad -\frac{1}{2}(||y-z+x||^2 + ||y-z-x||^2 - ||y+z||^2 + ||y-z||^2; \quad (8) \end{aligned}$$

Now let's take the expressions for $\Phi(x,y,z)$ in (5) and (8) and compute one half of their sum. As the result, we obtain the following expression for $\Phi(x,y,z)$:

$$\begin{aligned} \Phi(x,y,z) &= \frac{1}{2}(||y+z+x||^2 + ||y+z+x||^2) - \\ &\quad -\frac{1}{2}(||y-z+x||^2 + ||y-z-x||^2 - ||y+z||^2 + ||y-z||^2); \end{aligned}$$

By (6), the first term is equal to $||y+z||^2 + ||x||^2$, and the second one is $-||y+z||^2 - ||x||^2$. Hence, $\Phi(x,y,z) \equiv 0$, which proves property (a).

To prove (b), consider (for fixed x and y) the function

$$\phi(c) = \langle cx, y\rangle - c\langle x, y\rangle$$

By definition (2), $\phi(0) = \frac{1}{4}(||g||^2 - ||g||^2) = 0$ and $\phi(-1) = 0$, since $\langle -x, y\rangle = -\langle x, y\rangle$. Therefore, for an integer n we get

$$\langle nx, y\rangle = \langle(\mathrm{sign}(n)\cdot(x+\cdots+x), y\rangle = \mathrm{sign}(n)\cdot(\langle x, y\rangle + \ldots\langle x, y\rangle) = |n|\cdot\mathrm{sign}(n)\cdot\langle x, y\rangle = n\langle x, y\rangle$$

Hence, $\phi(n) = 0$ for an integer n.

Let p, q be integers, and $q \neq 0$. Then

$$\langle p/q\cdot x, y\rangle = p\langle 1/q\cdot x, y\rangle = p/q\cdot q\langle 1/q\cdot x, y\rangle = p/q\langle x, y\rangle$$

Hence, for a rational number $c = p/q$ we have $\varphi(c) = 0$. It remains to prove that $\varphi(c)$ is a continuous function, since it would then follow that it is equal to zero identically. This would imply that $\langle cx, y\rangle = c\langle x, y\rangle$ for all c (rational or not).

You were not really expected to show continuity, but here's the argument.

We must show that if $c_n \to c$ is a convergent sequence, then $\phi(c_n) \to \phi(c)$. It is clear that $c_n\langle x, y\rangle \to c\langle x, y\rangle$, since $\langle x, y\rangle$ is a fixed number, independent of n. So we must show that $\langle c_n x, y\rangle \to \langle cx, y\rangle$. We have

$$\langle c_n x, y\rangle = \frac{1}{4}\left(||c_n x + y||^2 - ||c_n x - y||^2\right).$$

It is therefore enough to prove that $||c_n x + y|| \to ||cx + y||$ for all y (since by replacing y with $-y$ we would get also that $||c_n x - y|| \to ||cx - y||$).

Now, by the triangle inequality for the norm, we have

$$||c_n x + y|| = ||c_n x - cx + cx + y|| \leq ||c_n x - cx|| + ||cx + y|| = |c_n - c|||x|| + ||cx + y||.$$

Similarly,

$$||cx + y|| = ||cx - c_n x + c_n x + y|| \leq ||cx - c_n x|| + ||c_n x + y|| = |c - c_n|||x|| + ||c_n x + y||.$$

Summarizing, we get

$$||cx + y|| \leq |c - c_n|||x|| + ||c_n x + y|| \leq 2|c - c_n|||x|| + ||cx + y||,$$

or, equivalently,

$$||cx + y|| - |c - c_n|||x|| \leq ||c_n x + y|| \leq |c - c_n|||x|| + ||cx + y||.$$

Since $c_n \to c$, we have that $|c - c_n| \to 0$, so that $|c - c_n|||x|| \to 0$. Thus $||cx + y|| - |c - c_n|||x|| \to ||cx + y||$ and $|c - c_n|||x|| + ||cx + y|| \to ||cx + y||$. Applying the squeeze theorem, we finally get that

$$||cx + y|| = \lim ||c_n x + y||.$$