Cocycle and orbit superrigidity for lattices in SL(n,$\mathbb{R}$) acting on homogeneous spaces

(joint work with Sorin Popa)

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Orbit equivalence superrigidity

Theorem (Popa - V, 2008)

Let $n \geq 5$ and $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ a lattice.

Any stable orbit equivalence of the linear action $\Gamma \curvearrowright \mathbb{R}^n$ and an arbitrary free, non-singular, a-periodic action $\Lambda \curvearrowright (Y, \eta)$ is

- either, a conjugacy of $\Gamma \curvearrowright \mathbb{R}^n$ and $\Lambda \curvearrowright Y$,
- or, a conjugacy of $\Gamma/\{\pm 1\} \curvearrowright \mathbb{R}^n/\{\pm 1\}$ and $\Lambda \curvearrowright Y$, (if $-1 \in \Gamma$).

- **Stable orbit equivalence of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$:**

  Isomorphism $\Delta : X_0 \to Y_0$ between non-negligible subsets such that $\Delta(X_0 \cap \Gamma \cdot x) = Y_0 \cap \Lambda \cdot \Delta(x)$ a.e.

- $\Lambda \curvearrowright Y$ is **a-periodic** = not induced from $\Lambda_1 \curvearrowright Y_1$ with $\Lambda_1 < \Lambda$ and $Y_1 \subset Y$
  = no factor $Y \to Y_2$ with Y_2 discrete.

$\rightsquigarrow$ **At the end of the talk :**
other actions with such orbit equivalence superrigidity.

Cocycle superrigidity

Zimmer 1-cocycle : Suppose that $\Delta : X \to Y$ is an orbit equivalence of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$. Then, $\omega : \Gamma \times X \to \Lambda : \Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x)$ is a 1-cocycle for $\Gamma \curvearrowright X$ with target group Λ.

Cohomology of 1-cocycles : $\omega_1 \sim \omega_2$ if there exists $\varphi : X \to \Lambda$ satisfying $\omega_2(g, x) = \varphi(g \cdot x)\omega_1(g, x)\varphi(x)^{-1}$.

Cocycle superrigidity for $\Gamma \curvearrowright X$, targeting $\mathcal{U}$: every 1-cocycle with target group in $\mathcal{U}$ is cohomologous to a group morphism.

Theorem (Popa - V, 2008)

The following actions are cocycle superrigid with countable target groups (and, more generally, targeting closed subgroups of $\mathcal{U}(N)$).

- $\Gamma \curvearrowright \mathbb{R}^n$ for $n \geq 5$ and $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ a lattice.
- $\Gamma \times H \curvearrowright \mathrm{M}_{n,k}(\mathbb{R})$ for $n \geq 4k + 1$, $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ a lattice and $H \subset \mathrm{GL}(k, \mathbb{R})$ an arbitrary closed subgroup.
- $\Gamma \ltimes \mathbb{Z}^n \curvearrowright \mathbb{R}^n$ for $n \geq 5$, $\Gamma \subset \mathrm{SL}(n, \mathbb{Z})$ of finite index.

Property (T) for equiv. relations and group actions

Group Γ	**Countable measured equivalence rel.** $\mathcal{R}$	**Group action** $\Gamma \curvearrowright (X, \mu)$
Unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ $\pi(gh) = \pi(g)\pi(h)$	1-cocycle $c : \mathcal{R} \to \mathcal{U}(H)$ $c(x,z) = c(x,y)c(y,z)$	1-cocycle $\omega : \Gamma \times X \to \mathcal{U}(H)$ $\omega(gh, x) = \omega(g, h \cdot x)\omega(h, x)$
Invariant vector $\xi \in H$ $\pi(g)\xi = \xi$	Invariant vector $\xi : X \to H$ $\xi(x) = c(x,y)\xi(y)$	Invariant vector $\xi : X \to H$ $\xi(g \cdot x) = \omega(g,x)\xi(x)$
Almost inv. vectors $\xi_n \in H$, $\|\xi_n\| = 1$ $\|\pi(g)\xi_n - \xi_n\| \to 0$	Almost inv. vectors $\xi_n : X \to H$, $\|\xi_n(x)\| = 1$ $\|\xi_n(x) - c(x,y)\xi_n(y)\| \to 0$ a.e.	Almost inv. vectors $\xi_n : X \to H$, $\|\xi_n(x)\| = 1$ $\|\xi_n(g \cdot x) - \omega(g,x)\xi_n(x)\| \to 0$ a.e.

Property (T) : every ... with almost invariant vectors admits a non-zero invariant vector.

Some properties of property (T)

The following results were proven by Zimmer and Anantharaman-Delaroche.

- If $\Gamma \curvearrowright (X, \mu)$ is probability measure preserving, then the action has property (T) iff the group has.
- If $\Gamma \curvearrowright (X, \mu)$ is a non-singular, ergodic, essentially free action, the action has property (T) iff the orbit equivalence relation has.
- If $\mathcal{R}$ is an ergodic, countable, measured equiv. relation on (X, μ) and $X_0 \subset X$ is non-negl., then $\mathcal{R}$ has property (T) iff $\mathcal{R}|_{X_0}$ has.

Furman, Popa : property (T) is a measure equivalence invariant.

- If $N \triangleleft G$ is a closed normal subgroup, $G \curvearrowright (X, \mu)$ a non-singular action such that N acts freely and properly, then $G \curvearrowright X$ has property (T) iff $G/N \curvearrowright X/N$ has.

Example of a property (T) action

Proposition

Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a lattice and $k < n$.

The diagonal action $\Gamma \curvearrowright \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$ has property (T) iff $n \geq k + 3$.

Proof. Write $e_i \in \mathbb{R}^n$, the standard basis vectors and
$H := \{A \in \mathrm{SL}(n, \mathbb{R}) \mid Ae_i = e_i \text{ for all } i = 1, \ldots, k\}$.

- Identify $\Gamma \curvearrowright \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$ with $\Gamma \curvearrowright \mathrm{SL}(n, \mathbb{R})/H$.
- The action $\Gamma \curvearrowright \mathrm{SL}(n, \mathbb{R})/H$ has property (T) iff
  $\Gamma \times H \curvearrowright \mathrm{SL}(n, \mathbb{R})$ has property (T) iff
  $H \curvearrowright \mathrm{SL}(n, \mathbb{R})/\Gamma$ has property (T) iff
  H has property (T).
- But, $H \cong \mathrm{SL}(n - k, \mathbb{R}) \ltimes \mathbb{R}^{n-k}$. QED

Application : property (T) and fundamental groups

Recall : the fundamental group of a II_1 equivalence relation $\mathcal{R}$ on (X, μ) consists of the numbers $\mu(Y)/\mu(Z)$ where $\mathcal{R}|_Y \cong \mathcal{R}|_Z$.

Theorem (Popa - V, 2008)

Let $n \geq 4$ and $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ a lattice.

Define $\mathcal{R}$ as the restriction of the orbit relation of $\Gamma \curvearrowright \mathbb{R}^n$ to a subset of finite measure.

- The equivalence relation $\mathcal{R}$ has property (T), but nevertheless fundamental group $\mathbb{R}_+$.
- The equivalence relation $\mathcal{R}$ cannot be realized
  - as the orbit relation of a freely acting group,
  - as the orbit relation of an action of a property (T) group,

(and neither can the amplifications $\mathcal{R}^t$, $t > 0$).

Proving cocycle superrigidity: Popa's malleability

Definition (Popa, 2001)

The finite or infinite m.p. action $\Gamma \curvearrowright (X, \mu)$ is called malleable if there exists a m.p. flow $\mathbb{R} \overset{\alpha}{\curvearrowright} X \times X$ satisfying

- α_t commutes with the diagonal action $\Gamma \curvearrowright X \times X$,
- $\alpha_1(x, y) \in \{y\} \times X$.

We call the action *s*-malleable if there is an involution β on $X \times X$:

- β commutes with the diagonal action,
- $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$ and $\beta(x, y) \in \{x\} \times Y$.

Examples.

- The Bernoulli action $\Gamma \curvearrowright [0, 1]^\Gamma$ is *s*-malleable.
- When $\Gamma \subset SL(n, \mathbb{R})$, the action $\Gamma \curvearrowright \mathbb{R}^n$ is *s*-malleable, through $\alpha_t(x, y) = (\cos(\pi t/2)x + \sin(\pi t/2)y, -\sin(\pi t/2)x + \cos(\pi t/2)y)$.

Cocycle superrigidity for malleable actions

Theorem (Popa, 2005)

Let $\Gamma \curvearrowright (X, \mu)$ be s-malleable and finite measure preserving. Assume that $H \lhd \Gamma$ is a normal subgroup with the relative property (T) such that $H \curvearrowright (X, \mu)$ is weakly mixing.

Then, $\Gamma \curvearrowright X$ is cocycle superrigid targeting closed subgr. of $\mathcal{U}(N)$.

Theorem (Popa - V, 2008)

Let $\Gamma \curvearrowright (X, \mu)$ be s-malleable and infinite measure preserving. Assume that the diagonal action $\Gamma \curvearrowright X \times X$ has property (T) and that the 4-fold diagonal action $\Gamma \curvearrowright X \times X \times X \times X$ is ergodic.

Then, $\Gamma \curvearrowright X$ is cocycle superrigid targeting closed subgr. of $\mathcal{U}(N)$.

$\rightsquigarrow$ **What follows :** a proof for countable target groups, in the spirit of Furman's proof for Popa's theorem.

Exploiting property (T)

Fix a non-singular action $\Lambda \curvearrowright (Y, \eta)$ and a countable group $\mathcal{G}$.

- We may assume that $\eta(Y) = 1$.
- Denote by $\mathcal{Z}^1(\Lambda \curvearrowright Y, \mathcal{G})$ the set of 1-cocycles for $\Lambda \curvearrowright Y$ with values in $\mathcal{G}$.
- Turn $\mathcal{Z}^1(\Lambda \curvearrowright Y, \mathcal{G})$ into a Polish space by putting $\omega_n \to \omega$ iff for every $g \in \Lambda$, we have $\eta(\{x \in X \mid \omega_n(g, x) \neq \omega(g, x)\}) \to 0$.
- Remember : equivalence relation on $\mathcal{Z}^1(\Lambda \curvearrowright Y, \mathcal{G})$ given by cohomology.

Lemma

If $\Lambda \curvearrowright (Y, \eta)$ is an action with property (T), then the cohomology equivalence classes are open in $\mathcal{Z}^1(\Lambda \curvearrowright Y, \mathcal{G})$.

Exploiting malleability

The theorem that we want to prove

Let $\Gamma \curvearrowright (X, \mu)$ be s-malleable and infinite measure preserving. Assume that the diagonal action $\Gamma \curvearrowright X \times X$ has property (T) and that the 4-fold diagonal action $\Gamma \curvearrowright X \times X \times X \times X$ is ergodic.

Then, $\Gamma \curvearrowright X$ is cocycle superrigid with countable target groups $\mathcal{G}$.

Take a 1-cocycle $\omega : \Gamma \times X \to \mathcal{G}$.

- Consider the diagonal action $\Gamma \curvearrowright X \times X$ and the flow $\alpha_t \curvearrowright X \times X$.
- Define a path of 1-cocycles in $\mathcal{Z}^1(\Gamma \curvearrowright X \times X, \mathcal{G})$:
  $$\omega_0(g, x, y) = \omega(g, x) \quad \text{and} \quad \omega_t(g, x, y) = \omega_0(g, \alpha_t(x, y)).$$
- By the Lemma, $\omega_0 \sim \omega_1$: $\omega(g, x) = \varphi(g \cdot x, g \cdot y)\omega(g, y)\varphi(x, y)^{-1}$.
- Writing $F(x, y, z) = \varphi(x, y)\varphi(y, z)$, we have
  $F(g \cdot x, g \cdot y, g \cdot z) = \omega(g, x)F(x, y, z)\omega(g, z)^{-1}$.
- Ergodicity of $\Gamma \curvearrowright X \times X \times X$ implies : $F(x, y, z) = H(x, z)$.
  But then, $\varphi(x, y) = \psi(x)\rho(y)$.

$\rightsquigarrow$ ω follows cohomologous to a group morphism.

Cocycle superrigidity for a few concrete actions

All cocycle superrigidity statements : arbitrary targets in $\mathcal{U}(N)$.

For $n \geq 4k+1$ and $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, the action $\Gamma \curvearrowright \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$ is cocycle superrigid.

A general principle

If the 1-cocycle $\omega : \Gamma \times X \to \mathcal{G}$ is a group morphism on $\Lambda < \Gamma$ and if the diagonal action of $\Lambda \cap g\Lambda g^{-1}$ on $X \times X$ is ergodic for every $g \in \Gamma$, then ω is a group morphism.

The following actions are cocycle superrigid.

- $\Gamma \times H \curvearrowright \mathrm{M}_{n,k}(\mathbb{R})$ for $n \geq 4k+1$, $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ a lattice and $H \subset \mathrm{GL}(k,\mathbb{R})$ an arbitrary closed subgroup.
- $\Gamma \ltimes \mathbb{Z}^n \curvearrowright \mathbb{R}^n$ for $n \geq 5$, $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$ of finite index.

OE superrigidity for actions on flag manifolds

Real flag manifold X of signature $(d_1, \ldots, d_l, n)$ is the space of

$$\text{flags} \quad V_1 \subset V_2 \subset \cdots \subset V_l \subset \mathbb{R}^n \quad \text{with } \dim V_i = d_i.$$

Note : $\mathrm{PSL}(n, \mathbb{R}) \curvearrowright X$.
$\mathrm{P}^{n-1}(\mathbb{R})$ is the real flag manifold of signature $(1, n)$.

Theorem

Let X be the real flag manifold of signature $(d_1, \ldots, d_l, n)$ with $n \geq 4d_l + 1$. Let $\Gamma < \mathrm{PSL}(n, \mathbb{R})$ be a lattice.

Then, $\Gamma \curvearrowright X$ is OE superrigid.

More precisely, any stable orbit equivalence of $\Gamma \curvearrowright X$ and an arbitrary non-singular, essentially free, a-periodic action $\Lambda \curvearrowright Y$ is a conjugacy of $\tilde{\Gamma}/\Sigma \curvearrowright \tilde{X}/\Sigma$ and $\Lambda \curvearrowright Y$ for some subgroup $\Sigma < \Sigma_l$.

Notations : $\tilde{X}$ is the space of oriented flags, $\Sigma_l \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus l}$ acts by changing orientations and $\tilde{\Gamma}$ is generated by Γ and Σ_l.

We have $e \to \Sigma_l \to \tilde{\Gamma} \to \Gamma \to e$ and $(\tilde{\Gamma} \curvearrowright \tilde{X}) \to (\Gamma \curvearrowright X)$.

OE superrigidity for $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{T}^n$

We find back a slightly more precise version of a theorem of Furman (but only for $n \geq 5$).

Theorem

Let $n \geq 5$ be odd and $\Gamma < \mathrm{SL}(n,\mathbb{Z})$ of finite index.

Any stable orbit equivalence of $\Gamma \curvearrowright \mathbb{T}^n$ and an arbitrary non-singular, essentially free, a-periodic action $\Lambda \curvearrowright Y$ is a conjugacy between $\Gamma \ltimes (\mathbb{Z}/k\mathbb{Z})^n \curvearrowright (\mathbb{R}/k\mathbb{Z})^n$ and $\Lambda \curvearrowright Y$, for some $k \in \{0, 1, 2, \ldots\}$.

Questions : Fix a compact abelian group K and $n \geq 3$.

- Which group actions are stably orbit equivalent to $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright K^n$?
- Can one describe all cocycles for $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright K^n$ with ... targets ?

Weak relation morphisms

Definition

Let $\mathcal{R}$ on (X,μ) and $\mathcal{S}$ on (Y,η) be countable, ergodic, measured equivalence relations.

A weak morphism from $\mathcal{R}$ to $\mathcal{S}$ is a measurable map $\theta : X' \subset X \to Y' \subset Y$ between non-negligible subsets such that $\theta_*\mu|_{X'} \sim \eta|_{Y'}$ and $(\theta(x),\theta(y)) \in \mathcal{S}$ for almost all $(x,y) \in \mathcal{R}|_{X'}$.

A result that could very well be true

Let $n \geq 5$ and $\mathcal{R} = \mathcal{R}(\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{T}^n)$.

Any weak morphism from $\mathcal{R}$ to $\mathcal{R}(\Lambda \curvearrowright Y)$, where $\Lambda \curvearrowright Y$ is a free, ergodic, non-singular action, comes from an embedding of $\mathrm{SL}(n,\mathbb{Z}) \ltimes (\mathbb{Z}/k\mathbb{Z})^n \curvearrowright (\mathbb{R}/k\mathbb{Z})^n$ into $\Lambda \curvearrowright Y$ or an embedding of everything mod $\{\pm 1\}$ into $\Lambda \curvearrowright Y$.

Missing ingredient : the only globally $\mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^n$ invariant von Neumann subalgebras of $\mathrm{L}^\infty(\mathbb{R}^n)$ are the obvious ones.