# Some Consequences of Martin's Conjecture

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# On the pronunciation of foreign words ...

**H.W. Fowler (1858-1933), *The King's English*, 2nd ed. 1908.**

Naïveté *is a word for which there is a clear use; and though the Englishman can pronounce it without difficulty if he chooses, he generally prefers doing without it altogether to attempting a precision that strikes him as either undignified or pretentious.*

# Countable Borel equivalence relations

## Definition

*The Borel equivalence relation $E$ on the standard Borel space $X$ is said to be countable iff every $E$-class is countable.*

## Standard Example

Let $G$ be a countable (discrete) group and let $X$ be a standard Borel $G$-space. Then the corresponding orbit equivalence relation $E_G^X$ is a countable Borel equivalence relation.

## Theorem (Feldman-Moore)

*If $E$ is a countable Borel equivalence relation on the standard Borel space $X$, then there exists a countable group $G$ and a Borel action of $G$ on $X$ such that $E = E_G^X$.*

# Borel reductions

## Definition

*Let $E$, $F$ be Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively.*

- $E \leq_B F$ *iff there exists a Borel map* $f : X \to Y$ *such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

*In this case,* $f$ *is called a Borel reduction from* $E$ *to* $F$.

- $E \sim_B F$ *iff both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *iff both* $E \leq_B F$ *and* $E \nsim_B F$.

## Definition

*More generally,* $f : X \to Y$ *is a Borel homomorphism from* $E$ *to* $F$ *iff*

$$x \, E \, y \implies f(x) \, F \, f(y).$$

# A Cantor-Bernstein Theorem

## Theorem

*If $E$, $F$ are countable Borel equivalence relations on the standard Borel spaces $X$, $Y$, then the following are equivalent:*

- $E \sim_B F$.
- *There exist complete Borel sections $A \subseteq X$ and $B \subseteq Y$ such that*

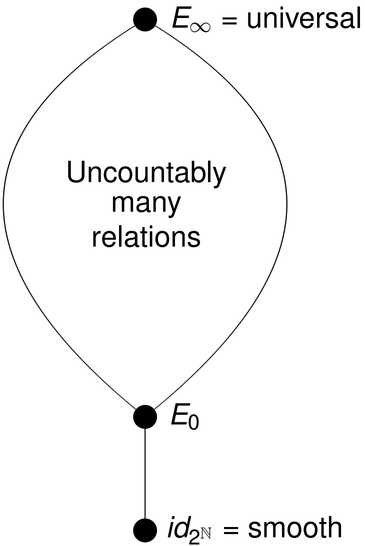
$$( A, E \restriction A ) \cong ( B, F \restriction B )$$

  *via a Borel isomorphism.*

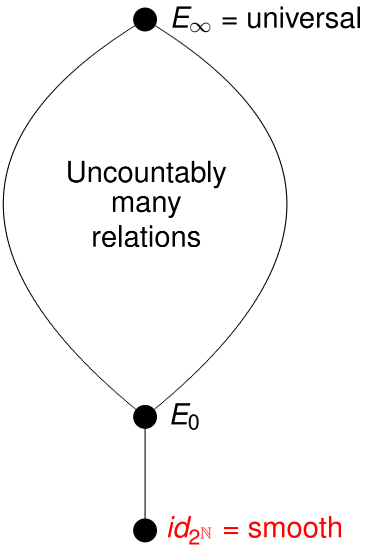
## Definition

*A Borel subset $A \subseteq X$ is a complete section iff $A$ intersects every $E$-class.*

# Countable Borel equivalence relations



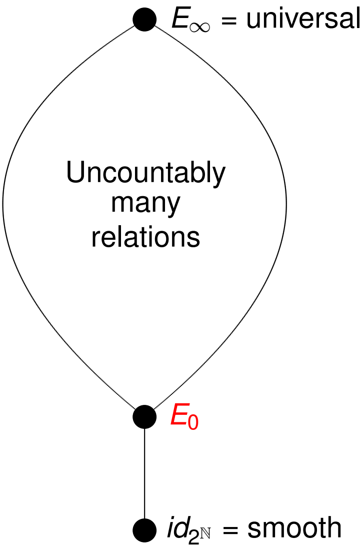
# Countable Borel equivalence relations




**Definition**

*The Borel equivalence relation $E$ is smooth iff $E \leq_B id_{2^{\mathbb{N}}}$, where $2^{\mathbb{N}}$ is the space of infinite binary sequences.*

# Countable Borel equivalence relations



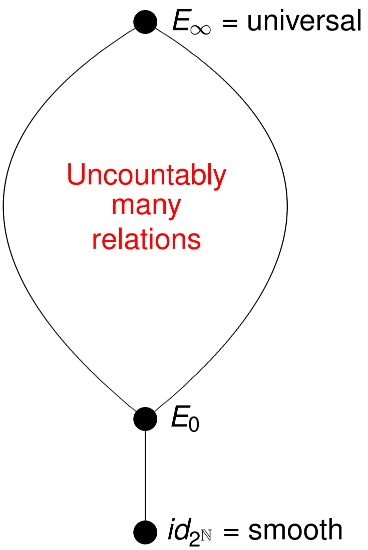

**Definition**

*The Borel equivalence relation $E$ is smooth iff $E \leq_B id_{2^{\mathbb{N}}}$, where $2^{\mathbb{N}}$ is the space of infinite binary sequences.*

**Definition**

*$E_0$ is the equivalence relation of eventual equality on the space $2^{\mathbb{N}}$ of infinite binary sequences.*

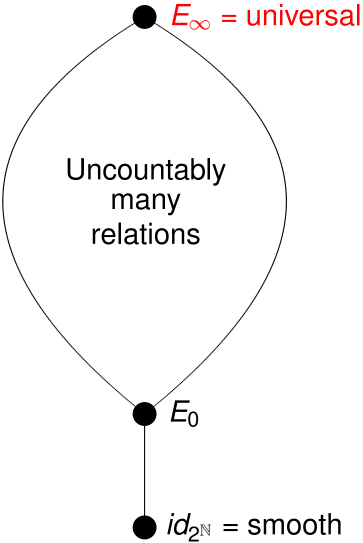
# Countable Borel equivalence relations




**Theorem (Adams-Kechris 2000)**

*There exist $2^{\aleph_0}$ many countable Borel equivalence relations up to Borel bireducibility.*

# Countable Borel equivalence relations




## Definition

*A countable Borel equivalence relation $E$ is universal iff $F \leq_B E$ for every countable Borel equivalence relation $F$.*

## Theorem (JKL)

*The orbit equivalence relation $E_\infty$ of the shift action of the free group $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is universal.*

# The measurable vs. Borel settings

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space.

## The Fundamental Question in the Borel setting

*To what extent does the data $( X, E_G^X )$ "remember" the group $G$ and its action on $X$?*

## Dirty Little Secret

We cannot possibly recover the group $G$ from the data $( X, E_G^X )$ unless we add the hypotheses that:

- $G$ acts freely on $X$; and
- there exists a $G$-invariant probability measure $\mu$ on $X$.

# Essentially free relations

## Definition

- *The countable Borel equivalence relation $E$ on $X$ is free iff there exists a countable group $G$ with a free Borel action on $X$ such that $E_G^X = E$.*
- *The countable Borel equivalence relation $E$ is essentially free iff there exists a free countable Borel equivalence relation $F$ such that $E \sim_B F$.*

## Theorem (Easy Consequence of Popa Superrigidity)

*The universal countable Borel equivalence relation $E_\infty$ is not essentially free.*

# Strongly universal relations

## Question (Thomas 2006)

*Does there exist a countable Borel equivalence relation $E$ on a standard Borel space $X$ such that:*

- *there exists an ergodic $E$-invariant probability measure $\mu$ on $X$;*
- *whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$, then $E \restriction Y$ is countable universal?*

## Main Theorem (*MC*)

- *Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a (not necessarily $E$-invariant) Borel probability measure on $X$.*
- *Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is not universal.*

# A slight digression ...

## Conjecture

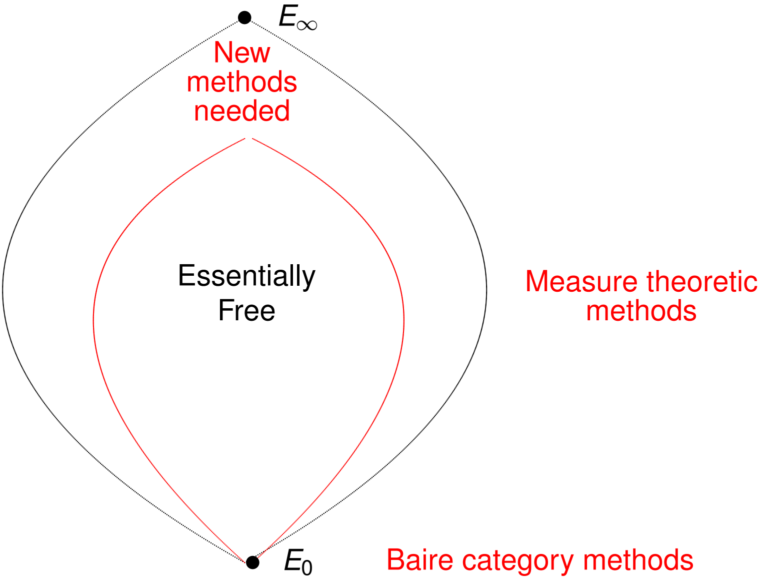
*There does not exist a f.g. group $G$ such that for every f.g. group $H$, there exists a f.g. group $K$ such that:*

- *$K$ is quasi-isometric to $G$, and*
- *$H \hookrightarrow K$.*

## Corollary (*MC*)

*The quasi-isometry relation on the space of f.g. groups is not essentially hyperfinite, essentially treeable, ...*

# Countable Borel Equivalence Relations



# Turing Reducibility

## Convention

Throughout the powerset $\mathcal{P}(\mathbb{N})$ will be identified with $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

## Definition

*If $x, y \in 2^{\mathbb{N}}$, then $x$ is Turing reducible to $y$, written $x \leq_T y$, iff there exists a $y$-oracle Turing machine which computes $x$.*

## Remark

In other words, there is an algorithm which computes $x$ modulo an oracle correctly answer questions of the form "Is $n \in y$?"

# A Notion of Largeness

## Definition

*For each $z \in 2^{\mathbb{N}}$, the corresponding cone is $C_z = \{ x \in 2^{\mathbb{N}} \mid z \leq_T x \}$.*

- Suppose $z_n = \{ a_{n,\ell} \mid \ell \in \mathbb{N} \} \in 2^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and define

$$\oplus z_n = \{ p_n^{a_{n,\ell}} \mid n, \ell \in \mathbb{N} \} \in 2^{\mathbb{N}},$$

  where $p_n$ is the $n$th prime.

- Then $z_m \leq_T \oplus z_n$ for each $m \in \mathbb{N}$ and so $C_{\oplus z_n} \subseteq \bigcap_n C_{z_n}$.

## Remark

It is well-known that if $C \subsetneq 2^{\mathbb{N}}$ is a proper cone, then $C$ is both null and meager.

# The Turing equivalence relation

## Definition

*The Turing equivalence relation* $\equiv_T$ *on* $2^{\mathbb{N}}$ *is defined by*

$$x \equiv_T y \quad \text{iff} \quad x \leq_T y \ \& \ y \leq_T x,$$

*where* $\leq_T$ *denotes Turing reducibility.*

## Remark

- Clearly $\equiv_T$ is a countable Borel equivalence relation on $2^{\mathbb{N}}$.
- However, $\equiv_T$ is not essentially free and is not induced by the action of any countable subgroup of $\mathrm{Sym}(\mathbb{N})$.

# Martin's Theorem

## Theorem (Martin)

*If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either $X$ or $2^{\mathbb{N}} \smallsetminus X$ contains a cone.*

## Remark

For later use, notice that if $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then the following are equivalent:

(i) $X$ contains a cone.

(ii) For all $z \in 2^{\mathbb{N}}$, there exists $x \in X$ with $z \leq_T x$.

# Ergodicity for Turing equivalence

## Theorem (Folklore)

*If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel map, then there exists a cone $C$ such that $\varphi \upharpoonright C$ is a constant map.*

## Proof.

- For each $n \in \mathbb{N}$, there exists $\varepsilon_n \in \{ 0, 1 \}$ such that $X_n = \{ x \in 2^{\mathbb{N}} \mid \varphi(x)(n) = \varepsilon_n \}$ contains a cone.
- Hence there exists a cone $C \subseteq \bigcap X_n$ and clearly $\varphi \upharpoonright C$ is a constant map.

$\square$

# Proof of Martin's Theorem

- Suppose that $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset.
- Consider the two player Borel game $G(X)$

$$s(0) \quad s(1) \quad s(2) \quad s(3) \quad \cdots$$

  where $I$ wins iff $s = (\, s(0)\, s(1)\, s(2) \, \cdots \,) \in X$.
- Then the Borel game $G(X)$ is determined. Suppose, for example, that $\sigma : 2^{<\mathbb{N}} \to 2$ is a winning strategy for $I$.
- Let $\sigma \leq_T t \in 2^{\mathbb{N}}$ and consider the run of $G(X)$ where
  - $II$ plays $t = (\, s(1)\, s(3)\, s(5) \, \cdots \,)$
  - $I$ responds with $\sigma$ and plays $(\, s(0)\, s(2)\, s(4) \, \cdots \,)$.
- Then $s \in X$ and $s \equiv_T t$. Hence $t \in X$ and so $C_\sigma \subseteq X$.

# Strong Ergodicity

## Definition

- *Suppose that $E$, $F$ are countable Borel equivalence relations on the standard Borel spaces $X$, $Y$ and that $\mu$ is an $E$-invariant Borel probability measure on $X$.*
- *Then $E$ is said to be $F$-ergodic iff for every Borel homomorphism $\varphi : X \to Y$ from $E$ to $F$, there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that $\varphi$ maps $Z$ into a single $F$-class.*

## Example (Jones-Schmidt)

$E_\infty$ is $E_0$-ergodic.

# Strong Ergodicity for Turing equivalence

## Definition

*Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$. Then $\equiv_T$ is said to be E-m-ergodic iff for every Borel homomorphism $\varphi : 2^{\mathbb{N}} \to X$ from $\equiv_T$ to $E$, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi$ maps $C$ into a single $E$-class.*

## Target

Classify the countable Borel equivalence relations $E$ such that $\equiv_T$ is $E$-$m$-ergodic.

## Question

*When is it “obvious” that $\equiv_T$ is not $E$-m-ergodic?*

# Weakly universal countable Borel equivalence relation

## Definition

- *The Borel homomorphism $\varphi : X' \to X$ from $E'$ to $E$ is said to be a weak Borel reduction iff $\varphi$ is countable-to-one. In this case, we write $E' \leq_B^w E$.*
- *A countable Borel equivalence relation $E$ is said to be weakly universal iff $F \leq_B^w E$ for every countable Borel equivalence relation $F$.*

## Some Examples

- If $E$ is universal, then $E$ is weakly universal.
- The Turing equivalence relation $\equiv_T$ is weakly universal.

## Observation

*If $E$ is weakly universal, then $\equiv_T$ is not $E$-m-ergodic.*

# Strong Ergodicity for Turing equivalence

## Strong Ergodicity Theorem (*MC*)

*If $E$ is any countable Borel equivalence relation, then exactly one of the following conditions holds:*

(a) *$E$ is weakly universal.*

(b) *$\equiv_T$ is $E$-$m$-ergodic.*

## Remark

- There are currently no nonsmooth countable Borel equivalence relations $E$ for which it has been proved that $\equiv_T$ is $E$-$m$-ergodic.
- In particular, it is not known whether $\equiv_T$ is $E_0$-$m$-ergodic, where $E_0$ denotes the eventual equality equivalence relation on $2^{\mathbb{N}}$.

# The Kechris-Miller Theorem

## Observation

*Let $E, F$ be countable Borel equivalence relations.*

- *If $E \leq_B F$, then $E \leq_B^w F$.*
- *If $E \subseteq F$, then $E \leq_B^w F$.*

## Theorem (Kechris-Miller)

*If $E, F$ are countable Borel equivalence relations on the uncountable standard Borel spaces $X, Y$ respectively, then the following conditions are equivalent:*

(i) $E \leq_B^w F$.

(ii) *There exists a countable Borel equivalence relation $S \subseteq F$ on $Y$ such that $S \sim_B E$.*

# The weak universality of Turing equivalence

## Proposition (Kechris)

*$\equiv_T$ is weakly universal.*

## Proof.

Identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$. $\square$

## Remark

If $C = \{x \in 2^{\mathbb{N}} \mid z \leq_T x\}$ is a cone, then the map $y \mapsto y \oplus z$ is a weak Borel reduction from $\equiv_T$ to $\equiv_T \restriction C$ and hence $\equiv_T \restriction C$ is also weakly universal.

# Martin's Conjecture

## The Martin Conjecture (*MC*)

*If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi$ maps $C$ into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T \varphi(x)$ for all $x \in C$.*

## Theorem (Slaman-Steel)

*Suppose that $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If $\varphi(x) <_T x$ on a cone, then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi$ maps $C$ into a single $\equiv_T$-class.*

# Some easy consequences of Martin's Conjecture

**Theorem (*MC*)**

*If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

- (i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi$ maps $C$ into a single $\equiv_T$-class.*
- (ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$.*

*Furthermore, in case (ii), if $D \subseteq 2^{\mathbb{N}}$ is any cone, then $[\,\varphi(D)\,]_{\equiv_T}$ contains a cone.*

# Some easy consequences of Martin's Conjecture

## Corollary (*MC*)

- $\equiv_T <_B (\equiv_T \sqcup \equiv_T)$.
- *In particular,* $\equiv_T$ *is not countable universal.*

## Corollary (*MC*)

*If* $A \subseteq 2^{\mathbb{N}}$ *is a* $\equiv_T$*-invariant Borel subset, then* $\equiv_T \restriction A$ *is weakly universal iff* $A$ *contains a cone.*

## Remark

There are currently no naturally occurring classes $D \subseteq 2^{\mathbb{N}}$ for which it is known that $\equiv_T \restriction D$ is not weakly universal.

# Proof of the Strong Ergodicity Theorem ($MC$)

- Let $E$ be any countable Borel equivalence relation.
- Since $E \leq_B^w \equiv_T$, we can suppose that $E \subseteq \equiv_T$.
- Suppose that $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $E$ and that $\varphi$ does not map any cone to a single $E$-class.
- Then $\varphi$ is also a Borel homomorphism from $\equiv_T$ to $\equiv_T$ and clearly $\varphi$ does not map any cone to a single $\equiv_T$-class.
- Hence there exists a cone $C$ such that $\varphi \restriction C$ is countable-to-one.
- Since $\equiv_T \restriction C$ is weakly universal and $(\equiv_T \restriction C) \leq_B^w E$, it follows that $E$ is weakly universal.

# Strongly universal relations

## Question (Thomas 2006)

*Does there exist a countable Borel equivalence relation $E$ on a standard Borel space $X$ such that:*

- *there exists an ergodic $E$-invariant probability measure $\mu$ on $X$;*
- *whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$, then $E \restriction Y$ is countable universal?*

## Main Theorem (*MC*)

- *Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a (not necessarily $E$-invariant) Borel probability measure on $X$.*
- *Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is not weakly universal.*

# The Heart of the Matter

## Main Lemma

- *Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a (not necessarily $E$-invariant) Borel probability measure on $X$.*
- *Let $\varphi : X \to 2^{\mathbb{N}}$ be a weak Borel reduction from $E$ to $\equiv_T$.*
- *Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $\varphi[Y]$ is disjoint from a cone.*

## Proof of Main Theorem (*MC*).

(*MC*) implies that if $C$ is any cone, then $\equiv_T \upharpoonright (2^{\mathbb{N}} \setminus C)$ is not weakly universal. $\square$

# Growth Rates

## Definition

*Identifying each* $r \in 2^{\mathbb{N}}$ *with the corresponding subset of* $\mathbb{N}$*, define the Borel map* $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *by:*

- $\theta(r)$ *is the increasing enumeration of* $r \cap 2\mathbb{N}$*, if* $r \cap 2\mathbb{N}$ *is infinite;*
- $\theta(r)$ *is the zero function, otherwise.*

## Observation

*For each* $h \in \mathbb{N}^{\mathbb{N}}$*, the* $\equiv_T$*-invariant Borel set*

$$D_h = \{\, r \in 2^{\mathbb{N}} \mid (\,\exists s \in 2^{\mathbb{N}}\,)\ s \equiv_T r \text{ and } h < \theta(s)\,\}$$

*contains a cone.*

# Growth Rates

## Definition

*If $g, h \in \mathbb{N}^{\mathbb{N}}$, then $g \leq^* h$ iff $g(n) \leq h(n)$ for all but finitely many $n \in \mathbb{N}$.*

## Observation (Folklore)

*If $(X, \mu)$ is a standard Borel probability space and $\pi : X \to \mathbb{N}^{\mathbb{N}}$ is a Borel map, then there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that*

$$\mu(\{ x \in X \mid \pi(x) \leq^* h \}) = 1.$$

- For each $n \in \mathbb{N}$, there exists $h(n) \in \mathbb{N}$ such that

$$\mu(\{ x \in X \mid \pi(x)(n) > h(n) \}) \leq (1/2)^{n+1}.$$

- By the Borel-Cantelli Lemma, we have that

$$\mu(\{ x \in X \mid \pi(x)(n) > h(n) \text{ for infinitely many } n \}) = 0.$$

# Yet another application of Feldman-Moore

## Lemma

*Suppose that* $\sigma : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *is a Borel map. Then there exists a Borel map* $\psi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *such that for all* $r \in 2^{\mathbb{N}}$,

$$\sigma(s) \leq^* \psi(r) \qquad \textit{for all } s \equiv_T r$$

## Proof.

- By Feldman-Moore, we can realize $\equiv_T$ by a Borel action of a countable group $G = \{\, \gamma_m \mid m \in \mathbb{N} \,\}$.
- Define $\psi(r)(n) = \max\{\, \sigma(\gamma_m \cdot r)(n) \mid m \leq n \,\}$.

□

# Proof of Main Lemma

- Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a Borel probability measure on $X$.
- Let $\varphi : X \to 2^{\mathbb{N}}$ be a weak Borel reduction from $E$ to $\equiv_T$.
- Let $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined earlier.
- By Feldman-Moore, there exists a Borel map $\psi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if $r \equiv_T s$, then $\theta(s) \leq^* \psi(r)$.
- Let $\pi : X \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by $\pi = \psi \circ \varphi$.
- Then there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that the Borel set $Y = \{ x \in X \mid \pi(x) \leq^* h \}$ satisfies $\mu(Y) = 1$.
- Clearly $\varphi[Y] \cap D_h = \emptyset$.

# Some Open Problems

**Problem**

*Prove that $\equiv_T$ is $E_0$-m-ergodic.*

**Problem**

- *Find a naturally occurring class of Turing degrees $D \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction D$ is not weakly universal.*
- *For example, how about the classes of minimal degrees, hyperimmune-free degrees, ... ?*