Some appications of free stochastic calculus to II_1 factors.

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Free Entropy Dimension (Voiculescu)

$G = \langle S \rangle$ finitely generated discrete group, $\tau_G : \mathbb{C}\mathbb{F}_{|S|} \to \mathbb{C}$.
$\prod^{\omega} M_{n\times n} = \prod_{k=1}^{\infty} M_{k\times k}/\{(x_k) : \lim_{k\to\omega} \frac{1}{k} Tr(x_k^* x_k) = 0\}$

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Properties of $\delta(G)$

You can think of $\Gamma(G)$ as an analog of the set $\mathcal{A}(G) = \{\text{homomorphisms } G \to Aut(X, \mu)\}$.

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More on $\delta(G)$

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Vanishing results.

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$$\delta(G) = \beta_1^{(2)}(G) - \beta_0^{(2)}(G) + 1.$$

To prove the conjecture one needs:

$(G) \leq \dim_{L(G)}\{$space of $\ell^2(G)$-valued cocycles$\}$ (i

on-cojugate "actions" in $\Gamma(G)$, then there are non-

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On the other hand, the second direction:

$$\delta(G) \geq \dim_{L(G)}\{\text{space of } \ell^2(G)\text{-valued cocycles}\}$$

looks constructive/easy.
Given $c : G \to \ell^2(G)$ construct elements in $\Gamma(G)$.

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Deformations and estimate on δ.

Let $S_1, \ldots, S_m$ be a free semicircular family in $L(\mathbb{F}_\infty)$.

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Stochastic Calculus

Main input: Brownian motion process: family of random variables $B([s,t))$, $t > s \geq 0$ so that $B([s,t))$ are Gaussian, $B([s,t)) + B([t,r)) = B([s,r))$ if $s < t < r$ and $B([s,t))$ is independent from $B([s',t'))$ if $[s,t) \cap [s',t'] = \emptyset$.
Then one can write

$$B([0,t)) = \int_0^t dB_t.$$

Furthermore, for nice enough probability measures μ, there exists a process X_t with the properties that:

X_t is stationary, i.e., X_t has distribution μ for all t

X_t satisfies the stochastic differential equation $dX_t = \phi(X_t) \cdot dB_t - \zeta(X_t)dt$

(Note: $dB_t \sim O(t^{1/2})$).

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Very roughly, this means that

$$X_{t+\epsilon} = X_t + \underbrace{\phi(X_t)B[t, t+\epsilon]}_{O(\varepsilon^{1/2})} - \zeta(X_t)\epsilon + O(\epsilon^{3/2})$$

In particular, the map

$$f(X) \mapsto f(X_{t^2})$$

gives rise to an isomoprhism $\alpha_t : L^\infty(\mathbb{R}, \mu) \to W^*(X_t) \subset W^*(X, B[s, t] : s < t)$ which satisfies

$$\alpha_t(f(X)) = f(X) + \underbrace{\phi(X)f'(X)B([0, t^2))}_{O(t)} + O(t^2)$$

(i.e., it "exponentiates" the derivation $\partial(f) = \phi f'$, $\partial : \text{polynomials} \to L^2(\mathbb{R}, \mu)$).

Free Stochastic Differential Equations

Main fact: $L(\mathbb{F}_\infty)$ is generated by a family of self-adjoint elements $\vec{S}([0,t)) = \{S_k([0,t))\}_{k=1}^n$ associated to intervals $[0,t) \subset \mathbb{R}$, $t \geq 0$. For all t, $\vec{S}([0,t))$ is a free semicircular family and $\vec{S}([0,t))$ is free from $\vec{S}([0,t')) - \vec{S}([0,t))$ if $t' > t$.
$\vec{S}([0,t))$ is the free analog of Brownian motion on $\mathbb{R}^n$ measured at time t.

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Stationary solutions.

If X_t is a stationary solution (this means that $\partial_t \tau(f(\vec{X}_t)) = 0$ for all non-commutative polynomials f), then the map

$$\alpha_{t^2} : f(\vec{X}) \mapsto f(\vec{X}_{t^2})$$

extends to an isomorphism from $W^*(\vec{X})$ to $W^*(X_t) \subset W^*(\vec{X}, \vec{S}([s,t)) : s < t)$. Moreover,

$$\alpha_t(X_j) = X_j + \partial(X_j) \# d\vec{S}_{t^2} + O(t^2).$$

so we can apply the estimate on δ.

Existence of stationary solutions.

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Group case.

As a corollary you get:

$$\delta(G) = \beta_1^{(2)}(G) - \beta_0^{(2)}(G) - 1$$

provided that $H^1(G; \ell^2(G))$ can be generated by a cocycle whose values at the generators of G are "analytic" (i.e., are convergent non-commutative power series rather than arbitrary elements of ℓ^2).

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