Measure Equivalence Rigidity and Bi-exactness of Groups

Hiroki Sako

Contents

- Gromov's Measure Equivalence
- Ozawa's class $\mathcal{S}$
- Bi-exactness and Key Prop.
- Results : Direct Product of $\mathcal{S}$ groups
  Wreath Product
  Amalgamated Free Product

Gromov's Measure Equivalence

Defn $G \sim_{ME} \Gamma$ Measure Equivalent

$\overset{\text{defn}}{\Longleftrightarrow}$ ${}^{\exists} G \times \Gamma \curvearrowright \Sigma$ Std. Ms Sp

${}^{\exists} X, Y \subset \Sigma$ s.t

- $\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma X = \bigsqcup_{g \in G} g Y$
- $m(X), m(Y) < +\infty$

Class $\mathcal{S}$ group

Defn $\cdot$ $\underset{\text{discrete}}{\Lambda} \curvearrowright \overset{\text{Compact Sp}}{V}$ is amenable

$\underset{d}{\Leftrightarrow}$ $^{\exists}\mu_i : V \longrightarrow \mathrm{Prob}(\Lambda)$ s.t.

$^{\forall}\lambda \in \Lambda$, $\lim_i \sup_{v \in V} \| \mu_i^{\lambda v} - \lambda \mu_i^{v} \|_1 = 0$

$\cdot$ $G \in \mathcal{S} \underset{d}{\Leftrightarrow} G \times G \curvearrowright \beta G \setminus G$ is amenable

Ozawa's Thm $G \in \mathcal{S} \Rightarrow LG$ is solid, i.e.,
$^{\forall}\mathcal{A} \subset LG$ diffuse, $\mathcal{A}' \cap LG$ is injective.

Examples of $\mathcal{S}$

$\{\text{amenable groups}\} \subset \mathcal{S} \subset \{\text{exact groups}\}$,

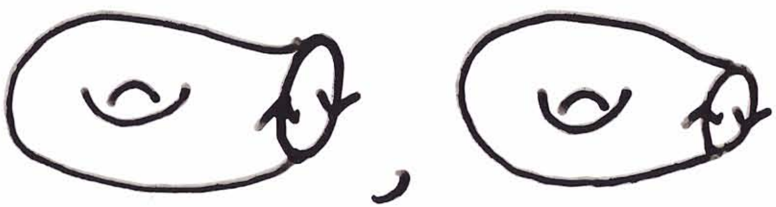
$\{\text{Word-hyp groups}\} \subset \mathcal{S}$, $\mathbb{Z}^2 \rtimes SL_2\mathbb{Z} \in \mathcal{S}$

$(\text{Non-ame}) \times (\text{infinite}) \notin \mathcal{S}$

$(\mathbb{F}_2 \times \mathbb{Z}) * \mathbb{Z} \notin \mathcal{S}$

<u>Thm</u>: $\Gamma \sim_{ME} G$, $G \in \mathcal{S} \Rightarrow \Gamma \in \mathcal{S}$

<u>Cor</u>: $\Gamma \sim_{ME} \mathbb{F}_n \Rightarrow L\Gamma$ is solid

π_1 of  , ...

(Gaboriau)

Bi-exactness (Brown-Ozawa)

Defn $\mathcal{L}$ a family of subgroups of Γ

- $T \subset \Gamma$ is small if $T \subset \bigcup^{\text{fin}} s_i \Lambda_i t_i$ for $^\exists s_i, t_i \in \Gamma$, $\Lambda_i \in \mathcal{L}$
- $c_0(\Gamma; \mathcal{L}) = \overline{\bigcup_{T:\text{small}} \ell_\infty T}^{\text{norm}} \subset \ell_\infty \Gamma$
- Γ is bi-exact rel to $\mathcal{L}$ if $\Gamma \times \Gamma \curvearrowright \ell_\infty \Gamma / c_0(\Gamma; \mathcal{L})$ is amenable.

<u>Example</u>: $\Gamma_1, \Gamma_2 \in \mathcal{S} \Rightarrow \Gamma_1 \times \Gamma_2$ is b-e. rel to $\{\Gamma_1, \Gamma_2\}$

<u>Defn</u> $H < G$, $\Lambda < \Gamma$, Σ coupling for $G \underset{ME}{\sim} \Gamma$

$H \prec_\Sigma \Lambda$ if $^\exists H \times \Lambda$-inv $\Omega \subset \Sigma$

s.t. $m(\Lambda\text{-fund. dom. of } \Omega) < +\infty$

$\boxed{\text{Key Prop}}$ Γ bi-exact rel to $\mathcal{L}$, $H < G$

Σ ergodic ME coupling for $G \underset{ME}{\sim} \Gamma$

$Z_G(H)$ non-ame $\Rightarrow {}^\exists \Lambda \in \mathcal{L}$, $H \prec_\Sigma \Lambda$

Direct Product of Sgroups

Thm $\underset{\text{non-ame}}{G_1} \times \underset{\text{non-ame}}{G_2} \underset{ME}{\sim} \underset{\in S}{\Gamma_1} \times \underset{\in S}{\Gamma_2}$

$\Rightarrow G_1 \underset{ME}{\sim} \Gamma_1,\ G_2 \underset{ME}{\sim} \Gamma_2$ (or $\Gamma_1 \leftrightarrow \Gamma_2$)

Pf Σ erg coupling for $G_1 \times G_2 \underset{ME}{\sim} \Gamma_1 \times \Gamma_2$

G_2 is non-ame $\rightsquigarrow G_1 <_\Sigma \Gamma_i$

G_1 is non-ame $\rightsquigarrow G_2 <_\Sigma \Gamma_j$

Since $G_1 \times G_2 \not<_\Sigma \Gamma_i$, $j \neq i$. ...... $\square$

- Furman's Tech
- ME $\rightsquigarrow$ WOE

<u>OE rigidity Thm</u> Free, MP Actions

$$\alpha : \underset{\text{mildly mixing}}{G \curvearrowright X}, \quad \beta : \underset{\text{irreducible}}{\Gamma_1 \times \Gamma_2 \curvearrowright Y}$$

$\Gamma_i \in \mathcal{S}$, non-ame, ICC

$$\alpha \underset{\text{WOE}}{\simeq} \beta \implies \alpha \text{ and } \beta \text{ are virtually conjugate.}$$

Wreath Product

ame exact

Lem (Ozawa) $A \wr G = (\bigoplus A) \rtimes G$ is b-e rel to $\{G\}$

Thm: $A \wr (G_1 \times G_2) \underset{ME}{\sim} B \wr (\Gamma_1 \times \Gamma_2)$

ame inf non-ame the same cond

$$\Rightarrow G_1 \times G_2 \underset{ME}{\sim} \Gamma_1 \times \Gamma_2$$

Pf Σ erg coupling for two wreath products

G_2 is non-ame $\rightsquigarrow G_1 \prec_\Sigma (\Gamma_1 \times \Gamma_2)$

$\exists$ Max. Embedding Ω_1. $\rightsquigarrow \Omega_1$ is G_2-inv.

$\rightsquigarrow \Omega_1$ gives $(G_1 \times G_2) \prec_\Sigma (\Gamma_1 \times \Gamma_2)$ $\cdots$ $\square$

AFP

Lem (Brown-Ozawa) $\underset{\text{ame}}{\overset{\text{exact}}{G_1} *_A \overset{\text{exact}}{G_2}}$ is b-e rel to $\{G_1, G_2\}$

Thm $\underset{\text{ame}}{G_1 *_A G_2} \sim_{ME} \underset{\text{ame}}{\Gamma_1 *_B \Gamma_2}$

G_i, Γ_j : exact, non-ame $\times$ non-ame

$\Rightarrow G_1 \sim_{ME} \Gamma_1,\ G_2 \sim_{ME} \Gamma_2$ (or $\Gamma_1 \leftrightarrow \Gamma_2$)

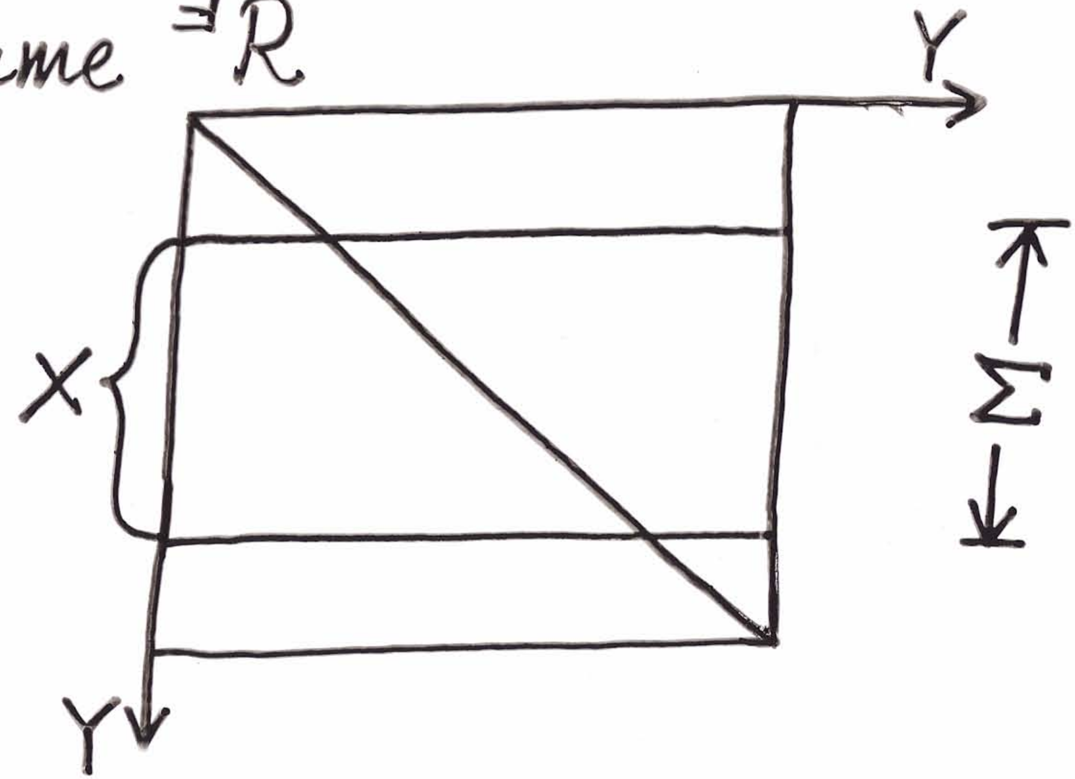
Pf $G_1^0 < G_1$ subgroup with non-ame centralizer

$\exists$ Max. Emb Ω for $G_1^0 \prec_\Sigma \Gamma_i$. Ω is G_1-inv

$\rightsquigarrow \Omega$ gives $G_1 \prec_\Sigma \Gamma_i$ $\square$

Pf for Key Prop

We may assume $^{\exists}R$



$\Gamma \curvearrowright Y$

$G \curvearrowright X$

$\mathcal{H} := L^2 R$, $\quad T \subset \Gamma$ small $\rightsquigarrow$ Proj e_T

$\ell^\infty \Gamma$, $B = L^\infty Y \rtimes_{red} \Gamma$, $C = JBJ \curvearrowright \mathcal{H}$

$$K := \overline{\bigcup e_T \mathcal{B}(\mathcal{H}) e_T}^{norm}$$

<u>Claim</u> : $B \otimes_{\mathbb{C}} C \longrightarrow (C^*(B,C)+K)/K$

is min $\otimes$ conti.

$\because$ $A := \ell_\infty \Gamma / c_0(\Gamma;\mathcal{L}) \otimes_{\min} L^\infty Y \otimes_{\min} J L^\infty Y J$

$^\exists$Natural $A \rtimes_{red} (\Gamma \times \Gamma) \longrightarrow \dfrac{C^*(B,C,\ell^\infty\Gamma)+K}{K}$ ⌟

∘ Suppose $^\forall \Lambda \in \mathcal{L}$, $H \not\prec_{\Sigma} \Lambda$

$\Downarrow$

∘ $^\forall T \subset \Gamma$ small, $^\exists h \in H$ s.t. $h e_T$ "almost $\perp$" e_T.

$\Downarrow$

∘ $C^*_{red}(Z_G(H)) \otimes_{\mathbb{C}} J \sim J \longrightarrow \mathcal{B}(L^2(R|x))$

is conti. w.r.t. $\otimes_{\min}$ □