Entropy in Measurable Dynamics

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The triple (G, X, μ) is a *dynamical system* .

Two systems (G, X_1, μ_1) and (G, X_2, μ_2) are *isomorphic* if there exists a measure-space isomorphism $\phi : X_1 \to X_2$ with $\phi(gx) = g\phi(x)$ for a.e. $x \in X_1$ and for all $g \in G$.

Main Problem: Classify systems up to isomorphism.

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- G acts on K^G by shifting. $(gx)(f) = x(g^{-1}f)$ for all $x \in K^G, g, f \in G$.
- (G, K^G, κ^G) is the *Bernoulli shift* over G with *base measure* κ.

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von Neumann's question: Is the full 2-shift over $\mathbb{Z}$ isomorphic to the full 3-shift over $\mathbb{Z}$?

Ideas from Information Theory

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$I(t)$ for $0 \leq t \leq 1$ should satisfy:

1. $I(t) \geq 0$.
2. $I(t)$ is continuous.
3. $I(ts) = I(t) + I(s)$.

So $I(t) = -\log_b(t)$ for some $b > 1$.

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The *Shannon entropy* of ϕ is the average amount of information one gains by learning the value of ϕ. I.e.,

$$H(\phi) = \sum_{a \in A} \mu(\phi^{-1}(a)) I(\phi^{-1}(a)) = -\sum_{a \in A} \mu(\phi^{-1}(a)) \log \Big(\mu(\phi^{-1}(a)) \Big).$$

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Let $T : X \to X$ be measure-preserving. The *entropy rate* of ϕ w.r.t T is:

$$h(T, \phi) = \lim_{n \to \infty} \frac{1}{2n+1} H\Big(\bigvee_{i=-n}^{n} \phi \circ T^i \Big).$$

Coding

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ϕ is a finite generator if Φ is an isomorphism from (G, X, μ) to $(G, A^G, \Phi_*\mu)$.

Kolmogorov's entropy

Theorem (Kolmogorov, 1958)

Let $T : X \to X$ be an automorphism of (X, μ) . If ϕ and ψ are finite generators for $(\mathbb{Z}, X, \mu) = (\langle T \rangle, X, \mu)$ then $h(T, \phi) = h(T, \psi)$.

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Theorem (Sinai, 1959)

If ϕ is any finite observable then $h(T, \phi) \leq h(\mathbb{Z}, X, \mu)$. Hence we may define the entropy of $(\mathbb{Z}, X, \mu)$ to be $\sup_\phi h(T, \phi)$.

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Theorem (Kolmogorov, 1958)

If $(\mathbb{Z}, K^{\mathbb{Z}}, \kappa^{\mathbb{Z}})$ is isomorphic to $(\mathbb{Z}, L^{\mathbb{Z}}, \lambda^{\mathbb{Z}})$ then $H(\kappa) = H(\lambda)$. So the full 2-shift is not isomorphic to the full 3-shift.

Questions

- Does the converse hold?
- What if $\mathbb{Z}$ is replaced with some other group G?

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Definition

A group G is *Ornstein* if whenever (K, κ), (L, λ) are two standard probability spaces with $H(\kappa) = H(\lambda)$ then (G, K^G, κ^G) is isomorphic to (G, L^G, λ^G).

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- Is every countably infinite group Ornstein?

Classification

Theorem (Ornstein, 1970)

Bernoulli shifts over $\mathbb{Z}$ are completely classified by their entropy.

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J. C. Kieffer. *A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space*. Ann. Probability 3 (1975), no. 6, 1031–1037.

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Theorem

If G is infinite and amenable then Bernoulli shifts over G are completely classified by their entropy (which equals their base measure entropy).

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What if G is nonamenable?

Factor maps

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Let (G, X, μ), (G, Y, ν) be two systems and $\phi : X \to Y$ a measurable map with $\phi_*\mu = \nu$, $\phi(gx) = g\phi(x)$ for a.e. $x \in X$ and all $g \in G$. Then ϕ is a *factor map* from (G, X, μ) to (G, Y, ν).

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Theorem (Sinai, 1959)

Entropy is nonincreasing under factor maps.

If G is amenable then the full n-shift over G has entropy $\log(n)$.

So the full 2-shift over G cannot factor onto the full 4-shift over G.

The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $\mathbb{F} = \langle a, b \rangle$ is the rank 2 free group then the full 2-shift over $\mathbb{F}$ factors onto the full 4-shift over $\mathbb{F}$.

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Theorem

If G contains a subgroup isomorphic to $\mathbb{F}$ then every nontrivial Bernoulli shift over G factors onto every other Bernoulli shift over G.

New Results

Theorem

If G is a sofic group (e.g., a residually finite group) then Kolmogorov's direction holds. I.e., if (G, K^G, κ^G) is isomorphic to (G, L^G, λ^G) then $H(\kappa) = H(\lambda)$.

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The idea: For $n > 0$, count the number of sequences $(a_1, a_2, \ldots, a_n)$ with elements $a_i \in A$ that approximate the above sequence.

Approximations in the case $G = \mathbb{Z}$

First idea: For each $a \in A$, we'd like the frequency of a in $(a_1, a_2, \dots, a_n)$ to be close to $\mu(\phi^{-1}(a))$.

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Let $\phi^m : X \to A^m$ be the function $\phi^m(x) = (\phi(x), \phi(Tx), \ldots, \phi(T^{m-1}x))$. Let $\phi^m_* \mu$ be the pushforward measure on A^m.

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Fix $m \geq 0$. Given a sequence $\alpha = (a_1, a_2, \dots, a_n)$, define $\pi(\alpha) : A^m \to [0, 1]$ by

$$\pi(\alpha)(b_1, \dots, b_m) = \frac{\#\text{appearances of } (b_1, \dots, b_m) \text{ in } (a_1, \dots, a_n)}{n}.$$

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Let $d_m(\alpha, \phi)$ be the l^1-distance between $\pi(\alpha)$ and $\phi^m_*\mu$.

$$d_m(\alpha, \phi) = \sum_{(b_1,\ldots,b_m)\in A^m} \Big| \pi(\alpha)(b_1, \ldots, b_m) - \phi_*^m \mu\big((b_1, \ldots, b_m)\big) \Big|.$$

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Theorem

$$h(T,\phi) = \lim_{m\to\infty} \inf_{\epsilon>0} \lim_{n\to\infty} \frac{1}{n} \log \Big|\{\alpha = (a_1,\ldots,a_n) \;:\; d_m(\alpha,\phi) < \epsilon\}\Big|.$$

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For $F \subset G$, let $V_F \subset \{1, \dots, m\}$ be the set of all v such that

$$\begin{aligned} \sigma(fg)v &= \sigma(f)\sigma(g)v \ \forall f, g \in F \text{ with } fg \in F, \\ \sigma(f)v &\neq \sigma(g)v \ \Leftarrow f \neq g \in F. \end{aligned}$$

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σ is a *(F,ϵ)-approximation* to G if $|V_F| \geq (1-\epsilon)m$.

Sofic Groups

A sequence $\Sigma = \{\sigma_i\}_{i=1}^{\infty}$ of maps $\sigma_i : G \to \mathrm{Sym}(m_i)$ is a *sofic approximation* if σ_i is an (F_i, ϵ_i)-approximation with $\epsilon_i \to 0$ and $F_i \to G$ (i.e., $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} F_i = G$).

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- Is every countable group sofic?

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The idea: Let ζ be the uniform measure on $\{1, \ldots, m_i\}$. Count the number of observables $\psi : \{1, \ldots, m_i\} \to A$ that approximate ϕ.

Approximating

If $F \subset G$ is finite, let $\phi^F \colon X \to A^F$ be the map $\phi^F(x) := \Big(\phi(fx)\Big)_{f \in F}$.

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Let $d_F(\phi, \psi)$ be the l^1-distance between $\phi^F_* \mu$ and $\psi^F_* \zeta$.

Entropy for sofic groups

$$h(\Sigma, \phi) := \inf_{F \subset G} \inf_{\epsilon > 0} \limsup_{i \to \infty} \frac{\log \left(\left| \{ \psi : \{1, \ldots, m_i\} \to A \ : \ d_F(\phi, \psi) \leq \epsilon \} \right| \right)}{m_i}.$$

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Theorem

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Theorem

$h(\Sigma,G,K^G,\kappa^G)=H(\kappa)$.

Proof sketch

We identify $\phi : X \to A$ with the partition $\{\phi^{-1}(a) \mid a \in A\}$ it induces on X (up to measure zero).

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ϕ and ψ are *equivalent* if there exists finite subsets $K, L \subset G$ such that ϕ^K refines ψ and ψ^L refines ϕ.

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ϕ is a *simple splitting* of ψ if there exists $f \in G$ and an observable ω refined by ψ such that

$$\phi = \psi \vee \omega \circ f.$$

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Proposition

If ϕ is a simple splitting of ψ then $h(\Sigma, \phi) = h(\Sigma, \psi)$.

Applications: von Neumann algebras

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Major problem: classify these algebras up to isomorphism in terms of the group/action data.

Theorem (Connes, 1976)

If G is infinite and amenable and the action $G \curvearrowright (X, \mu)$ is free and ergodic then $L^\infty(X, \mu) \rtimes G$ is hyperfinite. In particular, all such algebras are isomorphic.

Rigidity

Definition

(G_1, X_1, μ_1) and (G_2, X_2, μ_2) are *von Neumann equivalent (vNE) if* $L^\infty(X_1, \mu_1) \rtimes G_1 \cong L^\infty(X_2, \mu_2) \rtimes G_2$.

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Theorem (Popa, 2006)

If G is a countably infinite ICC property T group then any two von Neumann equivalent Bernoulli shifts over G are isomorphic.

Corollary

If, in addition, G is sofic and Ornstein then Bernoulli shifts over G are classified up to vNE by base measure entropy. E.g., this occurs when $G = PSL_n(\mathbb{Z})$ for $n > 2$.

Applications: orbit equivalence

Definition

(G_1, X_1, μ_1) is orbit equivalent (OE) to (G_2, X_2, μ_2) if there exists a measure-space isomorphism $\phi : X_1 \to X_2$ such that $\phi(G_1 x) = G_2 \phi(x)$ for a.e. $x \in X_1$.

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Theorem (Dye 1959, Connes-Feldman-Weiss 1981)

If G_1 and G_2 are amenable and infinite and their respective actions are ergodic and free then (G_1, X_1, μ_1) is OE to (G_2, X_2, μ_2).

OE rigidity

Theorem (Kida, 2008)

Let G be the mapping class group of a genus g surface with n holes. Assume $3g + n - 4 > 0$ *and* $(g, n) \notin \{(1,2),(2,0)\}$. *If* (G, X, μ) *is free and ergodic then it is strongly orbitally rigid. I.e., if* (G_2, X_2, μ_2) *is free, ergodic and OE to* (G, X, μ) *then it is isomorphic to* (G, X, μ).

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If G is as above then Bernoulli shifts over G are classified up to OE by base measure entropy.

Free Groups: a special case

For each $i \geq 1$, let $\sigma_i : \mathbb{F} = \langle s_1, \ldots s_r \to \mathrm{Sym}(i)$ be chosen uniformly at random.
Let $\Sigma = \{\sigma_i\}$.
Define

$$h(\Sigma, \phi) := \inf_{F \subset G} \inf_{\epsilon > 0} \limsup_{i \to \infty} \frac{\log \left(\mathbb{E}\left[\left| \{\psi : \{1, \ldots, i\} \to A \ : \ d_F(\phi, \psi) \leq \epsilon\} \right| \right] \right)}{i}$$

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If

$$F(\phi) := -(2r-1)H(\phi) + \sum_{i=1}^{r} H(\phi \vee \phi \circ s_i).$$

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then $h(\Sigma, \phi) = f(\phi)$.

Systems of algebraic origin

Let $\mathcal{G}$ be a compact separable group and let $T : \mathcal{G} \to \mathcal{G}$ be a group automorphism fixing a closed normal subgroup $\mathcal{N}$.

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$h(T, \mathcal{G}, Haar_{\mathcal{G}}) = h(T, \mathcal{N}, Haar_{\mathcal{N}}) + h(T, \mathcal{G}/\mathcal{N}, Haar_{\mathcal{G}/\mathcal{N}})$.

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If $\mathcal{G}$ is totally disconnected and $\mathbb{F}$ acts by automorphisms on $\mathcal{G}$ with closed normal subgroup $\mathcal{N}$ then
$f(\mathbb{F}, \mathcal{G}, Haar_{\mathcal{G}}) = f(\mathbb{F}, \mathcal{N}, Haar_{\mathcal{N}}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, Haar_{\mathcal{G}/\mathcal{N}})$.

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Let $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{F}}$. Let $\mathcal{N} = \{\mathbf{0}, \mathbf{1}\}$. By Ornstein-Weiss' example,

$$\mathcal{G}/\mathcal{N} \cong \mathcal{G} \times \mathcal{G} = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\mathbb{F}}.$$

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$$\begin{aligned} f(\mathbb{F}, \mathcal{G}, \mathrm{Haar}_{\mathcal{G}}) &= f(\mathbb{F}, \mathcal{N}, \mathrm{Haar}_{\mathcal{N}}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, \mathrm{Haar}_{\mathcal{G}/\mathcal{N}}) \\ \log(2) &= -\log(2) + \log(4). \end{aligned}$$

Further Results & Open Questions

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- Entropy for hyperlinear group actions.