Some results and problems in W*-rigidity

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Notations:

Γ, Λ countable (discrete infinite) groups.

$(X,\mu), (Y,\nu)$ probability measure spaces.

$\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ measure preserving actions (in general *free ergodic*).

$M = L^\infty(X) \rtimes \Gamma$ the *group measure space* vN algebra of $\Gamma \curvearrowright X$. $\{u_g\}_g \subset M$ the *canonical unitaries* (implement $\Gamma \curvearrowright L^\infty(X)$ by $u_g a u_g^* = g(a)$, $a \in L^\infty(X) \subset M$).

$\mathcal{L}(\Gamma) = \{u_g\}''$ the *group vN algebra* of Γ.

$\mathcal{R}_\Gamma = \{(t, gt) \mid t \in X\}$ the (countable) *equivalence relation* implemented by $\Gamma \curvearrowright X$;

Note: $A = L^\infty(X)$ maximal abelian in $M = \mathcal{L}(\mathcal{R}_\Gamma)$ and its normalizer generates M, i.e. A is *Cartan subalgebra* in M.

Fact: $\Gamma \curvearrowright X$ free ergodic $\Rightarrow L^\infty(X) \rtimes \Gamma$ II_1 factor; $\mathcal{L}(\Gamma)$ II_1 factor iff Γ is ICC (...).

If $t > 0$, then $\mathcal{R}^t$, M^t denote the *amplification of* $\mathcal{R} = \mathcal{R}_\Gamma$, resp $M = L^\infty(X) \rtimes \Gamma$ by t.

$\mathcal{F}(\mathcal{R}) = \{t > 0 \mid \mathcal{R}^t \simeq \mathcal{R}\}$, $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$ the *fundamental group* of $\mathcal{R}$, resp M.

Conjugacy of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ means $\Delta : (X, \mu) \simeq (Y, \nu)$ and $\delta : \Gamma \simeq \Lambda$ with $\Delta(gt) = \delta(g)\Delta(t)$, $\forall g \in \Gamma, t \in X$.

Note: Conjugacy implements isomorphism $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ by $\Sigma a_g u_g \mapsto \Sigma \Delta(a_g) v_{\delta(g)}$

Fact: An iso $\Delta : (X, \mu) \simeq (Y, \nu)$ extends to $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ iff Δ is an *orbit equivalence* (*OE*), i.e. $\Delta(\mathcal{R}_\Gamma) = \mathcal{R}_\Lambda$, or $\Delta(\Gamma t) = \Lambda\Delta(t)$, $\forall t$.

Thus: *Conjugacy* $\Rightarrow$ *OE* $\Rightarrow$ *iso of vN algebras* (W^*-equivalence)

- **The general W*-rigidity question**

Recover "as much as possible" of $\Gamma \curvearrowright X$ from its OE class $\mathcal{R}_\Gamma$, or merely from its W*eq class $L^\infty(X) \rtimes \Gamma$. Ideally, describe all iso $L^\infty(X) \rtimes \Gamma \simeq (L^\infty(Y) \rtimes \Lambda)^t$ (or $\mathcal{R}_\Gamma \simeq (R_\Lambda)^t$).

In particular, calculate the "symetry groups" of $\mathcal{R} = \mathcal{R}_\Gamma$, $M = L^\infty(X) \rtimes \Gamma$, i.e. Out($\mathcal{R}$), $\mathcal{F}(\mathcal{R})$, resp Out(M), $\mathcal{F}(M)$

- **Deformation/Rigidity approach** P 01-04

$Q, N \subset M$ vN subalg, with: (a) Q "rigid" in M; (b) M "soft" relative to N. Using *intertwining techniques*, deduce $Q \prec_M N$: $\exists u \in \mathcal{U}(M)$ with $uQu^* \subset N$ (roughly...)

Examples: 1° If $Q \subset N \otimes R$ has *spectral gap*, then $Q \prec N$. Thus, if $N_1 \otimes R_1 = N_2 \otimes R_2$ with N_i non-Gamma, then $\exists u$ with $uN_1u^* = N_2$.

2° $\mathcal{L}(\Gamma_1) \otimes \mathcal{L}(\mathbb{F}_{n_1}) = \mathcal{L}(\Gamma_2) \otimes \mathcal{L}(\mathbb{F}_{n_2})$, with Γ_i Kazhdan, then similar conclusion

3° $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ and Γ Kazhdan, $\Lambda \curvearrowright Y$ profinite or Bernoulli, then $\mathcal{L}(\Gamma) \prec \mathcal{L}(\Lambda)$

4° $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ and $\Gamma = \Gamma_1 \times \Gamma_2$, or $\Gamma = \Gamma_1 * \Gamma_2$, with Γ_2 amenable. If $H \subset \Lambda$ has T then $\mathcal{L}(H) \prec L^\infty(X) \rtimes \Gamma_1$.

- **OE superrigidity results**

Furman-99: Many actions $\Gamma \curvearrowright X$ of h.r.l. (such as $SL(n,\mathbb{Z}) \curvearrowright \mathbb{T}^n$) are *OE-superrigid*, i.e. $\forall$ $\Gamma \curvearrowright X \sim_{OE} \Lambda \overset{free}{\curvearrowright} Y$, "comes" from a conjugacy (...).

Popa-05,06: Bernoulli actions $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ of prop. (T) groups are OE Superrigid. Same true for $\Gamma \curvearrowright X$ *sub-malleable mixing* (e.g. quotients of Bernoulli & Gaussians) with Γ satisfying one of the following:

- $\exists$ $H \subset \Gamma$ infinite w-normal with rel prop (T)

- $\exists H \subset \Gamma$ infinite w-normal with non-amenable commutant (e.g. $\Gamma = H \times H'$, H' non-amen)

In fact: P05, P06 shows Bernoulli $\Gamma \curvearrowright X$ of such Γ is $\mathcal{U}_{fin}$-Cocycle Superrigid (CSR).

Ioana 07: Profinite actions $\Gamma \curvearrowright (X, \mu)$ of property (T) groups Γ are "virtually" OE superrigid (...). In fact, they are $\mathcal{U}_{dis}$-CSR.

Kida 07: Γ mapping class group, then $\forall \Gamma \curvearrowright X$ free ergodic is OE superrigid. K09: $\forall$ mixing action of $\Gamma = SL(n, \mathbb{Z}) *_{T_n} SL(n, \mathbb{Z})$, $n \geq 3$, is OE-superrigid.

Ozawa-Popa 08: $\mathbb{F}_n \times \mathbb{F}_m \curvearrowright X$ profinite are $U(n)$-CSR, $\forall n$.

Q1 Find other classes of OE superrigid & cocycle superrigid (CSR) group actions ($\mathcal{U}_{fin}$, $\mathcal{U}_{dis}$, etc). What are the groups Γ for which $\exists\, \Gamma \curvearrowright X$ CSR ($\mathcal{U}_{fin}$, $\mathcal{U}_{dis}$, etc)?

Q2 Find larger classes $\mathcal{U}$ of "target" groups with the property that any Bernoulli action of a Kazhdan (or other) group is $\mathcal{U}$-CSR.

Q3 Find the class $\mathcal{CS}$ (resp. $\mathcal{OES}$) of groups Γ such that any Bernoulli Γ-action is $\mathcal{U}_{fin}$-CSR (resp $\mathcal{U}_{dis}$-CSR, resp OE superrigid).

Conjecturally (Peterson, Chifan, Ioana, Popa): $\Gamma \in \mathcal{CS}$ iff $\beta_1^{(2)}(\Gamma) = 0$ (Peterson-Sinclair: $\beta_1^{(2)}(\Gamma) \neq 0$ implies Bernoulli $\Gamma \curvearrowright X$ are not $\mathbb{T}$-CSR)

Q4 Calculate $\mathsf{H}^2(\mathcal{R}_\Gamma)$ more generally $\mathsf{H}^n(\mathcal{R}_\Gamma)$ for some $\Gamma \curvearrowright X$, e.g. for Bernoulli. (No such calculations exist for $n \geq 2$; for Γ Kazhdan and action Bernoulli, one expects $\mathsf{H}^n(\mathcal{R}_\Gamma) = \mathsf{H}^n(\Gamma)$.)

Q5* Is it true that $\forall \Gamma, \Lambda$ non-amenable, any OE of Bernoulli actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ comes from a conjugacy ? For free groups ? Can one have $\mathbb{F}_2 \curvearrowright \{0,1\}^{\mathbb{F}_2} \sim_{OE_{1/2}} \mathbb{F}_3 \curvearrowright X_0^{\mathbb{F}_3}$, for some X_0? (Bowen 09: No/Yes)

RelatedQ Extend Bowen '07 entropy invariant to Bernoulli actions of arbitrary non-amenable groups.

- **W*-rigidity & unique Cartan decomp**

Sample W*-rigidity [P04, P06]:
Γ, Λ ICC groups, with Γ either: Kazhdan; or $\exists H \subset \Gamma$ w-normal with rel prop (T); or $\exists H \subset \Gamma$, $|H| = \infty$, $H' \cap \Gamma$ non-amenable. If $\Gamma \curvearrowright X$ free mixing and $\Lambda \curvearrowright Y$ Bernoulli, then any $\theta : L^\infty(X) \rtimes \Gamma \simeq (L^\infty(Y) \rtimes \Lambda)^t$ comes from a conjugacy (*Strong W*-Rigidity* result).

Q1 Find group actions $\Gamma \curvearrowright X$ that are W*-Superrigid, i.e. given any other free ergodic action $\Lambda \curvearrowright Y$, any isomorphism $L^\infty(X) \rtimes \Gamma \simeq (L^\infty(Y) \rtimes \Lambda)^t$ comes from a conjugacy.

Obs: To "upgrade" OE-superr to W^*-superr requires *unique Cartan decomposition* results for the corresponding grp meas space factors.

Obs: If Γ Kazhdan (or product group) & $\Gamma \curvearrowright X$ Bernoulli implies $L^\infty(X) \rtimes \Gamma$ has unique Cartan (or merely unique crossed product dec), then $\Gamma \curvearrowright X$ follows W*-Superrigid (by [P05, P06])

Q1 Does Γ Kazhdan, $\Gamma \curvearrowright X$ Bernoulli and $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ imply Λ Kazhdan? **Obs**: If so, then Bernoulli actions of Kazhdan groups follow W*-Superrigid (by [P04]).

Related Obs: If $PSL(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$ gives a factor with unique Cartan, then this action would follow W*-Superrigid (by [Fu99]).

Q2 Find classes of factors $L^\infty(X) \rtimes \Gamma$ with unique Cartan, or merely unique grp meas sp Cartan.

Ozawa-Popa 07: If $\Gamma = \mathbb{F}_{n_1} \times ... \times \mathbb{F}_{n_k}$ and $\Gamma \curvearrowright X$ profinite, then $L^\infty(X) \rtimes \Gamma$ has unique Cartan, up to unitary conjugacy. Also: $\mathcal{L}(\Gamma)$ has no Cartan.

OP 08: Same holds if $\Gamma = \Gamma_1 \times ... \times \Gamma_k$, with Γ_i lattices in either $SO(n, 1)$, $n \geq 2$, or $SU(n, 1)$.

Q3 Show some $\Gamma \curvearrowright X$ as above is OE-superrigid. (Then $\Gamma \curvearrowright X$ follows W*-superrigid.)

Q4 Show for some Γ as before, uniqueness of Cartan holds for any action (Note: For $\Gamma = \mathbb{F}_n \times \mathbb{F}_m$, this would imply Bernoulli $\mathbb{F}_n \times \mathbb{F}_m \curvearrowright X$ are W*-superrigid).

Q5 If Γ non-amenable & $\Gamma \curvearrowright X$ Bernoulli, then $L^\infty(X) \rtimes \Gamma$ has unique Cartan ?

Conjecture : If $\beta_1^{(2}(\Gamma) \neq 0$, then $L^\infty(X) \rtimes \Gamma$ has unique Cartan $\forall$ $\Gamma \curvearrowright X$. (Maybe even for $\beta_n^{(2}(\Gamma) \neq 0$, for some $n \geq 1$.)

Recent progress along these lines:

Peterson 09: If $\Gamma = \Gamma_1 * \Gamma_2$, where $\Gamma_1 \neq 1$ & Γ_2 doesn't have Haagerup prop, and $\Gamma \curvearrowright X$ profinite, then $L^\infty(X) \rtimes \Gamma$ has unique grp meas sp Cartan. Also, there exist virtually superrigid profinite $\Gamma \curvearrowright X$.

Popa-Vaes 09: Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, with: Γ_1 either contains a Kazhdan grp, or two commuting non-amenable grps; Σ amenable, $\neq \Gamma_2$, "w-malnormal" in Γ. Let $\Gamma \curvearrowright X$ arbitrary. Then $L^\infty(X) \rtimes \Gamma$ has unique grp meas sp Cartan.

Popa-Vaes 09: The following actions are W*superrigid:

- $\forall$ free mixing $PSL(n,\mathbb{Z}) *_{T_n} PSL(n,\mathbb{Z}) \curvearrowright X$, where $n \geq 3$ and T_n the subgroup of upper diagonal matrices.

- $\forall$ Bernoulli action of: (a) $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, with Γ_1 Kazhdan, Σ infinite, amenable, proper normal in Γ_2, "w-malnormal" in Γ_1;
(b) $\Gamma = (H \times H) *_\Sigma \Gamma_2$ with H ICC non-amenable, Σ infinite amenable embedded diagonally in $H \times H$, proper normal in Γ_2.

• **Questions on the fundamental group**

Q1* Is any fund. group $\mathcal{F}(\mathcal{R}_\Gamma)$, $\mathcal{F}(M)$, either countable or $\mathbb{R}^*_+$? ($\forall \mathcal{R}$ OE rel, $\forall M$ separable II_1 factor) No: P-Vaes 08.

Q2* $\exists\, \Gamma \curvearrowright X$ with $\mathcal{F}(\mathcal{R}_\Gamma) \neq 1, \mathbb{R}^*_+$? Can $\mathcal{F}(\mathcal{R}_\Gamma)$ contain irrationals (when $\neq \mathbb{R}^*_+$) if $\Gamma \curvearrowright X$ free? Yes: P-Vaes 08.

Q3* $\exists \mathbb{F}_\infty \curvearrowright X$ free ergodic with $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty}) = 1$, resp with $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty}) = \mathbb{R}^*_+$? (By Gaboriau, $\mathcal{F}(\mathcal{R}_{\mathbb{F}_n}) = 1$, $\forall \mathbb{F}_n \curvearrowright X$ free ergodic , $n < \infty$). Yes: P-Vaes 08.

Q4* $\exists \mathbb{F}_n \curvearrowright X$ free ergodic with $\mathrm{Out}(\mathcal{R}_{\mathbb{F}_n}) = 1$? Yes: Popa-Vaes for $n = \infty$, Gaboriau for $2 \leq n < \infty$

Popa-Vaes 08: For Γ countable group, denote $\mathcal{S}_{factor}(\Gamma) = \{\mathcal{F} \subset \mathbb{R}_+ \mid \exists \Gamma \curvearrowright X$ free erg with $\mathcal{F}(L^\infty(X) \rtimes \Gamma) = \mathcal{F}\}$. Similarly $\mathcal{S}_{eqrel}(\Gamma)$. Then

- If Λ_1, Λ_2 fin. gen. ICC, one of which has (T) and $\Gamma = \Lambda_1 * \Lambda_2$, then $\mathcal{S}_{factor}(\Gamma) = \{1\}$.

- $\mathcal{S}_{eqrel}(\mathbb{F}_\infty), \mathcal{S}_{factor}(\mathbb{F}_\infty)$ are "huge" (...).

Q4 $\mathcal{S}_{factor}(\mathbb{F}_n) = \{1\}$, $2 \leq k < \infty$? For all Γ with $\beta_1^{(2)}(\Gamma) \neq 0, \infty$? Note: It is known that $\mathcal{F}(L^\infty(X) \rtimes \mathbb{F}_k) = 1$, for many $\mathbb{F}_k \curvearrowright X$ ([P01], [OP07]).

Q5 (Gaboriau) Show that if $\mathcal{L}(\mathcal{R}) = \mathcal{L}(\mathcal{S})$, for some eq rel $\mathcal{R}$, $\mathcal{S}$, then $\beta_n^{(2)}(\mathcal{R}) = \beta_n^{(2)}(\mathcal{R})$, $\forall n$.

Q6 Is it true that $\{1\} \in \mathcal{S}_{factor}(\Gamma)$, $\forall\Gamma$ non-amenable ? (Note: If Γ amenable then $S_{factor}(\Gamma) = S_{eqrel}(\Gamma) = \{\mathbb{R}_+\}$). If $\Gamma \curvearrowright X$ Bernoulli, then $\mathcal{F}(L^\infty(X) \rtimes \Gamma) = 1$, $\forall\Gamma$ non-amenable ?

Q7 Axiomatization of subgroups $\mathcal{F} \subset \mathbb{R}_+$ for which $\exists$ separable II_1 factor M, (resp eq rel $\mathcal{R}$) such that $\mathcal{F}(M) = \mathcal{F}$ (resp ...).

Q8 $\mathcal{S}_{factor}(\Gamma) \subset \mathcal{S}_{factor}(\mathbb{F}_\infty) = \mathcal{S}_{eqrel}(\mathbb{F}_\infty)$, $\forall\Gamma$. In fact, $\mathcal{F}(M) \in \mathcal{S}_{factor}(\mathbb{F}_\infty)$, $\forall M$ sep. II_1.

Q9 $\mathcal{S}_{factor}(\Gamma) \subset \mathcal{P}(\mathbb{Q}_+)$, $\forall\Gamma$ with (T)?

- **On relative property (T)**

Q1* Give a "non-vNAlgebra" def. of *relative property* (T) (or *rigidity*, as defined in [P01]) for group actions $\Gamma \curvearrowright X$. Answered: Ioana 09.

Q2* Denote $\mathcal{R}$ the OE relation of $SL(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$. $\forall \mathcal{R}_0 \subset \mathcal{R}$ non-amenable is rigid? Yes: Ioana 09. Is any $\Gamma \curvearrowright \mathcal{G}/\Lambda$ rigid ? Yes: Ioana-Shalom 09.

Q3 What are the groups Γ for which $\exists\ \Gamma \curvearrowright X$ rigid? Progress by Ioana 07, Gaboriau 08.

Q4 $\Gamma \curvearrowright X$ rigid $\Rightarrow$ strongly ergodic ?

- **Connes' Rigidity Conjecture (CRC)**

If Γ, Λ *ICC groups with property (T), does* $\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Lambda)$ *imply* $\Gamma \simeq \Lambda$?

CRCStrongVersion : *If* Γ *ICC with prop (T) and* Λ *ICC, then any* $\theta : L(\Gamma) \simeq \mathcal{L}(\Lambda)^t$ *forces* $t = 1$ *and* $\exists \delta : \Gamma \to \Lambda$, $\gamma \in \mathrm{Hom}(\Gamma, \mathbb{T})$ *such that* $\theta(\Sigma_g c_g u_g) = \Sigma_g \gamma(g) c_g u_{\delta(g)}$*?*

Q1 $\mathcal{L}(\Gamma_n) \simeq \mathcal{L}(\Gamma_m) \implies n = m$? For $\Gamma_n = PSL(n, \mathbb{Z})$; for $\Gamma_n = \mathbb{Z}^n \rtimes SL(n, \mathbb{Z})$. True for $\Gamma_n \subset Sp(n, 1)$ by Cowling-Haagerup.

Q2 Is $\mathcal{L}(SL(3, \mathbb{Z}))$ solid (in Ozawa's sense)?

RelatedQ*: Γ, Λ are called *measure equivalent* if $\exists$ free ergodic $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ that are (stably) OE. Does OE of ICC groups Γ, Λ imply (or is implied by) $\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Lambda)^t$, for some $t > 0$ (Shlyakhtenko)? Chifan-Iona 09: No.

- **Free Group Factor Problems**

$\mathbf{Non-isomorphism Problem:}$
$\mathcal{L}(\mathbb{F}_n) \simeq \mathcal{L}(\mathbb{F}_m) \Rightarrow n = m$? Sufficient to prove: $\mathcal{L}(\mathbb{F}_\infty) \neq \mathcal{L}(\mathbb{F}_n)$ for some n (cf. Voiculescu, Radulescu, Dykema). Related to this:

$\mathbf{Finite Generation Problem:}$ Can $\mathcal{L}(\mathbb{F}_\infty)$ be fin gen as vN Alg ? Do there exist $\mathcal{L}(\Gamma)$ which cannot be fin gen ? (Obs: Any factor $\mathcal{L}(\mathcal{R}_\Gamma)$ can be generated by two unitaries)

Abstract Characterization of $\mathcal{L}(\mathbb{F}_n)$

Facts: $\mathcal{L}(\mathbb{F}_n)$ has no Cartan (Voiculescu 94); it is prime (Ge 96), even *solid*, i.e. $P' \cap \mathcal{L}(\mathbb{F}_n)$ amenable $\forall P \subset \mathcal{L}(\mathbb{F}_n)$ diffuse (Ozawa 03). In fact $\mathcal{F}(\mathbb{F}_n)$ is *strongly-solid*: $P \subset \mathcal{F}(\mathbb{F}_n)$ amenable diffuse $\Rightarrow \mathcal{N}(P)''$ amenable (OP 07).

Q1* If M II$_1$ factor is s-solid and $\Lambda_{cb}(M) = 1$ then $M \simeq \mathcal{L}(\mathbb{F}_n)^t$? (No: Houdayer 09.) What if "s-solid" is replaced by "if $B \subset M$ amenable diffuse and $B \subset B_i \subset M$ amenable then $\vee_i B_i$ amenable" (Peterson-Thom). Is any non-amenable $M \subset \mathcal{L}(\mathbb{F}_n)$ iso to some $\mathcal{L}(\mathbb{F}_n)^t$

Q2 Assume a II$_1$ factor M satisfies (FF) : The "free flip" $x * y \mapsto y * x$ is path connected to id in Aut$(M * M)$. Does (FF) imply $M \simeq \mathcal{L}(\mathbb{F}_t)$, some $1 < t \leq \infty$?

Related Charac. of hyperfiniteness

Q1 If M factor and classic flip (CF) on $M \otimes M$ is path connected to id, then $M \simeq R$?

Obs P09 If M satisfies (CF), then M has Γ and if (FF) holds true as well, then $M \simeq R$. Also, if $\exists Q \subset M$ with $Q' \cap M^\omega = (Q' \cap M)^\omega$ then M cannot satisfy (CF).

Q2 $(M' \cap M^\omega)' \cap M^\omega = M \Rightarrow M \simeq R$?