CLASSIFICATION OF ACTIONS OF DISCRETE AMENABLE GROUPS ON AMENABLE SUBFACTORS OF TYPE II

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Abstract. We prove a classification result for properly outer actions $\sigma$ of discrete amenable groups $G$ on strongly amenable subfactors of type II, $N \subset M$, a class of subfactors that were shown to be completely classified by their standard invariant $G_{N,M}$, in ([Po7]). The result shows that the action $\sigma$ is completely classified in terms of the action it induces on $G_{N,M}$. As an application of this, we obtain that inclusions of type III$_\lambda$ factors, $0 < \lambda < 1$, having discrete decomposition and strongly amenable graph, are completely classified by their standard invariant.

0. Introduction. In ([C1]) A.Connes classified the amenable semifinite factors showing that, up to isomorphism, there is only one of type II$_1$, the unique approximately finite dimensional II$_1$ factor $R$ of Murray and von Neumann, also called the hyperfinite factor, and one of type II$_\infty$ ( $R \otimes \mathcal{B}(\mathcal{H})$ ). Then, motivated by the problem of classifying infinite amenable factors of type III, automorphisms of amenable factors of type II were classified in ([C2,5]). Further classification results were proved for actions of finite and general amenable groups on $R$ and $R \otimes \mathcal{B}(\mathcal{H})$ in ([J1]) and respectively ([Oc]).

For inclusions of factors $N \subset M$ of finite Jones index $[M : N] < \infty$, the suitable notion of amenability was introduced in ([Po7]). Also, it was proved in ([Po7]) that the amenable subfactors coincide with the subfactors that can be approximated by the finite dimensional subalgebras of their higher relative commutants. In the case of a trivial inclusion $N = M \subset M$ this corresponds to the uniqueness of the amenable type II$_1$ factor. In general, this shows that amenable inclusions are completely classified by their standard invariant $G_{N,M}$, the graph type combinatorial object that encodes the lattice of higher relative commutants in the Jones tower ([Po7]). $G_{N,M}$ consists of a pair of weighted, pointed, bipartite graphs $(\Gamma_{N,M}, \vec{s})$, $(\Gamma'_{N,M}, \vec{s}')$ called the standard (or the principal) graphs of $N \subset M$ ([J2]), with some additional structure. The invariant gives rise to a canonical model $N^{st} \subset M^{st}$ and in fact the theorem in ([Po7]) states that $N \subset M$ is strongly amenable, i.e., it is amenable and its standard graph is ergodic, if and only if $N \subset M$ is isomorphic to its canonical model.

We will prove in this paper a classification result for properly outer actions of amenable groups on strongly amenable inclusions of type II$_1$ and II$_\infty$ factors. The result can be regarded as an equivariant version of ([Po7]). The main motivation for studying this problem is, as in the single von Neumann algebra case, the classification of inclusions of type III$_\lambda$ factors, $0 < \lambda < 1$, for which a similar Connes type
discrete decomposition holds as shown in ([Lo1]). Thus, by our results it follows that the classification of type III_λ inclusions, 0 < \lambda < 1, reduces to the classification of trace scaling actions of \( \mathbb{Z} \) on inclusions of type II_\infty factors.

More precisely, our first result (see Theorem 2.1) shows that if an inclusion \( N^\infty_\infty \subset M^\infty_\infty \) of hyperfinite II_\infty factors is extremal and strongly amenable, then a trace scaling automorphism \( \sigma \) on it splits into the tensor product between the action \( \sigma^{st} \), implemented by \( \sigma \) on the model inclusion of II_1 factors \( N^{st} \subset M^{st} \), and a model action \( \sigma_0 \) on a commonly splitted II_\infty factor. Thus, \( (N^\infty_\infty \subset M^\infty_\infty, \sigma) = ((N^{st} \subset M^{st}) \otimes R^\infty, \sigma^{st} \otimes \sigma_0) \). More generally, we prove that for a trace scaling action of \( \mathbb{Z}^n \) on \( N^\infty_\infty \subset M^\infty_\infty \) which is diagonalizable (in the obvious sense) a similar splitting result holds true.

An important application of the above result is the classification of inclusions of hyperfinite type III_λ factors with discrete decomposition and strongly amenable graph, i.e. of the form \( (N \subset M) = (N^\infty_\infty \rtimes \sigma \subset M^\infty_\infty \rtimes \sigma) \), with \( N^\infty_\infty \subset M^\infty_\infty \) a strongly amenable inclusion of type II_\infty factors and \( \sigma \) a \( \lambda \)-scaling automorphism of \( M^\infty_\infty \) leaving \( N^\infty_\infty \) globally invariant. Thus, our theorem implies that \( N \subset M \) is isomorphic to the inclusion \( (N^{st} \subset M^{st}) \rtimes \sigma^{st} \otimes \sigma_0 \subset (M^{st} \subset R^\infty) \rtimes \sigma^{st} \otimes \sigma_0 \), where \( N^{st} \subset M^{st} \) is the canonical model associated with \( N^\infty_\infty \subset M^\infty_\infty \), \( \sigma^{st} \) is the action implemented on it by \( \sigma \), and \( \sigma_0 \) is a model \( \lambda \)-scaling automorphism on the hyperfinite II_\infty factor \( R^\infty \).

Although it is not needed for the classification of type III_λ inclusions, we also prove a classification result for properly outer actions \( \theta \) of arbitrary discrete amenable groups \( G \) on strongly amenable inclusions of type II_1 factors \( N \subset M \) as well (see Theorem 3.1). It shows that \( \theta \) is cocycle conjugate to an action of the form \( \theta^{st} \otimes \sigma_0 \) on \( (N^{st} \subset M^{st}) \otimes R \), with \( \theta^{st} \) the “standard” part of \( \theta \) and \( \sigma_0 \) a properly outer action of \( G \) on the hyperfinite II_1 factor \( R \).

Thus, in all these cases the actions are completely classified (up to outer conjugacy) by the actions they implement on the standard invariant \( G_{N,M} \). Due to its rigid combinatorial structure \( G_{N,M} \) generally admits only a few (finitely many) actions, oftenly just the trivial action. In the case of the index \( n = 4 \) all such actions were listed ([Lo1,2]).

The proofs of both theorems rely on non-commutative ergodic theory techniques, much in the spirit of ([Po1,2,4,7]). The idea we use is to build “local Rohlin towers” for larger and larger finite parts of the acting group \( \mathbb{Z}^n, G \), indexed by corresponding Følner sets. We then “glue them together” by using maximality arguments, very much the same way we did in [P07], inspired by similar arguments in [C3]. As a result of this argument, we obtain that \( \sigma \) splits into a tensor product of \( \sigma^{st} \) and a trace scaling action \( \sigma_0 \). In case \( n = 1 \), we can then further apply Connes Theorem in [C2] to derive the final result. In turn, for the proof of the classification of (trace preserving) actions of arbitrary amenable groups \( G \) on \( N \subset M \simeq R \), we ultimately use Ocneanu’s uniqueness (up to cocycle conjugacy) of cocycle actions of \( G \) on the hyperfinite II_1 factor [Oc]. However, as pointed out in [Po7,8], note that our classification of strongly amenable subfactors in [P07] does imply both the uniqueness of the trace scaling automorphisms of \( R^\infty \) in [C2] and the case \( G \) strongly amenable of the results in [Oc], for which it thus provides alternative proofs.

Part of the results in this paper have been presented by the author in a number of lectures during 1991-1992 and in a C. R. Acad. Sci. Paris note ([Po8]). A preliminary form of the paper has been circulated by the author since the fall of

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Added in the proof, July 2009. While the work presented in this paper has been completed some 18 years ago, its typing proved to be an agonizing process. Thus, my traveling schedule during 1991-1992 resulted into several people being involved with this task. They had to cope with a manuscript full of complicated formulas and a handwriting unfamiliar to them, under limited time-frame. The outcome was, alas, a 1992 preprint with hundreds and hundreds of typos, a nightmare to correct. As the results in the paper were fully accepted by the mathematical community and amply used and cited in subsequent papers, I felt little incentive to go through the necessary proof reading ordeal, for several years. Then other mathematical interests prevailed, all the way until the Summer of 2009, when I needed some of the results and techniques in this paper, to study rigidity properties of certain inductive limits of $\text{II}_1$ factors. This gave me the energy to carefully go through the paper and make the corrections. Other than that, I have chosen to leave the original 1992 preprint essentially unchanged, although I certainly would have written the paper quite differently today...

While initially the main interest in this paper was due to the application to the classification of type $\text{III}_\lambda$ subfactors, a hot topic at the time, the techniques used here seem to be of a broader interest in von Neumann algebras, including deformation/rigidity theory. Hence my decision to revive the paper and seek its publication in a refereed journal, despite so many years of neglect. Some of the ideas used in this approach to the classification of actions of amenable groups on inclusions of hyperfinite type II factors, are quite novel. Thus, unlike the usual strategy for classifying trace scaling automorphisms of the hyperfinite $\text{II}_\infty$ factor $R^\infty$ due to Connes ([C2]), which consists in reducing to the trace preserving case (resulting into a “type $\text{II}_1$ treatment”), one works here directly in the $\text{II}_\infty$ setting. Actually, one relies heavily on the trace scaling property of the automorphism, to prove it coincides with the model. Also, in the classification (up to cocycle conjugacy) of actions $\theta$ of amenable groups $G$ on strongly amenable $\text{II}_1$ subfactor $N \subset M \simeq R$, the model considered is a cocycle action $\theta_0$ on $R$, rather than a genuine action (see Sec. 3.2)!

It is worth mentioning that the techniques and ideas in this paper have later inspired me in isolating the concept of central freeness and approximate innerness for subfactors in [P9,10] and in the proof of the classification of approximately inner, centrally free subfactors with amenable graph in [P9,10,11] (note that these are the only papers I have added to the reference list of the initial 1992 version). In particular, [Po10] provided a new proof to the classification of hyperfinite $\text{III}_\lambda$ subfactors with strongly amenable graph, in case the standard part of the action of the trace scaling automorphism in the common discrete decomposition is trivial (i.e., with the above notations, $\sigma^{st} = id$).

1. The standard invariant of an equivariant inclusion. We begin by explaining the standard invariant and model associated to a triple $(N \subset M, \theta)$, where $\theta \in \text{Aut}(M, N)$ def $\{\sigma \in \text{Aut}M | \sigma(N) = N\}$, as considered by Loi in [Lo1], for inclusions of type $\text{II}_1$ or $\text{II}_\infty$ factors $N \subset M$ of finite index.
1.1. Inclusions of type $\Pi_\infty$ factors. If $N^\infty \subset M^\infty$ are type $\Pi_\infty$ factors then there exists a normal conditional expectation of $M^\infty$ onto $N^\infty$ if and only if $N^\infty \cap (N^\infty' \cap M^\infty)$ contains finite projections of $M^\infty$ (see e.g. [Po6]). If $N^\infty$ contains finite projections of $M^\infty$ then there exists a unique trace preserving conditional expectation of $M^\infty$ onto $N^\infty$, $E_{N^\infty}$ and, in fact, if $N^\infty = N \overline{\otimes} B(\mathcal{H})$ is any splitting of $N^\infty$ with $N$ a type $\Pi_1$ factor and $M = B(\mathcal{H})'$, in the same way one does for type $\Pi_2$ standard invariant $G$ of the tunnel. Moreover this invariant, that we denote $\lambda$, exists, there exist projections $(\infty u N_0)$ independently on the choice of $G$ factors of commuting squares of higher relative commutants $\infty u N_0$. We have

$$[M^\infty : N^\infty] = (\max\{\lambda \geq 0 | E_{N^\infty} (x) \geq \lambda x, x \in M^\infty_+\})^{-1}$$

in case $N^\infty$ contains finite projections of $M^\infty$. Alternatively, with the above notations, we can define the index of $N^\infty \subset M^\infty$ by $[M^\infty : N^\infty] \defeq [M : N]$. If the index is finite then we can associate to $N^\infty \subset M^\infty$ the tower of embeddings ([Po7]):

$$N^\infty \subset M^\infty \subset M^\infty_1 \subset \cdots$$

where $M^\infty_k = M_k \overline{\otimes} B(\mathcal{H})$ and $e_k \in M_k \simeq M_k \otimes 1 \subset M_k \otimes B(\mathcal{H})$ are the usual Jones projections for the Jones’ tower $N \subset M \subset M^\infty_1 \subset \cdots$.

Moreover, due to the splitting of $N^\infty \subset M^\infty$ into a type $\Pi_1$ inclusion $\otimes B(\mathcal{H})$, there exist projections $e_0 \in N^\infty \subset M^\infty$ so that $E_{N^\infty} (e_0) = \lambda 1 = [M^\infty : N^\infty]^{-1}$. The proof of [PiPo1] then shows that any two such projections are conjugate by a unitary element of $N^\infty$. If $e_0, e'_0 \in M^\infty$ are as above, with $E_{N^\infty} (e_0) = E_{N^\infty} (e'_0) = \lambda 1$, and $N^\infty_1 = \{e_0\}' \cap N^\infty$, $N^\infty_1 = \{e_0\}' \cap N^\infty$, with $e_0 \in N \otimes 1 \subset N^\infty$ and $N^\infty_1 = \{e_0\}' \cap N$, then the unitary element $u \in N^\infty$, $ue_0 u^* = e_0$, also satisfies $uN^\infty u^* = N^\infty_1$. We have $N^\infty_1 = N_1 \otimes B(\mathcal{H})$ and $N^\infty_1 = N_1 \otimes B(\mathcal{H})$. Also, we denote by $\lambda$ the conjugation by $u$ changes the splitting of $N^\infty \subset M^\infty \subset M^\infty_1 \subset \cdots$ by $B(\mathcal{H})$ into a splitting by $B(\mathcal{H})^\infty$.

It thus follows that we may choose recursively a tunnel of type $\Pi_\infty$ factors

$$M^\infty \supset N^\infty \supset N^\infty_1 \supset N^\infty_2 \supset N^\infty_3 \supset \cdots$$

so that for each $k$ there exists $B(\mathcal{H}_k^0) \subset N^\infty_k$ factors with the property that $N^\infty_0 = B(\mathcal{H}_0^0)' \cap N^\infty_0$ are type $\Pi_1$ factors and $M \supset N \supset N^\infty_k \supset \cdots \supset N^\infty_{k+1} \supset N^\infty_k$ is a tunnel of type $\Pi_1$ factors for $M \supset N$.

It thus follows that one can define for a type $\Pi_\infty$ inclusion $N^\infty \subset M^\infty$ its standard invariant, in the same way one does for type $\Pi_1$ inclusion, as the sequence of commuting squares of higher relative commutants $N^\infty_0 \cap N^\infty \subset N^\infty_0 \cap M^\infty$, on which one has a canonical finite trace, and which doesn’t depend on the choice of the tunnel. Moreover this invariant, that we denote $\mathcal{G}_{N^\infty, M^\infty}$, coincides with the standard invariant $\mathcal{G}_{N, M}$ of the corresponding inclusion of type $\Pi_1$ factors $N \subset M$, independently on the choice of $N \subset M$ (e.g., by [Po7]). Also, we denote by $M^\infty = \cup M_k^\infty$ with the closure being taken in the strong topology of the common trace.
Note that $M^\infty$ coincides with $M^\infty \otimes B(\mathcal{H})$. It is called the *enveloping algebra* of $N^\infty \subset M^\infty$.

Finally, let us define the notion of amenability for inclusions of type $\text{II}_\infty$ factors. In order to avoid lengthy discussions and too abstract statements (which, unlike in the $\text{II}_1$ case where they were quite necessary, are here practically useless) we will adopt the simple minded point of view of reducing to the type $\text{II}_1$ case. We thus put:

**Definition.** $N^\infty \subset M^\infty$ is amenable if $N \subset M$ is amenable. It is strongly amenable (respectively, has *ergodic care*) if $N \subset M$ is strongly amenable (respectively, has ergodic core).

1.2. *The standard part of an automorphism.* Let $N \subset M$ be both type $\text{II}_1$ factors and $(N^\infty \subset M^\infty) = N \otimes B(\mathcal{H}) \subset M \otimes B(\mathcal{H})$, the corresponding type $\text{II}_\infty$ amplification. Assume $[M : N] = [M^\infty : N^\infty] < \infty$. Denote by $\text{Aut}(M^\alpha, N^\alpha) = \{ \theta \in \text{Aut}(M^\alpha) | \theta(N^\alpha) = N^\alpha \}$ where $\alpha = 1$ or $\alpha = \infty$.

Let us first point out that the automorphisms in $\text{Aut}(M^\infty, N^\infty)$ always commute with the trace-preserving expectation.

**Lemma.** If $\theta \in \text{Aut}(M^\infty, N^\infty)$ then $\theta E_{N^\infty} = E_{N^\infty} \theta$ and the action of $\theta$ on $N^\infty \cap M^\infty$, equipped with its canonical trace (see 1.1), is trace preserving.

**Proof.** Since an expectation of $M^\infty$ onto $N^\infty$ is uniquely determined by its values on $N^\infty \cap M^\infty$, by comparing $E_{N^\infty}$ and $\theta E_{N^\infty} \theta^{-1}$ we see that it is sufficient to prove that they agree on $N^\infty \cap M^\infty$. For this it is sufficient to show that $\theta|_{N^\infty \cap M^\infty}$ is trace preserving. This is clear if mod $\theta = 1$. If mod $\theta = \lambda \neq 1$ then let $R^\infty$ be a hyperfinite type $\text{II}_\infty$ factor and $\sigma$ an automorphism of $R^\infty$ with mod $\sigma = \lambda^{-1}$. Then clearly $(N^\infty \overline{\otimes} R^\infty)' \cap M^\infty \overline{\otimes} R^\infty = N^\infty \cap R^\infty \otimes 1$ and the action of $\theta \otimes \sigma$ on $(N^\infty \cap M^\infty) \otimes 1$ coincides with that of $\theta|_{N^\infty \cap M^\infty} \otimes 1$. But by the above case $\theta \otimes \sigma$ acts trace preservingly on $(N^\infty \otimes R^\infty)' \cap M^\infty \otimes R^\infty$. Q.E.D.

We can now describe Loi’s construction ([Lo2]), based on ([PiPo1]), of the action implemented by $\theta$ on the higher relative commutants, for arbitrary actions $\theta$ on inclusions of type $\text{II}$ factors.

(i) Given any choice of the tunnel

$$M^\alpha \supset N^\alpha \supset \cdots$$

there exist unitary elements $u_k \in N^\alpha_k$, $k \geq 0$, such that $\text{Ad}(u_k \cdots u_0) \theta(e_\infty) = e_\infty$ (cf.[PiPo1]).

(ii) For $x \in \bigcup_k (N^\alpha_k \cap M^\alpha)$ define

$$\theta^{st}(x) \overset{\text{def}}{=} \text{Ad}(\cdots u_k \cdots u_0) \theta(x).$$

Then $\theta^{st}(N^\alpha_k \cap M^\alpha) = N^\alpha_k \cap M^\alpha$, $\theta^{st}(N^\alpha_k \cap N^\alpha) = N^\alpha_k \cap N^\alpha$, $\theta^{st}(e_\infty) = e_\infty$, for all $k \geq 0$. Moreover, in case $\alpha = \infty$, if $N^\infty_k \cap M^\infty$ are interpreted as higher relative commutants of the corresponding type $\text{II}_1$ inclusions obtained by splitting
all $N^\infty, i \geq k \geq -1$, by a common $\mathcal{B}(\mathcal{H}) \subset N^\infty$, then $\theta^{st}$ as defined above is trace preserving (with respect to the corresponding type $\Pi_1$ trace).

Thus, both in the $\Pi_1$ and $\Pi_\infty$ case, $\theta^{st}$ implements a trace preserving automorphism on the union algebra $\cup(N^{\alpha}_k \cap M^\infty)$, leaving $\cup(N^{\alpha}_k \cap M^{\alpha}_k)$, $N^{\alpha}_k \cap M^{\alpha}$, $N^{\alpha}_k \cap N^{\alpha}$, globally invariant and all the Jones’ projections fixed.

(iii) $\theta^{st}$ implements a trace preserving automorphism on the standard invariant $G_{N^{\alpha}_\ast, M^{\alpha}} = \{N^{\alpha}_k \cap N^{\alpha} \subset N^{\alpha}_k \cap M^\infty\}_{k \geq 0}$, which is independent on the choice of the tunnel $N^{\alpha}_k$, on the choice of the unitaries $u_k \in N^{\alpha}_k$ and on perturbations of $\theta$ by inner automorphisms $\text{Ad}_s$, with $s \in U(N^{\alpha})$.

(iv) If one also denotes by $\theta^{st}$ the trace preserving automorphism of $M^{st} = \cup(N^{\alpha}_k \cap M^{\alpha})$ leaving $N^{st} = \cup(N^{\alpha}_k \cap N^{\alpha})$ globally fixed, then $\theta \mapsto \theta^{st}$ implements a group homomorphism, from $\text{Aut}(M^{\alpha}, N^{\alpha})/\text{Int}N^{\alpha}$ to $\text{Aut}(M^{st}, N^{st})$, leaving all Jones projections $e_{-k}$ fixed and all finite dimensional algebras $N^{\alpha}_k \cap N^{\alpha}_j$ globally invariant.

(v) Define recursively $\theta_{k+1} : M^{\alpha}_{k+1} \to M^{\alpha}_{k+1}$ by

$$
\theta_{k+1}(\Sigma x_ie_{k+1}y_i) = \Sigma_i \theta_k(x_i)e_{k+1}\theta_k(y_i), k \geq 0, x_i, y_i \in M^{\alpha}_k.
$$

Then $\theta_{k+1}|M^{\alpha}_1 = \theta_k$ and $\theta_{k+1} \in \text{Aut}(M^{\alpha}_{k+1}, M^{\alpha}_k)$. Also denote $\theta_{\infty} : M^{\alpha}_{\infty} \to M^{\alpha}_{\infty}$ the unique automorphism which on the dense set $\cup M^{\alpha}_k$ acts by $\theta_{\infty}|M^{\alpha}_k = \theta_k$ and by $\theta^{st,0} : M^{\alpha}_k \cap M^{\alpha}_{\infty} \to M^{st}_k \cap M^{\alpha}_{\infty}$ its restriction to $M^{\alpha}_k \cap M^{\alpha}_{\infty}$. Then $\theta \mapsto \theta^{st,0}$ is a group morphism from $\text{Aut}(M^{\alpha}, N^{\alpha})$ to $\text{Aut}(M^{\alpha}_k \cap M^{\alpha}_{\infty}, M^{\alpha}_1 \cap M^{\alpha}_{\infty})$, with $\theta^{st,0}$ leaving each Jones projection $e_k$ fixed and $M^{st}_k \cap M^{\alpha}_k \subset M^{\alpha}_k \cap M^{\alpha}_k$ globally invariant. Note that if $N^{\alpha} \subset M^{\alpha}$ is extremal ([P7]) and $\theta^{st}$ denotes the canonical antiisomorphism of $N^{\alpha}_{k-1} \subset M$ onto $M^{\prime} \cap M_k$ ([Po7]), then $\theta^{op}$ intertwines $\theta^{st}$ and $\theta^{st,0}$.

**Definition.** The automorphism $\theta^{st}$ on $N^{st} \subset M^{st}$ is called the standard part of the automorphism $\theta$ (on $N \subset M$). The automorphism $\theta^{st,0}$ on $M^{\prime}_1 \cap M^{\infty}_{\infty}$ is called the opposite standard part of $\theta$. The action implemented by $\theta^{st}$ on $G_{N^{\alpha}, M^{\alpha}}$ (i.e., on the lattice of higher relative commutants) will be denoted by $\gamma_\theta$ and $(G_{N^{\alpha}, M^{\alpha}}, \gamma_\theta)$ is called the standard invariant of the equivariant inclusion $(N^{\alpha} \subset M^{\alpha}, \theta)$.

We will now summarize some of the properties of $\theta^{st}$ that are more or less implicit in the above considerations.

1.3. PROPOSITION. (i) The application $^{st}$ factors to a group morphism from $\text{Aut}(M^{\alpha}, N^{\alpha})/\text{Int}N^{\alpha}$ into $\text{Aut}(M^{st}, N^{st})$.

(ii) The group morphism $^{st}$ is continuous from $\text{Aut}(M^{\alpha}, N^{\alpha})/\text{Int}N^{\alpha}$ with the quotient topology into $\text{Aut}(M^{st}, N^{st})$ with its usual topology.

(iii) Assume $N^{\alpha} \subset M^{\alpha}$ is strongly amenable standard invariant (equivalently, $N^{st}, M^{st}$ are factors and have same higher relative commutants as $N^{\alpha} \subset M^{\alpha}$, see [Po7]). If $\sigma \in \text{Aut}(M^{\alpha}, N^{\alpha})$ then $(\sigma^{st})^{st} = \sigma^{st}$.

**Proof.** We already proved (i) above. Then (ii), and (iii) are trivial by the definitions.

Q.E.D.
1.4. Actions with trivial standard part. As it turns out in certain situations the standard part of an action follows automatically trivial. Since the standard part of a cocycle action on an inclusion \((N^\alpha \subset M^\alpha, \theta)\) coincides with the standard part of the cocycle action it implements on the associated \(\Pi_1\) inclusion, \(N \subset M\), we can reduce our discussion to the case of \(\Pi_1\) inclusions, i.e. when \(\alpha = 1\).

PROPOSITION. (i) If \(\sigma\) is inner on all relative commutants \(M' \cap M_k\) and \(\Gamma_{N,M}\) is a tree (equivalently \(a_{k\ell} \in \{0,1\}, \forall k, \ell\), and \(\Gamma_{N,M}\) has no cycles), then \(\sigma^{st} = id\).

(ii) If \(\Gamma_{N,M}\) is a tree and the canonical weights \((s_k)_{k \in K}\), resp. \((t_l)_{l \in L}\), are distinct then, \(\sigma^{st,0} = id \forall \sigma \in Aut(M,N)\).

(iii) If \(\text{sup dim} Z((M' \cap M_k)^\sigma) < \infty\) then \(\sigma^{st}\) is periodic, \(\forall \sigma \in Aut(M^\alpha, N^\alpha)\), i.e., \((\sigma^{st})^n = id\) for some \(n < \infty\), \((\sigma^{st})^k\) properly outer for \(0 < k < n\).

Proof (i) If \(\sigma^{st,0}\) is inner on \(M' \cap M_k\) (equivalently, \(\sigma^{st}\) inner on all \(N^\eta_{k-1} \cap M\)) then it acts trivially on its center. If \(\Gamma_{N,M}\) is a tree then \(M' \cap M_{k+1} = \text{sp}(M' \cap M_k) + B_{k+1}\), with \(B_{k+1}\) abelian. Since \(\sigma(e_{k+1}) = e_{k+1}, B_{k+1} \subset Z(M' \cap M_{k+1})\) and \(\sigma|_{Z(M' \cap M_{k+1})} = id\), we obtain that \(\sigma|_{M' \cap M_k} = id\) implies \(\sigma|_{M' \cap M_{k+1}} = id\).

Then (ii) is trivial by (i), since for distinct \((s_k)_k, (t_l)_l, \sigma\) is forced to act trivially on the center of \(M' \cap M_k\), \(\forall k\).

To prove (iii) note first that if \(N^{st} \subset M^{st}\) are factors and if for some \(k_0\) we have \(ax = \sigma^{st}(x)a, x \in N_{k_0}^{st}, a \in M^{st}\), then given any \(\varepsilon > 0\) there exists \(m\) such that \(a \in N_{m+1}^{st} \cap M^{st}\). Thus, if \(a \neq 0\) then for \(x \in N_{m}^{st}\) we have \(\|x, a\|_2 \leq 2\varepsilon\|x\|\) and \(\|x - \sigma^{st}(x)\|_2 < f(\varepsilon)\|x\|\) with \(f(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Thus \(\sigma^{st}\) is inner on \(N_{m}^{st}\) for \(m\) large enough (cf. e.g. [Ch]), say \(\sigma^{st}_{N_{m}^{st}} = Adu, u \in N_{m}^{st}\). But \(\sigma^{st} = (Adu^{st})^{st}\) for \(u \in N_{m}^{st}\) that \(\sigma^{st} = id\) on \(N_{m}^{st}\). Since \(\sigma^{st}(e_{-i}) = e_{-i}, \forall i \geq 0\), it follows that \(\sigma^{st} = id\), unless \(a = 0\).

Now, if \(\text{dim} Z((M' \cap M_k)^\sigma)\) is uniformly bounded then by (1.1 in [Po7]), \((M' \cap M_{k-1})^{\sigma} \subset (M' \cap M_k)^{\sigma} \subset (M' \cap M_{k+1})^{\sigma}\) is a basic construction for \(k\) large enough and the support of \(e_{k+1}\) in \((M' \cap M_{k+1})^{\sigma}\) is 1. Thus the support of \(e_{k+1}\) in \(M' \cap M_{k+1}\) is also 1 and \(N \subset M\) has finite depth (cf. e.g. 1.1 in [Po7]), in particular it is extremal and \(N^{st} \subset M^{st}\) are factors (e.g. by [Po4]). Also, we have a sequence of commuting squares

\[
M' \cap M_{k-1} \subset M' \cap M_k \subset \bigcup_{i=0}^{e_{k+1}} M' \cap M_{k+1} \bigcup_{i=0}^{e_{k+1}} (M' \cap M_{k+1})^{\sigma}
\]

in which \(e_{k+1}\) implements a basic construction on both rows (for \(k\) large enough). Also \((M' \cap M_\infty)^{\sigma} = (\bigcup_{k} (M' \cap M_k)^{\sigma})\) is a factor (e.g., by [Po4, 5], [We]) and by ([GHJ]), if \(T\) is the inclusion matrix of \((M' \cap M_{k-1})^{\sigma} \subset M' \cap M_k\) then \([M' \cap M_\infty : (M' \cap M_{k-1})^{\sigma}] = \|T\|^2.\)

By the arguments above, we see that if \((\sigma^{st})^n\) is inner for some \(n\) then \((\sigma^{st})^n = id\) on \(N^{st}\) and thus on all \(M^{st}\). Thus, either \((\sigma^{st})^n = id\) and \((\sigma^{st})^k\) is outer for \(0 < k < n\), where \(n\) is the period of \(\sigma^{st}\), or \(\sigma^{st}\) acts freely on \(M^{st}\). In the latter case though the fixed point algebra \((M^{st})^{\sigma^{st}}\) would have infinite index in \(M^{st}\) (cf. e.g. [Lo1]), a contradiction.

Q.E.D.
COROLLARY. (i) If $\Gamma_{N,M}$ is a tree then $\ker^{(*)}$ contains the connected component of the identity in $\text{Aut}(M,N)$.

(ii) If $\sigma : G \to \text{Aut}(M,N)$ is a continuous action of a connected group $G$ and either $\Gamma_{N,M}$ is a tree, or if $\sup \dim Z((M' \cap M_k)\sigma) < \infty$, then $\sigma^{st} = id$.

Proof. (i) If $\sigma$ is in the connected component of the identity in $\text{Aut}(M,N)$ then $\sigma^{st}_{|N_k \cap M}$ is inner $\forall k$ and part (i) of the previous proposition applies.

(ii) If $\Gamma_{N,M}$ is a tree then part (i) applies. Also, since $G$ is connected, if for some $k \sigma_{|N_k \cap M}^{st} \neq id$ then there exists $g \in G$ such that $\sigma^{st}(g)^n \neq id \forall n$ (even on $N_k' \cap M$), which contradicts (iii) in the previous proposition.

Q.E.D.

1.5. Proper outerness for actions on inclusions. In order to be able to prove that the standard invariant of an equivariant inclusion is a complete invariant, we need the actions to satisfy some proper outerness condition.

Definition 1. An automorphism $\sigma$ of a $\Pi_1$ inclusion $N \subset M$ is called properly outer if for any $k$ and any choice of the tunnel $M \supset N \supset \cdots \supset N_k$ we have the implication:

(*) If $a \in M$ is such that $ax = \sigma(x)a$, $\forall x \in N_k$, then $a = 0$.

An automorphism $\sigma$ of a $\Pi_\infty$ inclusion $N^\infty \subset M^\infty$ is properly outer if the corresponding automorphism of its associated $\Pi_1$ inclusion is properly outer.

Remarks. 1°. If $N^\alpha = M^\alpha$ then the above condition coincides with the usual definition of proper outerness of an action on the single algebra $M^\alpha$.

2°. If $\sigma$ is a properly outer action on $N^\alpha \subset M^\alpha$ then it is properly outer on both $N^\alpha$ and $M^\alpha$, more generally, if $M^\alpha \supset N^\alpha \supset \cdots \supset N_k^\alpha$ is a tunnel and $u \in U(N^\alpha)$ is so that $\text{Ad}uN_j^\alpha = N_j^\alpha$, $j \leq k$, then $\text{Ad}u\sigma$ is properly outer on each $N_j^\alpha$, $j \leq k$.

3°. The above definition doesn’t depend on the outer conjugacy class of $\sigma$, i.e., if $\sigma$ satisfies the above condition and $u \in U(M^\alpha)$ is so that $\text{Ad}uN^\alpha = N^\alpha$ (e.g., if $u \in N^\alpha$ or if $u \in N^\alpha \cap M^\alpha$) then $\text{Ad}u\sigma$ satisfies it. Also if $p \in N^\alpha$ is a nonzero projection then $\sigma$ is properly outer if $\sigma_p$ is properly outer.

4°. If the implication (*) in the above definition holds true for some $k$ and some choice of the tunnel up to $k$ then it holds true for any $j \leq k$ and any choice of the tunnel up to $j$. Thus, in order for $\sigma$ to be properly outer it is sufficient that there exists a tunnel $M^\alpha \supset N^\alpha \supset \cdots$ such that if for some $k \geq 0$ and $a \in M$ we have $ax = \sigma(x)a$, $\forall x \in N_k$, then $a = 0$.

PROPOSITION. Let $\theta$ be a cocycle action of a discrete group $G$ on $N^\alpha \subset M^\alpha$. The following conditions are equivalent:

(i) $\theta(g)$ is properly outer on $N^\alpha \subset M^\alpha$, $\forall g \neq e$.

(ii) There exists a tunnel $M^\alpha \supset N^\alpha \supset \cdots$ such that $N_{k}^{\alpha'} \cap (M^\alpha \rtimes_{\theta} G) = N_{k}^{\alpha'} \cap M^\alpha$ for all $k \geq 0$.

(i') For any tunnel $M^\alpha \supset N^\alpha \supset \cdots$ and any $k$, $N_{k}^{\alpha'} \cap (M^\alpha \rtimes G) = N_{k}^{\alpha'} \cap M^\alpha$.

Moreover, if $N^\alpha \subset M^\alpha$ is extremal, then these conditions are also equivalent to the following:

(iii) $M^{\alpha'} \cap (M_{k}^\alpha \rtimes G) = M^{\alpha'} \cap M_{k}^\alpha$, $\forall k$.

(iii') $M^{\alpha'} \cap (M_{\infty}^\alpha \rtimes G) = M^{\alpha'} \cap M_{\infty}^\alpha$.

Proof. (i) $\Rightarrow$ (ii). If $X = \Sigma x_g u_g \in M^\alpha \rtimes G$ is so that $[X, N_k^\alpha] = 0$ for some $k$ and if $x_g \neq 0$ for some $g \neq e$ then $x_g \theta(g)(x) = xx_g$ for all $x \in N_k$. Thus $\theta(g)$ for
that $g$ will not be properly outer.

(ii) $\Rightarrow$ (i). If $\theta(g)$ is not properly outer then for some $k$ and $a \in M^\alpha$, $a \neq 0$, we have $a\sigma(x) = xa, \forall x \in N_k^\alpha$. Thus $X = au_g \in N_k^\alpha \cap (M^\alpha \times G)$ but $X \notin N_k^\alpha \cap M^\alpha$.

(ii) $\Leftrightarrow$ (iii). By 1.2(v) there exists an antiisomorphism of $N_{2k}^{\alpha'} \cap (M^\alpha \times G)$ onto $M^{\alpha'} \cap (M_2^\alpha \times G)$ carrying $N_{2k}^{\alpha'} \cap M^\alpha$ onto $M^{\alpha'} \cap M_{2k}^\alpha$.

(iii) $\Leftrightarrow$ (iii'). If $M^{\alpha'} \cap (M_2^\alpha \times G) = M^{\alpha'} \cap M_{2k}^\alpha$ then by the commuting square relation we have $M^{\alpha'} \cap (M_k^\alpha \times G) = E_{M_k^\alpha \times G}(M^{\alpha'} \cap (M_\infty^\alpha \times G)) = E_{M_k^\alpha \times G}(M^{\alpha'} \cap M_{\infty}^\alpha) = M^{\alpha'} \cap M_k^\alpha$.

Q.E.D.

**Definition 2.** A cocycle action $\theta$ of a discrete group $G$ on $N^\alpha \subset M^\alpha$ is properly outer if the equivalent conditions in the above proposition are satisfied. A faithful action $\theta$ of a locally compact group $G$ on an extremal inclusion $N^\alpha \subset M^\alpha$ is properly outer if $M^{\alpha'} \cap (M_k^\alpha \times G) = M^{\alpha'} \cap M_k^\alpha$, $\forall k, \forall$. (It has been pointed out to us by Y. Kawahigashi that a similar property has been independently considered by M. Choda and H. Kosaki in [ChK]).

In ([EvKa]) there are examples of periodic automorphisms on $N \subset M$ that are properly outer on both $N$ and $M$ but not on $N \subset M$ in the sense of the above 2 definitions. There in fact do exist aperiodic ones as well:

**Example.** Let $P^\alpha$ be a type II$_1$ or II$_\infty$ factor and $\sigma = (\sigma_1, \ldots, \sigma_n)$ some $n$ automorphisms acting on $P^\alpha$. Let $M^\alpha = M_{(n+1) \times (n+1)}(P^\alpha)$ and $N^\alpha = \{\Sigma_j \sigma_j(x) e_{jj} \mid x \in P^\alpha\}$, where $\sigma_0 = id$ and $\{e_{ij}\}_{0 \leq i, j \leq n}$ is a matrix unit for $M_{(n+1) \times (n+1)}(\sigma)$ (see [Po5]). An automorphism $\theta$ of $M^\alpha$ fixing $\{e_{ij}\}$ will leave $N^\alpha$ globally invariant iff $\theta$ commutes with all $\sigma_i$. It is easy to see that such a $\theta$ acts properly outer on $N^\alpha \subset M^\alpha$ iff $\theta$ doesn’t belong to the group generated by the $\sigma_i$’s and $\text{Int}P^\alpha$ in $\text{Aut}P^\alpha$. In particular, if we take $n = 1$ and $\sigma_1 = \theta$ aperiodic, one obtains an example of an automorphism which is aperiodic on $M^\alpha$ but is not properly outer on the inclusion $N^\alpha \subset M^\alpha$ that was first pointed out by Y. Kawahigashi (private communication).

1.6. **Sufficient conditions for proper outerness.** We will now show that in certain situations an action on an inclusion follows automatically properly outer once it is properly outer on one of the algebras. But first, we will introduce an invariant that measures the “distance” from proper outerness.

**LEMMA.** Let $\theta$ be a cocycle action of a discrete group $G$ on the inclusion $N^\alpha \subset M^\alpha$ and assume that the action is properly outer on each of the algebras $N^\alpha, M^\alpha$ (in the usual sense). If $\{N_k^\alpha\}_{k \geq 1}$ is a choice of a tunnel then the algebras $\{N_k^\alpha \cap (M^\alpha \times G)\}_{k \geq 1}$ are finite dimensional and there exists a unique normalized trace $\tau_0$ on $\bigcup (N_k^\alpha \cap (M^\alpha \times G))$ such that $E_{N_k^\alpha}^\alpha \times G(x) = \tau_0(x)1$, for all $k \geq -1$ and all $x \in N_k^\alpha \cap (M^\alpha \times G)$.

Moreover, the sequence of inclusions $C = M^\alpha \cap (M^\alpha \times G) \subset N^\alpha \cap (M^\alpha \times G) \subset \cdots$ and respectively the trace $\tau_0$ are described by a pointed matrix (or pointed bipartite graph) $\Gamma_\theta$ and its transpose and respectively a positive vector $\overline{s}_\theta$ satisfying $\Gamma_\theta^t \overline{s}_\theta = [M^\alpha : N^\alpha] \overline{s}_\theta$. Also, up to trace preserving isomorphism this sequence of inclusions (and thus $\Gamma_\theta, s_\theta$) doesn’t depend on the choice of the tunnel.
Proof. Let $R^\infty$ be a copy of the hyperfinite type $\Pi_\infty$ factor and let $\theta'$ be an action of the group $G_0 = G / \ker(\text{mod}\theta)$ on $R^\infty$ such that $\text{mod}\theta'(\pi(g)) = \text{mod}\theta(g)^{-1}$, $g \in G$, where $\pi : G \to G_0$ is the quotient map. Let $\theta_0 : G \to \text{Aut}R^\infty, \theta_0(g) \in \theta'(\pi(g))$. Then $\theta \otimes \theta_0 : G \to \text{Aut}(\mathcal{M}_\alpha \otimes R^\infty, \mathcal{N}_\alpha \otimes R^\infty)$ is a cocycle action that will still be properly outer on each algebra and $\text{mod}\theta \otimes \theta_0(g) = 1$ for all $g$. Also the higher relative commutants and the state $\tau_0$ do not change if we replace $\theta$ by $\theta \otimes \theta_0$. So we may assume from the beginning that $\theta$ is trace preserving and then, by splitting off some $\mathcal{B}(H)$, that $(\mathcal{N}_\alpha \subset \mathcal{M}_\alpha) = (\mathcal{N} \subset \mathcal{M})$ are type $\Pi_1$ factors. The trace $\tau_0$ is then simply the restriction of the unique trace on the type $\Pi_1$ factor $M \rtimes G$. Since $M' \cap (M \rtimes G) = \mathbb{C}$ and since $N_k' \cap (M \rtimes G) \subset N_{k+1}' \cap (M \rtimes G)$ has [PiPo1] index $\leq [M : N]$ (cf. e.g., [Po3]), the algebras are indeed finite dimensional. The fact that the inclusions are determined by a unique pointed matrix and an eigenvector follows then by ([Po7], §1.2). Obviously, all this is independent on the tunnel. Q.E.D.

Definition. The weighted graph $(\Gamma_\theta, s_\theta)$ of the above lemma is called the standard graph of the action $\theta$. Note that, by ([Po7]), $\|\Gamma_\theta\|^2 \leq [M : N]$ and that if $\Gamma_\theta$ is finite we have equality, by the Peron-Frobenius theorem. Note also that $\theta$ is properly outer iff $\Gamma_\theta = \Gamma_{N,M}$. In general, this may not be the case though (see [Ka] for examples). However we have:

**THEOREM.** Let $\theta$ be a cocycle action of a discrete group $G$ on an extremal inclusion of type $\Pi_\alpha$ factors $\mathcal{N}_\alpha \subset \mathcal{M}_\alpha$. If one of the following conditions $1^\circ$ -- $4^\circ$ holds true, then the action $\theta$ is properly outer on $\mathcal{N}_\alpha \subset \mathcal{M}_\alpha$.

1. $\alpha = \infty$ and $\theta$ is trace scaling, i.e., $\text{Tr}\theta(g) \neq \text{Tr}, g \neq e$ (so that necessarily $G \subset \mathbb{R}$).

2. $\mathcal{N}_\alpha \subset \mathcal{M}_\alpha$ has finite depth, $G$ is torsion free and the action $\theta$ is properly outer on either $\mathcal{M}_\alpha$ or on $\mathcal{N}_\alpha$.

3. $\Gamma_\theta$ is a tree (i.e., it has only multiplicities 0 and 1 and it has no cycles), $G$ is torsion free and the action $\theta$ is properly outer on $\mathcal{M}_\alpha$.

4. The standard vectors $(s_k)_k$, resp. $(t_k)_l$, have distinct entries and $\theta$ is properly outer on $\mathcal{M}_\alpha$.

5. $\alpha = 1$ and $(\mathcal{N} \subset \mathcal{M}) = (\mathcal{N}^{st} \subset \mathcal{M}^{st})$ is a standard inclusion of type $\Pi_1$ factors and $\theta = \theta^{st}$ is a nontrivial standard action on it such that $\theta$ is properly outer on $\mathcal{M}^{st}$.

Moreover, in the case $\alpha = 1$, we have:

6. If $M' \cap N^\omega$ is nontrivial, and if $\sigma \in \text{Aut}(M, N)$ is properly outer on $M' \cap N^\omega$, then $\sigma$ is properly outer on $N \subset M$. If in addition $N \subset M$ is strongly amenable then, conversely, if $\sigma$ is properly outer on $N \subset M$ then it is properly outer on $M' \cap N^\omega$.

Proof. To prove $1^\circ$, it is clearly sufficient to treat the case $\theta$ is a single automorphism. Let $\lambda = \text{mod}\theta$, i.e. $\text{Tr}\theta = \lambda \text{Tr}, \lambda \neq 1$. Assume there exists $k \geq 0$ and $0 \neq a \in M^\infty$ such that $\theta(x) = 0$. Assume there exists $k \geq 0$ and $0 \neq a \in M^\infty$ such that $\theta(x) = ax, x \in N_k^\infty$. By conjugating if necessary with a unitary $u \in N^\infty$ we may suppose $\theta N_k^\infty = N_k^\infty$. Thus $\theta(N_k^\infty \cap M^\infty) = N_k^\infty \cap M^\infty$. Taking polar decomposition of $a$, it follows that there exists a partial isometry $0 \neq v \subset M^\infty$ such that $\theta(x)v = vx, x \in N_k^\infty$. Also, $v^*v, vv^* \in N_k^\infty \cap M^\infty$. If $p \leq v^*v, q \leq vv^*$ are minimal projections in $N_k^\infty \cap M^\infty$ such that $qv \neq 0$, then, by multiplying the equation $\theta(x)y = vx$ by $q$ on the left and $p$ on the right, we may suppose $q = v^*v, p = vv^*$ are minimal projections in $N_k^\infty \cap M^\infty$. 

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Since $N^∞ \subset M^∞$ is extremal, $N^∞_k \subset M^∞$ is extremal (cf. [PiPo2]) so that by [PiPo1], $[qM^∞_k : N^∞_k]E_{N^∞_k}(p) = [pM^∞_k : N^∞_k]E_{N^∞_k}(q)$. But since $\theta(x)v = vx$, $x \in N_k$, we have $N_kv = vN_k$, so that $(N^∞_k \subset qM^∞_k) = v(N^∞_k \subset pM^∞_k)v$. Thus $[qM^∞_k : N^∞_k] = [pM^∞_k : N^∞_k]$ and we get $E_{N^∞_k}(p) = E_{N^∞_k}(q).

It follows that if $f \in N^∞_k, Trf < ∞$, then $Tr(pf) = Tr(qf) = (Trf)E_{N^∞_k}(q) = (Trf)E_{N^∞_k}(p)$. Thus we get for $\lambda = \mod\theta$

$$\lambda Tr(f) = \lambda Tr(f)E_{N^∞_k}(p) = Tr(\sigma(f))E_{N^∞_k}(q) = Tr(\sigma(f)q) = Tr(\sigma(f)v)$$
$$= Tr(vfu^*) = Tr(fu^*v) = Tr(pf).$$

This contradiction shows that $a$ must be zero and thus $\theta$ acts properly outer on $N^∞ \subset M^∞$.

To prove 2°, it is sufficient to prove the case $G = Z$. To this end, denote $P_k = N^∞_k \cap (M^∞ \rtimes Z)$ and $Q_k = N^∞_k \cap M^∞$. Reasoning as in the proofs of the previous lemma and of Lemma 1.2, it is sufficient to prove the case $(N^∞_k \subset M^∞) = (N \subset M)$ is of type $Π_1$.

Let $\hat{θ}$ be the action of $T = \hat{Z}$ on $M \rtimes θ Z$ dual to $θ$ and note that $\hat{θ}(P_k) = P_k, \hat{θ}|Q_k = id$. Since $M = (M \rtimes θ Z)^θ$ we also have $Q_k = P_k^{\hat{θ}}$, by commuting squares. Since $T$ is simply connected, $\hat{θ}$ acts innerly on each $P_k$ and thus trivially on $Z(P_k)$. Thus, $\sup \dim Z(Q_k) ≥ \sup \dim Z(P_k)$, so that if $N \subset M$ has finite depth then $Γθ$ is finite. Thus, for $k$ large enough, the commuting square

$$P_k \subset P_{k+1}$$
$$\cup \quad \cup$$
$$Q_k \subset Q_{k+1}$$

is just the basic construction of

$$P_{k-1} \subset P_k$$
$$\cup \quad \cup$$
$$Q_{k-1} \subset Q_k$$

so that by [GHJ] or [Wel], we have $[P_∞ : Q_∞] < ∞$, where $P_∞ = \bigcup_k P_k, Q_∞ = \bigcup_k Q_k$.

Since $P_∞^{\hat{θ}} = Q_∞$, if we can show that $Q'_∞ \cap P_∞ = C$ we will get a contradiction, unless $Q_∞ = P_∞$. In turn, this equality implies $θ$ is properly outer, by Proposition 1.5.

If $x \in Q'_∞ \cap P_∞$ then let $ε > 0, k₀ ≤ 0$ and $x_{k₀} \in P_{k₀} = N'_{k₀} \cap (M \rtimes Z)$ be so that $∥x − x_{k₀}∥_2 < ε$. Thus $∥ux_{k₀}u^* − x∥_2 < ε, u \in U(Q_∞)$. Since $\sup \dim Z(N'_j \cap (M \rtimes Z)) < ∞$ we have $\sup \dim Z(N'_j \cap (M_i \rtimes Z)) < ∞$ (see e.g. the proof of the previous Proposition). Thus, there exists $ℓ$ large enough such that $N'_{j+1} \cap (M_ℓ \rtimes Z)$ is the basic construction of $N'_{j+1} \cap (M_ℓ \rtimes Z)$ by $N'_{j-1} \cap (M_ℓ \rtimes Z)$ with Jones projection $e_j$, for all $j$. Thus $x_{k₀} \in N'_{k₀} \cap (M \rtimes Z) \subset N'_{k₀} \cap (M_ℓ \rtimes Z) = sp(N'_{k₀-1} \cap (M_ℓ \rtimes Z))e_{k₀-1}(N'_{k₀-1} \cap (M_ℓ \rtimes Z))$. But since $N \subset M$ has finite depth $N_{k₀+1} \subset N_{k₀+2}$ has
finite depth, so that if \( Q_{j,\infty} = \overline{\bigcup_i (N_j^i \cap N_j)} \) then \( E_{Q_{j,\infty}} \cap Q_{\infty} (e_{-j}) \in \mathbb{C} \). Averaging over \( u \in Q_{j,\infty}, j = k_0 + 1, k_0 + 2, \ldots \) in the relation \( \| u x_{k_0} u^* - x \| < \varepsilon \), we obtain recursively that there exists \( x_0 \in \bigcap_{j \leq k_0 - 1} (N_j^i \cap (M_k \times \mathbb{Z})) \cap M = M' \cap M = \mathbb{C}1 \) such that \( \| x_0 - x \| < \varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, \( x \in \mathbb{C}1 \).

To prove 3°, it is again sufficient to consider the case \( G = \mathbb{Z} \) and we can use the same notations as above. Also, note that if \( \Gamma \theta \) is a tree for the \( G \) action, then it is a tree for the \( Z \) action as well. In this case, \( P_{k+1} \) is obtained from \( P_k \) by the basic construction (obtained with the Jones’ projection which is in \( M \) and thus on which \( \hat{\theta} \) acts trivially) adding in direct sum an abelian algebra. By induction, since \( P_0 = C \), we get by the triviality of \( \hat{\theta} \) on \( P_k \) and \( Z(P_{k+1}) \) that \( \hat{\theta} \) acts trivially on \( P_{k+1} \), thus \( Q_j = P_j \) for all \( j \) and \( \theta \) follows properly outer by Proposition 1.5.

To prove 4°, we may assume \( \theta \) is a single automorphism, which is outer on \( M^\alpha \). We may also clearly assume \( \alpha = 1 \). If for some \( k \) and \( 0 \neq a \in M_k \) we have \( \theta(x) a = a x \) for all \( x \in M \), then taking the polar decomposition of \( a \) we may assume \( a = v \) is a partial isometry with the right and left supports being minimal projections in \( M' \cap M_k \). By the hypothesis, it follows that these supports are in the same direct summand of \( M' \cap M_k \). This implies that the normalizer of \( M \rho \) in \( p M_k \rho p \) is non-trivial, where \( p \) is a minimal projection in that direct summand of \( M' \cap M_k \).

But this implies that \( (s_k) \) has an entry \( s_k = 1 \) for some \( k \neq *, \) a contradiction.

Part 5° has already been proved in the first part of the proof of (iii) in Proposition 1.4.

To prove 6°, let \( N \subset M \) be an inclusion of type II\(_1\) factors such that \( M' \cap N = M' \cap N^\omega \neq C1 \), i.e., \( N \) contains nontrivial central consequences of \( M \). Note that by arguing as in [McD], it follows that \( M' \cap N^\omega \) has no atoms. Let \( \theta \in \text{Aut}(M, N) \). Then \( \theta(M' \cap N^\omega) = M' \cap N^\omega \). Since \( M' \cap N^\omega \) has no atoms, if \( \theta \) is properly outer on \( M' \cap N^\omega \) then by Connes’ local Rohlin lemma ([C2], see also [P2]), given any \( \varepsilon > 0 \) there exists a partition \( \{ p_i \} \) of 1 with projections in \( M' \cap N^\omega \) such that \( \| \sum p_i \theta(p_i) \| < \varepsilon \). If \( \theta(x) a = a x \) for all \( x \in N_k \), for some \( k \geq 0 \) and \( a \in M \), then, since \( [p_i, a] = 0 \) and \( p_i \in N_k \) (because \( p_i \in N^\omega \) and \( [p_i, e_j] = 0, j \geq 0 \)), we get \( \sum p_i \theta(p_i) p_i a = \sum p_i a, \) while \( \| \sum \theta(p_i) p_i a \| \leq \varepsilon \| a \| \). Since \( \varepsilon > 0 \) was arbitrary \( a = 0 \).

Thus \( \theta \) is properly outer on \( N \subset M \).

Conversely, if we assume \( N \subset M \) is a strongly amenable inclusion and if \( \theta \) is properly outer on \( N \subset M \) then let \( M' \supset M' \supset N' \supset \cdots \) be a tunnel such that \( N_k' \cap M = M' \cap M \) and let \( p \in \mathcal{P}(M' \cap N^\omega), p \neq 0 \). Since \( p \in N^\omega \) and \( [p, e_k] = 0, k \geq 0 \), \( p \subset N_k^\omega \). Thus we may assume \( p \) is represented by a sequence \( p = (p_n)_n \), with \( p_n \in \mathcal{P}(N_k) \) and \( k_n \to \infty \). Note that, since \( N_k' \cap M = M' \cap M \), any sequence \( x = (x_n) \) with \( x_n \in N_k \) is in \( M' \cap N^\omega \). Since \( \theta \) is properly outer on \( N \subset M \), the proof of the Rohlin lemma in [Po2] shows that given any \( \varepsilon > 0 \) and any \( n \) there exists \( \delta = \delta(\varepsilon) \) (independent on \( n \)) and \( q_n = \mathcal{P}(N_k) \), \( q_n \leq p_n \), \( \tau(q_n) \geq \delta \tau(p_n) \), such that \( \| \theta(q_n) q_n \| < \varepsilon \| q_n \| \). But then \( q = (q_n) \) is in \( M' \cap N^\omega, 0 \neq q \leq p \) and \( \| \theta(q) q \| < \varepsilon \| q \| \). Thus \( \theta \) is properly outer on \( M' \cap N^\omega \). This proves the converse implication in 6°.

Q.E.D.

We mention that the finite depth case of part 6° in the above theorem was also shown independently by Y. Kamahigashi ([Ka2]). Thus, all the examples in [EvKa] of actions of finite groups on \( N \subset M \) that act trivially on \( M' \cap N^\omega \) (i.e., which are “centrally trivial”) are also examples of non-outer automorphisms of \( N \subset M \).
2. Classification of trace scaling actions on type $\Pi_\infty$ subfactors. We will prove in this section that a trace-scaling action of $\mathbb{Z}^n$ on a strongly amenable inclusion of type $\Pi_\infty$ factors splits into the tensor product between its standard part (as defined in 1.2) and a trace-scaling action on a common type $\Pi_\infty$ factor. More precisely we will prove:

2.1. THEOREM. Let $N^\infty \subset M^\infty$ be a strongly amenable inclusion of type $\Pi_\infty$ factors. Let $\theta$ be a properly outer action of $\mathbb{Z}^n$ on $N^\infty \subset M^\infty$ such that $\text{mod}\theta(g) \neq 1$, if $g \neq e = (0, \ldots, 0)$. Assume there exists a partition of the unity $\{e_g\}_g \in \mathbb{Z}^n$ with finite projections in $N^\infty$ such that $\theta(h)e_g = e_{hg}$, $h, g \in \mathbb{Z}^n$. Then there exists an isomorphism $\alpha$ from $M^\infty$ onto $M^{st} \otimes R^\infty$, with $\alpha(N^\infty) = N^{st} \otimes R^\infty$, such that $\alpha \theta \alpha^{-1} = \theta^{st} \otimes \sigma$, where $R^\infty$ is a copy of the hyperfinite type $\Pi_\infty$ factor and $\sigma$ is an action of $\mathbb{Z}^n$ on $R^\infty$ with $\text{mod}\sigma(g) = \text{mod}\theta(g)$, $g \in \mathbb{Z}^n$.

When applied to the case $n = 1$, i.e., for actions by one automorphism that scale the trace, by using Connes' theorem showing that all automorphisms acting on the hyperfinite type $\Pi_\infty$ factor and scaling the trace by the same number are conjugate, we get:

2.2. COROLLARY. Let $N^\infty \subset M^\infty$ be a strongly amenable inclusion of type $\Pi_\infty$ factors and $\theta$ a properly outer automorphism on $N^\infty \subset M^\infty$, scaling the trace (i.e., $\theta N^\infty = N^\infty$, $\text{mod}\theta \neq 1$). Then there exists an isomorphism $\alpha$ of $M^\infty$ onto $M^{st} \otimes R^\infty$, with $\alpha(N^\infty) = N^{st} \otimes R^\infty$, such that $\alpha \theta \alpha^{-1} = \theta^{st} \otimes \sigma_0$, where $R^\infty$ is a copy of the hyperfinite type $\Pi_\infty$ factor and $\sigma_0$ is the model action on $R^\infty$, with $\text{mod}\sigma_0 = \text{mod}\theta$. Moreover, if $N \subset M$ is extremal then the proper outerness condition is automatically satisfied.

2.3. COROLLARY. Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of hyperfinite $\Pi_\lambda$ factors, $0 < \lambda < 1$. Assume there exists a conditional expectation of finite index from $\mathcal{M}$ onto $\mathcal{N}$ which has discrete decomposition, i.e., if $N^\infty$ is the $\Pi_\infty$ core of $\mathcal{N}$ and $\phi$ is a normal semifinite weight on $\mathcal{N}$ whose centralizer is $N^\infty$, then the centralizer of $\phi \circ \mathcal{E}$ is the $\Pi_\infty$ core of $\mathcal{M}$, $M^\infty$. Let $(\mathcal{N} \subset \mathcal{M}) = (N^\infty \rtimes \sigma \subset M^\infty \rtimes \sigma)$ be the associated discrete decomposition. Assume also that $N^\infty \subset M^\infty$ has strongly amenable graph. Then $(\mathcal{N} \subset \mathcal{M}) \simeq (N^{st} \otimes R^\infty \rtimes \sigma^{st} \otimes \sigma_0 \subset M^{st} \otimes R^\infty \rtimes \sigma^{st} \otimes \sigma_0)$, where $\sigma^{st}$ is the action implemented by $\sigma$ on the model $\Pi_1$ inclusion $N^{st} \subset M^{st}$ associated with $N^\infty \subset M^\infty$ and $\sigma_0$ is a $\lambda$-scaling automorphism of the hyperfinite $\Pi_\infty$ factor $R^\infty$.

We will prove 2.1 by building a tunnel for $M^\infty \supset N^\infty$ that is invariant to all $\theta(g)$, $g \in \mathbb{Z}^n$, and so that the algebra of higher relative commutants splits its commutant in $N^\infty$, being a hyperfinite type $\Pi_\infty$ factor containing the "diagonal" $\{e_g\}_{g \in \mathbb{Z}^n}$.

To do this we need a technical lemma, which uses the noncommutative local Rohlin theorem ([Po1,2,7]) and some maximality arguments inspired from ([C3]). We will consistently denote by multiplication the operation in a discrete group, including $\mathbb{Z}^n$.

2.4. LEMMA. Assume $\text{mod}\theta(g) > 1$ if $g \neq e = (0, \ldots, 0)$ and $g$ has only nonnegative entries. Let $v_i$, $1 \leq i \leq n$, be partial isometries in $N^\infty$ such that
$v_i v_i^* = e_e$, $v_i^* v_i \leq e_{\delta_i}$, $1 \leq i \leq n$, where $\delta_i = (0, \ldots, 1, 0, \ldots, 0)$, $1$ appearing only on the $i$'th entry. Let $m \geq 0$ and denote by $K_m = \{ g \in \mathbb{Z}^n \mid 0 \leq g_i \leq m \}$. Let $\epsilon > 0$ and $X \subset e_e M^\infty e_e$ be a finite set. Then there exist unitary elements $w_i^0 \in e_0 N^\infty e_0$, such that one has the estimates

$$\{ \theta \} \subset N^\infty.$$ 

(iii) Assume $\epsilon > 0$ and $X \subset e_e M^\infty e_e$ be a finite set. Then there exist unitary elements $w_i^0 \in e_0 N^\infty e_0$, and a partition of the unity $\{ \theta \} \subset N^\infty$ with projections in $e_e N^\infty e_e$ of trace $\frac{1}{|K_m|}$ such that

1. $\| w_i^0 - e_0 \|_{1, \text{Tr}} \leq 3/m$, $1 \leq i \leq n$.

2. $\text{Ad}(w_i^0 \theta(\delta_i)(q_0^0) = q_{\theta(w_i)}$, $1 \leq i \leq n$, $g \in K_m$, where $s_i(g) = \delta_i g$, if the $i$'th entry $g_i$ of $g$ is less than $m$ and $s_i(g) = (g_1, \ldots, g_{i-1}, 0, g_{i+1}, \ldots, g_n)$ if $g_i = m$.

3. $\| x - \sum q_g x q_g \| < \epsilon$, $x \in X$.

**Proof.** Let $\mathcal{F}$ be the set of all $(n + |K_m|)$-tuples $((w_i)_{1 \leq i \leq n}, (q_g)_{g \in K_m})$ in which $(q_g)_{g \in K_m}$ are mutually orthogonal, mutually equivalent projections in $e_e N^\infty e_e$ and $w_i$ are unitary elements in $e_0 N^\infty e_0$, such that

(i) $\| w_i^0 - e_0 \|_{1, \text{Tr}} \leq 3/m \text{Tr}(\theta(\delta_i)(\sum q_g))$, $1 \leq i \leq n$.

(ii) $\text{Ad}(w_i \theta(\delta_i)(q_0) = q_{\theta(w_i)}, 1 \leq i \leq n, g \in K_m$.

(iii) $\| x - (1 - \sum q_g)x(1 - \sum q_g)\|_{2, \text{Tr}} \leq \text{Tr}(\sum q_g), x \in X$.

We define on $\mathcal{F}$ a (strict) order $<$ by letting $((w_i), (q_g)) < ((w_i'), (q'_g))$ if $q_g \leq q'_g, \forall g \in K_m$.

$$\sum q_g \neq \sum q'_g,$$

$$w_i = w_i' \theta(\delta_i)(\sum q_g),$$

$$\| w_i^0 - w_i^0 \|_{1, \text{Tr}} \leq 3/m \text{Tr}(\theta(\delta_i)(\sum q_g)), \forall 1 \leq i \leq n.$$ 

Then $(\mathcal{F}, <)$ is clearly inductively ordered. Let $((w_i^0), (q_g^0))$ be a maximal element in $\mathcal{F}$. Assume $\sum q_g \neq e_e$ and let $s = e_e - \sum q_g^0$. Note that $\text{Ad}(w_i \theta(\delta_i)(s) = s$.

To prove the lemma we only need to show that the assumption $s \neq 0$ leads to a contradiction.

For each $g \in K_m$ let $v_g$ be a partial isometry in $N^\infty$ with $v_g v_g^* = s$, $v_g^* v_g \leq \theta(g)$, $e_g \leq e_g$.

Let $\epsilon_0 > 0$. Since $s N^\infty s$ is hyperfinite, there exists a finite dimensional subfactor $B_0 \subset s N^\infty s$ such that:

a) $\theta(g)^{-1}(v_g v_g^*) \in B_0$, $g \in K_m$.

b) $\theta(\delta_i)^{-1}(v_{\delta_i g} v_i \theta(\delta_i)^{-1}(v_i v_i^*)) \in B_0$, $g \in K_m$, $1 \leq i \leq n$.

Since $B_0' \cap s N^\infty s \subset B_0' \cap s M^\infty s$ is strongly amenable, given any $\epsilon_0' > 0$ there exists a tunnel $M^\infty \supset N^\infty \supset N_1^\infty \supset \cdots \supset N_k^\infty$ such that $s \in N_k, B_0 \subset s N_k s$, and such that one has the estimates

c) $\theta(g)^{-1}(v_g x v_g^*) \subset_{\epsilon_0} B_0 \cap (s N_k \cap s M^\infty s), g \in K_m$.

Next let $\epsilon_1 > 0$. By the noncommutative local Rohlin theorem ([Po1,2,7]) from the proper outerness condition 1.3 on $\theta$ it follows that there exists a partition of the unity $\{ \eta_j \}$ with projections in $B_0' \cap s N^\infty s$ such that

\[ g \in K_m. \]
d) $\sum_{g \neq g' \in K_m} \left\| \sum_{j} \theta(g)(r_j)v_g^*v_{g'}^\theta(g')(r_j) \right\|_{2,\text{Tr}}^2 < \varepsilon_1^4 \left\| \sum_{j} r_j \right\|_{2,\text{Tr}}^2$.

It follows by d) that the set $J_1$ of all the $j$’s for which

$$\sum_{g \neq g'} \left\| v_g\theta(g)(r_j)v_g^*v_{g'}^\theta(g')(r_j)v_{g'}^* \right\|_{2,\text{Tr}}^2 < \varepsilon_1^2 \left\| r_j \right\|_{2,\text{Tr}}^2$$

satisfies $\text{Tr}(\sum_{j \in J_1} r_j) \geq (1 - \varepsilon_1^2)\text{Tr}(s)$.

Moreover, for each $g \in K_m$, by a) we have:

$$\sum_{j} \text{Tr} \left( v_g\theta(g)(r_j)v_g^* - v_g\theta(g)(r_j)v_{g'}^\theta(g')(r_j)v_{g'}^* \right)$$

$$= \text{Tr} \left( \theta(g)(s)v_g^*v_g - \sum_{j} \theta(g)(r_j)v_g^*v_{g'}^\theta(g')(r_j)v_{g'}^* \right) - \left\| \sum_{j} \theta(g)(r_j)v_g^*v_{g'}^\theta(g')(r_j) \right\|_{2,\text{Tr}}^2$$

$$\leq \text{Tr}(v_g^*v_g) - \left\| \theta(g)(E_{B'_0 \cap sP^\infty_s}(\theta(g)^{-1}(v_g^*v_g))) \right\|_{2,\text{Tr}}^2$$

$$\leq \text{Tr}(s) - \left\| \theta(g)(\theta(g)^{-1}(v_g^*v_g)) \right\|_{2,\text{Tr}}^2 + \left\| \theta(g)(\theta(g)^{-1}(v_g^*v_g) - E_{B'_0 \cap sP^\infty_s}(\theta(g)^{-1}(v_g^*v_g))) \right\|_{2,\text{Tr}}^2$$

$$= \left\| \theta(g)(\theta(g)^{-1}(v_g^*v_g) - E_{B'_0 \cap sP^\infty_s}(\theta(g)^{-1}(v_g^*v_g))) \right\|_{2,\text{Tr}}^2$$

$$\leq \varepsilon_0^2 \text{mod}(g) \text{Tr}(\sum_j r_j)$$.

This shows that if $J_2$ denotes the set of all $j$’s for which

$$\sum_{g} \left\| v_g\theta(g)(r_j)v_g^* - (v_g\theta(g)(r_j)v_g^*)^2 \right\|_{1,\text{Tr}} < \varepsilon_0 \text{Tr} r_j$$

then $\Sigma_{j \in J_2} \text{Tr} r_j \geq (1 - \varepsilon_0 \sum_{g} \text{mod}(g)) \text{Tr}s$. Indeed, for if not then $\Sigma_{j \in J_2} \text{Tr} r_j \geq (1 - \varepsilon_0 \sum_{g} \text{mod}(g)) \text{Tr}s$, so

$$\Sigma_{j \notin J_2} \text{Tr} r_j > \varepsilon_0 (\sum_{g} \text{mod}(g)) \text{Tr}s$$

which together with the inequality

$$\sum_{j \notin J_2} \sum_{g} \left\| v_g\theta(g)(r_j)v_g^* - (v_g\theta(g)(r_j)v_g^*)^2 \right\|_{1,\text{Tr}} \geq \varepsilon_0 \sum_{j \notin J_2} \text{Tr} r_j$$

implies

$$\sum_{j \notin J_2} \sum_{g} \left\| v_g\theta(g)(r_j)v_g^* - (v_g\theta(g)(r_j)v_g^*)^2 \right\|_{1,\text{Tr}} \geq \varepsilon_0 \Sigma_{j \notin J_2} \text{Tr} r_j > \varepsilon_0^2 (\sum_{g} \text{mod}(g)) \text{Tr}s$$
contradicting the inequality (2).

Also by first applying the Cauchy-Schwartz inequality and then a), we get:

\[
\sum_j |\text{Tr}(v_g \theta(g)(r_j)v^*_g) - \text{Tr}(r_j)|
\]

\[
= \sum_j |\text{Tr}(\theta(g)(r_j)(v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g))\theta(g)(r_j))|
\]

\[
\leq \| \sum_j \theta(g)(r_j)(v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g))\theta(g)(r_j) \|_{1,\text{Tr}}
\]

\[
\leq \| \sum_j \theta(g)(r_j)(v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g))\theta(g)(r_j) \|_{2,\text{Tr}} \| \sum_j \theta(g)(r_j) \|_{2,\text{Tr}}
\]

\[
\leq \| v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g) \|_{2,\text{Tr}} \| \theta(g)(s) \|_{2,\text{Tr}} \leq \varepsilon_0 \text{mod} \theta(g) \sum_j \text{Tr}r_j.
\]

Thus, reasoning exactly as above, it follows that if \( J_3 \) denotes the set of all \( j \)'s for which

\[
\sum_g |\text{Tr}(v_g \theta(g)(r_j)v^*_g) - \text{Tr}(r_j)| < \varepsilon_0^{1/2}\text{Tr}r_j
\]

then \( \sum_{j \in J_3} \text{Tr}r_j \geq (1 - \varepsilon_0^{1/2} \sum_g \text{mod} \theta(g))\text{Trs} \).

Further, we have for all \( g \in K_m \) for which \( \delta_i g \in K_m \):

\[
\sum_j \| \text{Ad}(v_i w_i^0)\theta(\delta_i)(v_g \theta(g)(r_j)v^*_g) - v_{\delta_i g} \theta(\delta_i g)(r_j)v_{\delta_i g}^* \|_{2,\text{Tr}}^2 =
\]

\[
= \sum_j \text{Tr}((v_i w_i^0 \theta(\delta_i)(v_g \theta(g)(r_j)(v^*_g)(v_i^0 w_i^0 v^*_i))^2 + \sum_j \text{Tr}((v_{\delta_i g} \theta(\delta_i g)(r_j)v^*_g)^2)
\]

\[
- 2\text{Tr}(\theta(\delta_i g)(r_j)(v^*_g v_i \theta(\delta_i)(v_g))\theta(\delta_i g)(r_j)(\theta(\delta_i)(v^*_g w_i^0 v_i^* v^*_i v_{\delta_i g}))
\]

\[
= \| E_{A_0}(\delta_i)(v^*_g w_i^0 v_i^* v_{\delta_i g})\theta(\delta_i)(v_g) \|_{2,\text{Tr}}^2 + \| E_{A_0}(v^*_g v_i w_i^0 \theta(\delta_i)(v_g)) \|_{2,\text{Tr}}^2
\]

where we denoted by \( A_0 \subset \theta(\delta_i g)(B_0 \cap sN^g_{\infty} s) \) the algebra generated by the partition \( \{ \theta(\delta_i g)(r_j) \}_j \). But then b) shows that this last term is majorized by:

\[
\| \theta(\delta_i)(v_g^* v_{\delta_i g}^* v_i w_i^0 \theta(\delta_i)(v_g)) \|_{2,\text{Tr}}^2
\]

\[
+ \| v_{\delta_i g} v_{\delta_i g}^* \|_{2,\text{Tr}}^2 - 2\| v_{\delta_i g} v_i w_i^0 \theta(\delta_i)(v_g) \|_{2,\text{Tr}} + 4\varepsilon_0 \text{mod} \theta(\delta_i)\text{Tr}(s)
\]

\[
= \| v_{\delta_i g} v_{\delta_i g}^* - v_i w_i^0 \theta(\delta_i)(v_g v_g^* w_i^0 v_i^* v_i^* \theta(\delta_i)(v_g) \|_{2,\text{Tr}} + 4\varepsilon_0 \text{mod} \theta(\delta_i)
\]

\[
= \| s - s \|_{2,\text{Tr}}^2 + 4\varepsilon_0 \text{mod} \theta(\delta_i)\text{Tr}(s) = 4\varepsilon_0 \text{mod} \theta(\delta_i) \sum_j \text{Tr}r_j.
\]

Thus, if we denote by \( J_4 \) the set of all the \( j \)'s for which

\[
\sum_g \| \text{Ad}(v_i w_i^0)\theta(\delta_i)(v_g \theta(g)(r_j)v^*_g) - v_{\delta_i g} \theta(\delta_i g)(r_j)v^*_g \|_{2,\text{Tr}}^2 < \varepsilon_0^{1/2}\text{Tr}r_j
\]
then \( \sum_{j \in J} \text{Tr}(r_j) \geq (1 - 4\varepsilon_0^{1/2}\text{mod}(\delta_i))\text{Tr}_s. \)

Finally, c) shows that if \( \varepsilon_0' \) is sufficiently small then

\[
\sum_{x \in X} \sum_{g \in K_m} \|v_g^*xv_g - \sum_j \theta(g)(r_j)v_g^*xv_g\theta(g)(r_j)\|_{2,\text{Tr}}^2 < \varepsilon_0'\text{Tr} \left( \sum r_j \right).
\]

Thus, if we denote by \( J_5 \) the set of all \( j \)'s for which

\[
(5) \quad \sum_{x} \sum_{g} \|\theta(g)(s) - \theta(g)(r_j)\|_{\text{Tr}}^2 < \varepsilon_0\text{Tr}r_j
\]

then \( \text{Tr} r_j \geq (1 - \varepsilon_0)\text{Tr}_s. \)

From all this, we see that if \( \varepsilon_0, \varepsilon_1 \) are chosen sufficiently small then \( \bigcap_{i=1}^{5} J_i \neq \emptyset. \)

Let \( j \in \bigcap J_i \) be fixed. Let \( a_g = v_g\theta(g)(r_j)v_g^* \), \( g \in K_m \). If \( \alpha = \varepsilon_1 + 2\varepsilon_0^{1/4} \) then by (1) - (5) we get:

(1') \( \|a_g^*a_g\|_{2,\text{Tr}} < \alpha\|a_g\|_{1,\text{Tr}}, \forall g, g' \in K_m, g \neq g'. \)

(2') \( \|a_g^2 - a_g\|_{1,\text{Tr}} < \alpha\|a_g\|_{1,\text{Tr}}, \forall g \in K_m. \)

(3') \( |\text{Tr}(a_g) - \text{Tr}(a_e)| < \alpha\text{Tr}(a_e), \forall g \in K_m, e = (0, \ldots, 0). \)

(4') \( \|\text{Ad}(v_i^0\theta(\delta_i))(a_g) - a_{\delta_i}g\|_{2,\text{Tr}} < \alpha\|a_g\|_{2,\text{Tr}}, g \in K_m, 1 \leq i \leq n, \delta_i g \in K_m. \)

(5') \( \sum_{g} \|sxs\,a_g^2\|_2^2 < \alpha\|a_g\|_{2,\text{Tr}}, \forall x \in X. \)

But then a standard perturbation argument shows that (1') - (3') imply the existence of mutually orthogonal projections \( \{p_g\}_{g \in K_m} \) in \( sN^\infty s \) such that \( \|p_g - a_g\|_{2,\text{Tr}} < f_0(\alpha)\|p_g\|_{2,\text{Tr}} \), \( \text{Tr}p_g = \text{Tr}p_e, g \in K_m \), where \( f_0(\alpha) \to 0 \) as \( \alpha \to 0 \) (\( m \) is fixed!). By (5') we then have for any \( x \in X \) the estimate:

\[
\left\| \left(sxs - \left(s - \sum p_g\right)x \left(s - \sum p_g\right) - \sum p_gxp_g\right) \right\|_{2,\text{Tr}}^2 \leq f_1(\alpha) \left\| \sum p_g \right\|_{2,\text{Tr}}^2,
\]

with \( f_1(\alpha) \to 0 \) as \( \alpha \to 0. \)

By (4') it follows the existence of unitary elements \( w_i' \) in \( \theta(\delta_i)(s)N^\infty \theta(\delta_i)(s) = \theta(\delta_i)(sN^\infty s) \) such that

\[
\text{Ad}(v_i^0w_i^0w_i')\theta(\delta_i)(p_g) = p_{\delta_i}g, g \in K_m \text{with } \delta_i g \in K_m
\]

\[
\|w_i' - \theta(\delta_i)(s)\|_{1,\text{Tr}} < f_2(\alpha)\text{Tr} \left( \sum_{g} \theta(\delta_i)(p_g) \right)
\]

where \( f_2(\alpha) \to 0 \) as \( \alpha \to 0. \)

Since the set \( \{ g \in K_m \mid \delta_i g \notin K_m \} \) has cardinality \( |K_m|/m \) it follows that there exist unitary elements \( v_i' \) in \( sN^\infty s \) such that \( \|v_i' - s\|_{1,\text{Tr}} < 2/m\text{Tr}(\sum p_g) \) and such that \( \text{Ad}(v_i^0w_i^0w_i')\theta(\delta_i)(p_g) = p_{s_i(g)} \) (with \( s_i(g) \) the bijection on \( K_m \) defined in the statement of the Lemma). But then, \( w_i^*w_i^0v_i^0w_i'w_i^0w_i' \) can be completed to a unitary \( w_i'' \) in \( \theta(\delta_i)(sN^\infty s) \), by defining it to be the identity on the complement of
its support. This unitary will still satisfy \( \|w'_i - \theta(\delta_i)(s)\|_{1,\text{Tr}} \leq 2/m \text{Tr}(\sum g p_g) < 2/m \text{Tr}(\theta(\delta_i) \sum p_g) \). If we thus define \( w_i = w'_i - \theta(\delta_i)(s) \) then we have:

\[
\text{Adv}_i w_i^0 w_i \theta(\delta_i)(p_g) = p_{s_i(g)}, \ g \in K_m;
\]

\[
\|w_i - \epsilon_\delta\|_{1,\text{Tr}} < (f_2(\alpha) + 2/m) \text{Tr}(\sum g \theta(\delta_i)(p_g));
\]

where the inequality follows from the estimate

\[
\|w'_i w''_i - \epsilon_\delta\|_{1,\text{Tr}} \leq \|w'_i w''_i - w'_i\|_{1,\text{Tr}} + \|w'_i - \epsilon_\delta\|_{1,\text{Tr}} = \|w''_i - \epsilon_\delta\|_{1,\text{Tr}} \leq (f_2(\alpha) + 2/m) \text{Tr}(\sum g \theta(\delta_i)(p_g)).
\]

Thus, if \( \alpha \) is sufficiently small then \( \|w_i - \epsilon_\delta\|_{1,\text{Tr}} < 3/m \text{Tr}(\sum_g \theta(\delta_i)(p_g)) \).

Define \( w_i^1 = w_i^0 w_i \) and \( q^1_g = q^0_g + p_g, 1 \leq i \leq n, g \in K_m \). Then by the definitions we have:

(i) \( \text{Adv}_i w_i^1 \theta(\delta_i)(q^1_g) = q^1_{s_i(g)}, \ g \in K_m, 1 \leq i \leq n \).

(ii) \( \|w_i - \epsilon_\delta\|_{1,\text{Tr}} \leq \|w_i^0 w_i - w_i^1\|_{1,\text{Tr}} + \|w_i - \epsilon_\delta\|_{1,\text{Tr}} \leq 3/m \text{Tr}(\theta(\delta_i) (\sum q^1_g)) \).

(iii) \( w_i^1 \sum_g q^0_g = w_i^1 (1 - s) = w_i^0 \).

(iv) \( \|w_i^1 - w_i^0\|_{1,\text{Tr}} = \|w_i - \epsilon_\delta\|_{1,\text{Tr}} \leq 3/m \text{Tr}(\theta(\delta_i) (\sum q^1_g - \sum q^0_g)) \).

(v)

\[
\left\| \left( x - \left( e_e - \sum q^1_g \right) x \left( e_e - \sum q^0_g \right) \right) - \sum q^1_g x q^0_g \right\|_{2,\text{Tr}}^2 \\
\leq \left\| \left( x - \left( e_e - \sum q^0_g \right) x \left( e_e - \sum q^0_g \right) \right) - \sum q^0_g x q^0_g \right\|_{2,\text{Tr}}^2 + \\
\| (sx - (s - \sum p_g)x (s - \sum p_g) - \sum p_g x p_g \right\|_{2,\text{Tr}} \leq \epsilon \text{Tr}(\sum q^1_g).
\]

But this shows that \( ((w_i^1)_i, (q^0_g)_g) \) is in \( \mathcal{F} \) and majorizes \( ((w_i^0)_i, (q^0_g)_g) \), thus contradicting the maximality of the latter.

Q.E.D.

We can now obtain the existence of equivariant tunnels for which the higher relative commutant approximate well a given finite set of elements.

### 2.5. LEMMA

Let \( N^\infty \subset M^\infty \) be a strongly amenable inclusion of type II\( _\infty \) factors, \( \theta : \mathbb{Z}^n \to \text{Aut}(N^\infty, N^\infty) \) an action scaling the trace and \( \{e_g\}_{g \in \mathbb{Z}^n} \subset N^\infty \) a partition of the unity like in the hypothesis of 2.2. Let \( X \subset M = e_e M^e e_e \) be a finite set, \( A_0 \subset e_e N^\infty e_e = N \) be a finite dimensional factor and \( w_i \in N^\infty \) be partial isometries such that \( w_i w_i^* = e_e, w_i^* w_i \leq e_\delta \). Given any \( \delta > 0 \) there exist a choice of the tunnel \( M \supset \tau \supset N \supset \cdots \supset \tau_{\ell+1} \supset N_{\ell-1} \supset N_\ell \), up to some \( \ell, \) for \( M \supset N \), and a finite dimensional subfactor \( B_0 \subset N_\ell \) containing \( A_0 \) such that if \( e_{-\delta} = \sum_g \theta(g)(e^0_{-\delta}) \) then

(i) \( x \in B_0 \vee (N_\ell \cap M), \ x \in X \).

(ii) \( w_i \in N^\infty_\delta, 1 \leq i \leq n, \) where \( N^\infty_\delta = \{e_0, e_{-1}, \ldots, e_{-\ell+1}\}' \cap N^\infty \).
Proof. By Lemma 2.4, if $\varepsilon > 0$ then there exist $m > 3\varepsilon^{-1}$, unitary elements $\omega_i \in e_i, N^\infty e_i$, $1 \leq i \leq n$, and a partition of the identity with mutually orthogonal, mutually equivalent projections $(q_g)_{g \in K_m}$ in $e_i N^\infty e_i$ such that

1. $\text{Ad}(w_i \omega_i)\theta(\delta_i)(q_g) = q_{s_i(g)}, g \in K_m, 1 \leq i \leq n$.
2. $\|\omega_i - e_i\|_{2,\text{Tr}} \leq \varepsilon, 1 \leq i \leq n$.
3. $\|x - \sum_g q_g x q_g\|_{2,\text{Tr}} \leq \varepsilon, x \in X$ or $x = e_{ij} \in A_0$, with $\{e_{ij}\}$ a fixed matrix unit of $A_0$.

Note that (1) and the fact that $\{q_g\}_g$ have equal traces implicitly means that $[\theta(\delta_i)(q_g), \omega_i^* w_i^* w_i \omega_i] = 0$ and $\text{Tr}(\theta(\delta_i)(q_g)\omega_i^* w_i^* w_i \omega_i) = \text{Tr}(q_g) = \text{Tr}(q_g)$, for all $g \in K_m, 1 \leq i \leq n$.

For each $g \in K_m$ let $w_g \in N^\infty$ with $w_g w_g^* = q_g, w_g^* w_g \leq \theta(g)(q_g)$ and $w_e = e$. Like in the proof of the previous lemma, given any $\varepsilon_0 > 0$ there exists a finite dimensional subfactor $B \subset q_e N^\infty q_e$ such that:

(a) $\theta(g)^{-1}(w_g w_g^*) \subset e_0 B, g \in K_m$.
(b) $\theta(\delta_i g)^{-1}(w_{\delta_i g}^* w_{\delta_i g} \omega_{\delta_i g} \omega_{\delta_i g}^*) \subset B, g \in K_m \cap \delta_i^{-1} K_m$.

By a small perturbation depending only on $\varepsilon_0$ it follows that there are partial isometries $w'_g$ in $N^\infty$ such that:

(a') $w^*_g w'_g \in \theta(g)(B), w_g^* w_g^* \leq q_g, \|w_g^* - w'_g\|_{2,\text{Tr}} \leq f'_0(\varepsilon_0), g \in K_m$.

Moreover, by enlarging if necessary $B$, we will also have:

(a'') $q_g e_{ij} q_g \in \text{Ad} w'_g \theta(g)(B), g \in K_m, e_{ij} \in A_0$.

It then follows that there are partial isometries $\omega'_{i,g} \in q_{\delta_i g} N^\infty q_{\delta_i g}$ such that:

(b') $b_{i,g} = w^*_{\delta_i g} \omega'_{i,g} w_{\delta_i g} \omega_{\delta_i g} \omega'_{i,g}^* \in \theta(\delta_i g)(B)$ is a partial isometry in $\theta(\delta_i g)(B)$ and we have the estimates

$$
\omega'_{i,g} \omega_{i,g} \omega_{i,g}^* \omega'_{i,g} \leq \text{Ad}(w_i \omega_i)\theta(\delta_i)(w'_g w_g^*), \text{omega}'_{i,g} \omega_{i,g}^* \leq w'_g w_g^*
$$

$$
\|\omega'_{i,g} - q_{\delta_i g}\|_{2,\text{Tr}} \leq f''_0(\varepsilon_0), g \in K_m \cap \delta_i^{-1} K_m
$$

where $f'_0(\varepsilon_0), f''_0(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$.

Now, since $N^e = B' \cap q_e N^\infty q_e \subset B' \cap q_e M^\infty q_e = M^e$, and more generally $(w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^* \nsubseteq (w'_g \theta(g)(B))w_g^*$ are strongly amenable and also since

$$(w'_g \theta(g)(B))w_g^* = \text{Ad} w'_g \theta(g)(N^e),
$$

$$(w'_g \theta(g)(B))w_g^* = \text{Ad} w'_g \theta(g)(M^e),
$$

it follows that given any $\varepsilon_1 > 0$ there exists a choice of the tunnel up to some $\ell, M^e \subset N^e \subset N^e \subset \cdots \subset N^e \subset N^e \subset N^e$ for $M^e \subset N^e$, so that to have:

(c') $w_g^* w_g^* X w_g^* w_g^* \subset \text{Ad} w'_g \theta(g)(B) \vee \text{Ad} w'_g \theta(g)(N^e \cap M^e)$.

Put $p_e = e_e - \sum_g w_g^* w_g^*$ and let $\{e_j\}_{0 \leq j \leq \ell}$ be a set of Jones projections in $p_e M^\infty p_e$, with $e_0$ projecting on the scalar $[M^\infty : N^\infty]^{-1} p_e$ when expected onto $p_e N^\infty p_e$ and with $e_1, \ldots, e_{\ell-1} \in p_e N^\infty p_e$. 

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Then define $e^0_{-j} \overset{\text{def}}{=} \sum_{g \in K_m} \text{Ad} w^*_g \theta(g)(e^{e^i}_{-j})$ and $e_{-j} \overset{\text{def}}{=} \sum_{g \in \mathbb{Z}^n} \theta(g)(e^{0}_{-j})$.

$N^\infty_j \overset{\text{def}}{=} \{ q_0, \ldots, q_{-\ell + 1} \} \cap N^\infty$. Since $\sum_{g \in K_m} \text{Ad} w^*_g \theta(g)(B) \subset e_{e^i}N^\infty \subset e_{\varepsilon}$ it follows that given any $\varepsilon_2 > 0$ there exists a finite dimensional subfactor $B_0 \subset e_{e^i}N^\infty \subset e_{\varepsilon}$ such that $\{ q_g \}_{g \in K_m} \subset B_0, B \subset q_e B_0 q_e, \text{Ad} w^*_g \theta(g)(B) \subset B_0, \forall g \in K_m$.

Since by (a') we have $\| e_{i,g} - \sum_{g \in K_m} w^*_g w^g \|_{\ell^2} \leq |K_m| f'(\varepsilon_0)$, with $|K_m| f'(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, and since by (3) we have $X \subset \sum_{g \in K_m} q_g X q_g$, it follows by (c') and by the definition of $B_0$ that if we let $\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2$ sufficiently small then condition (i) of the statement is satisfied.

To show that (ii) is also satisfied note that if we let

$$q' \overset{\text{def}}{=} \sum_{g \in K_m} w^*_g w^g \in e_{e^i}N^\infty \subset e_{\varepsilon},$$

$$q'' \overset{\text{def}}{=} \sum_{g \in K_m} \omega^*_i \omega^i = \sum_{g \in K_m} w^*_g b_i g^* b^i_g w^i_g,$$

$$\omega^i = \sum_{g \in K_m} \omega^i g,$$

then $[q', e_{-j}] = 0, [q'', e_{-j}] = 0$. By (b') we then have:

$$\text{Ad}(\omega^i w^i; \omega^i) \theta(\delta_i)(q'e_{-j}) = \sum_{g \in K_m} \text{Ad}(\omega^i w^i; \omega^i)(\delta_i g)(\omega^i e_{-j})$$

$$= \sum_{g \in K_m} \text{Ad}(w^i g b_i, g \delta_i g)(\omega^i e_{-j})$$

$$= \sum_{g \in K_m} \text{Ad}(w^i g \delta_i g)(\omega^i e_{-j}) b_i g^* b_i g$$

$$= e_{-j} q''.$$

Thus, since $\theta(\delta_i)(e_{-j}) = e_{-j}$, it follows that

$$(q'' \omega^i w^i; \omega^i)(q')(q') e_{-j} = e_{-j}(q'' \omega^i w^i; \omega^i)(q')(q').$$

Since $q''$ is close to $e_{e^i}, \theta(\delta_i)(q')$ is close to $e_{\delta_i}, \omega^i$ is close to $\omega^i$, it follows that there exist $w^i_1 (= q'' \omega^i w^i; \omega^i)(\delta_i)(q'))$ in $N^\infty$ that is arbitrarily close to $w^i_1$, say $\| w^i_1 - w^i \|_{\ell^2} < \delta$, so that $\| w^i_1 - w^i \|_{\ell^2} < \delta$, i.e., so that $w^i_1 \in N^\infty_\delta$. This shows that for appropriately small $\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2$ condition (ii) of the statement will also be satisfied.

Still, the condition $A_0 \subset B_0$ is not yet achieved. But by the way $B_0$ was defined, condition (a'') shows that $A_0 \subset B_0$ with $\alpha$ as small as we please. Since both $A_0, B_0 \subset e_{e^i}N^\infty \subset e_{\varepsilon}$ it follows by [Ch] that there exists a unitary element $u_0 \in e_{e^i}N^\infty \subset e_{\varepsilon}$ close to $e_{e^i}$ so that $A_{u_0}(B_0) \subset A_0$. But then, if we conjugate spatially all the previous choices, $\{ e_{-j} \}_{0 \leq j \leq \varepsilon_1}, \{ N^\infty_i \}_{1 \leq i \leq \ell}$, by $A_{u_0}$, where $u = \sum_{g \in \mathbb{Z}^n} \theta(g)(u_0)$, then for $\alpha$ small enough the estimates (i), (ii) will still hold true. Q.E.D.
Proof of Theorem 2.1. By changing if necessary the positive cone of $\mathbb{Z}^n$ we may assume $\text{Tr} \theta(g) \geq \text{Tr}$ for all the $g$ having only nonnegative coordinates.

Let $\{x_n\}_n$ be a sequence of elements in $e_e M^\infty e_e$, dense in the strong operator topology in $e_e M^\infty e_e$. We construct recursively an increasing sequence of integers, $i_1 < i_2 < \ldots$, continuations of the tunnel $M^\infty \supset N^\infty \supset N^\infty_1 \supset \cdots \supset N^\infty_{i_k} \cdots$ and an finite dimensional factors $B_k \in N^\infty_{i_k} = \{e_0, e_{-1}, \ldots, e_{-i_k}\}' \cap N^\infty$, such that:

(i) $x_j \in B_k \lor e_e (N^\infty_{i_k} \cap M^\infty)e_e, 1 \leq j \leq k$.

(ii) $\|w_{k}^j - w_{k}^{j-1}\|_{2, \text{Tr}} \leq 2^{-k}, w_{k}^j w_{k}^j = e_e, w_{k}^j w_{k}^j \leq e_{\delta_i}, 1 \leq i \leq n$.

Assume we made this construction up to some $k$.

By [C3] there exists $\delta' > 0$ such that if $P^\infty \subset N^\infty_k$ is a subfactor containing the projections $\{e_g\}_{g \in \mathbb{Z}^n}$ and if $b_i \in P^\infty$ are so that $\|b_i\| \leq 1$, $b_i = e_e b_i e_{\delta_i}, \|w_{k}^j - b_i\|_{2, \text{Tr}} < \delta$, then there exists partial isometries $w_{k}^j \in P^\infty$ such that $w_{k}^j w_{k}^{j*} = e_e, w_{k}^j w_{k}^{j*} \leq e_{\delta_i}, \|w_{k}^j - w_{k}^{j-1}\|_{2, \text{Tr}} \leq 2^{-k-1}$.

Also, there exist a finite subset $X \subset e_e N^\infty_{i_k-1} e_e$ and a $\delta'' > 0$ such that if $N^\infty_{i_k-1} \supset N^\infty_k \supset \cdots \supset N^\infty_{i_{k+1}}$ is a continuation of the tunnel up to some $i_{k+1}$ such that $B_k \subset N^\infty_{i_{k+1}}, \{e_g\}_{g \in \mathbb{Z}^n} \subset N^\infty_{i_{k+1}}$ and if $X \subset B_{k+1} \lor e_e (N^\infty_{i_{k+1}} \cap N^\infty_{i_k}) e_e$, for some finite dimensional subfactor $B_{k+1} \subset e_e N^\infty_{i_{k+1}} e_e$ with $B_{k+1} \supset B_k$, then $x_j \in B_{k+1} \lor e_e (N^\infty_{i_{k+1}} \cap M^\infty)e_e, 1 \leq j \leq k + 1$. Indeed if one takes $\{m_j\}_j$ to be an orthonormal basis of $e_e N^\infty_{i_k-1} e_e$ over $e_e N^\infty_{i_k} e_e$ then one may construct an orthonormal basis $\{m_j\}_i$ of $e_e M^\infty e_e$ over $e_e N^\infty_{i_k} e_e$ as words in the $m_j$’s and in $e_{0}, e_{-1}, \ldots, e_{-i_k}$ (see [PiPo1,2]). Writing $x_1, \ldots, x_{k+1}$ in the basis $\{m_j\}_i$ of $e_e M^\infty e_e$ over $e_e N^\infty_{i_k} e_e$ we obtain a finite set $X \subset e_e N^\infty_{i_k} e_e$ which for suitably small $\delta''$ will yield the above estimates.

If we now take $\delta = \min\{\delta', \delta''\}$ and apply Lemma 2.5 for $N^\infty_{i_k} \subset N^\infty_{i_{k-1}}$ (as $N^\infty \subset M^\infty$), $B_k \subset e_e N^\infty_{i_k} e_e$ (as $A_0$), the restriction of $\theta$ to $N^\infty_{i_{k-1}}$ and the above $X$ and $\delta$ then we get the $i_{k+1}$ the $\theta$-equivariant continuation of the tunnel up to $i_{k+1}$, the algebra $B_{k+1}$ and the partial isometry $w_{k+1}$ satisfying (i), (ii) for $k + 1$.

Now, since all $N^\infty_{i_k}$ contain the projections $\{e_g\}_{g \in \mathbb{Z}^n}$, we have $\{e_g\}_{g \in \bigcap_j N^\infty_{i_k}}$. By condition (ii) we have $\{w_j\}_k$ is Cauchy in the norm $\|\cdot\|_{2, \text{Tr}}$ and if $w_i = \lim w_{k}^i$ then $w_i$ are also contained in $\bigcap_j N^\infty_j$. Moreover $\bigcup_j B_k \subset \bigcap_{e_j N^\infty_j e_e}$ and if $P = \bigcup_k B_k$ then by (i) we have $e_e M^\infty e_e = P \lor e_e (N^\infty_k \cap M^\infty)e_e$. Thus $e_e (\bigcap_j N^\infty_j)e_e = P$ so that $e_e (\bigcap_j N^\infty_j)e_e$ and all $\theta(g)(e_e \cap N^\infty_j e_e) = e_g (\bigcap_j N^\infty_j)e_g$ are type $\text{II}_1$ factors. But then the fact that the $w_i$ (and thus all $\theta(g)(w_i)$) are in $\bigcap_k N^\infty_k$ shows that the projections $\{e_g\}_{g \in \mathbb{Z}^n} \subset \bigcap_j N^\infty_j$ are comparable (in the Murray-von Neumann sense) in $\bigcap_j N^\infty_j$. Thus $\bigcap_j N^\infty_j$ is a type $\text{II}_\infty$ factor.

It follows that if we denote by $R^\infty = \bigcap_j N^\infty_j, M^{st} = \bigcup_j (N^\infty_j \cap M)$, $N^{st} = \bigcup_j (N^\infty_j \cap N^\infty)$ then all the conditions in the theorem are satisfied. Q.E.D.

3. Classification of actions on type $\text{II}_1$ subfactors. In this section we will use ([Po7]) and noncommutative ergodic theory techniques to prove that a properly outer cocycle action $\theta$ of a discrete amenable group $G$ on a strongly amenable
inclusion of type II\(_1\) factors \(N \subset M\) is (cyclic) conjugate to the tensor product of the canonical action \(\theta^{st}\) on \(N^{st} \subset M^{st}\) and a commonly split properly outer
model cocycle action \(\sigma\) of \(G\) on a single hyperfinite type II\(_1\) factor \(R_0\). When applied to the case \(N \subset M = M_{2 \times 2}(\mathbb{C})\), for which the standard part of any action is trivial, this shows that any properly outer cocycle action of \(G\) on a hyperfinite II\(_1\) factor \(R_0\) is cocycle conjugate to a cocycle action of the form \(id \otimes \sigma\). Altogether, this gives:

3.1. THEOREM. Let \(N \subset M\) be a strongly amenable inclusion of type II\(_1\) factors and \(G\) a countable discrete amenable group. Let \(\theta : G \to \text{Aut}(M, N)\) be such that \(\theta(g)\) is properly outer on \(N \subset M\) for each \(g \neq e\) and such that \(\theta(g)\theta(h) = \text{Ad}(u(g, h))\theta(gh)\), for \(g, h \in G\), where \(u(g, h) \in \mathcal{U}(N)\) satisfy

\[
u(g, h)u(gh, h') = \theta(g)(u(h, h'))u(g, hh')\]

\[
u(e, g) = u(g, e) = 1.
\]

If \(R_0\) is a copy of the hyperfinite type II\(_1\) factor and \(\sigma_0\) is a properly outer action of \(G\) on \(R_0\), then \((N \subset M, \theta)\) is cocycle conjugate to \((N^{st} \otimes R_0 \subset M^{st} \otimes R_0, \theta^{st} \otimes \sigma_0)\), i.e., there exists an isomorphism \(\alpha : M \to M^{st} \otimes R_0\) such that \(\alpha(N^{st} \otimes R_0) = N\) and such that \(\text{Adv}(g)\theta(g) = \alpha^{-1}(\theta^{st}(g) \otimes \sigma_0(g))\alpha\), for all \(g \in G\) and for some unitaries \(v(g) \in \mathcal{U}(N)\).

The rest of this section is devoted to the proof of this theorem. The idea is to construct a tunnel of subfactors \(M \supseteq e_0 \supseteq e_{-1} \supseteq N_1 \supseteq \cdots\) with unitaries \(v_k(g) \in \mathcal{U}(N_k)\)

such that \(\|v_k(g) - 1\| < 2^{-k}\), for \(g \in F_k \subset G\), with \(F_k\) finite sets satisfying \(F_k \subset F_{k-1}\), \(F_k = G\), and \(\text{Ad}(v_k(g) \cdots v_0(g)\theta(g)(e_{-k}) = e_{-k}\), for all \(g \in G\), and such that \(S = \bigcup (N_k \cap \mathcal{M})\), \(\bigcup (N_k' \cap \mathcal{M}) = R\) splits \(N \subset M\), i.e., \(S' \cap R = R' \cap \mathcal{M}\) and \((N \subset M) \approx (R \cap \mathcal{M}) \subset (R' \cap \mathcal{M})\). Then \(v(g) = \lim v_k(g) \cdots v_0(g)\), \((N^{st} \subset M^{st}) \cong (S \subset R)\), \(\theta^{st}(g) = \text{Adv}(g)\theta(g)|R, R_0 = R' \cap M, \sigma(g) = \text{Adv}(g)\theta(g)|R_0\)

will be of the form \(\theta^{st} \otimes \sigma_0\) on \((N^{st} \subset M^{st}) \otimes R_0\), with \(\sigma_0\) an action of \(G\) on \(R_0\).

Moreover, we will construct this split off of \(\theta\) so that \(\text{Adv}\theta_{R_0}\) is a prescribed model properly outer cocycle action of \(G\) on the hyperfinite II\(_1\) factor \(R_0\).

3.2. Some model cocycle actions. To construct the above perturbations \(v_k(g)\) and prove the Theorem, we first need to introduce some notations. We will then construct a “model” for properly outer cocycle actions of amenable groups on the hyperfinite II\(_1\) factor.

Notations.

(1) Let \(G\) be a countable discrete amenable group. For each finite subset \(F \subset G\) and \(\varepsilon > 0\) we choose a finite (Hofner) set \(K \subset G, K = K(F, \varepsilon)\), with the property that \(|FK\Delta K| < \varepsilon|K|\). Moreover, for each \(h \in G\) we choose once for all a permutation \(s_K(h)\) of the set \(K\), such that for \(g \in K\) with \(hg \in K\) we have \(s_K(h)(g) = hg\).

(2) Let \(\{F_n\}_{n \geq 1}\) be an increasing sequence of finite dimensional subsets of \(G\) with \(G = \bigcup F_n\) and \(K_n = K(F_n, 2^{-n})\). Denote by \(w_n(h)\) the unitary element on \(\ell^2(K_n)\) defined on the orthonormal basis \(\{\delta_g\}_{g \in K_n}\) of \(\ell^2(K_n)\) by \(w_n(h)\delta_g = \delta_{s_{K_n}(h)(g)}, h \in G, g \in K_n\).
(3) Let $R = \bigotimes_{n \geq 1} B(\ell^2(K_n), \tau_n)$ be the hyperfinite type II$_1$ factor realized as an infinite tensor product with respect to the unique normalized traces $\tau_n$ of the finite dimensional factors $B(\ell^2(K_n))$. For each $k \geq 1$ and $h \in G$ let $\theta_k(h) \in \text{Aut}R$ be defined as the product type automorphism $\theta_k(h) = \text{Ad}(1 \otimes \cdots \otimes 1 \otimes w_k(h) \otimes w_{k+1}(h) \otimes \cdots$ where $w_k(h)$ appears exactly at the $k$’th position.

**LEMMA.** For each $k \geq 1$ and $h, g \in G$, the element

$$u_k(h, g) = 1 \otimes \cdots \otimes 1 \otimes w_k(g)w_k(h)w_k(gh)^{-1} \otimes w_{k+1}(g)w_{k+1}(h)w_{k+1}(gh)^{-1} \otimes \cdots$$

is a unitary in $R$ and $\theta_k : G \rightarrow \text{Aut}R$ is a properly outer cocycle action of $G$ with 2-cocycle $u_k$.

**Proof.** Due to the condition $|gK_n \setminus K_n| < 2^{-n}|K_n|$, $|hK_n \setminus K_n| < 2^{-n}|K_n|$ for $h, g \in F_n$, it follows by the definition of $s_{K_n}(g)$, $s_{K_n}(h)$, $s_{K_n}(gh)$ that $s_{K_n}(gh)$ coincides with $s_{K_n}(g)s_{K_n}(h)$ on “most” of $K_n$, more precisely $\|w_k(g)w_k(h)w_k(gh)^{-1} - 1\|_2^2 < 2^{-n+2}$. This shows that $u_k(g, h)$ are all unitaries, $k \geq 1$, $g, h \in G$, and that

$$\lim_{k \to \infty} \|\theta_k(g) - 1\|_2 = 0, \ g, h \in G.$$ 

Clearly we have $\text{Ad}u_k(g, h) = \theta_k(g)\theta_k(h)\theta_k(gh)^{-1}$ and $u_k(g, e) = u_k(e, g) = 1$ and the identity:

$$u_k(g, h)u_k(gh, h') = 1 \otimes \cdots \otimes 1 \otimes w_k(g)w_k(h)w_k(gh)^{-1}w_k(gh)w_k(h')w_k(ghh')^{-1}$$

$$= 1 \otimes \cdots \otimes 1 \otimes w_k(g)w_k(h)w_k(h')w_k(ghh')^{-1} \otimes 1 \otimes \cdots$$

$$= \cdots \otimes 1 \otimes w_k(g)(w_k(h)w_k(h')w_k(hh')^{-1})w_k(g)^{-1}(w_k(g)w_k(hh')w_k(ghh')^{-1}) \otimes \cdots$$

shows that $u_k(g, h)u_k(gh, h') = \theta_k(g)(u_k(h, h'))u_k(g, hh')$, and thus $u_k(g, h)$ is a 2-cocycle for $\theta_k$. Q.E.D.

**COROLLARY.** Let $\theta : G \rightarrow R$ be a cocycle action of the discrete amenable group $G$ on the hyperfinite type II$_1$ factor $R$. Given any $\varepsilon > 0$, $x_1, \ldots, x_n \in R$, $F \subset R$ a finite dimensional factor, there exist a finite dimensional subfactor $B \subset R$ containing $B_0$ and unitary elements $v(h) \in R, h \in F$, such that

1. $\|E_B(x_i) - x_i\|_2 < \varepsilon, 1 \leq i \leq n$.
2. $\|v(h) - 1\|_2 < \varepsilon, h \in F$.
3. $\text{Adv}(h)\theta(h)(B) = B, h \in F$.

**Proof.** By the previous Lemma and [Oc], there are unitary elements $v_0(h) \in R, h \in G$, such that $\text{Adv}_0(h)\theta(h)$ is identical to the cocycle model action $\theta_1(h), h \in G$, for some splitting of $R$ in an infinite tensor product, as in (3) above. But then there exists $m$ large enough so that if $B = B(\ell^2(K_1)) \otimes \cdots \otimes B(\ell^2(K_m))$, then $\|E_B(x_i) - x_i\|_2 < \delta, 1 \leq i \leq n$, $\|E_B(v_0(h)) - v_0(h)\|_2 < \delta, h \in F$, with $\delta$ to be chosen later. Since $\theta_1(h)B = B$, from the last set of inequalities it follows that for $x \in B, \|x\| \geq 1$, we have:

$$\|E_B(\theta(h)(x)) - \theta(h)(x)\|_2 = \|E_B(v_0(h)^*\theta_1(h)(x)v_0(h)) - v_0(h)^*\theta_1(h)(x)v_0(h)\|_2 \leq 3\delta.$$
By [Ch] it follows that if $\delta$ is small enough then there exist unitary elements $v(h) \in R$ such that $\text{Adv}(h)\theta(h)B = B$ and $\|v(h) - 1\|_2 < \varepsilon$, $h \in F$.

Q.E.D.

3.3. **Construction of Rohlin towers.** We will first construct partitions of the unity in $N$ on which small perturbations of a given finite set of automorphisms $\theta_g$ act by permutations, as in the model action 3.2, i.e., like on a Rohlin tower.

**LEMMA.** Under the hypothesis of 3.1 and with the notations in 3.2, let $\varepsilon > 0$, $X = X^* \subset M$, $F \subset G$ be finite sets and $K = K(F, \varepsilon/2)$. Then, there exist projections $\{e_g^0\}_{g \in K} \subset N$, with $\Sigma e_g^0 = 1$, and unitary elements $\{v_0(h)\}_{h \in F} \subset N$ such that

$$
\left\| x - \sum_{g \in K} e_g^0 xx_g^0 \right\|_2 \leq \varepsilon, \quad x \in X
$$

$$
\left\| v_0(h) - 1 \right\|_1 \leq \varepsilon, \quad h \in F
$$

$$
\text{Adv}(h)\theta_h(e_g^0) = e_{s_h(h)}(g), \quad h \in F, \quad g \in K
$$

Proof. Let $F$ be the set of all $|F| + |K|$ tuples $((v(h))_{h \in F}, (e_g)_{g \in K})$ in which $(e_g)_{g \in K}$ are mutually orthogonal projections in $N$ and $(v(h))_{h \in F}$ are unitary elements in $N$ such that

$$
\left\| (x - (1 - \sum_K e_{g'})x(1 - \sum_K e_{g'})) - \sum_K e_gxx_g \right\|_2^2 \leq \varepsilon^2 \sum_K e_g^2
$$

$$
\left\| v(h) - 1 \right\|_1 \leq \varepsilon \sum \tau(e_g)
$$

$$
\text{Adv}(h)\theta_h(e_g) = e_{s_h(h)}(g), \quad g \in K, \quad h \in F.
$$

We define on $F$ the (strict) order $<$ given by

$$(v(h))_{h \in F}, (e_g)_{g \in K} < ((v'(h))_{h \in F}, (e_g')_{g \in K})$$

if $e_g < e_g'$, $g \in K, \Sigma e_g \neq \Sigma e_g', v(h) = v'(h) \Sigma e_g$ and $\|v'(h) - v(h)\|_n \leq \varepsilon \tau(\Sigma e_g' - \Sigma e_g)$. Then $(F, <)$ is clearly inductively ordered. Let $((v_0(h))_{h \in F}, (e_g^0)_{g \in K})$ be a maximal element in $F$. Assume $\Sigma e_g^0 \neq 1$ and let $s = 1 - \sum_{g \in K} e_g^0$. Note that

$$\text{Adv}(h)\theta_h(s) = s.$$

Now denote $sNs = Q$, $sMs = P$ and $\sigma : G \to \text{Aut}(P, Q)$ the cocycle action defined by $\sigma_g = \text{Adv}(g)\theta_{g|P}$ for each $g \in G$ and note that $\sigma_g\sigma_g = \text{Adv}(h, g)\sigma_h$ for some unitary elements $w(h, g) \in Q$, $g, h \in G$. By 1.3 $\sigma$ is properly outer on $Q \subset P$. Let $Y = \{\sigma_g^{-1}(sx) \mid x \in X, g \in K\} \cup \{\sigma_g^{-1}(w(h, g)) \mid h \in F, g, g' \in K\}$ and let $\delta_0 > 0$. Since $Q \subset P$ has the generating property, there exists a choice of the tunnel up to some $i$, $P \supset Q \supset Q_1 \supset Q_2 \supset \cdots \supset Q_i$ such that $\|E_{Q_i\cap P}(y) - y\|_2 < \delta_0, \ y \in Y$.

By 1.2 there are unitary elements $w'(g) \in Q$ such that $\text{Adv}(g)\sigma_g(Q_j) = Q_j, \ j \leq i$. It follows by the proper outerness of $\sigma$ that if we denote by $L$ the crossed product algebra $P \rtimes G$ and by $u_g \in L$ the unitary elements implementing the action $\sigma_g$ on $P$, then $E_{Q_i\cap L}(u_g^{-1}u_g') = 0$, $g \neq g'$.

Given any $\delta > 0$ there exists by (A.1.4 in [Po7]) a projection $q \in Q_i$ such that

$$
\|qy - E_{Q_i\cap P}(y)q\|_2 < \delta_0/2\|q\|_2, \quad y \in Y
$$

$$
\|qu_g^{-1}u_g'q\|_2 < \delta\|q\|_2, \quad g, g' \in K, \quad g \neq g'.
$$
Since $Y = Y^*$ the first relation implies
\[ \| [q, y] \|_2 < \delta_0 \| q \|_2, \quad y \in Y \]
so that one has:

1. $\| [\sigma_g(q), x] \|_2 < \delta_0 \| q \|_2, \quad x \in X \cup \{ w(h, g) \mid h \in F, g \in K \}, g \in K$.

The second relation implies $\sigma_g(q), g \in K$, are $\delta$-mutually orthogonal projections.

2. $\| \sigma_g(q) \sigma_g'(q) \|_2 < \delta\| q \|_2, \quad g, g' \in K, \quad g \neq g'$.

When applied to $x = w(h, g)$, (1) implies that

3. $\| \sigma_h \sigma_g(q) - \sigma_h q \|_2 < \delta_0 \| q \|_2, \quad h \in F, g \in K \cap h^{-1}K$.

Since $|K \setminus FK| < \varepsilon/2|K|$ and since $\delta, \delta_0$ are arbitrarily small, it follows that there exist mutually orthogonal projections $e_g \in Q, g \in K$, such that

4. $\| e_g, x \|_2 < f_1(\delta_0) \| e_g \|_2, \quad \tau(e_g) = \tau(q), g \in K$;

5. $\| e_g - \sigma_g(q) \|_2 < f_2(\delta) \| e_g \|_2, \quad g \in K$;

6. $\| \sigma_h(e_g) - e_h q \|_2 < f_3(\delta) \| e_g \|_2, \quad h \in F, g \in K \cap h^{-1}K$.

where $f_i(\varepsilon) \to 0$ as $\varepsilon \to 0, i = 1, 2, 3$.

Thus, if $w(h) = \sum_{g \in K \cap h^{-1}K} e_h g \tau(e_g)$, then

\[ \| w(h) - \sum_{g \in K \cap h^{-1}K} e_g \|_2^2 < f_3(\delta)^2 \| \sum_{g \in K \cap h^{-1}K} e_g \|_2^2. \]

Since $\sum_{g \in K \cap hK} \tau(e_g) < \varepsilon/2 \sum_{g \in K} \tau(e_g)$, it follows by letting $\delta, \delta_0$ very small and by taking the polar decomposition of $w(h)$ and suitably extending it to a partial isometry $w_1(h)$ from $\sum_{g \in K} \sigma_h(e_g)$ to $\sum_{g \in K} e_g$ carrying $\sigma_h(e_g)$ onto $e_h g$ when $g \in K \cap h^{-1}K$ and more generally $\sigma_h(e_g)$ onto $e_{\tau(h)(g)}$, that we have $w_1(h) \in Q = sNa, h \in F$ and

\[ \| w_1(h) - \sum_{g \in K} e_g \|_1 \leq (\varepsilon/2 + f(\delta_0, \delta)) \| \sum_{g \in K} e_g \|_1. \]

But then, by ([3]), there exists a suitable extension of $w_1(h)$ to a unitary $w_0(h)$ in $Q$ which will satisfy

\[ \| w_0(h) - 1 \| \leq \varepsilon \| \sum_{g \in K} e_g \|_1. \]

Thus, if we take now the $|F| + |K|$ tuple $((v_1(h))_{h \in F}, (e^1_g)_{g \in K})$, with $v_1(h) = ((1 - s) + w_0(h)) v_0(h)$ and $e^1_g = e^0_g + e_g$, then it is easy to see that it satisfies the necessary conditions to be contained in $\mathcal{F}$. But $((v_1(h))_h, (e^1_g)_{g \in K}) > ((v_0(h))_h, (e^0_g)_{g \in K})$, which contradicts the maximality of $((v_0(h))_{h \in F}, (e^0_g)_{g \in K})$.

This shows that $\sum_{g \in K} e^0_g = 1$ and that $v_0(h), h \in F$, are unitary elements satisfying the requirements.

Q.E.D.

3.4. Existence of equivariant Jones projections. We now prove that given any finitely many automorphisms of a cocycle action of an amenable group, there exist Jones projections that are almost invariant to all of them.
LEMMA. Assume the conditions in the hypothesis of 3.1 are satisfied. Let $X \subset M$, $F \subset G$ be finite sets and $\varepsilon > 0$. Denote $K = K(F, |F|^{-1}/4)$. There exist a choice of the tunnel up to some $i$, $M \supset N \supset N_1 \supset \cdots \supset N_i$, a partition of the unity $\{e^0_g\}_{g \in K}$ in $N_i$ and unitary elements $\{v(h)\}_{h \in F}$ in $N$ such that

1. $\|v(h) - 1\|_2 < \varepsilon, h \in F$
2. $\text{Ad}(h)\theta(h)(e^0_g) = e^0_{sK(h)(g)}$, $h \in F, g \in K$
3. $\text{Ad}(h)\theta(h)(e_{-j}) = e_{-j}, 0 \leq j \leq i - 1, h \in G$
4. $\|E_{N_i \cap \{N_i \cap M\}}(x) - x\|_2 < \varepsilon, x \in X$.

Proof: By Lemma 3.3, there exist some unitary elements $\{v_0(h)\}_{h \in F} \subset N$ and a partition of the unity by projections $\{e^0_g\}_{g \in K} \subset N$ such that

$$\|v_0(h) - 1\|_2 < \varepsilon/2, \quad h \in F$$

$$\text{Ad}(h)\theta(h)(e^0_g) = e^0_{sK(h)(g)}, \quad h \in F, g \in K$$

$$\|x - \sum_g e^0_g xe^0_g\|_2 < \varepsilon/2, \quad x \in X.$$

Let $\{w_g\}_{g \in K}$ be partial isometries such that $w_g\theta(g)(e^0_g)w^*_g = e^0_g, g \in K$. For any $\delta_0 > 0$, there exists a finite dimensional factor $B_0 \subset e^0_N Ne^0_N$ such that if $h \in F, g \in K$, and $hg \in K$ then $\|((\text{Ad}w_h)\theta(hg))^{-1}\text{Ad}(h)\theta(h)\text{Ad}(g)(\theta(0) - x)\|_2 < \delta_0/2$ for all $x \in B_0 \cap e^0_N Ne^0_N$, $\|x\| \leq 1$. This is possible because $\theta$ is a 2-cocycle action, so that any product $\theta(h)\theta(g)$ differs from $\theta(hg)$ by an inner automorphism and also because $N$ (and thus $e^0_N Ne^0_N$) can be approximated by finite dimensional subfactors. Let $Y = \{(\text{Ad}w_h)\theta(hg)\}^{-1}(x) | x \in X, g \in K \} \subset e^0_M e^0_e$. Since the inclusion $B_0 \cap e^0_N Ne^0_N \subset B_0 \cap e^0_M e^0_e$ has the generating property, it follows that there exist some $i$ and some choice of the tunnel up to $i$ for this inclusion, coming from some tunnel $M \supset N \supset N_1 \supset \cdots \supset N_i$, with $e^0_e \in N_i$, $B_0 \subset e^0_N e^0_N$, such that $\sum \|E_{B_0 \cap e^0_N e^0_N}(y) - y\|_2 < 3e^2/4$, the sum being taken over all $y \in Y$.

Finally, let $e_{-j} = \sum_{g \in K} w_g\theta(g)(e^0_{-j}e^0_g)w^*_g, 0 \leq j \leq i - 1$. If $B = \Sigma_g \text{Ad}(w_g)(\theta(g)(B_0) \subset \Sigma_g \text{Ad}(w_g)(\theta(g)(B_0) \subset \Sigma_g \text{Ad}(w_g)(\theta(g)(B_0)$ then $e^0_e \in N_i, B \subset N_i$ and for each $x \in X$ we have

$$\|E_{N_i \cap \{N_i \cap M\}}(x) - x\|_2 \leq \|E_{\Sigma_g \text{Ad}(w_g)(\theta(g)(B_0 \cap e^0_N e^0_N)}(x) - x\|_2$$

$$\leq \sum_g \|E_{B_0 \cap e^0_N e^0_N}(e^0_g xe^0_g) - e^0_g xe^0_g\|_2 + \|x - \sum e^0_g xe^0_g\|^2$$

$$\leq \sum_g \|E_{B_0 \cap e^0_N e^0_N}(e^0_g xe^0_g) - (\text{Ad}(w_0)(\theta(g))^{-1}(e^0_g xe^0_g))\|^2$$

$$- (\text{Ad}(w_0)(\theta(g))^{-1}(e^0_g xe^0_g))^2$$

This proves that the tunnel $M^e \supset N^{-1} \supset N_1 \supset \cdots \supset N_{i-1} \supset N_i$ satisfies (4).

Next, due to the choice of $B_0$, we have that $\text{Ad}(v_0(h)\theta(h)\text{Ad}(w_0)(\theta(g))\text{Ad}(w_0)(\theta(g))^{-1})$ differs from $\text{Ad}(w_h)\theta(hg)(\text{Ad}(w_0)(\theta(g))^{-1})$ by an inner automorphism $\text{Ad}(w_0)(\theta(g))^{-1}$ with $u'_{h',g} \in e^0_{-j}e^0_g \text{Ad}(w_0)(\theta(g))(B_0 \cap N) e^0_{-j}e^0_g$, $\|u'_{h',g} - 1\|_2 < f(\delta_0)$ and $f(\delta_0) \to 0$ as $\delta_0 \to 0$.

We now let $u'_h = \Sigma u'_{h',g}$, with the sum taken over $g \in K' = \cap_{h \in F} h^{-1}K$ and define $u_h = u'_h + u''_h$, where $u''_h = \Sigma u''_{h',g}$ the sum being taken over all $g \in K \setminus K'$, with $u''_{h,g}$ satisfying $\text{Ad}(w_0)(\theta(g))(e^0_{-j}e^0_g) = \text{Ad}(w_0)(\theta(g))(e^0_{-j}e^0_g)$. Then $v(h) =$
u_h v_0(h) is a unitary element in N that satisfies (1) and (3). Since u_h lies in \{ e_g \}_{g \in K} \cap N, condition (2) will still be satisfied, since v_0(h) satisfies it. Q.E.D.

3.5. Proof of Theorem 3.1. Let \{ x_n \}_n be a dense sequence (in the norm \| \|_2) in M. Let F_n be an increasing sequence of finite subsets of G such that \bigcup F_n = G.

We construct recursively an increasing sequence of integers 0 = i_0 < i_1 < i_2 < \cdots < i_k < \cdots, for each k a choice of the tunnel \{ e^{-i_j} \} N_{i_k-1} \supset \cdots \supset e^{-i_{k+1}} N_{i_k} \supset N_{i_k}, finite dimensional subfactors \{ B_k \}_{k \geq 0}, with B_k \subset \bigcap_j N_j, B_k = 0, k \neq k', and unitary elements \{ v_k(h) \}_{k \geq 0}, h \in G, with v_k(h) \in (B_0 \vee \cdots \vee B_{k-1})' \cap N_{i_k}, and satisfying:

1. \| x_j - E_{B_0 \vee B_1 \vee \cdots \vee B_k} v(N'_{i_k} \cap M)(x_j) \|_2 < 2^{-k}, j \leq k.
2. Ad(v_k(h) \cdots v_0(h)) \theta(h)(e_{-j}) = e_{-j}, 0 \leq i_k, h \in G.
3. Ad(v_k(h) \cdots v_0(h)) \theta(h)(B_k) = B_k, h \in G with B_k \simeq B(\ell^2(K_k)) \otimes M_{n_k \times n_k}(\mathbb{C}), where K_k = K(F_k, 2^{k-2} | F_k |^{-1}) and Ad(v_k(h) \cdots v_0(h)) \theta(h) e_g = e_{sK_k(h)(g)} for h \in F_k, \{ e_g \}_{g \in K}, being the diagonal of B(\ell^2(K_k)).
4. \| v_k(h) - 1 \|_2 < 2^{-k}, h \in F_k.

Assume we made this construction up to some k \geq 0. We claim that there exist \varepsilon > 0 and a finite subset X \subset (B_0 \vee \cdots \vee B_k)' \cap N_{i_k-1} such that if N_{i_k-1} \supset \cdots \supset N_{m-1} \supset N_i \supset \cdots \supset e^{-i_{k+1}} N_{i_k} \supset \cdots \supset e^{-i_k} N_{i_k-1} \supset e^{-i_k} N_{i_k-1} = \text{a continuation of the tunnel satisfying } B = B_0 \vee \cdots \vee B_k \subset N_i and X \subset N_i \cap \bigvee_j N_{i_{k+1}}, then x_j \in N_i \cap \bigvee_j N_{i_{k+1}}.

To see this, we argue exactly as in the proof of Theorem 2.1. Thus, we take \{ m_j \} to be an orthonormal basis of \{ m_i \}, of M over N_i which for suitably small \varepsilon will yield the estimates.

We can thus apply Lemma 3.4 to the inclusion B' \cap N_{i_k} \subset B' \cap N_{i_{k-1}} to the above \varepsilon > 0 and X \subset B' \cap N_{i_{k-1}} to the automorphisms \{ Ad(v_k(h) \cdots v_0(h)) \theta(h) \}_{B' \cap N_{i_{k-1}}} and to the finite set F_{k+1} \subset G, to get a continuation of the tunnel up to some i_{k+1}' \cdots i_{k-1}' \supset i_k \supset \cdots \supset \supset i_{k+1}' \supset \cdots \supset \supset N_{i_{k+1}} \supset \cdots \supset \supset N_{i_k-1} \supset N_{i_k+1}, with B \subset N_{i_k+1}, unitary elements v_k^0(h) \in B' \cap N_k and a partition of the unity \{ e_{g}^{k+1} \}_{g \in K_{k+1}} in B' \cap N_{i_{k+1}} indexed by the Folner set K_{k+1} = K(F_{k+1}, | F_{k+1} |^{-12} | k^{-3}), such that

1. \| E_{(B_0 \vee \cdots \vee B_k) \cap N_{i_{k+1}} \cap N_{i_{k-1}}} \theta(h)(x) - x \|_2 < \varepsilon, x \in X.
2. Ad(v_k^0(h) \cdots v_0(h)) \theta(h)(e_{-j}) = e_{-j}, i_k \leq j < i_{k+1}, h \in G.
3. Ad(v_k^0(h) v_k(h) \cdots v_0(h)) \theta(h)(e_g^{k+1}) = e_{sK_{k+1}(h)(g)}, h \in F_{k+1}, g \in K_{k+1}.
4. \| v_k^0(h) - 1 \|_2 < 2^{-1 - 2^{-k-1}}, h \in F_{k+1}.

Indeed, in order to be able to apply Lemma 3.4 we only need to show that Ad(v_k(h) \cdots v_0(h)) \theta(h), h \notin e, are all properly outer when restricted to both N_{i_{k-1}} and N_{i_k}. To see this we only need to show that if \sigma \in \text{Aut}(M, N), \sigma, \sigma|_N are properly outer, M_{e_0} N \supset N_1 is a downward basic construction and \nu \in N is a unitary such that Ad\sigma(e_0) = e_0, Ad\sigma(N_1) = N_1, then Ad\sigma|_{N_1} is also properly outer. But Ad\sigma properly outer on M and Ad\sigma(e_0) = e_0 implies Ad\sigma|_{e_0 M e_0} properly outer and since e_0 M e_0 = N_1 e_0 \simeq N_1, it follows that Ad\sigma|_{N_1} is also properly outer.
Now, since $N_{i_k+1}$ is hyperfinite, there exists a finite dimensional factor $B_{k+1}^0 \subset (B_0 \vee \ldots \vee B_k) \cap N_{i_k+1}$ containing the projections $\{e_{g}^{k+1}\}_{g \in K_{k+1}}$ such that

$$\lim_{k \to \infty} \|Ex^{k+1}_{g} \vee (N_{i_k+1} \cap N_{i_{k-1}}) - x\|_2 < \varepsilon, x \in X.$$  

But then, by Corollary 3.2, there exist a finite dimensional subfactor $B_{k+1} \subset B' \cap N_{i_k+1}$ containing $B_{k+1}^0$ and unitary elements $v_{k+1}(h) \in B' \cap N_{i_k+1}, h \in G$, such that

$$(3'') \quad \text{Ad} v_{k+1}^1(h) v_{k+1}^0(h) \theta(h)(B_{k+1}) = B_{k+1}, h \in G.$$  

Thus, if we define $v_{k+1}(h) = v_{k+1}^1(h) v_{k+1}^0(h)$ then, by the above properties of $X, \varepsilon$, it follows that all the conditions (1) – (4) are satisfied at step $k + 1$.

We now let $R = \bigcup_k (N_k' \cap M), S = \bigcup_k (N_k' \cap N), R_0 = B_0 \vee B_1 \vee \ldots, v(h) = \lim_k (v_k(h) \ldots v_0(h)), \sigma_0(h) = \theta(h)|_{R_0}, h \in G$.

Condition (1) implies that $M = R \vee R_0 \simeq R \otimes R_0$. Condition (4) shows that $v(h)$ are all unitary elements in $N$. Then (2) shows that Ad$v(h)\theta(h)$ restricted to $S \subset R$ (which is isomorphic to $N^{st} \subset M^{st}$) is nothing but $\theta^{st}(h)$. Finally, by (3), we have that Ad$v(h)\theta(h)|(R_0) = R_0$ and also that Ad$v(h)\theta(h)$ is of the form $\theta_0 \otimes \theta_1$, for some splitting $R_0 = R \otimes R_1$, with $\theta_0$ a model properly outer cocycle action of $G$ on the hyperfinite $\text{II}_1$ factor $R$, as in Section 3.2. Thus, Ad$v(h)\theta(h)|_{R_0}, h \in G$, is a cocycle action of $G$ on $R_0$. But then by Ocneanu’s theorem in [Oc], Ad$v(h)\theta(h)|_{R_0}$ is the same as the chosen action $\sigma_0$ on $R_0$, after perturbing if necessary with unitaries in $R_0$.

Q.E.D.

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*Added references, July 2009:*

