259A Winter 2023: "Introduction to C*- and W*-algebras"

Instructor: **Sorin Popa**. Meetings: **MWF 3-4pm in MS5147**

This is an introductory course in algebras of operators on Hilbert space, which are, roughly speaking, noncommutative extensions of the algebras of continuous complex functions on compact spaces (the C*-*algebras*) and respectively of the bounded measurable functions on a measure space (the W*-*algebras*).

Operator Algebras enable to extend many topics from Topology, Measure Theory, Geometry and Analysis to the noncommutative setting, creating powerful tools for the investigation of several problems in Mathematics and Theoretical Physics.

The prerequisite for this class are the Real Analysis 245 and Functional Analysis 255 classes. Here is the material I would like to cover:

1. Revisiting the geometry of Hilbert space. You have all done this in the 245 and 255 classes, but I will nevertheless go through the definitions and basic results, without proofs.

2. The space $\mathcal{B}(\mathcal{H})$ of all linear bounded operators on a Hilbert space $\mathcal{H}$: (a) the adjoint (or *) operation; (b) the operator norm, the *wo* and *so*-topologies; (c) special classes of operators (self-adjoint, positive, normal, unitaries, compact, etc);

3. Definition of C*-algebras and W*-algebras (or von Neumann algebras) as subalgebras of $\mathcal{B}(\mathcal{H})$; (a) first examples; (b) the commutant of a self-adjoint set in $\mathcal{B}(\mathcal{H})$ is a vN algebra; (c) vN's bicommutant theorem.

4. Abstract C*-algebras: (a) defintion and examples; (b) characterization of commutative C*-algebras as $C(X)$ and continuous functional calculus; (c) ideals and morphisms; (d) positivity; (e) positive functionals; (f) GNS construction and representation of an abstract C*-algebra as algebra of operators on Hilbert space.

5. More examples of C*-algebras: group C*-algebras (reduced and universal), crossed product constructions, inductive limits, UHF-algebras, AF-algebras.

6. Some basic facts on tensor products of C*-algebras.

7. Completely positive maps: definition, conditional expectations and Tomyiama's theorem, Arveson's extension theorem, Stinespring dilation theorem.

8. Back to W*- (or vN) algebras: Borel functional calculus and polar decomposition in $\mathcal{B}(\mathcal{H})$ and in vN algebras; Kaplansky density theorem; geometry of projections; finite and properly infinite vN algebras; classification by type; vN algebras with trace and II$_1$ factors.

9. Examples of II$_1$ factors: (a) group factors; (b) group measure space factors;

All registered students will get an A, but will have to make a presentations on Mondays 4-6pm, towards the end of quarter.

Useful texts: J. Dixmier *C*-algebras and their representations*; S. Sakai *C*-algebras and W*-algebras*; G. Pedersen *C*-algebras and their automorphism groups.*
My notes on II1 factors: https://www.math.ucla.edu/popa/Books/IIunV15.pdf
Notes from my 259A Fall 2020:
https://drive.google.com/drive/folders/1RGsIvFo2-ZfSRXecovKqiyUvY9drjVtj