Manifold Theory

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Abstract. These notes are a supplement to a first year graduate course in manifold theory.

Contents

CHAPTER 1

Manifolds

1.1. Smooth Manifolds

1.1.1. Basic Definitions. An manifold M, is a topological space with a *maximal atlas* or a *maximal smooth structure*.

There are two virtually identical definitions. The standard definition is as follows: There is an *atlas* $\mathcal{A}$ consisting of maps $F_\alpha : U_\alpha \to \mathbb{R}^{n_\alpha}$ such that

(1) U_α is an open covering of M.
(2) F_α is a homeomorphism on to its image.
(3) The transition functions $F_\alpha \circ F_\beta^{-1} : F_\beta (U_\alpha \cap U_\beta) \to F_\alpha (U_\alpha \cap U_\beta)$ are diffeomorphisms.

In condition 3 it suffices to show that the transition functions are smooth as they are already forced to be homeomorphisms.

A *smooth structure* is a collection $\mathcal{D}$ consisting of continuous functions whose domains are open subsets of M with the property that: For each $p \in M$, there is an open ngbd $U \ni p$ and functions $x^i \in \mathcal{D}$, $i = 1, ..., n$ such that

(1) The domains of x^i contain U.
(2) The map $F = \left(x^1, ..., x^n\right) : U \to \mathbb{R}^n$ is a homeomorphism onto its image $V \subset \mathbb{R}^n$.
(3) For each $f : O \to \mathbb{R}$ in $\mathcal{A}$ there is a smooth function $h : V \cap F(O) \to \mathbb{R}$ such that $f = h\left(x^1, ..., x^n\right)$ on $U \cap O$.

Note that $h = f \circ F^{-1}$ in condition 3, but it is usually possible to find h without having to invert F. The map in 2 in both definitions is called a *chart* or *coordinate system* on U. The topology of M is recovered by these maps.

Note that it is very easy to see that these two definitions are equivalent. Both have advantages. The first in certain proofs, the later is generally easier to work with when showing that a concrete space is a manifold. It is also often easier to work with when it comes to defining key concepts.

A continuous function $f : O \to \mathbb{R}$ is said to be smooth wrt $\mathcal{D}$ if $\mathcal{D} \cup \{f\}$ is also a smooth structure. In other words we only need to check that condition 3 still holds when we add f to our collection $\mathcal{D}$.

The space of all smooth functions is now a maximal smooth structure.

It is often the case that all the functions in $\mathcal{D}$ have domain M. In fact it is possible to always select the smooth structure such that this is the case. We shall also show that it is possible to always use a finite collection $\mathcal{D}$.

A manifold of dimension n or an n-manifold is a manifold such that coordinate charts always use n functions.

PROPOSITION 1.1.1. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets that are diffeomorphic, then $m = n$.*

PROOF. The differential of the diffeomorphism is forced to be a linear isomorphism which shows that $m = n$. □

COROLLARY 1.1.2. *A connected manifold is an n-manifold for some integer n.*

PROOF. It is not possible to have coordinates around a fixed point into different Euclidean spaces. Let $A^n \subset M$ be the set of points that have coordinates using n functions. This is clearly an open set. Moreover if $p_i \to p$ and $p_i \in A^n$ then we see that if p has a chart that uses m functions then p_i will also have this property showing that $m = n$. □

1.1.2. Examples. Spheres and their coordinates, both orthogonal projection coordinates and stereographic projection coordinates.

Projective spaces.

Matrices of constant rank.

Tangent spaces to spheres:

$$TS^n \simeq \left\{(p, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |p| = 1 \text{ and } (p|v) = 0\right\}$$

1.1.3. Topological Properties of Manifolds. The goal is to show that we can construct partitions of unity. This means that we have to start by showing that the space is paracompact. The minimal topological assumptions for this to work is that the space is locally compact, second countable, and Hausdorff. For a manifold as defined above this means that the topology additionally has to be separable (contains a dense countable subset) and Hausdorff. The Hausdorff property is essential for much that we do, but it will also seem as if we rarely use it explicitly. Two essential properties come from the Hausdorff axiom. First, that limits of sequences are uniquely defined. Second, that compact subsets are closed sets. A compact set which is not closed is often called quasi-compact.

We now proceed to the constructions that are directly related to what we shall use later.

THEOREM 1.1.3. *A Hausdorff, second countable and locally compact space is σ-compact and paracompact.*

PROOF. The fact that the space is locally compact and has a countable dense subset immediately shows that it is σ-compact. In fact the space has a countable open cover A_α where each closure $\bar{A}_\alpha$ is compact. From such a cover we can construct what is known as a compact exhaustion: $K_1 \subset K_2 \subset \cdots$, where each K_i is compact, $K_i \subset \text{int} K_{i+1}$, and $M = \bigcup K_i$. The construction is inductive: Let K_1 be any of the sets $\bar{A}_\alpha$, and define K_i as the union of a finite collection of $\bar{A}_\beta$ such that A_β cover K_{i-1}.

To show that the space is paracompact let $V_i = \text{int} K_i - K_{i-1}$. Then $C_i = \bar{V}_i$ are compact "annuli" with $C_i \cap C_j = \emptyset$ when $|i - j| > 1$. Extend this to a covering of open sets $U_i = V_{i-1} \cup V_i \cup V_{i+1}$ and note that $U_i \cap U_j = \emptyset$ when $|i - j| > 4$. In other words these are locally finite covers. Given an open cover B_α we can consider the refinement $B_\alpha \cap U_i$. For fixed i we can then extract a finite collection of $B_\alpha \cap U_i$ that cover the compact set V_i. This leads us to a locally finite refinement of the original cover. □

The fundamental lemma we need is a smooth version of Urysohn's lemma.

LEMMA 1.1.4. (Smooth Urysohn) *If M is a smooth manifold and $C_0, C_1 \subset M$ are disjoint closed sets, then there exists a smooth function $f : M \to [0,1]$ such that $C_0 = f^{-1}(0)$ and $C_1 = f^{-1}(1)$.*

PROOF. First we claim that for each open set $O \subset M$ there is a smooth function $f : M \to [0,\infty)$ such that $M - O = f^{-1}(0)$.

We start by proving this in Euclidean space. First note that for any open cube

$$O = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

there is a bump function $\mathbb{R}^n \to [0,\infty)$ that is positive on O and vanishes on the complement. Next write a general open set O as a union of open cubes such that for all $p \in O$ there is a neighborhood U that intersects only finitely many open cubes. Using bump functions on each of the cubes we can then add them up to get a function that is positive only on O.

Next note that if $U \subset M$ is open and admits a chart, then this construction gives us a function that is positive on U. If we extend this function to vanish on $M - U$ we certainly obtain a continuous function. Since all partial derivatives converge to zero as we approach the boundary of U this also becomes a smooth function.

More generally we can now cover any open set $O \subset M$ by a locally finite cover of charts. On each chart construct a function as just explained and then add all of these functions to obtain a smooth function that is positive on O and vanishes on $M - O$.

Finally, the Urysohn function is constructed by selecting $f_i : M \to [0,\infty)$ such that $f_i^{-1}(0) = C_i$ and defining

$$f(x) = \frac{f_0(x)}{f_0(x) + f_1(x)}.$$

This function is well-defined as $C_0 \cap C_1 = \emptyset$ and is the desired Urysohn function. □

We can now easily construct the partitions of unity we need.

LEMMA 1.1.5. *Let M be a separable Hausdorff manifold. Any countable locally finite covering U_α of open sets has partition of unity subordinate to this covering, i.e., there are smooth functions $\phi_\alpha : M \to [0,1]$ such that $\operatorname{supp}\phi_\alpha \subset U_\alpha$ and $1 = \sum_\alpha \phi_\alpha$.*

PROOF. The previous result gives us functions $\lambda_\alpha : M \to [0,1]$ such that $\lambda_\alpha^{-1}(0) = M - U_\alpha$. As the cover is locally finite the sum $\sum_\alpha \lambda_\alpha$ is well-defined. Moreover it is always positive as U_α cover M. We can then define

$$\phi_\alpha = \frac{\lambda_\alpha}{\sum_\alpha \lambda_\alpha}$$

□

Finally we can use this to show

PROPOSITION 1.1.6. *A separable Hausdorff manifold admits a proper smooth function.*

PROOF. Select a compact exhaustion $K_1 \subset K_2 \subset \cdots$, where each K_i is compact, $K_i \subset \operatorname{int} K_{i+1}$, and $M = \bigcup K_i$. Then choose functions $\phi_i : M \to [0,i]$ such that $\phi_i(K_{i-1}) = 0$ and $\phi_i(M - K_i) = 1$. Then consider $\rho = \sum \phi_i$. □

One can additionally show that a connected metric space that is locally compact is also σ-compact and hence separable. On the other hand the Urysohn metrization theorem asserts that a separable normal Hausdorff space is metrizable. The proof of this result is remarkably simple.

THEOREM 1.1.7. *A second countable normal Hausdorff space is metrizable.*

PROOF. We shall only use that the space is completely regular. In fact Tychonoff's Lemma shows that a regular Lindelöf space is normal. So it suffices to assume that the space is separable and regular. There are second countable Hausdorff spaces that are not regular (79 in [Steen & Seebach]). Note that such spaces can't be locally compact.

The key is to use that the Hilbert cube: $\times_{i=1}^{\infty} I_i$ where $I_i = [0,1]$ is metrizable and so the goal is simply to show that our space is homeomorphic to a subset in this space. The metric comes from identifying I_i with the interval of length 2^{-i}.

Choose a countable collection of closed sets $\mathcal{C}$ such that their complements generate the topology of M and countable dense subset $A \subset M$. Enumerate the all pairs $(x_i, C_i) \in A \times \mathcal{C}$ with $x_i \notin C_i$, and for each such pair select a function $\phi_i : M \to [0,1]$ such that $\phi_i(x_i) = 0$ and $\phi_i(C_i) = 1$. Then we obtain a map $\Phi : M \to \times_{i=1}^{\infty} I_i$ by $\Phi(x) = \times_{i=1}^{\infty} \phi_i(x)$. This map is injective since distinct points can be separated by open sets whose complements are in $\mathcal{C}$. Next one can show that for each $C \in \mathcal{C}$ the image $\Phi(C)$ is also closed. This shows that the map is an embedding on to its image.

A metric can be explicitly given by

$$d(x,y) = \sum_i 2^{-i} |f_i(x) - f_i(y)|$$

□

1.1.4. Submanifolds. A subset $S \subset M$ is a submanifold if admits a topology such that the restriction of the differentiable structure on M to S is a differentiable structure. The dimension of the structure on S will generally be less than that of M unless S is an open subset with the induced topology. Note that the topology on S can be different that the induced topology, but it has to be finer as we require all smooth functions on M to be smooth on S.

1.1.5. Whitney Embeddings.

PROPOSITION 1.1.8. *If $U \subset M$ is an open set in a smooth manifold and $f : U \to \mathbb{R}^n$ is smooth, then λf defines a smooth function on M if λ and all of its derivatives vanish on $M - U$.*

PROOF. Clearly λf is smooth away from the boundary of U. On the boundary λ and all it derivatives vanish so the product rule shows that λf is also smooth there. □

THEOREM 1.1.9. (Pre-Whitney Embedding) *If M^m is covered by finitely many coordinate charts, then it admits a proper embedding into Euclidean space $\mathbb{R}^n$ for some $n >> m$.*

PROOF. Cover M by a finite number of coordinate charts $F_i : U_i \to \mathbb{R}^m$, $i = 1, ..., N$. Next select bump functions $\lambda_i : M \to [0,1]$ such that $\lambda_i^{-1}(0) =$

$M - U_i$. Then $\lambda_i F_i$ define smooth functions on M. Finally select a proper function $\rho : M \to [0, \infty)$. We can then consider the smooth map

$$\begin{aligned} F &: M \to \mathbb{R} \times (\mathbb{R}^m)^N \times \mathbb{R}^N \\ F(x) &= (\rho(x), \lambda_1(x) F_1(x), ..., \lambda_N(x) F_N(x), \lambda_1(x), ..., \lambda_N(x)) \end{aligned}$$

This is our desired embedding. First we show that it is injective and that the differential is injective.

If $F(x) = F(y)$, then $\lambda_i(x) = \lambda_i(y)$ for all i. Selecting i so that $\lambda_i(x) > 0$ then shows that $F_i(x) = F_i(y)$. This shows that $x = y$ as F_i is bijective.

If $DF(v) = DF(w)$ for $v, w \in T_p M$, then again $d\lambda_i(v) = d\lambda_i(w)$. The product rule implies

$$D(\lambda_i F_i)|_p = (d\lambda_i)|_p F_i(p) + \lambda_i(p) DF_i|_p$$

Selecting i so that $\lambda_i(p) > 0$ then gives

$$DF_i|_p(v) = DF_i|_p(w)$$

showing that $v = w$.

Finally note that the map is also proper and therefore a closed map (see section on flows). This implies that it is a homeomorphism onto its image and in particular an embedding. □

Theorem 1.1.10. (Whitney Embedding, Intermediate Version) *If $F : M^m \to \mathbb{R}^n$ is an injective immersion, then there is also an injective immersion $M^m \to \mathbb{R}^{2m+1}$. Moreover, if one of the coordinate functions of F is proper, then we can keep this property. In particular, when M is compact we obtain an embedding.*

Proof. For each $v \in \mathbb{R}^n - \{0\}$ consider the orthogonal projection onto the orthogonal complement

$$f_v(x) = x - \frac{(x|v)\, v}{|v|^2}$$

The image is an $n-1$ dimensional subspace. So if we can show that $f_v \circ F$ is an injective immersion then the ambient dimension has been reduced by 1.

Note that $f_v \circ F(x) = f_v \circ F(y)$ iff $F(x) - F(y)$ is proportional to v. Similarly $D(f_v \circ F)(w) = 0$ iff $DF(w)$ is proportional to v.

As long as $2m+1 < n$ Sard's theorem implies that the union of the two images

$$\begin{aligned} H &: M \times M \times \mathbb{R} \to \mathbb{R}^n \\ h(x, y, t) &= t(F(x) - F(y)) \end{aligned}$$

$$\begin{aligned} G &: TM \to \mathbb{R}^n \\ G(w) &= dF(w) \end{aligned}$$

has measure zero. Therefore, we can select $v \in \mathbb{R}^n - (H(M \times M \times \mathbb{R}) \cup G(TM))$.

Assuming $f_v \circ F(x) = f_v \circ F(y)$, we have $F(x) - F(y) = sv$. If $s = 0$ this shows that $F(x) = F(y)$ and hence $x = y$. Otherwise $s \neq 0$ showing that $s^{-1}(F(x) - F(y)) = v$ and hence that $v \in H(M \times M \times \mathbb{R})$.

Assuming $D(f_v \circ F)(w)$ we get that $DF(w) = sv$. If $s = 0$, this shows that $DF(w) = 0$ and hence that $w = 0$. Otherwise $DF(s^{-1} w) = v$ showing that $v \in G(TM)$.

Note that the v we selected could be taken from $O - (H(M \times M \times \mathbb{R}) \cup G(TM))$, where $O \subset \mathbb{R}^n$ is any open subset. This gives us a bit of extra information. While we can't get the ultimate map $M^m \to \mathbb{R}^{2m+1}$ to map into a specific $2m+1$ dimensional subspace of $\mathbb{R}^n$, we can map it into a subspace arbitrarily close to a fixed subspace of dimension $2m+1$. To be specific simply assume that $\mathbb{R}^{2m+1} \subset \mathbb{R}^n$ consists of the first $2m+1$ coordinates. By selecting $v \in (-\varepsilon, \varepsilon)^{2m+1} \times (1-\varepsilon, 1+\varepsilon)^{n-2m-1}$ we see that f_v changes the first coordinates with an error that is small.

This can be used to obtain proper maps $f_v \circ F$. To see this assume that the first coordinate for F is proper. Then $(F(x)|w)$ is still proper as a function of x as long as w is close to the first basis vector e_1. This in turn shows that $(f_v \circ F(x)|w)$ is proper when $w \perp v$. □

Note also that if F starts out being only an immersion, then we can find an immersion into $\mathbb{R}^{2m}$. This is because $G(TM) \subset \mathbb{R}^n$ has measure zero as long as $n > 2m$.

THEOREM 1.1.11. (Whitney Embedding, Final Version) *An m-dimensional manifold M admits a proper embedding into $\mathbb{R}^{2m+1}$.*

PROOF. We only need to find a proper embedding into some Euclidean space. This uses both of the previous results in a surprising way. The key observation is that any open subset of M with compact closure admits an embedding into $\mathbb{R}^{2m+1}$. Write $M = \bigcup U_i$ where each U_i is open with compact closure and $U_i \cap U_j = \emptyset$ when $|i-j| > 1$. Then note that each of the disjoint unions $\bigcup U_{2i}$ and $\bigcup U_{2i+1}$ can be embedded into $\mathbb{R}^{2m+1}$ by mapping disjoint components into disjoint balls in Euclidean space. Then do a partition of unity to create an injective immersion into $(\mathbb{R}^{2m+1})^2 \times \mathbb{R}^2$ and finally a proper embedding into $\mathbb{R} \times (\mathbb{R}^{2m+1})^2 \times \mathbb{R}^2$. Then use the previous Theorem to obtain the final proper embedding. □

1.1.6. Extending Embeddings. >Local canonical forms for immersions and nonsingular maps.

LEMMA 1.1.12. *Let $F : M \to N$ be an immersion, such that it is an embedding when restricted to the embedded submanifold $S \subset M$, then F is an embedding on a neighborhood of S.*

PROOF. We only do the case where $\dim M = \dim N$. It is a bit easier and also the only case we actually need.

By assumption F is an open mapping as it is a local diffeomorphism. Thus it suffices to show that it is injective on a neighborhood of S. If it is not injective on any neighborhood, then we can find sequences x_i and y_i that approach S with $F(x_i) = F(y_i)$. If both sequences have accumulation points, then those points will lie in S and we can, by passing to subsequences, assume that they converge to points x and y in S. Then $F(x) = F(y)$ so $x = y$ and $x_i = y_i$ for large i as they lie in a neighborhood of $x = y$ where F is injective. If one or both of these sequences have no accumulation points, then it is possible to find a neighborhood of S that doesn't contain the sequence. This shows that we don't have to worry about the sequence. □

LEMMA 1.1.13. *Let $M \subset \mathbb{R}^n$ be an embedded submanifold. Then some neighborhood of the normal bundle of M in $\mathbb{R}^n$ is diffeomorphic to a neighborhood of M in $\mathbb{R}^n$.*

PROOF. The normal bundle is defined as

$$\nu(M \subset \mathbb{R}^n) = \{(v,p) \in T_p\mathbb{R}^n \times M : v \perp T_pM\}$$

There is a natural map

$$\begin{aligned} \nu\left(M \subset \mathbb{R}^n\right) &\to \mathbb{R}^n, \\ (v,p) &\to v+p \end{aligned}$$

One checks easily that this is a local diffeomorphism on some neighborhood of the zero section M and that it is clearly an embedding when restricted to the zero section. The previous lemma then shows that it is a diffeomorphism on a neighborhood of the zero section. □

THEOREM 1.1.14. *Let $M \subset N$ be an embedded submanifold. Then some neighborhood of the normal bundle of M in N is diffeomorphic to a neighborhood of M in N.*

PROOF. Any subbundle of $TN|_M$ that is transverse to TM is a normal bundle. It is easy to see that all such bundles are isomorphic. One specific choice comes from embedding $N \subset \mathbb{R}^n$ and then defining

$$\nu\left(M \subset N\right) = \left\{(v,p) \in T_pN \times M : v \perp T_pM\right\}$$

We don't immediately get a map $\nu\left(M \subset N\right) \to N$. What we do is to select a neighborhood $N \subset U \subset \mathbb{R}^n$ as in the previous lemma. The projection $\pi : U \to N$ that takes $w+q \in U$ to $q \in N$ is a submersion deformation retraction. We then select a neighborhood $M \subset V \subset \nu\left(M \subset N\right)$ such that $v+p \in U$ if $(v,p) \in V$. Now we get a map

$$\begin{aligned} V &\to N \\ (v,p) &\to \pi\left(v+p\right) \end{aligned}$$

that is a local diffeomorphism near the zero section and an embedding on the zero section. □

1.1.7. Flows and Submersions.

We start with something very basic.

PROPOSITION 1.1.15. *Let $F : M^m \to N^n$ be a smooth map.*
If F is proper, then it is closed.
If F is a submersion, then it is open.
If F is a proper submersion and N is connected then it is surjective.

PROOF. 1. Let $C \subset M$ be a closed set and assume $F\left(x_i\right) \to y$, where $x_i \in C$. The set $\left\{y, F\left(x_i\right)\right\}$ is compact. Thus the preimage is also compact. This implies that $\left\{x_i\right\}$ has an accumulation point. If we assume that $x_{i_j} \to x \in C$, then continuity shows that $F\left(x_{i_j}\right) \to F\left(x\right)$. Thus $y = F\left(x\right) \in F\left(C\right)$.

2. Consequence of local coordinate representation of F.

3. Follows directly from the two other properties. □

Before delving in to the more general theory we present an important basic result for maps with nonsingular differential.

LEMMA 1.1.16. *Let $F : M^m \to N^m$ be a smooth proper map. If $y \in N$ is a regular value, then there exists a neighborhood V around y such that $F^{-1}(V) = \bigcup_{k=1}^{n} U_k$ where U_k are mutually disjoint and $F : U_k \to V$ is a diffeomorphism.*

PROOF. First use that F is proper to show that $F^{-1}(y) = \{x_1, \ldots, x_n\}$ is a finite set. Next use that y is regular to find mutually disjoint neighborhoods W_k around each x_k such that $F : W_k \to F(W_k)$ is a diffeomorpism. If the desired V does not exist then we can find a sequence $z_i \in M - \bigcup_{k=1}^{n} W_k$ such that $F(z_i) \to y$. Using again that F is proper it follows that (z_i) must have an accumulation point z. Continuity of F then shows that $z \in F^{-1}(y)$. This in turn shows that infinitely many z_i must lie in $\bigcup_{k=1}^{n} W_k$, a contradiction. □

>Main theorem on flows and integral curves, with emphasis on integrals curves being defined for all time or leaving every compact set.

We use the general notation that Φ_X^t is the flow corresponding to a vector field X, i.e.

$$\frac{d}{dt}\Phi_X^t = X|_{\Phi_X^t} = X \circ \Phi_X^t$$

Let $F : M^m \to N^n$ be a smooth map between manifolds. If X is a vector field on M and Y a vector field on N, then we say that X and Y are F*-related* provided $DF\left(X|_p\right) = Y|_{F(p)}$, or in other words $DF\left(X\right) = Y \circ F$.

PROPOSITION 1.1.17. *X and Y are F-related iff $F \circ \Phi_X^t = \Phi_Y^t \circ F$ for sufficiently small t.*

PROOF. Assuming that $F \circ \Phi_X^t = \Phi_Y^t \circ F$ we have

$$\begin{aligned} DF\left(X\right) &= DF\left(\frac{d}{dt}|_{t=0}\Phi_X^t\right) \\ &= \frac{d}{dt}|_{t=0}\left(F \circ \Phi_X^t\right) \\ &= \frac{d}{dt}|_{t=0}\left(\Phi_Y^t \circ F\right) \\ &= Y \circ \Phi_Y^0 \circ F \\ &= Y \circ F \end{aligned}$$

Conversely $DF\left(X\right) = Y \circ F$ implies that

$$\begin{aligned} \frac{d}{dt}\left(F \circ \Phi_X^t\right) &= DF\left(\frac{d}{dt}\Phi_X^t\right) \\ &= DF\left(X \circ \Phi_X^t\right) \\ &= Y \circ F \circ \Phi_Y^t \\ &= \frac{d}{dt}\left(\Phi_Y^t \circ F\right) \end{aligned}$$

Since the two curves $t \to F \circ \Phi_X^t$ and $t \to \Phi_Y^t \circ F$ clearly agree when $t = 0$, this shows that they are the same. In fact we just showed that $t \to F \circ \Phi_X^t$ is an integral curve for Y. □

>Local canonical form for submersions

In case F is a submersion it is possible to construct vector fields in M that are F-related to a given vector field in N.

PROPOSITION 1.1.18. *Assume that F is a submersion. Given a vector field Y in N, there are vector fields X in M that are F-related to Y.*

PROOF. First we do a local construction of X. Since F is a submersion we can always find charts in M and N so that in these charts F looks like

$$F\left(x^1,...,x^m\right)=\left(x^1,...,x^n\right).$$

Note that $m\geq n$ so the RHS just consists of the first n coordinate from $\left(x^1,...,x^m\right)$. If we write $Y=a^i\partial_i$, then we can simply define $X=\sum_{i=1}^n a^i\partial_i$. This gives the local construction.

For the global construction assume that we have a covering U_α, vector fields X_α on U_α that are F-related to Y, and a partition of unity λ_α subordinate to U_α. Then simply define $X=\sum\lambda_\alpha X_\alpha$ and note that

$$\begin{aligned} DF(X) &= DF\left(\sum\lambda_\alpha X_\alpha\right) \\ &= \sum\lambda_\alpha DF(X_\alpha) \\ &= \sum\lambda_\alpha Y\circ F \\ &= Y\circ F. \end{aligned}$$

□

Finally we can say something about the maximal domains of definition for the flows of F-related vector fields given F is proper.

PROPOSITION 1.1.19. *Assume that F is proper and that X and Y are F-related vector fields. If $F(q)=p$ and $\Phi_Y^t(p)$ is defined on $[0,b)$, then $\Phi_X^t(q)$ is also defined on $[0,b)$. In other words the relation $F\circ\Phi_X^t=\Phi_Y^t\circ F$ holds for as long as the RHS is defined.*

PROOF. Assume $\Phi_X^t(q)$ is defined on $[0,a)$. If $a<b$, then the set

$$\begin{aligned} K &= \left\{x\in M : F(x)=\Phi_Y^t(p) \text{ for some } t\in[0,a]\right\} \\ &= F^{-1}\left(\left\{\Phi_Y^t(p) : t\in[0,a]\right\}\right) \end{aligned}$$

is compact in M since F is proper. The integral curve $t\to\Phi_X^t(q)$ lies in K since $F\left(\Phi_X^t(q)\right)=\Phi_Y^t(p)$. It is now a general result that maximally defined integral curves are either defined for all time or leave every compact set. Thus $\Phi_X^t(q)$ must be defined on $[0,b)$. □

These relatively simple properties lead to some very general and tricky results.

A *fibration* $F:M\to N$ is a smooth map which is locally trivial in the sense that for every $p\in N$ there is a neighborhood U of p such that $F^{-1}(U)$ is diffeomorphic to $U\times F^{-1}(p)$. This diffeomorphism must commute with the natural maps of these sets on to U. In other words $(x,y)\in U\times F^{-1}(p)$ must be mapped to a point in $F^{-1}(x)$. Note that it is easy to destroy the fibration property by simply deleting a point in M. Note also that in this context fibrations are necessarily submersions.

Special cases of fibrations are covering maps and vector bundles. The Hopf fibration $S^3\to S^2=\mathbb{P}^1$ is a more non trivial example of a fibration, which we shall study further below. Tubular neighborhoods are also examples of fibrations.

THEOREM 1.1.20. (Ehresman) *If $F:M\to N$ is a proper submersion, then it is a fibration.*

PROOF. As far as N is concerned this is a local result. In N we simply select a set U that is diffeomorphic to $\mathbb{R}^n$ and claim that $F^{-1}(U) \approx U \times F^{-1}(0)$. Thus we just need to prove the theorem in case $N = \mathbb{R}^n$, or more generally a coordinate box around the origin.

Next select vector fields $X_1, ..., X_n$ in M that are F-related to the coordinate vector fields $\partial_1, ..., \partial_n$. Our smooth map $G : \mathbb{R}^n \times F^{-1}(0) \to M$ is then defined by $G\left(t^1, ..., t^n, x\right) = \Phi_{X_1}^{t^1} \circ \cdots \circ \Phi_{X_n}^{t^n}(x)$. The inverse to this map is $G^{-1}(z) = \left(F(z), \Phi_{X_n}^{-t^n} \circ \cdots \circ \Phi_{X_1}^{-t^1}(z)\right)$, where $F(z) = \left(t^1, ..., t^n\right)$. □

The theorem also unifies several different results.

COROLLARY 1.1.21. (Basic Lemma in Morse Theory) *Let $F : M \to \mathbb{R}$ be a proper map. If F is regular on $(a, b) \subset \mathbb{R}$, then $F^{-1}(a, b) \simeq F^{-1}(c) \times (a, b)$ where $c \in (a, b)$.*

COROLLARY 1.1.22. *Let $F : M \to N$ be a proper nonsingular map with N connected, then F is a covering map.*

COROLLARY 1.1.23. (Hadamard) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a proper nonsingular map, then F is a diffeomorphism.*

COROLLARY 1.1.24. (Reeb) *Let M be a closed manifold that admits a map with two critical points, then M is homeomorphic to a sphere. (This is a bit easier to show if we also assume that the critical points are nondegenerate.)*

Finally we can extend the fibration theorem to the case when M has boundary.

THEOREM 1.1.25. *Assume that M is a manifold with boundary and that N is a manifold without boundary, if $F : M \to N$ is proper and a submersion on M as well as on ∂M, then it is a fibration.*

PROOF. The proof is identical and reduced to the case when $N = \mathbb{R}^n$. The assumptions allow us to construct the lifted vector fields so that they are tangent to ∂M. The flows will then stay in ∂M or $\text{int}M$ for all time if they start there. □

This theorem is sometimes useful when we have a submersion whose fibers are not compact. It is then occasionally possible to add a boundary to M so as to make the map proper. A good example is a tubular neighborhood around a closed submanifold $S \subset U$. By possibly making U smaller we can assume that it is a compact manifold with boundary such that the fibers of $U \to S$ are closed discs rather than open discs.

There is also a very interesting converse problem: If M is a manifold and $\sim$ an equivalence relation on M when is $M/\sim$ a manifold and $M \to M/\sim$ a submersion? Clearly the equivalence classes must form a foliation and the leaves/equivalence classes be closed subsets of M. Also their normal bundles have to be trivial as preimages of regular values have trivial normal bundle. The most basic and still very nontrivial case is that of a Lie group G and a subgroup H. The equivalence classes are the cosets gH in G and the quotient space is G/H. When H is dense in G the quotient topology is not even Hausdorff. However one can prove that if H is closed in G then the quotient is a manifold and the quotient map a submersion.

A nasty example is $\mathbb{R}^2 - \{0\}$ with the equivalence relation being that two points are equivalent if they have the same x-coordinate and lie in the same component of the corresponding vertical line. The quotient space is the line with double origin and so is not Hausdorff!

1.2. Projective Space

Given a vector space V we define $\mathbb{P}(V)$ as the space of 1-dimensional subspaces or lines through the origin. It is called the projective space of V. In the special case were $V = \mathbb{F}^{n+1}$ we use the notation $\mathbb{P}\left(\mathbb{F}^{n+1}\right) = \mathbb{FP}^n = \mathbb{P}^n$. This is a bit confusing in terms of notation. The point is that $\mathbb{P}^n$ is an n-dimensional space as we shall see below.

One can similarly develop a theory of the space of subspaces of any given dimension. The space of k-dimensional subspaces is denoted $G_k(V)$ and is called the Grassmannian.

1.2.1. Basic Geometry of Projective Spaces. The space of operators or endomorphisms on V is denoted $\text{end}(V)$ and the invertible operators or automorphisms by $\text{aut}(V)$. When $V = \mathbb{F}^n$ these are represented by matrices $\text{end}(\mathbb{F}^n) = \text{Mat}(\mathbb{F})$ and $\text{aut}(\mathbb{F}^n) = Gl_n(\mathbb{F})$. Since invertible operators map lines to lines we see that $\text{aut}(V)$ acts in a natural way on $\mathbb{P}(V)$. In fact this action is homogeneous, i.e., if we have $p, q \in \mathbb{P}(V)$, then there is an operator $A \in \text{aut}(V)$ such that $A(p) = q$. Moreover, as we know that any two bases in V can be mapped to each other by invertible operators we have that any collection of k independent lines $p_1, ..., p_k$, i.e., $p_1 + \cdots + p_k = p_1 \oplus \cdots \oplus p_k$ can be mapped to any collection of k independent lines $q_1, ..., q_k$. This means that the action of $\text{aut}(V)$ on $\mathbb{P}(V)$ is k-point homogeneous for all $k \leq \dim(V)$. Note that this action is not effective, i.e., some transformations act trivially on $\mathbb{P}(V)$. Specifically, the maps that act trivially are precisely the homotheties $A = \lambda 1_V$.

Since an endomorphism might have a kernel it is not true that it maps lines to lines, however, if we have $A \in \text{end}(V)$, then we do get a map $A : \mathbb{P}(V) - \mathbb{P}(\ker A) \to \mathbb{P}(V)$ defined on lines that are not in the kernel of A.

Let us now assume that V is an inner product space with an inner product $(v|w)$ that can be real or complex. The key observation in relation to subspaces is that they are completely characterized by the orthogonal projections onto the subspaces. Thus the space of k-dimensional subspaces is the same as the space of orthogonal projections of rank k. It is convenient to know that an endomorphism $E \in \text{end}(V)$ is an orthogonal projection iff it is a projection, $E^2 = E$ that is self-adjoint, $E^* = E$. In the case of a one dimensional subspace $p \in \mathbb{P}(V)$ spanned by a unit vector $v \in V$, the orthogonal projection is given by

$$\text{proj}_p(x) = \langle x, v \rangle v.$$

Clearly we get the same formula for all unit vectors in p. Note that the formula is quadratic in v. Using this we get a map $\mathbb{P}(V) \to \text{end}(V)$. This gives $\mathbb{P}(V)$ a natural topology and even a metric. One can also easily see that $\mathbb{P}(V)$ is compact.

We can define two natural metrics on $\mathbb{P}(V)$. One is simply the angle between the lines. Another related metric uses that $\text{end}(V)$ is itself an inner product space with inner product $\langle A, B \rangle = \text{tr}(AB^*)$. Let just compute this inner product for

$\mathrm{proj}_p(x) = \langle x, v\rangle v$ and $\mathrm{proj}_q(x) = \langle x, w\rangle w$ where v and w are unit vectors:

$$\begin{aligned}
\langle \mathrm{proj}_p, \mathrm{proj}_q \rangle &= \mathrm{tr}\left(\mathrm{proj}_p \circ \mathrm{proj}_q\right) \\
&= \sum \left\langle \mathrm{proj}_p \circ \mathrm{proj}_q(e_i), e_i \right\rangle \\
&= \sum \left\langle \left\langle \mathrm{proj}_q(e_i), v \right\rangle v, e_i \right\rangle \\
&= \sum \langle \langle \langle e_i, w\rangle w, v\rangle v, e_i \rangle \\
&= \sum \langle e_i, w\rangle \langle w, v\rangle \langle v, e_i\rangle \\
&= \sum \langle \langle v, e_i\rangle e_i, w\rangle \langle w, v\rangle \\
&= \sum \langle v, w\rangle \langle w, v\rangle \\
&= |\langle v, w\rangle|^2 .
\end{aligned}$$

Note again the quadratic nature of this formula. Since $|\langle v, w\rangle|^2 \leq |v|^2 |w|^2 = 1$ we can define the angle between $p, q \in \mathbb{P}(V)$ as the unique $\measuredangle(p, q) \in \left[0, \frac{\pi}{2}\right]$ such that

$$\cos \measuredangle(p, q) = |\langle v, w\rangle|^2 .$$

Our first observation is that this angle is zero iff $|\langle v, w\rangle|^2 = |v|^2 |w|^2$, which we know is equivalent to v and w being proportional, and hence defining the same line.

Note that this angle concept isn't quite what we might expect geometrically although it does recapture our intuitive notion of perpendicularity, e.g., $p \perp q$ iff $v \perp w$. A more geometric angle concept would be defined via

$$\cos \measuredangle(p, q) = |\langle v, w\rangle| .$$

Automorphisms clearly do not preserve angles between lines and so are not necessarily isometries. However if we restrict attention to unitary or orthogonal transformations $U \subset \mathrm{aut}(V)$, then we know that they preserve inner products of vectors. Therefore, they must also preserve angles between lines. Thus U acts by isometries on $\mathbb{P}(V)$. This action is again homogeneous so $\mathbb{P}(V)$ looks the same everywhere.

1.2.2. Coordinates. We are now ready to coordinatize $\mathbb{P}(V)$. Select $p \in \mathbb{P}(V)$ and consider the set of lines $\mathbb{P}(V) - \mathbb{P}(p^\perp)$ that are not perpendicular to p. This is clearly an open set in $\mathbb{P}(V)$ and we claim that there is a coordinate map $G_p : \hom(p, p^\perp) \to \mathbb{P}(V) - \mathbb{P}(p^\perp)$. To construct this map decompose $V \simeq p \oplus p^\perp$ and note that any line not in $p^\perp$ is a graph over p given by a unique homomorpism in $\hom(p, p^\perp)$. The next thing to check is that G_p is a homeomorphism onto its image and is differentiable as a map into $\mathrm{end}(V)$. Neither fact is hard to verify. Finally observe that $\hom(p, p^\perp)$ is a vector space of dimension $\dim V - 1$. In this way $\mathbb{P}(V)$ becomes a manifold of dimension $\dim V - 1$.

In case we are considering $\mathbb{P}^n$ we can construct a more explicit coordinate map. First we introduce homogenous coordinates: select $z = (z^0, ..., z^n) \in \mathbb{F}^{n+1} - \{0\}$ denote the line by $[z^0 : \cdots : z^n] \in \mathbb{P}^n$, thus $[z^0 : \cdots : z^n] = [w^0 : \cdots : w^n]$ iff and only if z and w are proportional and hence generate the same line. If we let $p = [1 : 0 : \cdots : 0]$, then $\mathbb{F}^n \to \mathbb{P}^n$ is simply $G_p(z^1, ..., z^n) = [1 : z^1 : \cdots : z^n]$.

Keeping in mind that p is the only line perpendicular to all lines in $p^{\perp}$ we see that $\mathbb{P}^n - p$ can be represented by

$$\mathbb{P}^n - p = \left\{ \left[z : z^1 : \cdots : z^n \right] : \left(z^1, ..., z^n \right) \in \mathbb{F}^n - \{0\} \text{ and } z \in \mathbb{F} \right\}.$$

Here the subset

$$\mathbb{P}\left(p^{\perp}\right) = \left\{ \left[0 : z^1 : \cdots : z^n \right] : \left(z^1, ..., z^n \right) \in \mathbb{F}^n - \{0\} \right\}$$

can be identified with $\mathbb{P}^{n-1}$. Using the transformation

$$\begin{aligned} R_0 &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \\ \ker(R_0) &= p \end{aligned}$$

we get a retract $R_0 : \mathbb{P}^n - p \to \mathbb{P}^{n-1}$, whose fibers are diffeomorphic to $\mathbb{F}$. Using the transformations

$$R_t = \begin{bmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

we see that R_0 is in fact a deformation retraction.

Finally we check the projective spaces in low dimensions. When $\dim V = 1$, $\mathbb{P}(V)$ is just a point and that point is in fact V it self. Thus $\mathbb{P}(V) = \{V\}$. When $\dim V = 2$, we note that for each $p \in \mathbb{P}(V)$ the orthogonal complement $p^{\perp}$ is again a one dimensional subspace and therefore an element of $\mathbb{P}(V)$. This gives us an involution $p \to p^{\perp}$ on $\mathbb{P}(V)$ just like the antipodal map on the sphere. In fact

$$\begin{aligned} \mathbb{P}(V) &= (\mathbb{P}(V) - \{p\}) \cup \left(\mathbb{P}(V) - \left\{p^{\perp}\right\}\right), \\ \mathbb{P}(V) - \{p\} &\simeq \mathbb{F} \simeq \mathbb{P}(V) - \left\{p^{\perp}\right\}, \\ \mathbb{F} - \{0\} &\simeq (\mathbb{P}(V) - \{p\}) \cap \left(\mathbb{P}(V) - \left\{p^{\perp}\right\}\right). \end{aligned}$$

Thus $\mathbb{P}(V)$ is simply a one point compactification of $\mathbb{F}$. In particular, we have that $\mathbb{RP}^1 \simeq S^1$ and $\mathbb{CP}^1 \simeq S^2$, (you need to convince your self that this is a diffeomorphism.) Since the geometry doesn't allow for distances larger than $\frac{\pi}{2}$ it is natural to suppose that these projective "lines" are spheres of radius $\frac{1}{2}$ in $\mathbb{F}^2$. This is in fact true.

1.2.3. Bundles. Define the tautological or canonical line bundle

$$\tau(\mathbb{P}^n) = \left\{ (p, v) \in \mathbb{P}^n \times \mathbb{F}^{n+1} : v \in p \right\}.$$

This is a natural subbundle of the trivial vector bundle $\mathbb{P}^n \times \mathbb{F}^{n+1}$ and therefore has a natural orthogonal complement

$$\tau^{\perp}(\mathbb{P}^n) \simeq \left\{ (p, v) \in \mathbb{P}^n \times \mathbb{F}^{n+1} : p \perp v \right\}$$

Note that in the complex case we are using Hermitian orthogonality. These are related to the tangent bundle in an interesting fashion

$$T\mathbb{P}^n \simeq \hom\left(\tau(\mathbb{P}^n), \tau^{\perp}(\mathbb{P}^n)\right)$$

This identity comes from our coordinatization around a point $p \in \mathbb{P}^n$. We should check that these bundle are locally trivial, i.e., fibrations over $\mathbb{P}^n$. This is quite easy, for each $p \in \mathbb{P}^n$ we use the coordinate neighborhood around p and show that the bundles are trivial over these neighborhoods.

Note that the fibrations $\tau(\mathbb{P}^n) \to \mathbb{P}^n$ and $\mathbb{F}^{n+1} - \{0\} \to \mathbb{P}^n$ are suspiciously similar. The latter has fibers $p - \{0\}$ where the former as p. This means that the latter fibration can be identified with the nonzero vectors in $\tau(\mathbb{P}^n)$. This means that the missing 0 in $\mathbb{F}^{n+1} - \{0\}$ is replaced by the zero section in $\tau(\mathbb{P}^n)$ in order to create a larger bundle. This process is called a blow up of the origin in $\mathbb{F}^{n+1}$. Essentially we have a map $\tau(\mathbb{P}^n) \to \mathbb{F}^{n+1}$ that maps the zero section to 0 and is a bijection outside that. We can use $\mathbb{F}^{n+1} - \{0\} \to \mathbb{P}^n$ to create a new fibration by restricting it to the unit sphere $S \subset \mathbb{F}^{n+1} - \{0\}$.

The conjugate to the tautological bundle can also be seen internally in $\mathbb{P}^{n+1}$ as the map

$$\mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n$$

When $p = [1 : 0 : \cdots : 0]$ this fibration was given by

$$\left[z : z^0 : \cdots : z^n\right] \to \left[z^0 : \cdots : z^n\right].$$

This looks like a vector bundle if we use fiberwise addition and scalar multiplication on z.

The equivalence is obtained by mapping

$$\mathbb{P}^{n+1} - \{[1 : 0 : \cdots : 0]\} \to \tau(\mathbb{P}^n),$$

$$\left[z : z^0 : \cdots : z^n\right] \to \left(\left[z^0 : \cdots : z^n\right], \bar{z}\frac{\left(z^0, ..., z^n\right)}{\left|\left(z^0, ..., z^n\right)\right|^2}\right)$$

It is necessary to conjugate z to get a well-defined map. This is why the identification is only conjugate linear. The conjugate to the tautological bundle can also be identified with the dual bundle $\hom(\tau(\mathbb{P}^n), \mathbb{C})$ via the natural inner product structure coming from $\tau(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{F}^{n+1}$. The relevant linear functional corresponding to $\left[z : z^0 : \cdots : z^n\right]$ is given by

$$v \to \left\langle v, \bar{z}\frac{\left(z^0, ..., z^n\right)}{\left|\left(z^0, ..., z^n\right)\right|^2}\right\rangle$$

This functional appears to be defined on all of $\mathbb{F}^{n+1}$, but as it vanishes on the orthogonal complement to $\left(z^0, ..., z^n\right)$ we only need to consider the restriction to $\operatorname{span}\left\{\left(z^0, ..., z^n\right)\right\} = \left[z^0 : \cdots : z^n\right]$.

Finally we prove that these bundles are not trivial. In fact, we show that there can't be any smooth sections $F : \mathbb{P}^n \to S \subset \mathbb{F}^{n+1} - \{0\}$ such that $F(p) \in p$ for all p, i.e., it is not possible to find a smooth (or continuous) choice of basis for all 1-dimensional subspaces. Should such a map exist it would evidently be a lift of the identity on $\mathbb{P}^n$ to a map $\mathbb{P}^n \to S$. In case $\mathbb{F} = \mathbb{R}$, the map $S \to \mathbb{RP}^n$ is a nontrivial two fold covering map. So it is not possible to find $\mathbb{RP}^n \to S$ as a lift of the identity. In case $\mathbb{F} = \mathbb{C}$ the unit sphere S has larger dimension than $\mathbb{CP}^n$ so Sard's theorem tells us that $\mathbb{CP}^n \to S$ isn't onto. But then it is homotopic to a constant, thus showing that the identity $\mathbb{CP}^n \to \mathbb{CP}^n$ is homotopic to the constant map. We shall see below that this is not possible.

In effect, we proved that a fibration of a sphere $S \to B$ is nontrivial if either $\pi_1(B) \neq \{1\}$ or $\dim B < \dim S$.

1.2.4. Lefschetz Numbers. Finally we are going to study Lefschetz numbers for linear maps on projective spaces. The first general observation is that a map $A \in \operatorname{aut}(V)$ has a fixed point $p \in \mathbb{P}(V)$ iff p is an invariant one dimensional subspace for A. In other words fixed points for A on $\mathbb{P}(V)$ correspond to eigenvectors, but without information about eigenvalues.

We start with the complex case as it is a bit simpler. The claim is that any $A \in \operatorname{aut}(V)$ with distinct eigenvalues is a Lefschetz map on $\mathbb{P}(V)$ with $L(A) = \dim V$. Since such maps are diagonalizable we can restrict attention to $V = \mathbb{C}^{n+1}$ and the diagonal matrix

$$A = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

By symmetry we need only study the fixed point $p = [1 : 0 : \cdots : 0]$. Note that the eigenvalues are assumed to be distinct and none of then vanish. To check the action of A on a neighborhood of p we use the coordinates introduced above $\left[1 : z^1 : \cdots : z^n\right]$. We see that

$$\begin{aligned} A\left[1 : z^1 : \cdots : z^n\right] &= \left[\lambda_0 1 : \lambda_1 z^1 : \cdots : \lambda_n z^n\right] \\ &= \left[1 : \frac{\lambda_1}{\lambda_0} z^1 : \cdots : \frac{\lambda_n}{\lambda_0} z^n\right]. \end{aligned}$$

This is already (complex) linear in these coordinates so the differential at p must be represented by the complex $n \times n$ matrix

$$DA|_p = \begin{bmatrix} \frac{\lambda_1}{\lambda_0} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n}{\lambda_0} \end{bmatrix}.$$

As the eigenvalues are all distinct 1 is not an eigenvalue of this matrix, showing that A really is a Lefschetz map. Next we need to check the differential of $\det\left(I - DA|_p\right)$. Note that in **[Guillemin-Pollack]** the authors use the sign of $\det\left(DA|_p - I\right)$, but this is not consistent with Lefschetz' formula for the Lefschetz number as we shall see below. Since $Gl_n(\mathbb{C})$ is connected it must lie in $Gl_{2n}^{+}(\mathbb{R})$ as a real matrix, i.e., complex matrices always have positive determinant when viewed as real matrices. Since $DA|_p$ is complex it must follow that $\det\left(I - DA|_p\right) > 0$. So all local Lefschetz numbers are 1. This shows that $L(A) = n+1$. Since $Gl_{n+1}(\mathbb{C})$ is connected any linear map is homotopic to a linear Lefschetz map and must therefore also have Lefschetz number $n+1$.

In particular, we have shown that all invertible complex linear maps must have eigenvectors. Note that this fact is obvious for maps that are not invertible. This could be the worlds most convoluted way of proving the Fundamental Theorem of Algebra. We used the fact that $Gl_n(\mathbb{C})$ is connected. This in turn follows from the polar decomposition of matrices, which in turn follows from the Spectral Theorem. Finally we observe that any reasonable proof of the Spectral Theorem will not use the Fundamental Theorem of Algebra.

The alternate observation that the above Lefschetz maps are dense in $Gl_n(\mathbb{C})$ is quite useful as the density statement normally uses the Fundamental Theorem of Algebra.

The real projective spaces can be analyzed in a similar way but we need to consider the parity of the dimension as well as the sign of the determinant of the linear map.

For $A \in GL^+_{2n+2}(\mathbb{R})$ we might not have any eigenvectors whatsoever as A could be $n+1$ rotations. Since $GL^+_{2n+2}(\mathbb{R})$ is connected this means that $L(A) = 0$ on $\mathbb{RP}^{2n+1}$ if $A \in GL^+_{2n+2}(\mathbb{R})$. When $A \in GL^-_{2n+2}(\mathbb{R})$ it must have at least two eigenvalues of opposite sign. Since $GL^-_{2n+2}(\mathbb{R})$ is connected we just need to check what happens for a specific

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & & & & & \\ 0 & -1 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & R \end{bmatrix} \end{aligned}$$

We have two fixed points

$$\begin{aligned} p &= [1 : 0 : \cdots : 0], \\ q &= [0 : 1 : \cdots : 0]. \end{aligned}$$

For p we can quickly guess that

$$DA_p = \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}.$$

This matrix doesn't have 1 as an eigenvalue and

$$\begin{aligned} \det\left(I - \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}\right) &= \det \begin{bmatrix} 2 & 0 \\ 0 & I - R \end{bmatrix} \\ &= \det \begin{bmatrix} 2 & & & & & \\ & 1 & 1 & & & \\ & -1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & -1 & 1 \end{bmatrix} \\ &= 2^{n+1}. \end{aligned}$$

So we see that the determinant is positive. For q we use the coordinates $\left[z^0 : 1 : z^2 : \cdots : z^n\right]$ and easily see that the differential is

$$\begin{bmatrix} -1 & 0 \\ 0 & -R \end{bmatrix}$$

which also doesn't have 1 as an eigenvalue and again gives us positive determinant for $I - DA_q$. This shows that $L(A) = 2$ if $A \in GL^-_{2n+2}(\mathbb{R})$.

In case $A \in Gl_{2n+1}(\mathbb{R})$ it is only possible to compute the Lefschetz number mod 2 as $\mathbb{RP}^{2n}$ isn't orientable. We can select

$$A^{\pm} = \begin{bmatrix} \pm 1 & 0 \\ 0 & R \end{bmatrix} \in GL^{\pm}_{2n+1}(\mathbb{R})$$

with R as above. In either case we have only one fixed point and it is a Lefschetz fixed point since $DA^{\pm}_p = \pm R$. Thus $L(A^{\pm}) = 1$ and all $A \in Gl_{2n+1}(\mathbb{R})$ have $L(A) = 1$.

1.3. Matrix Spaces

Lie groups and spaces of matrices of constant rank.

CHAPTER 2

Basic Tensor Analysis

2.1. Lie Derivatives and Its Uses

Let X be a vector field and F^t the corresponding locally defined flow on a smooth manifold M. Thus $F^t(p)$ is defined for small t and the curve $t \to F^t(p)$ is the integral curve for X that goes through p at $t = 0$. The Lie derivative of a tensor in the direction of X is defined as the first order term in a suitable Taylor expansion of the tensor when it is moved by the flow of X.

2.1.1. Definitions and Properties. Let us start with a function $f : M \to \mathbb{R}$. Then

$$f\left(F^t(p)\right) = f(p) + t\left(L_X f\right)(p) + o(t),$$

where the Lie derivative $L_X f$ is just the directional derivative $D_X f = df(X)$. We can also write this as

$$\begin{aligned} f \circ F^t &= f + tL_X f + o(t), \\ L_X f &= D_X f = df(X). \end{aligned}$$

When we have a vector field Y things get a little more complicated. We wish to consider $Y|_{F^t}$, but this can't be directly compared to Y as the vectors live in different tangent spaces. Thus we look at the curve $t \to DF^{-t}\left(Y|_{F^t(p)}\right)$ that lies in T_pM. Then we expand for t near 0 and get

$$DF^{-t}\left(Y|_{F^t(p)}\right) = Y|_p + t\left(L_X Y\right)|_p + o(t)$$

for some vector $\left(L_X Y\right)|_p \in T_pM$. This Lie derivative also has an alternate definition.

Proposition 2.1.1. *For vector fields X, Y on M we have*

$$L_X Y = [X, Y].$$

Proof. We see that the Lie derivative satisfies

$$DF^{-t}\left(Y|_{F^t}\right) = Y + tL_X Y + o(t)$$

or equivalently

$$Y|_{F^t} = DF^t(Y) + tDF^t\left(L_X Y\right) + o(t).$$

It is therefore natural to consider the directional derivative of a function f in the direction of $Y|_{F^t} - DF^t(Y)$.

$$\begin{aligned} D_{Y|_{F^t} - DF^t(Y)} f &= D_{Y|_{F^t}} f - D_{DF^t(Y)} f \\ &= (D_Y f) \circ F^t - D_Y \left(f \circ F^t\right) \\ &= D_Y f + t D_X D_Y f + o(t) \\ &\quad - D_Y (f + t D_X f + o(t)) \\ &= t (D_X D_Y f - D_Y D_X f) + o(t) \\ &= t D_{[X,Y]} f + o(t). \end{aligned}$$

This shows that

$$\begin{aligned} L_X Y &= \lim_{t \to 0} \frac{Y|_{F^t} - DF^t(Y)}{t} \\ &= [X, Y]. \end{aligned}$$

□

We are now ready to define the Lie derivative of a $(0,p)$-tensor T and also give an algebraic formula for this derivative. We define

$$\left(F^t\right)^* T = T + t(L_X T) + o(t)$$

or more precisely

$$\begin{aligned} \left(\left(F^t\right)^* T\right)(Y_1, ..., Y_p) &= T\left(DF^t(Y_1), ..., DF^t(Y_p)\right) \\ &= T(Y_1, ..., Y_p) + t(L_X T)(Y_1, ..., Y_p) + o(t). \end{aligned}$$

PROPOSITION 2.1.2. *If X is a vector field and T a $(0,p)$-tensor on M, then*

$$(L_X T)(Y_1, ..., Y_p) = D_X(T(Y_1, ..., Y_p)) - \sum_{i=1}^{p} T(Y_1, ..., L_X Y_i, ..., Y_p)$$

PROOF. We restrict attention to the case where $p = 1$. The general case is similar but requires more notation. Using that

$$Y|_{F^t} = DF^t(Y) + t DF^t(L_X Y) + o(t)$$

we get

$$\begin{aligned} \left(\left(F^t\right)^* T\right)(Y) &= T\left(DF^t(Y)\right) \\ &= T\left(Y|_{F^t} - t DF^t(L_X Y)\right) + o(t) \\ &= T(Y) \circ F^t - t T\left(DF^t(L_X Y)\right) + o(t) \\ &= T(Y) + t D_X(T(Y)) - t T\left(DF^t(L_X Y)\right) + o(t). \end{aligned}$$

Thus

$$\begin{aligned} (L_X T)(Y) &= \lim_{t \to 0} \frac{\left(\left(F^t\right)^* T\right)(Y) - T(Y)}{t} \\ &= \lim_{t \to 0} \left(D_X(T(Y)) - T\left(DF^t(L_X Y)\right)\right) \\ &= D_X(T(Y)) - T(L_X Y). \end{aligned}$$

□

Finally we have that Lie derivatives satisfy all possible product rules. From the above propositions this is already obvious when multiplying functions with vector fields or $(0,p)$-tensors. However, it is less clear when multiplying tensors.

PROPOSITION 2.1.3. *Let T_1 and T_2 be $(0,p_i)$-tensors, then*

$$L_X\left(T_1\cdot T_2\right)=\left(L_XT_1\right)\cdot T_2+T_1\cdot\left(L_XT_2\right).$$

PROOF. Recall that for 1-forms and more general $(0,p)$-tensors we define the product as

$$T_1\cdot T_2\left(X_1,...,X_{p_1},Y_1,...,Y_{p_2}\right)=T_1\left(X_1,...,X_{p_1}\right)\cdot T_2\left(Y_1,...,Y_{p_2}\right).$$

The proposition is then a simple consequence of the previous proposition and the product rule for derivatives of functions. □

PROPOSITION 2.1.4. *Let T be a $(0,p)$-tensor and $f:M\to\mathbb{R}$ a function, then*

$$L_{fX}T\left(Y_1,...,Y_p\right)=fL_XT\left(Y_1,...,Y_p\right)+df\left(Y_i\right)\sum_{i=1}^{p}T\left(Y_1,...,X,...,Y_p\right).$$

PROOF. We have that

$$\begin{aligned}L_{fX}T\left(Y_1,...,Y_p\right) &= D_{fX}\left(T\left(Y_1,...,Y_p\right)\right)-\sum_{i=1}^{p}T\left(Y_1,...,L_{fX}Y_i,...,Y_p\right)\\ &= fD_X\left(T\left(Y_1,...,Y_p\right)\right)-\sum_{i=1}^{p}T\left(Y_1,...,\left[fX,Y_i\right],...,Y_p\right)\\ &= fD_X\left(T\left(Y_1,...,Y_p\right)\right)-f\sum_{i=1}^{p}T\left(Y_1,...,\left[X,Y_i\right],...,Y_p\right)\\ &\quad +df\left(Y_i\right)\sum_{i=1}^{p}T\left(Y_1,...,X,...,Y_p\right)\end{aligned}$$

□

The case where $X|_p=0$ is of special interest when computing Lie derivatives. We note that $F^t\left(p\right)=p$ for all t. Thus $DF^t:T_pM\to T_pM$ and

$$\begin{aligned}L_XY|_p &= \lim_{t\to 0}\frac{DF^{-t}\left(Y|_p\right)-Y|_p}{t}\\ &= \frac{d}{dt}\left(DF^{-t}\right)|_{t=0}\left(Y|_p\right).\end{aligned}$$

This shows that $L_X=\frac{d}{dt}\left(DF^{-t}\right)|_{t=0}$ when $X|_p=0$. From this we see that if θ is a 1-form then $L_X\theta=-\theta\circ L_X$ at points p where $X|_p=0$.

Before moving on to some applications of Lie derivatives we introduce the concept of interior product, it is simply evaluation of a vector field in the first argument of a tensor:

$$i_XT\left(X_1,...,X_k\right)=T\left(X,X_1,...,X_k\right)$$

We can now list 4 general properties of Lie derivatives and how they are related to interior products.

$$\begin{aligned} L_{[X,Y]} &= L_X L_Y - L_Y L_X, \\ L_X (fT) &= L_X (f) T + f L_X T, \\ L_X [Y, Z] &= [L_X Y, Z] + [Y, L_X Z], \\ L_X (i_Y T) &= i_{L_X Y} T + i_Y (L_X T), \end{aligned}$$

2.1.2. Lie Groups. Lie derivatives also come in handy when working with Lie groups. For a Lie group G we have the inner automorphism $\mathrm{Ad}_h : x \to hxh^{-1}$and its differential at $x = e$ denoted by the same letters

$$\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}.$$

LEMMA 2.1.5. *The differential of* $h \to \mathrm{Ad}_h$ *is given by* $U \to \mathrm{ad}_U (X) = [U, X]$

PROOF. If we write $\mathrm{Ad}_h (x) = R_{h^{-1}} L_h (x)$, then its differential at $x = e$ is given by $\mathrm{Ad}_h = DR_{h^{-1}} DL_h$. Now let F^t be the flow for U. Then $F^t (g) = gF^t (e) = L_g (F^t (e))$ as both curves go through g at $t = 0$ and have U as tangent everywhere since U is a left-invariant vector field. This also shows that $DF^t = DR_{F^t(e)}$. Thus

$$\begin{aligned} \mathrm{ad}_U (X)|_e &= \frac{d}{dt} DR_{F^{-t}(e)} DL_{F^t(e)} (X|_e)|_{t=0} \\ &= \frac{d}{dt} DR_{F^{-t}(e)} \left(X|_{F^t(e)} \right)|_{t=0} \\ &= \frac{d}{dt} DF^{-t} \left(X|_{F^t(e)} \right)|_{t=0} \\ &= L_U X = [U, X]. \end{aligned}$$

□

This is used in the next Lemma.

LEMMA 2.1.6. *Let* $G = Gl(V)$ *be the Lie group of invertible matrices on* V. *The Lie bracket structure on the Lie algebra* $\mathfrak{gl}(V)$ *of left invariant vector fields on* $Gl(V)$ *is given by commutation of linear maps. i.e., if* $X, Y \in T_I Gl(V)$, *then*

$$[X, Y]|_I = XY - YX.$$

PROOF. Since $x \to hxh^{-1}$ is a linear map on the space $\hom(V, V)$ we see that $\mathrm{Ad}_h (X) = hXh^{-1}$. The flow of U is given by $F^t (g) = g (I + tU + o(t))$ so we have

$$\begin{aligned} [U, X] &= \frac{d}{dt} \left(F^t (I) X F^{-t} (I) \right)|_{t=0} \\ &= \frac{d}{dt} ((I + tU + o(t)) X (I - tU + o(t)))|_{t=0} \\ &= \frac{d}{dt} (X + tUX - tXU + o(t))|_{t=0} \\ &= UX - XU. \end{aligned}$$

□

2.1.3. The Hessian. Lie derivatives are also useful for defining Hessians of functions.

We start with a Riemannian manifold (M^m, g). The Riemannian structure immediately identifies vector fields with 1-froms. If X is a vector field, then the corresponding 1-form is denoted ω_X and is defined by

$$\omega_X(v) = g(X, v).$$

In local coordinates this looks like

$$\begin{aligned} X &= a^i \partial_i, \\ \omega_X &= g_{ij} a^i dx^j. \end{aligned}$$

This also tells us that the inverse operation in local coordinates looks like

$$\begin{aligned} \phi &= a_j dx^j \\ &= \delta_j^k a_k dx^j \\ &= g_{ji} g^{ik} a_k dx^j \\ &= g_{ij} \left(g^{ik} a_k \right) dx^j \end{aligned}$$

so the corresponding vector field is $X = g^{ik} a_k \partial_i$. If we introduce an inner product on 1-forms that makes this correspondence an isometry

$$g(\omega_X, \omega_Y) = g(X, Y).$$

Then we see that

$$\begin{aligned} g\left(dx^i, dx^j\right) &= g\left(g^{ik}\partial_k, g^{jl}\partial_l\right) \\ &= g^{ik} g^{jl} g_{kl} \\ &= \delta_l^i g^{jl} \\ &= g^{ji} = g^{ij}. \end{aligned}$$

Thus the inverse matrix to g_{ij}, the inner product of coordinate vector fields, is simply the inner product of the coordinate 1-forms.

With all this behind us we define the gradient $\text{grad}f$ of a function f as the vector field corresponding to df, i.e.,

$$\begin{aligned} df(v) &= g(\text{grad}f, v), \\ \omega_{\text{grad}f} &= df, \\ \text{grad}f &= g^{ij} \partial_i f \partial_j. \end{aligned}$$

This correspondence is a bit easier to calculate in orthonormal frames $E_1, ..., E_m$, i.e., $g(E_i, E_j) = \delta_{ij}$, such a frame can always be constructed from a general frame using the Gram-Schmidt procedure. We also have a dual frame $\phi^1, ..., \phi^m$ of 1-forms, i.e., $\phi^i(E_j) = \delta_j^i$. First we observe that

$$\phi^i(X) = g(X, E_i)$$

thus

$$\begin{aligned} X &= a^i E_i = \phi^i(X) E_i = g(X, E_i) E_i \\ \omega_X &= \delta_{ij} a^i \phi^j = a^i \phi^i = g(X, E_i) \phi^i \end{aligned}$$

In other words the coefficients don't change. The gradient of a function looks like

$$\begin{aligned} df &= a_i\phi^i = (D_{E_i}f)\,\phi^i, \\ \mathrm{grad}f &= g(\mathrm{grad}f, E_i)\,E_i = (D_{E_i}f)\,E_i. \end{aligned}$$

In Euclidean space we know that the usual Cartesian coordinates ∂_i also form an orthonormal frame and hence the differentials dx^i yield the dual frame of 1-forms. This makes it particularly simple to calculate in $\mathbb{R}^n$. One other manifold with the property is the torus T^n. In this case we don't have global coordinates, but the coordinates vector fields and differentials are defined globally. This is precisely what we are used to in vector calculus, where the vector field $X = P\partial_x + Q\partial_y + R\,\partial_z$ corresponds to the 1-form $\omega_X = Pdx + Qdy + Rdz$ and the gradient is given by $\partial_x f\partial_x + \partial_y f\partial_y + \partial_z f\partial_z$.

Having defined the gradient of a function the next goal is to define the Hessian of F. This is a bilinear form, like the metric, $\mathrm{Hess}f\,(X,Y)$ that measures the second order change of f. It is defined as the Lie derivative of the metric in the direction of the gradient. Thus it seems to measure how the metric changes as we move along the flow of the gradient

$$\mathrm{Hess}f\,(X,Y) = \frac{1}{2}\left(L_{\mathrm{grad}f}\,g\right)(X,Y)$$

We will calculate this in local coordinates to check that it makes some sort of sense:

$$\begin{aligned}
\mathrm{Hess}f\,(\partial_i,\partial_j) &= \frac{1}{2}\left(L_{\mathrm{grad}f}\,g\right)(\partial_i,\partial_j) \\
&= \frac{1}{2}L_{\mathrm{grad}f}\,g_{ij} - \frac{1}{2}g\left(L_{\mathrm{grad}f}\partial_i,\partial_j\right) - \frac{1}{2}g\left(\partial_i, L_{\mathrm{grad}f}\partial_j\right) \\
&= \frac{1}{2}L_{\mathrm{grad}f}\,g_{ij} - \frac{1}{2}g\left([\mathrm{grad}f,\partial_i],\partial_j\right) - \frac{1}{2}g\left(\partial_i,[\mathrm{grad}f,\partial_j]\right) \\
&= \frac{1}{2}L_{g^{kl}\partial_l f\partial_k}\,g_{ij} - \frac{1}{2}g\left(\left[g^{kl}\partial_l f\partial_k,\partial_i\right],\partial_j\right) - \frac{1}{2}g\left(\partial_i,\left[g^{kl}\partial_l f\partial_k,\partial_j\right]\right) \\
&= \frac{1}{2}g^{kl}\partial_l f\partial_k\left(g_{ij}\right) + \frac{1}{2}g\left(\partial_i\left(g^{kl}\partial_l f\right)\partial_k,\partial_j\right) + \frac{1}{2}g\left(\partial_i,\partial_j\left(g^{kl}\partial_l f\right)\partial_k\right) \\
&= \frac{1}{2}g^{kl}\partial_l f\partial_k\left(g_{ij}\right) + \frac{1}{2}\partial_i\left(g^{kl}\partial_l f\right)g_{kj} + \frac{1}{2}\partial_j\left(g^{kl}\partial_l f\right)g_{ik} \\
&= \frac{1}{2}g^{kl}\partial_k\left(g_{ij}\right)\partial_l f + \frac{1}{2}\partial_i\left(g^{kl}\right)g_{kj}\partial_l f + \frac{1}{2}\partial_j\left(g^{kl}\right)g_{ik}\partial_l f \\
&\quad + \frac{1}{2}g^{kl}\partial_i\left(\partial_l f\right)g_{kj} + \frac{1}{2}g^{kl}\partial_j\left(\partial_l f\right)g_{ik} \\
&= \frac{1}{2}g^{kl}\partial_k\left(g_{ij}\right)\partial_l f - \frac{1}{2}g^{kl}\partial_i\left(g_{kj}\right)\partial_l f - \frac{1}{2}g^{kl}\partial_j\left(g_{ik}\right)\partial_l f \\
&\quad + \frac{1}{2}\delta^l_j\partial_i\partial_l f + \frac{1}{2}\delta^l_i\left(\partial_j\partial_l f\right) \\
&= \frac{1}{2}g^{kl}\left(\partial_k g_{ij} - \partial_i g_{kj} - \partial_j g_{ik}\right)\partial_l f + \partial_i\partial_j f.
\end{aligned}$$

So if the metric coefficients are constant, as in Euclidean space, or we are at a critical point, this gives us the old fashioned Hessian.

It is worth pointing out that these more general definitions and formulas are useful even in Euclidean space. The minute we switch to some more general coordinates, such as polar, cylindrical, spherical etc, the metric coefficients are no longer all constant. Thus the above formulas are our only way of calculating the gradient and Hessian in such general coordinates. We also have the following interesting result that is often used in Morse theory.

LEMMA 2.1.7. *If a function $f : M \to \mathbb{R}$ has a critical point at p then the Hessian of f at p does not depend on the metric.*

PROOF. Assume that $X = \nabla f$ and $X|_p = 0$. Next select coordinates x^i around p such that the metric coefficients satisfy $g_{ij}|_p = \delta_{ij}$. Then we see that

$$\begin{aligned} L_X\left(g_{ij}dx^i dx^j\right)|_p &= L_X\left(g_{ij}\right)|_p + \delta_{ij}L_X\left(dx^i\right)dx^j + \delta_{ij}dx^i L_X\left(dx^j\right) \\ &= \delta_{ij}L_X\left(dx^i\right)dx^j + \delta_{ij}dx^i L_X\left(dx^j\right) \\ &= L_X\left(\delta_{ij}dx^i dx^j\right)|_p. \end{aligned}$$

Thus $\mathrm{Hess} f|_p$ is the same if we compute it using g and the Euclidean metric in the fixed coordinate system. □

2.2. Operations on Forms

2.2.1. General Properties. Given p 1-forms $\omega_i \in \Omega^1(M)$ on a manifold M we define

$$(\omega_1 \wedge \cdots \wedge \omega_p)(v_1, ..., v_p) = \det\left(\left[\omega_i\left(v_j\right)\right]\right)$$

where $\left[\omega_i\left(v_j\right)\right]$ is the matrix with entries $\omega_i\left(v_j\right)$. We can then extend the wedge product to all forms using linearity and associativity. This gives the *wedge product* operation

$$\begin{aligned} \Omega^p(M) \times \Omega^q(M) &\to \Omega^{p+q}(M), \\ (\omega, \psi) &\to \omega \wedge \psi. \end{aligned}$$

This operation is bilinear and antisymmetric in the sense that:

$$\omega \wedge \psi = (-1)^{pq}\, \psi \wedge \omega.$$

The wedge product of a function and a form is simply standard multiplication.

The exterior derivative of a form is defined by

$$\begin{aligned} d\omega\left(X_0, ..., X_k\right) &= \sum_{i=0}^{k} (-1)^i L_{X_i}\left(\omega\left(X_0, ..., \widehat{X}_i, ..., X_k\right)\right) \\ &\quad - \sum_{i<j} (-1)^i \omega\left(X_0, ..., \widehat{X}_i, ..., L_{X_i}X_j, ..., X_k\right) \\ &= \sum_{i=0}^{k} (-1)^i L_{X_i}\left(\omega\left(X_0, ..., \widehat{X}_i, ..., X_k\right)\right) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega\left(L_{X_i}X_j, X_0, ..., \widehat{X}_i, ..., \widehat{X}_j, ..., X_k\right) \\ &= \frac{1}{2}\sum_{i=0}^{k} (-1)^i \left(\begin{array}{c} \left(L_{X_i}\omega\right)\left(X_0, ..., \widehat{X}_i, ..., X_k\right) \\ +L_{X_i}\left(\omega\left(X_0, ..., \widehat{X}_i, ..., X_k\right)\right) \end{array} \right) \end{aligned}$$

Lie derivatives, interior products, wedge products and exterior derivatives on forms are related as follows:

$$\begin{aligned} d(\omega\wedge\psi) &= (d\omega)\wedge\psi+(-1)^p\,\omega\wedge(d\psi),\\ i_X(\omega\wedge\psi) &= (i_X\omega)\wedge\psi+(-1)^p\,\omega\wedge(i_X\psi),\\ L_X(\omega\wedge\psi) &= (L_X\omega)\wedge\psi+\omega\wedge(L_X\psi),\end{aligned}$$

and the composition properties

$$\begin{aligned} d\circ d &= 0,\\ i_X\circ i_X &= 0,\\ L_X &= d\circ i_X+i_X\circ d,\\ L_X\circ d &= d\circ L_X,\\ i_X\circ L_X &= L_X\circ i_X.\end{aligned}$$

The third property $L_X = d\circ i_X + i_X\circ d$ is also known a H. Cartan's formula (son of the geometer E. Cartan). It is behind the definition of exterior derivative we gave above in the form

$$i_{X_0}\circ d = L_{X_0} - d\circ i_{X_0}.$$

2.2.2. The Volume Form. We are now ready to explain how forms are used to unify some standard concepts from differential vector calculus. We shall work on a Riemannian manifold (M,g) and use orthonormal frames $E_1,...,E_m$ as well as the dual frame $\phi^1,...,\phi^m$ of 1-forms.

The local volume form is defined as:

$$d\mathrm{vol} = d\mathrm{vol}_g = \phi^1\wedge\cdots\wedge\phi^m.$$

We see that if $\psi^1,...,\psi^m$ is another collection of 1-forms coming from an orthonormal frame $F_1,...,F_m$, then

$$\begin{aligned} \psi^1\wedge\cdots\wedge\psi^m(E_1,...,E_m) &= \det\left(\psi^i(E_j)\right)\\ &= \det\left(g(F_i,E_j)\right)\\ &= \pm 1.\end{aligned}$$

The sign depends on whether or not the two frames define the same orientation. In case M is oriented and we only use positively oriented frames we will get a globally defined volume form. Next we calculate the local volume form in local coordinates assuming that the frame and the coordinates are both positively oriented:

$$\begin{aligned} d\mathrm{vol}(\partial_1,...\partial_m) &= \det\left(\phi^i(\partial_j)\right)\\ &= \det\left(g(E_i,\partial_j)\right).\end{aligned}$$

As E_i hasn't been eliminated we have to work a little harder. To this end we note that

$$\begin{aligned} \det\left(g(\partial_i,\partial_j)\right) &= \det\left(g\left(g(\partial_i,E_k)E_k,\, g(\partial_j,E_l)E_l\right)\right)\\ &= \det\left(g(\partial_i,E_k)\,g(\partial_j,E_l)\,\delta_{kl}\right)\\ &= \det\left(g(\partial_i,E_k)\,g(\partial_j,E_k)\right)\\ &= \det\left(g(\partial_i,E_k)\right)\det\left(g(\partial_j,E_k)\right)\\ &= \left(\det\left(g(E_i,\partial_j)\right)\right)^2.\end{aligned}$$

Thus

$$\begin{aligned} d\mathrm{vol}\left(\partial_1, \ldots\partial_m\right) &= \sqrt{\det g_{ij}}, \\ d\mathrm{vol} &= \sqrt{\det g_{ij}}dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

2.2.3. Divergence. The divergence of a vector field is defined as the change in the volume form as we flow along the vector field. Note the similarity with the Hessian.

$$L_X d\mathrm{vol} = \mathrm{div}\left(X\right) d\mathrm{vol}$$

In coordinates using that $X = a^i \partial_i$ we get

$$\begin{aligned} L_X d\mathrm{vol} &= L_X \left(\sqrt{\det g_{kl}} dx^1 \wedge \cdots \wedge dx^m\right) \\ &= L_X \left(\sqrt{\det g_{kl}}\right) dx^1 \wedge \cdots \wedge dx^m \\ &\quad + \sqrt{\det g_{kl}} \sum_i dx^1 \wedge \cdots \wedge L_X \left(dx^i\right) \wedge \cdots \wedge dx^m \\ &= a^i \partial_i \left(\sqrt{\det g_{kl}}\right) dx^1 \wedge \cdots \wedge dx^m \\ &\quad + \sqrt{\det g_{kl}} \sum_i dx^1 \wedge \cdots \wedge d\left(L_X x^i\right) \wedge \cdots \wedge dx^m \\ &= a^i \partial_i \left(\sqrt{\det g_{kl}}\right) dx^1 \wedge \cdots \wedge dx^m \\ &\quad + \sqrt{\det g_{kl}} \sum_i dx^1 \wedge \cdots \wedge d\left(a^i\right) \wedge \cdots \wedge dx^m \\ &= a^i \partial_i \left(\sqrt{\det g_{kl}}\right) dx^1 \wedge \cdots \wedge dx^m \\ &\quad + \sqrt{\det g_{kl}} \sum_i dx^1 \wedge \cdots \wedge \left(\partial_j a^i dx^j\right) \wedge \cdots \wedge dx^m \\ &\quad a^i \partial_i \left(\sqrt{\det g_{kl}}\right) dx^1 \wedge \cdots \wedge dx^m \\ &\quad + \sqrt{\det g_{kl}} \sum_i dx^1 \wedge \cdots \wedge \left(\partial_i a^i dx^i\right) \wedge \cdots \wedge dx^m \\ &= \left(a^i \partial_i \left(\sqrt{\det g_{kl}}\right) + \sqrt{\det g_{kl}} \partial_i a^i\right) dx^1 \wedge \cdots \wedge dx^m \\ &= \frac{\partial_i \left(a^i \sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}} \sqrt{\det g_{kl}} dx^1 \wedge \cdots \wedge dx^m \\ &= \frac{\partial_i \left(a^i \sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}} d\mathrm{vol} \end{aligned}$$

We see again that in case the metric coefficients are constant we get the familiar divergence from vector calculus.

H. Cartan's formula for the Lie derivative of forms gives us a different way of finding the divergence

$$\begin{aligned} \mathrm{div}\left(X\right) d\mathrm{vol} &= L_X d\mathrm{vol} \\ &= d i_X \left(d\mathrm{vol}\right) + i_X d\left(d\mathrm{vol}\right) \\ &= d i_X \left(d\mathrm{vol}\right), \end{aligned}$$

in particular $\mathrm{div}\left(X\right) d\mathrm{vol}$ is always exact.

This formula suggests that we should study the correspondence that takes a vector field X to the $(n-1)$-form $i_X\left(d\mathrm{vol}\right).$ Using the orthonormal frame this correspondence is

$$\begin{aligned} i_X\left(d\mathrm{vol}\right) &= i_{g(X,E_j)E_j}\left(\phi^1\wedge\cdots\wedge\phi^m\right)\\ &= g\left(X,E_j\right)i_{E_j}\left(\phi^1\wedge\cdots\wedge\phi^m\right)\\ &= \sum\left(-1\right)^{j+1}g\left(X,E_j\right)\phi^1\wedge\cdots\wedge\widehat{\phi^j}\wedge\cdots\wedge\phi^m \end{aligned}$$

while in coordinates

$$\begin{aligned} i_X\left(d\mathrm{vol}\right) &= i_{a^j\partial_j}\left(\sqrt{\det g_{kl}}dx^1\wedge\cdots\wedge dx^m\right)\\ &= \sqrt{\det g_{kl}}\sum a^j i_{\partial_j}\left(dx^1\wedge\cdots\wedge dx^m\right)\\ &= \sqrt{\det g_{kl}}\sum\left(-1\right)^{j+1}a^j dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^m \end{aligned}$$

If we compute $di_X\left(d\mathrm{vol}\right)$ using this formula we quickly get back our coordinate formula for $\mathrm{div}\left(X\right).$

In vector calculus this gives us the correspondence

$$\begin{aligned} i_{(P\partial_x+Q\partial_y+R\partial_z)}dx\wedge dy\wedge dz &= Pi_{\partial_x}dx\wedge dy\wedge dz\\ &\quad +Qi_{\partial_y}dx\wedge dy\wedge dz\\ &\quad +Ri_{\partial_z}dx\wedge dy\wedge dz\\ &= Pdy\wedge dz-Qdx\wedge dz+Rdx\wedge dy\\ &= Pdy\wedge dz+Qdz\wedge dx+Rdx\wedge dy \end{aligned}$$

If we compose the grad and div operations we get the Laplacian:

$$\mathrm{div}\left(\mathrm{grad}f\right)=\Delta f$$

For this to make sense we should check that it is the "trace" of the Hessian. This is most easily done using an orthonormal frame E_i. On one hand the trace of the Hessian is:

$$\begin{aligned} \sum_i\mathrm{Hess}f\left(E_i,E_i\right) &= \sum_i\frac{1}{2}\left(L_{\mathrm{grad}f}g\right)\left(E_i,E_i\right)\\ &= \sum_i\frac{1}{2}L_{\mathrm{grad}f}\left(g\left(E_i,E_i\right)\right)-\frac{1}{2}g\left(L_{\mathrm{grad}f}E_i,E_i\right)-\frac{1}{2}g\left(E_i,L_{\mathrm{grad}f}E_i\right)\\ &= -\sum_i g\left(L_{\mathrm{grad}f}E_i,E_i\right). \end{aligned}$$

While the divergence is calculated as

$$\begin{aligned} \mathrm{div}\left(\mathrm{grad}f\right) &= \mathrm{div}\left(\mathrm{grad}f\right)d\mathrm{vol}\left(E_1,...,E_m\right)\\ &= \left(L_{\mathrm{grad}f}\phi^1\wedge\cdots\wedge\phi^m\right)\left(E_1,...,E_m\right)\\ &= \sum\left(\phi^1\wedge\cdots\wedge L_{\mathrm{grad}f}\phi^i\wedge\cdots\wedge\phi^m\right)\left(E_1,...,E_m\right)\\ &= \sum\left(L_{\mathrm{grad}f}\phi^i\right)\left(E_i\right)\\ &= \sum L_{\mathrm{grad}f}\left(\phi^i\left(E_i\right)\right)-\phi^i\left(L_{\mathrm{grad}f}E_i\right)\\ &= -\sum\phi^i\left(L_{\mathrm{grad}f}E_i\right)\\ &= -\sum g\left(L_{\mathrm{grad}f}E_i,E_i\right). \end{aligned}$$

2.2.4. Curl. While the gradient and divergence operations work on any Riemannian manifold, the curl operator is specific to oriented 3 dimensional manifolds. It uses the above two correspondences between vector fields and 1-forms as well as 2-forms:

$$d(\omega_X) = i_{\mathrm{curl}X}(d\mathrm{vol})$$

If $X = P\partial_x + Q\partial_y + R\partial_z$ and we are on $\mathbb{R}^3$ we can easily see that

$$\mathrm{curl}X = (\partial_y R - \partial_z Q)\,\partial_x + (\partial_z P - \partial_x R)\,\partial_y + (\partial_x Q - \partial_y P)\,\partial_z$$

Taken together these three operators are defined as follows:

$$\begin{aligned} \omega_{\mathrm{grad}f} &= df, \\ i_{\mathrm{curl}X}(d\mathrm{vol}) &= d(\omega_X), \\ \mathrm{div}(X)\,d\mathrm{vol} &= d i_X(d\mathrm{vol}). \end{aligned}$$

Using that $d \circ d = 0$ on all forms we obtain the classical vector analysis formulas

$$\begin{aligned} \mathrm{curl}(\mathrm{grad}f) &= 0, \\ \mathrm{div}(\mathrm{curl}X) &= 0, \end{aligned}$$

from

$$\begin{aligned} i_{\mathrm{curl}(\mathrm{grad}f)}(d\mathrm{vol}) &= d(\omega_{\mathrm{grad}f}) = ddf, \\ \mathrm{div}(\mathrm{curl}X)\,d\mathrm{vol} &= d i_{\mathrm{curl}X}(d\mathrm{vol}) = dd\omega_X. \end{aligned}$$

2.3. Orientability

Recall that two ordered bases of a finite dimensional vector space are said to represent the same orientation if the transition matrix from one to the other is of positive determinant. This evidently defines an equivalence relation with exactly two equivalence classes. A choice of such an equivalence class is called an orientation for the vector space.

Given a smooth manifold each tangent space has two choices for an orientation. Thus we obtain a two fold covering map $O_M \to M$, where the preimage of each $p \in M$ consists of the two orientations for T_pM. A connected manifold is said to be *orientable* if the orientation covering is disconnected. For a disconnected manifold, we simply require that each connected component be connected. A choice of sheet in the covering will correspond to a choice of an orientation for each tangent space.

To see that O_M really is a covering just note that if we have a chart $(x^1, x^2, ..., x^n) : U \subset M \to \mathbb{R}^n$, where U is connected, then we have two choices of orientations over U, namely, the class determined by the framing $(\partial_1, \partial_2, ..., \partial_n)$ and by the framing $(-\partial_1, \partial_2, ..., \partial_n)$. Thus U is covered by two sets each diffeomorphic to U and parametrized by these two different choices of orientation. Observe that this tells us that $\mathbb{R}^n$ is orientable and has a canonical orientation given by the standard Cartesian coordinate frame $(\partial_1, \partial_2, ..., \partial_n)$.

Note that since simply connected manifolds only have trivial covering spaces they must all be orientable. Thus S^n, $n > 1$ is always orientable.

An other important observation is that the orientation covering O_M is an orientable manifold since it is locally the same as M and an orientation at each tangent space has been picked for us.

THEOREM 2.3.1. *The following conditions for a connected n-manifold M are equivalent.*

1. M is orientable.

2. Orientation is preserved moving along loops.

3. M admits an atlas where the Jacobians of all the transitions functions are positive.

4. M admits a nowhere vanishing n-form.

PROOF. $1 \Leftrightarrow 2$: The unique path lifting property for the covering $O_M \to M$ tells us that orientation is preserved along loops if and only if O_M is disconnected.

$1 \Rightarrow 3$: Pick an orientation. Take any atlas (U_α, F_α) of M where U_α is connected. As in our description of O_M from above we see that either each F_α corresponds to the chosen orientation, otherwise change the sign of the first component of F_α. In this way we get an atlas where each chart corresponds to the chosen orientation. Then it is easily checked that the transition functions $F_\alpha \circ F_\beta^{-1}$ have positive Jacobian as they preserve the canonical orientation of $\mathbb{R}^n$.

$3 \Rightarrow 4$: Choose a locally finite partition of unity (λ_α) subordinate to an atlas (U_α, F_α) where the transition functions have positive Jacobians. On each U_α we have the nowhere vanishing form $\omega_\alpha = dx_\alpha^1 \wedge ... \wedge dx_\alpha^n$. Now note that if we are in an overlap $U_\alpha \cap U_\beta$ then

$$\begin{aligned} dx_\alpha^1 \wedge ... \wedge dx_\alpha^n \left(\frac{\partial}{\partial x_\beta^1}, ..., \frac{\partial}{\partial x_\beta^n} \right) &= \det \left(dx_\alpha^i \left(\frac{\partial}{\partial x_\beta^j} \right) \right) \\ &= \det \left(D \left(F_\alpha \circ F_\beta^{-1} \right) \right) \\ &> 0. \end{aligned}$$

Thus the globally defined form $\omega = \sum \lambda_\alpha \omega_\alpha$ is always nonnegative when evaluated on $\left(\frac{\partial}{\partial x_\beta^1}, ..., \frac{\partial}{\partial x_\beta^n} \right)$. What is more, at least one term must be positive according to the definition of partition of unity.

$4 \Rightarrow 1$: Pick a nowhere vanishing n-form ω. Then define two sets $O_\pm$ according to whether ω is positive or negative when evaluated on a basis. This yields two disjoint open sets in O_M which cover all of M. □

With this result behind us we can try to determine which manifolds are orientable and which are not. Conditions 3 and 4 are often good ways of establishing orientability. To establish non-orientability is a little more tricky. However, if we suspect a manifold to be non-orientable then 1 tells us that there must be a nontrivial 2-fold covering map $\pi : \hat{M} \to M$, where $\hat{M}$ is oriented and the two given orientations at points over $p \in M$ are mapped to different orientations in M via $D\pi$. A different way of recording this information is to note that for a two fold covering $\pi : \hat{M} \to M$ there is only one nontrivial deck transformation $A : \hat{M} \to \hat{M}$ with the properties: $A(x) \neq x$, $A \circ A = id_M$, and $\pi \circ IA = \pi$. With this is mind we can show

PROPOSITION 2.3.2. *Let $\pi : \hat{M} \to M$ be a non-trivial 2-fold covering and $\hat{M}$ an oriented manifold. Then M is orientable if and only if A preserves the orientation on $\hat{M}$.*

PROOF. First suppose A preserves the orientation of $\hat{M}$. Then given a choice of orientation $e_1, ..., e_n \in T_x\hat{M}$ we can declare $D\pi(e_1), ..., D\pi(e_n) \in T_{\pi(x)}M$ to be an orientation at $\pi(x)$. This is consistent as $DA(e_1), ..., DA(e_n) \in T_{I(x)}\hat{M}$ is mapped to $D\pi(e_1), ..., D\pi(e_n)$ as well (using $\pi \circ A = \pi$) and also represents the given orientation on $\hat{M}$ since A was assumed to preserve this orientation.

Suppose conversely that M is orientable and choose an orientation for M. Since we assume that both $\hat{M}$ and M are connected the projection $\pi : \hat{M} \to M$, being nonsingular everywhere, must always preserve or reverse the orientation. We can without loss of generality assume that the orientation is preserved. Then we just use $\pi \circ A = \pi$ as in the first part of the proof to see that A must preserve the orientation on $\hat{M}$. □

We can now use these results to check some concrete manifolds for orientability.

We already know that $S^n, n > 1$ are orientable, but what about S^1? One way of checking that this space is orientable is to note that the tangent bundle is trivial and thus a uniform choice of orientation is possible. This clearly generalizes to Lie groups and other parallelizable manifolds. Another method is to find a nowhere vanishing form. This can be done on all spheres S^n by considering the n-form

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$$

on $\mathbb{R}^{n+1}$. This form is a generalization of the 1-form $xdy - ydx$, which is $\pm$ the angular form in the plane. Note that if $X = x^i \partial_i$ denotes the radial vector field, then we have (see also the section below on the classical integral theorems)

$$i_X \left(dx^1 \wedge \cdots \wedge dx^{n+1} \right) = \omega.$$

From this it is clear that if $v_2, ..., v_n$ form a basis for a tangent space to the sphere, then

$$\begin{aligned} \omega(v_2, ..., v_n) &= dx^1 \wedge \cdots \wedge dx^{n+1} (X, v_2, ..., v_{n+1}) \\ &\neq 0. \end{aligned}$$

Thus we have found a nonvanishing n-form on all spheres regardless of whether or not they are parallelizable or simply connected. As another exercise people might want to use one of the several coordinate atlases known for the spheres to show that they are orientable.

Recall that $\mathbb{R}P^n$ has S^n as a natural double covering with the antipodal map as a natural deck transformation. Now this deck transformation preserves the radial field $X = x^i \partial_i$ and thus its restriction to S^n preserves or reverses orientation according to what it does on $\mathbb{R}^{n+1}$. On the ambient Euclidean space the map is linear and therefore preserves the orientation iff its determinant is positive. This happens iff $n + 1$ is even. Thus we see that $\mathbb{R}P^n$ is orientable iff n is odd.

Using the double covering lemma show that the Klein bottle and the Möbius band are non-orientable.

Manifolds with boundary are defined like manifolds, but modeled on open sets in $L^n = \left\{ x \in \mathbb{R}^n : x^1 \leq 0 \right\}$. The boundary ∂M is then the set of points that correspond to elements in $\partial L^n = \left\{ x \in \mathbb{R}^n : x^1 = 0 \right\}$. It is not hard to prove that if $F : M \to \mathbb{R}$ has $a \in \mathbb{R}$ as a regular value then $F^{-1}(-\infty, a]$ is a manifold with boundary. If M is oriented then the boundary is oriented in such a way that if we add the outward pointing normal to the boundary as the first basis vector then we get a positively oriented basis for M. Thus $\partial_2, ..., \partial_n$ is the positive orientation for

∂L^n since ∂_1 points away from L^n and $\partial_1, \partial_2, ..., \partial_n$ is the usual positive orientation for L^n.

2.4. Integration of Forms

We shall assume that M is an oriented n-manifold. Thus, M comes with a covering of charts $\varphi_\alpha = \left(x_\alpha^1, \ldots, x_\alpha^n\right) : U_\alpha \longleftrightarrow B(0,1) \subset \mathbb{R}^n$ such that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ preserve the usual orientation on Euclidean space, i.e., $\det\left(D\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)\right) > 0$. In addition, we shall also assume that a partition of unity with respect to this covering is given. In other words, we have smooth functions $\phi_\alpha : M \to [0,1]$ such that $\phi_\alpha = 0$ on $M - U_\alpha$ and $\sum_\alpha \phi_\alpha = 1$. For the last condition to make sense, it is obviously necessary that the covering be also locally finite.

Given an n-form ω on M we wish to define:

$$\int_M \omega.$$

When M is not compact, it might be necessary to assume that the form has compact support, i.e., it vanishes outside some compact subset of M.

In each chart we can write

$$\omega = f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

Using the partition of unity, we then obtain

$$\begin{aligned} \omega &= \sum_\alpha \phi_\alpha \omega \\ &= \sum_\alpha \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n, \end{aligned}$$

where each of the forms $\phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ has compact support in U_α. Since U_α is identified with $\bar{U}_\alpha \subset \mathbb{R}^n$, we simply declare that

$$\int_{U_\alpha} \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \int_{\bar{U}_\alpha} \phi_\alpha f_\alpha dx^1 \cdots dx^n.$$

Here the right-hand side is simply the integral of the function $\phi_\alpha f_\alpha$ viewed as a function on $\bar{U}_\alpha$. Then we define

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

whenever this sum converges. Using the standard change of variables formula for integration on Euclidean space, we see that indeed this definition is independent of the choice of coordinates.

With these definitions behind us, we can now state and prove Stokes' theorem for manifolds with boundary.

Theorem 2.4.1. *For any $\omega \in \Omega^{n-1}(M)$ with compact support we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

PROOF. If we use the trick

$$d\omega = \sum_{\alpha} d\left(\phi_{\alpha}\omega\right),$$

then we see that it suffices to prove the theorem in the case $M = L^n$ and ω has compact support. In that case we can write

$$\omega = \sum_{i=1}^{n} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

The differential of ω is now easily computed:

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} (df_i) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial x^i}\right) dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{L^n} d\omega &= \int_{L^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{L^n} \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int \left(\int \left(\frac{\partial f_i}{\partial x^i}\right) dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n. \end{aligned}$$

The fundamental theorem of calculus tells us that

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{\partial f_i}{\partial x^i}\right) dx^i &= 0, \text{ for } i > 1, \\ \int_{-\infty}^{0} \left(\frac{\partial f_1}{\partial x^1}\right) dx^1 &= f_1\left(0, x^2, ..., x^n\right). \end{aligned}$$

Thus

$$\int_{L^n} d\omega = \int_{\partial L^n} f_1\left(0, x^2, ..., x^n\right) dx^2 \wedge \cdots \wedge dx^n.$$

Since $dx^1 = 0$ on ∂L^n it follows that

$$\omega|_{\partial L^n} = f_1 dx^2 \wedge \cdots \wedge dx^n.$$

This proves the theorem. □

We get a very nice corollary out of Stokes' theorem.

THEOREM. (Brouwer) *Let M be a connected compact manifold with nonempty boundary. Then there is no retract $r : M \to \partial M$.*

PROOF. Note that if ∂M is not connected such a retract clearly can't exists so we need only worry about having connected boundary.

If M is oriented and ω is a volume form on ∂M, then we have

$$\begin{aligned} 0 &< \int_{\partial M} \omega \\ &= \int_{\partial M} r^* \omega \\ &= \int_M d\left(r^* \omega\right) \\ &= \int_M r^* d\omega \\ &= 0. \end{aligned}$$

If M is not orientable, then we lift the situation to the orientation cover and obtain a contradiction there. □

We shall briefly discuss how the classical integral theorems of Green, Gauss, and Stokes follow from the general version of Stokes' theorem presented above.

Green's theorem in the plane is quite simple.

THEOREM 2.4.2. (Green's Theorem) *Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$. If $X = P\partial_x + Q\partial_y$ is a vector field defined on a region containing Ω then*

$$\int_\Omega \left(\partial_x Q - \partial_y P\right) dxdy = \int_{\partial\Omega} Pdx + Qdy.$$

PROOF. Note that the integral on the right-hand side is a line integral, which can also be interpreted as the integral of the 1-form $\omega = Pdx^1 + Qdx^2$ on the 1-manifold $\partial\Omega$. With this in mind we just need to observe that $d\omega = \left(\partial_1 Q - \partial_2 P\right) dx^1 \wedge dx^2$ in order to establish the theorem. □

Gauss' Theorem is quite a bit more complicated, but we did some of the ground work when we defined the divergence above. The context is a connected, compact, oriented Riemannian manifold M with boundary, but the example to keep in mind is a domain $M \subset \mathbb{R}^n$ with smooth boundary

THEOREM 2.4.3. (The divergence theorem or Gauss' theorem) *Let X be a vector field defined on M and N the outward pointing unit normal field to ∂M, then*

$$\int_M (\mathrm{div} X)\, d\mathrm{vol}_g = \int_{\partial M} g\left(X, N\right) d\mathrm{vol}_{g|_{\partial M}}$$

PROOF. We know that

$$\mathrm{div} X d\mathrm{vol}_g = d\left(i_X\left(d\mathrm{vol}_g\right)\right).$$

So by Stokes' theorem it suffices to show that

$$i_X\left(d\mathrm{vol}_g\right)|_{\partial M} = g\left(X, N\right) d\mathrm{vol}_{\mathrm{g}|_{\partial\mathrm{M}}}$$

The orientation on $T_p\partial M$ is so that $v_2, ..., v_n$ is a positively oriented basis for $T_p\partial M$ iff $N, v_2, ..., v_n$ is a positively oriented basis for T_pM. Therefore, the natural volume form for ∂M denoted $d\text{vol}_{g|\partial M}$ is given by $i_N\left(d\text{vol}_g\right)$. If $v_2, ..., v_n \in T_p\partial M$ is a basis, then

$$\begin{aligned} i_X\left(d\text{vol}_g\right)|_{\partial M}\left(v_2, ..., v_n\right) &= d\text{vol}_g\left(X, v_2, ..., v_n\right) \\ &= d\text{vol}_g\left(g\left(X,N\right)N, v_2, ..., v_n\right) \\ &= g\left(X,N\right)d\text{vol}_g\left(N, v_2, ..., v_n\right) \\ &= g\left(X,N\right)i_N\left(d\text{vol}_g\right) \\ &= g\left(X,N\right)d\text{vol}_{g|\partial M} \end{aligned}$$

where we used that $X - g\left(X,N\right)X$, the component of X tangent $T_p\partial M$, is a linear combination of $v_2, ..., v_n$ and therefore doesn't contribute to the form. □

Stokes' Theorem is specific to 3 dimensions. Classically it holds for an oriented surface $S \subset \mathbb{R}^3$ with smooth boundary but can be formulated for oriented surfaces in oriented Riemannian 3-manifolds.

THEOREM 2.4.4. (Stokes' theorem) *Let $S \subset M^3$ be an oriented surface with boundary ∂S. If X is a vector field defined on a region containing S and N is the unit normal field to S, then*

$$\int_S g\left(\text{curl}X, N\right)d\text{vol}_{g|S} = \int_{\partial S}\omega_X.$$

PROOF. Recall that ω_X is the 1-form defined by

$$\omega_X\left(v\right) = g\left(X, v\right).$$

This form is related to curl$X$ by

$$d\left(\omega_X\right) = i_{\text{curl}X}\left(d\text{vol}_g\right).$$

So Stokes' Theorem tells us that

$$\int_{\partial S}\omega_X = \int_S i_{\text{curl}X}\left(d\text{vol}_g\right).$$

The integral on the right-hand side can now be understood in a manner completely analogous to our discussion of $i_X\left(d\text{vol}_g\right)|_{\partial M}$ in the Divergence Theorem. We note that N is chosen perpendicular to T_pS in such a way that $N, v_2, v_3 \in T_pM$ is positively oriented iff $v_2, v_3 \in T_pS$ is positively oriented. Thus we have again that

$$d\text{vol}_{g|S} = i_N d\text{vol}_g$$

and consequently

$$i_{\text{curl}X}\left(d\text{vol}_g\right) = g\left(\text{curl}X, N\right)d\text{vol}_{g|S}$$

□

CHAPTER 3

Basic Cohomology Theory

3.1. De Rham Cohomology

Throughout we let M be an n-manifold. Using that $d \circ d = 0$, we trivially get that the exact forms

$$B^p(M) = d\left(\Omega^{p-1}(M)\right)$$

are a subset of the closed forms

$$Z^p(M) = \{\omega \in \Omega^p(M) : d\omega = 0\}.$$

The de Rham cohomology is then defined as

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}.$$

Given a closed form ψ, we let $[\psi]$ denote the corresponding cohomology class.

The first simple property comes from the fact that any function with zero differential must be locally constant. On a connected manifold we therefore have

$$H^0(M) = \mathbb{R}.$$

Given a smooth map $F : M \to N$, we get an induced map in cohomology:

$$\begin{array}{rcl} H^p(N) & \to & H^p(M), \\ F^*([\psi]) & = & [F^*\psi]. \end{array}$$

This definition is independent of the choice of ψ, since the pullback F^* commutes with d.

The two key results that are needed for a deeper understanding of de Rham cohomology are the Meyer-Vietoris sequence and homotopy invariance of the pull back map.

LEMMA 3.1.1. (The Mayer-Vietoris Sequence) *If $M = A \cup B$ for open sets $A, B \subset M$, then there is a long exact sequence*

$$\cdots \to H^p(M) \to H^p(A) \oplus H^p(B) \to H^p(A \cap B) \to H^{p+1}(M) \to \cdots.$$

PROOF. The proof is given in outline, as it is exactly the same as the corresponding proof in algebraic topology. We start by defining a short exact sequence

$$0 \to \Omega^p(M) \to \Omega^p(A) \oplus \Omega^p(B) \to \Omega^p(A \cap B) \to 0.$$

The map $\Omega^p(M) \to \Omega^p(A) \oplus \Omega^p(B)$ is simply restriction $\omega \to (\omega|_A, \omega|_B)$. The second is given by $(\omega, \psi) \to (\omega|_{A\cap B} - \psi|_{A\cap B})$. With these definitions it is clear that $\Omega^p(M) \to \Omega^p(A) \oplus \Omega^p(B)$ is injective and that the sequence is exact at $\Omega^p(A) \oplus \Omega^p(B)$. It is a bit less obvious why $\Omega^p(A) \oplus \Omega^p(B) \to \Omega^p(A \cap B)$ is surjective. To see this select a partition of unity λ_A, λ_B with respect to the covering

A, B. Given $\omega \in \Omega^p(A \cap B)$ we see that $\lambda_A \omega$ defines a form on B, while $\lambda_B \omega$ defines a form on A. Then $(\lambda_B \omega, -\lambda_A \omega) \to \omega$.

These maps induce maps in cohomology

$$H^p(M) \to H^p(A) \oplus H^p(B) \to H^p(A \cap B)$$

such that this sequence is exact. The connecting homomorphisms

$$\delta : H^p(A \cap B) \to H^{p+1}(M)$$

are constructed using the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^{p+1}(M) & \to & \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) & \to & \Omega^{p+1}(A \cap B) & \to & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \to & \Omega^p(M) & \to & \Omega^p(A) \oplus \Omega^p(B) & \to & \Omega^p(A \cap B) & \to & 0 \end{array}$$

If we take a form $\omega \in \Omega^p(A \cap B)$, then $(\lambda_B \omega, -\lambda_A \omega) \in \Omega^p(A) \oplus \Omega^p(B)$ is mapped onto ω. If $d\omega = 0$, then

$$\begin{array}{rcl} d(\lambda_B \omega, -\lambda_A \omega) & = & (d\lambda_B \wedge \omega, -d\lambda_A \wedge \omega) \\ & \in & \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) \end{array}$$

vanishes when mapped to $\Omega^{p+1}(A \cap B)$. So we get a well-defined form

$$\begin{array}{rcl} \delta\omega & = & \left\{ \begin{array}{ll} d\lambda_B \wedge \omega & \text{on } A \\ -d\lambda_A \wedge \omega & \text{on } B \end{array} \right. \\ & \in & \Omega^{p+1}(M). \end{array}$$

It is easy to see that this defines a map in cohomology that makes the Meyer-Vietoris sequence exact.

The construction here is fairly concrete, but it is a very general homological construction. □

The first part of the Meyer-Vietoris sequence

$$0 \to H^0(M) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \to H^1(M)$$

is particularly simple since we know what the zero dimensional cohomology is. In case $A \cap B$ is connected it must be a short exact sequence

$$0 \to H^0(M) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \to 0$$

so the Meyer-Vietoris sequence for higher dimensional cohomology starts with

$$0 \to H^1(M) \to H^1(A) \oplus H^1(B) \to \cdots$$

To study what happens when we have homotopic maps between manifolds we have to figure out how forms on the product $[0,1] \times M$ relate to forms on M.

On the product $[0,1] \times M$ we have the vector field ∂_t tangent to the first factor as well as the corresponding one form dt. In local coordinates forms on $[0,1] \times M$ can be written

$$\omega = a_I dx^I + b_J dt \wedge dx^J$$

if we use summation convention and multi index notation

$$\begin{array}{rcl} a_I & = & a_{i \cdots i_k}, \\ dx^I & = & dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{array}$$

For each form the dt factor can be integrated out as follows

$$\mathcal{I}(\omega) = \int_0^1 \omega = \int_0^1 b_J dt \wedge dx^J = \left(\int_0^1 b_J dt\right) dx^J$$

Thus giving a map

$$\Omega^{k+1}([0,1] \times M) \to \Omega^k(M)$$

To see that this is well-defined note that it can be expressed as

$$\mathcal{I}(\omega) = \int_0^1 dt \wedge i_{\partial_t}\omega$$

since

$$i_{\partial_t}(\omega) = b_J dx^J.$$

LEMMA 3.1.2. *Let $j_t : M \to [0,1] \times M$ be the map $j_t(x) = (t, x)$, then*

$$\mathcal{I}(d\omega) + d\mathcal{I}(\omega) = j_1^*(\omega) - j_0^*(\omega)$$

PROOF. The key is to prove that

$$\mathcal{I}(d\omega) + d\mathcal{I}(\omega) = \mathcal{I}(L_{\partial_t}\omega)$$

Given this we see that the right hand side is

$$\begin{aligned}
\mathcal{I}(L_{\partial_t}\omega) &= \int_0^1 dt \wedge L_{\partial_t}\omega \\
&= \int_0^1 dt \wedge L_{\partial_t}\left(a_I dx^I + b_J dt \wedge dx^J\right) \\
&= \int_0^1 dt \wedge \left(\partial_t a_I dx^I + \partial_t b_J dt \wedge dx^J\right) \\
&= \int_0^1 dt \wedge (\partial_t a_I)\, dx^I \\
&= \left(\int_0^1 dt \partial_t a_I\right) dx^I \\
&= (a_I(1,x) - a_I(0,x))\, dx^I \\
&= j_1^*(\omega) - j_0^*(\omega)
\end{aligned}$$

The first formula follows by noting that

$$\begin{aligned}
\mathcal{I}(d\omega) + d\mathcal{I}(\omega) &= \int_0^1 dt \wedge i_{\partial_t} d\omega + d\left(\int_0^1 dt \wedge i_{\partial_t}\omega\right) \\
&= \int_0^1 dt \wedge i_{\partial_t} d\omega + \int_0^1 dt \wedge d i_{\partial_t}\omega \\
&= \int_0^1 dt \wedge (i_{\partial_t} d\omega + d i_{\partial_t}\omega) \\
&= \int_0^1 dt \wedge (L_{\partial_t}\omega)
\end{aligned}$$

The one tricky move here is the identity

$$d\left(\int_0^1 dt \wedge i_{\partial_t}\omega\right) = \int_0^1 dt \wedge d i_{\partial_t}\omega$$

On the left hand side it is clear what d does, but on the right hand side we are computing d of a form on the product. However, as we are wedging with dt this does not become an issue. More worrisome is the thought that one would expect

$$\begin{aligned} d\left(\int_0^1 dt\wedge i_{\partial_t}\omega\right) &= \int_0^1 d\left(dt\wedge i_{\partial_t}\omega\right) \\ &= \int_0^1 ddt\wedge i_{\partial_t}\omega-\int_0^1 dt\wedge di_{\partial_t}\omega \\ &= -\int_0^1 dt\wedge di_{\partial_t}\omega \end{aligned}$$

but this is not what is happening as the dt has in fact been integrated out. Specifically if d is exterior differentiation on $[0,1]\times M$ and d_x exterior differentiation on M, then

$$\begin{aligned} d_x\left(\int_0^1 dt\wedge i_{\partial_t}\omega\right) &= d_x\left(\int_0^1 b_J dt\right)\wedge dx^J \\ &= \left(\int_0^1 dt\wedge\left(d_x b_J\right)\right)\wedge dx^J \\ &= \left(\int_0^1 dt\wedge\left(db_J-\partial_t b_J dt\right)\right)\wedge dx^J \\ &= \left(\int_0^1 dt\wedge db_J\right)\wedge dx^J \\ &= \int_0^1 dt\wedge di_{\partial_t}\omega \end{aligned}$$

□

We can now establish homotopy invariance.

Proposition 3.1.3. *If $F_0,F_1:M\to N$ are smoothly homotopic, then they induce the same maps on de Rham cohomology.*

Proof. Assume we have a homotopy $H:[0,1]\times M\to N$, such that $F_0=H\circ j_0$ and $F_1=H\circ j_1$, then

$$\begin{aligned} F_1^*\left(\omega\right)-F_0^*\left(\omega\right) &= \left(H\circ j_1\right)^*\left(\omega\right)-\left(H\circ j_0\right)^*\left(\omega\right) \\ &= j_1^*\left(H^*\left(\omega\right)\right)-j_0^*\left(H^*\left(\omega\right)\right) \\ &= d\mathcal{I}\left(H^*\left(\omega\right)\right)+\mathcal{I}\left(H^*\left(d\omega\right)\right) \end{aligned}$$

So if $\omega\in\Omega^k\left(N\right)$ is closed, then we have shown that the difference

$$F_1^*\left(\omega\right)-F_0^*\left(\omega\right)\in\Omega^k\left(M\right)$$

is exact. Thus the two forms $F_1^*\left(\omega\right)$ and $F_0^*\left(\omega\right)$ must lie in the same de Rham cohomology class. □

Corollary 3.1.4. *If two manifolds, possibly of different dimension, are homotopy equivalent, then they have the same de Rham cohomology.*

Proof. This follows from having maps $F:M\to N$ and $G:N\to M$ such that $F\circ G$ and $G\circ F$ are homotopic to the identity maps. □

LEMMA 3.1.5. (The Poincaré Lemma) *The cohomology of a contractible manifold M is*

$$\begin{aligned} H^0(M) &= \mathbb{R}, \\ H^p(M) &= \{0\} \text{ for } p > 0. \end{aligned}$$

In particular convex sets in $\mathbb{R}^n$ have trivial de Rham cohomology.

PROOF. Being contractible is the same as being homotopy equivalent to a point. □

3.2. Examples of Cohomology Groups

For S^n we use that

$$\begin{aligned} S^n &= (S^n - \{p\}) \cup (S^n - \{-p\}), \\ S^n - \{\pm p\} &\simeq \mathbb{R}^n, \\ (S^n - \{p\}) \cap (S^n - \{-p\}) &\simeq \mathbb{R}^n - \{0\}. \end{aligned}$$

Since $\mathbb{R}^n - \{0\}$ deformation retracts onto S^{n-1} this allows us to compute the cohomology of S^n by induction using the Meyer-Vietoris sequence. We start with S^1, which a bit different as the intersection has two components. The Meyer-vietoris sequence starting with $p = 0$ looks like

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1(S^1) \to 0.$$

Showing that $H^1(S^1) \simeq \mathbb{R}$. For $n \geq 2$ the intersection is connected so the connecting homomorphism must be an isomorphism

$$H^{p-1}(S^{n-1}) \to H^p(S^n)$$

for $p \geq 1$. Thus

$$H^p(S^n) = \begin{cases} 0, & p \neq 0, n, \\ \mathbb{R}, & p = 0, n. \end{cases}$$

For $\mathbb{P}^n$ we use the decomposition

$$\mathbb{P}^n = (\mathbb{P}^n - \mathbb{P}^{n-1}) \cup (\mathbb{P}^n - p),$$

where

$$\begin{aligned} p &= [1 : 0 : \cdots : 0], \\ \mathbb{P}^{n-1} &= \mathbb{P}(p^\perp) = \left\{ [0 : z^1 : \cdots : z^n] : (z^1, ..., z^n) \in \mathbb{F}^n - \{0\} \right\}, \end{aligned}$$

and consequently

$$\begin{aligned} \mathbb{P}^n - p &= \left\{ [z : z^1 : \cdots : z^n] : (z^1, ..., z^n) \in \mathbb{F}^n - \{0\} \text{ and } z \in \mathbb{F} \right\} \simeq \mathbb{P}^{n-1}, \\ \mathbb{P}^n - \mathbb{P}^{n-1} &= \left\{ [1 : z^1 : \cdots : z^n] : (z^1, ..., z^n) \in \mathbb{F}^n \right\} \simeq \mathbb{F}^n, \\ (\mathbb{P}^n - \mathbb{P}^{n-1}) \cap (\mathbb{P}^n - p) &= \left\{ [1 : z^1 : \cdots : z^n] : (z^1, ..., z^n) \in \mathbb{F}^n - \{0\} \right\} \simeq \mathbb{F}^n - \{0\}. \end{aligned}$$

We have already identified $\mathbb{P}^1$ so we don't need to worry about having a disconnected intersection when $\mathbb{F} = \mathbb{R}$ and $n = 1$. Using that $\mathbb{F}^n - \{0\}$ deformation retracts to the unit sphere S of dimension $\dim_{\mathbb{R}} \mathbb{F}^n - 1$ we see that the Meyer-Vietoris sequence reduces to

$$\begin{aligned} 0 &\to H^1(\mathbb{P}^n) \to H^1(\mathbb{P}^{n-1}) \to H^1(S) \to \cdots \\ \cdots &\to H^{p-1}(S) \to H^p(\mathbb{P}^n) \to H^p(\mathbb{P}^{n-1}) \to H^p(S) \to \cdots \end{aligned}$$

for $p \geq 2$. To get more information we need to specify the scalars and in the real case even distinguish between even and odd n. First assume that $\mathbb{F} = \mathbb{C}$. Then $S = S^{2n-1}$ and $\mathbb{CP}^1 \simeq S^2$. A simple induction then shows that

$$H^p\left(\mathbb{CP}^n\right) = \begin{cases} 0, & p = 1, 3, ..., 2n-1, \\ \mathbb{R}, & p = 0, 2, 4, ..., 2n. \end{cases}.$$

When $\mathbb{F} = \mathbb{R}$, we have $S = S^{n-1}$ and $\mathbb{RP}^1 \simeq S^1$. This shows that $H^p\left(\mathbb{RP}^n\right) = 0$ when $p = 1, ..., n-2$. The remaining cases have to be extracted from the last part of the sequence

$$0 \to H^{n-1}\left(\mathbb{RP}^n\right) \to H^{n-1}\left(\mathbb{RP}^{n-1}\right) \to H^{n-1}\left(S^{n-1}\right) \to H^n\left(\mathbb{RP}^n\right) \to 0$$

where we know that

$$H^{n-1}\left(S^{n-1}\right) = \mathbb{R}.$$

This first of all shows that $H^n\left(\mathbb{RP}^n\right)$ is either 0 or $\mathbb{R}$. Moreover if $H^n\left(\mathbb{RP}^n\right) = 0$, then it follows that $H^{n-1}\left(\mathbb{RP}^{n-1}\right) = \mathbb{R}$ and $H^{n-1}\left(\mathbb{RP}^n\right) = 0$. While if $H^n\left(\mathbb{RP}^n\right) = \mathbb{R}$, then it is hard to conclude what happens without knowing anything about the middle arrow. we first obtain that $H^{n-1}\left(\mathbb{RP}^n\right) = 0$??? and then that $H^{n-1}\left(\mathbb{RP}^{n-1}\right) = 0$. Given that we know that $H^1\left(\mathbb{RP}^1\right) = \mathbb{R}$ we then obtain the cohomology groups:

$$H^p\left(\mathbb{RP}^{2n}\right) = \begin{cases} 0, & p \geq 1, \\ \mathbb{R}, & p = 0. \end{cases},$$

$$H^p\left(\mathbb{RP}^{2n+1}\right) = \begin{cases} 0, & 2n \geq p \geq 1, \\ \mathbb{R}, & p = 0, 2n+1. \end{cases}.$$

3.3. Poincaré Duality

The last piece of information we need to understand is how the wedge product acts on cohomology. It is easy to see that we have a map

$$\begin{aligned} H^p(M) \times H^q(M) &\to H^{p+q}(M), \\ ([\psi], [\omega]) &\to [\psi \wedge \omega]. \end{aligned}$$

We are interested in understanding what happens in case $p + q = n$. This requires a surprising amount of preparatory work. First we have

Theorem 3.3.1. *If M is an oriented closed n-manifold, then we have a well-defined isomorphism*

$$\begin{aligned} H^n(M) &\to \mathbb{R}, \\ [\omega] &\to \int_M \omega. \end{aligned}$$

Proof. That the map is well-defined follows from Stokes' theorem. It is also onto, since any form with the property that it is positive when evaluated on a positively oriented frame is integrated to a positive number. Thus, we must show that any form with $\int_M \omega = 0$ is exact. This is not easy to show, and in fact, it is more natural to show this in a more general context: If M is an oriented n-manifold, then any compactly supported n-form ω with $\int_M \omega = 0$ is exact.

The proof of this result is by induction on the number of charts it takes to cover the support of ω. But before we can start the inductive procedure, we must establish the result for the n-sphere.

Case 1: $M = S^n$. We know that $H^n(S^n) = \mathbb{R}$, so $\int : H^n(S^n) \to \mathbb{R}$ must be an isomorphism.

Case 2: $M = \mathbb{R}^n$. We can think of $M = S^n - \{p\}$. Any compactly supported form ω on M therefore yields a form on S^n. Given that $\int_M \omega = 0$, we therefore also get that $\int_{S^n} \omega = 0$. Thus, ω must be exact on S^n. Let $\psi \in \Omega^{n-1}(S^n)$ be chosen such that $d\psi = \omega$. Use again that ω is compactly supported to find an open disc U around p such that ω vanishes on U and $U \cup M = S^n$. Then ψ is clearly closed on U and must by the Poincaré lemma be exact. Thus, we can find $\theta \in \Omega^{n-2}(U)$ with $d\theta = \psi$ on U. This form does necessarily extend to S^n, but we can select a bump function $\lambda : S^n \to [0,1]$ that vanishes on $S^n - U$ and is 1 on some smaller neighborhood $V \subset U$ around p. Now observe that $\psi - d(\lambda\theta)$ is actually defined on all of S^n. It vanishes on V and clearly

$$d(\psi - d(\lambda\theta)) = d\psi = \omega.$$

Case 3: $\text{supp}\omega \subset A \cup B$ where the result holds on the open sets $A, B \subset M$. Select a partition of unity $\lambda_A + \lambda_B = 1$ subordinate to the cover $\{A, B\}$. Given an n-form ω with $\int_M \omega = 0$, we get two forms $\lambda_A \cdot \omega$ and $\lambda_B \cdot \omega$ with support in A and B, respectively. Using our assumptions, we see that

$$\begin{aligned} 0 &= \int_M \omega \\ &= \int_A \lambda_A \cdot \omega + \int_B \lambda_B \cdot \omega. \end{aligned}$$

On $A \cap B$ we can select an n-form $\tilde{\omega}$ with compact support inside $A \cap B$ such that

$$\int_{A\cap B} \tilde{\omega} = \int_A \lambda_A \cdot \omega.$$

Using $\tilde{\omega}$ we can create two forms,

$$\begin{aligned} &\lambda_A \cdot \omega - \tilde{\omega}, \\ &\lambda_B \cdot \omega + \tilde{\omega}, \end{aligned}$$

with support in A and B, respectively. From our constructions it follows that they both have integral zero. Thus, we can by assumption find ψ_A and ψ_B with support in A and B, respectively, such that

$$\begin{aligned} d\psi_A &= \lambda_A \cdot \omega - \tilde{\omega}, \\ d\psi_B &= \lambda_B \cdot \omega + \tilde{\omega}. \end{aligned}$$

Then we get a globally defined form $\psi = \psi_A + \psi_B$ with

$$\begin{aligned} d\psi &= \lambda_A \cdot \omega - \tilde{\omega} + \lambda_B \cdot \omega + \tilde{\omega} \\ &= (\lambda_A + \lambda_B) \cdot \omega \\ &= \omega. \end{aligned}$$

The theorem now follows by using induction on the number of charts it takes to cover the support of ω. □

The above proof indicates that it might be more convenient to work with compactly supported forms. This leads us to *compactly supported cohomology*, which is defined as follows: Let $\Omega_c^p(M)$ denote the compactly supported p-forms. With this we have the compactly supported exact and closed forms $B_c^p(M) \subset Z_c^p(M)$ (note that $d : \Omega_c^p(M) \to \Omega_c^{p+1}(M)$). Then define

$$H_c^p(M) = \frac{Z_c^p(M)}{B_c^p(M)}.$$

Needless to say, for closed manifolds the two cohomology theories are identical. For open manifolds, on the other hand, we have that the closed 0-forms must be zero, as they also have to have compact support. Thus $H_c^0(M) = \{0\}$ if M is not closed.

Note that only proper maps $F : M \to N$ have the property that they map $F^* : \Omega_c^p(N) \to \Omega_c^p(M)$. In particular, if $A \subset M$ is open, we do not have a map $H_c^p(M) \to H_c^p(A)$. Instead we observe that there is a natural inclusion $\Omega_c^p(A) \to \Omega_c^p(M)$, which induces

$$H_c^p(A) \to H_c^p(M).$$

The above proof, stated in our new terminology, says that

$$\begin{array}{rcl} H_c^n(M) & \to & \mathbb{R}, \\ [\omega] & \to & \int_M \omega \end{array}$$

is an isomorphism for oriented n-manifolds. Moreover, using that $\mathbb{R}^n = S^n - \{p\}$, Case 2 in the above proof shows that

$$H_c^n(\mathbb{R}^n) = \mathbb{R}$$

a similar but simpler argument can now be used to prove:

$$H_c^p(\mathbb{R}^n) = 0, \ p < n.$$

In order to carry out induction proofs with this cohomology theory, we also need a Meyer-Vietoris sequence:

$$\cdots \longrightarrow H_c^p(A \cap B) \longrightarrow H_c^p(A) \oplus H_c^p(B) \longrightarrow H_c^p(M) \longrightarrow H_c^{p+1}(A \cap B) \longrightarrow \cdots.$$

This is established in the same way as before using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_c^{p+1}(A \cap B) & \longrightarrow & \Omega_c^{p+1}(A) \oplus \Omega_c^{p+1}(B) & \longrightarrow & \Omega_c^{p+1}(M) & \longrightarrow & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \longrightarrow & \Omega_c^p(A \cap B) & \longrightarrow & \Omega_c^p(A) \oplus \Omega_c^p(B) & \longrightarrow & \Omega_c^p(M) & \longrightarrow & 0 \end{array}$$

where the horizontal arrows are defined by:

$$\begin{array}{rcl} \Omega_c^p(A \cap B) & \to & \Omega_c^p(A) \oplus \Omega_c^p(B) \\ [\omega] & \to & ([\omega], -[\omega]) \end{array}$$

and

$$\begin{array}{rcl} \Omega_c^p(A) \oplus \Omega_c^p(B) & \to & \Omega_c^p(M) \\ ([\omega_A], [\omega_B]) & \to & [\omega_A + \omega_B] \end{array}$$

THEOREM 3.3.2. *Let M be an oriented n-manifold. The pairing*

$$\begin{array}{rl} H^p(M) \times H_c^{n-p}(M) & \to \mathbb{R}, \\ ([\omega], [\psi]) & \to \int_M \omega \wedge \psi \end{array}$$

is well-defined and nondegenerate. In particular, the two cohomology groups $H^p(M)$ and $H_c^{n-p}(M)$ are dual to each other and therefore have the same dimension as finite-dimensional vector spaces.

PROOF. For the case $M = \mathbb{R}^n$, this theorem follows from the Poincaré lemma and the above calculation of $H_c^p(\mathbb{R}^n)$. Next note that if M is the disjoint union of a possibly infinite collection of open sets and the result holds for each open set then it must hold for M. In general suppose $M = A \cup B$, where the theorem is true for A, B, and $A \cap B$. Note that the pairing gives a natural map

$$H^p(N) \to \left(H_c^{n-p}(N)\right)^* = \hom\left(H_c^{n-p}(N), \mathbb{R}\right)$$

for any manifold N. We apparently assume that this map is an isomorphism for $N = A, B, A \cap B$. Using that taking duals reverses arrows, we obtain a diagram where the left- and right most columns have been eliminated

$$\begin{array}{ccccccccc}
\to & H^{p-1}(A \cap B) & \to & H^p(M) & \to & H^p(A) \oplus H^p(B) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & \left(H_c^{n-p+1}(A \cap B)\right)^* & \to & \left(H_c^{n-p}(M)\right)^* & \to & \left(H^{n-p}(A)\right)^* \oplus \left(H^{n-p}(B)\right)^* & \to
\end{array}$$

Each square in this diagram is either commutative or anticommutative (i.e., commutes with a minus sign.) As all vertical arrows, except for the middle one, are assumed to be isomorphisms, we see by a simple diagram chase (the five lemma) that the middle arrow is also an isomorphism.

It is now clear that the theorem holds for all open subsets of $\mathbb{R}^n$ that are finite unions of open convex sets. This in turn shows that we can prove the theorem for all open sets in $\mathbb{R}^n$. Use an exhaustion of compact sets to write such an open set as a union $\bigcup U_i$ where each U_i is a finite union of convex sets and $U_i \cap U_j = \emptyset$ when $|i - j| \geq 2$. Thus the theorem holds for $\bigcup U_{2i}$, $\bigcup U_{2i+1}$, and $(\bigcup U_{2i}) \cap (\bigcup U_{2i+1})$ and consequently for the entire union.

Finally we can establish the theorem in general. The argument is similar to the argument that worked for open sets in $\mathbb{R}^n$. Write $M = \bigcup U_i$where each U_i is a finite union of charts and $U_i \cap U_j = \emptyset$ when $|i - j| \geq 2$. This means the theorem holds for $\bigcup U_{2i}$, $\bigcup U_{2i+1}$, and $(\bigcup U_{2i}) \cap (\bigcup U_{2i+1})$ and consequently for the entire union. □

COROLLARY 3.3.3. *On a closed oriented n-manifold M we have that $H^p(M)$ and $H^{n-p}(M)$ are isomorphic.*

Note that $\mathbb{RP}^2$ does not satisfy this duality between H^0 and H^2. In fact we always have

THEOREM 3.3.4. *Let M be an n-manifold that is not orientable, then*

$$H_c^n(M) = 0.$$

PROOF. We use the two fold orientation cover $F : \hat{M} \to M$ and the involution $A : \hat{M} \to \hat{M}$ such that $F = F \circ A$. The fact that M is not orientable means that A is orientation reversing. The key now is that pull back by A changes integrals by a sign:

$$\int_{\hat{M}} \eta = -\int_{\hat{M}} A^*\eta, \; \eta \in \Omega_c^n\left(\hat{M}\right).$$

To prove the theorem select $\omega \in \Omega_c^n(M)$ and consider the pull-back $F^*\omega \in \Omega_c^n\left(\hat{M}\right)$. Since $F = F \circ A$ this form is invariant under pull back by A we have

$$\int_{\hat{M}} F^*\omega = \int_{\hat{M}} A^* \circ F^*\omega.$$

On the other hand as A reverses orientation we must also have

$$\int_{\hat{M}} F^*\omega = -\int_{\hat{M}} A^* \circ F^*\omega.$$

Thus

$$\int_{\hat{M}} F^*\omega = 0.$$

This shows that the pull back is exact

$$F^*\omega = d\psi,\ \psi \in \Omega_c^{n-1}\left(\hat{M}\right)$$

The form ψ need not be a pull back of a form on M, but we can average it

$$\bar{\psi} = \frac{1}{2}\left(\psi + A^*\psi\right) \in \Omega_c^{n-1}\left(\hat{M}\right)$$

to get a form that is invariant under A

$$\begin{aligned} A^*\bar{\psi} &= \frac{1}{2}\left(A^*\psi + A^*A^*\psi\right) \\ &= \frac{1}{2}\left(A^*\psi + \psi\right) \\ &= \bar{\psi}. \end{aligned}$$

The differential, however, stays the same

$$\begin{aligned} d\bar{\psi} &= \frac{1}{2}\left(d\psi + A^*d\psi\right) \\ &= \frac{1}{2}\left(F^*\omega + A^*F^*\omega\right) \\ &= F^*\omega. \end{aligned}$$

Now there is a unique $\phi \in \Omega_c^{n-1}(M)$, such that $F^*\phi = \bar{\psi}$. Moreover $d\phi = \omega$, since F is a local diffeomorphism and

$$\omega = F^*d\phi = dF^*\phi = d\bar{\psi}$$

□

The last part of this proof yields a more general result:

Corollary 3.3.5. *Let $F : M \to N$ be a two fold covering map, then*

$$F^* : H_c^p(N) \to H_c^p(M)$$

is an injection.

3.4. Degree Theory

Given the simple nature of the top cohomology class of a manifold, we see that maps between manifolds of the same dimension can act only by multiplication on the top cohomology class. We shall see that this multiplicative factor is in fact an integer, called the *degree* of the map.

To be precise, suppose we have two oriented n-manifolds M and N and also a proper map $F : M \to N$. Then we get a diagram

$$\begin{array}{ccc} H_c^n(N) & \stackrel{F^*}{\to} & H_c^n(M) \\ \downarrow \int & & \downarrow \int \\ \mathbb{R} & \stackrel{d}{\to} & \mathbb{R}. \end{array}$$

Since the vertical arrows are isomorphisms, the induced map f^* yields a unique map $d : \mathbb{R} \to \mathbb{R}$. This map must be multiplication by some number, which we call the degree of f, denoted by $\deg F$. Clearly, the degree is defined by the property

$$\int_M F^*\omega = \deg F \cdot \int_N \omega.$$

From the functorial properties of the induced maps on cohomology we see that

$$\deg (F \circ G) = \deg (F) \deg (G)$$

LEMMA 3.4.1. *If* $F : M \to N$ *is a diffeomorphism between oriented* n*-manifolds, then* $\deg F = \pm 1$*, depending on whether* F *preserves or reverses orientation.*

PROOF. Note that our definition of integration of forms is independent of coordinate changes. It relies only on a choice of orientation. If this choice is changed then the integral changes by a sign. This clearly establishes the lemma. □

THEOREM 3.4.2. *If* $F : M \to N$ *is a proper map between oriented* n*-manifolds, then* $\deg F$ *is an integer.*

PROOF. The proof will also give a recipe for computing the degree. First, we must appeal to Sard's theorem. This theorem ensures that we can find $y \in N$ such that for each $x \in F^{-1}(y)$ the differential $DF : T_xM \to T_yN$ is an isomorphism. The inverse function theorem then tells us that F must be a diffeomorphism in a neighborhood of each such x. In particular, the preimage $F^{-1}(y)$ must be a discrete set. As we also assumed the map to be proper, we can conclude that the preimage is finite: $\{x_1, \dots, x_k\} = F^{-1}(y)$. We can then find a neighborhood U of y in N, and neighborhoods U_i of x_i in M, such that $F : U_i \to U$ is a diffeomorphism for each i. NEED TO INVOKE STACK OF RECORDS AS WE NEED $F^{-1}(U) = \bigcup_i U_i$. Now select $\omega \in \Omega_c^n(U)$ with $\int \omega = 1$. Then we can write

$$F^*\omega = \sum_{i=1}^{k} F^*\omega|_{U_i},$$

where each $F^*\omega|_{U_i}$ has support in U_i. The above lemma now tells us that

$$\int_{U_i} F^*\omega|_{U_i} = \pm 1.$$

Hence,

$$\begin{aligned} \deg F &= \deg F \cdot \int_N \omega \\ &= \deg F \cdot \int_U \omega \\ &= \int_M F^*\omega \\ &= \sum_{i=1}^{k} \int_{U_i} F^*\omega|_{U_i} \end{aligned}$$

is an integer. □

Note that $\int_{U_i} F^*\omega|_{U_i}$ is ± 1, depending simply on whether F preserves or reverses the orientations at x_i. Thus, the degree simply counts the number of preimages for regular values with sign. In particular, a finite covering map has degree equal to the number of sheets in the covering.

We get several nice results using degree theory. Several of these have other proofs as well using differential topological techniques. Here we emphasize the integration formula for the degree. The key observation is that the degree of a map is a homotopy invariant. However, as we can only compute degrees of proper maps it is important that the homotopies are though proper maps. When working on closed manifolds this is not an issue. But if the manifold is Euclidean space, then all maps are homotopy equivalent, although not necessarily through proper maps.

COROLLARY 3.4.3. *Let $F : M \to N$ be a proper nonsingular map of degree ± 1 between oriented connected manifolds, then F is a diffeomorphism.*

PROOF. Since F is nonsingular everywhere it either reverses or preserves orientations and all points. If the degree is well defined it follows that it can only be ± 1 if the map is injective. On the other hand the fact that it is proper shows that it is a covering map, thus it must be a diffeomorphism. □

COROLLARY 3.4.4. *The identity map on a closed manifold is not homotopic to a constant map.*

PROOF. The constant map has degree 0 while the identity map has degree 1 on an oriented manifold. In case the manifold isn't oriented we can lift to the orientation cover and still get it to work. □

COROLLARY 3.4.5. *Even dimensional spheres do not admit nonvanishing vector fields.*

PROOF. Let X be a vector field on S^n we can scale it so that it is a unit vector field. If we consider it as a function $X : S^n \to S^n \subset \mathbb{R}^{n+1}$ then it is always perpendicular to its foot point. We can then create a homotopy

$$H(p,t) = p\cos(\pi t) + X_p \sin(\pi t).$$

Since $p \perp X_p$ and both are unit vectors the Pythagorean theorem shows that $H(p,t) \in S^n$ as well. When $t = 0$ the homotopy is the identity, and when $t = 1$ it is the antipodal map. Since the antipodal map reverses orientations on even dimensional spheres it is not possible for the identity map to be homotopic to the antipodal map. □

On an oriented Riemannian manifold (M, g) we always have a canonical volume form denoted by $d\mathrm{vol}_g$. Using this form, we see that the degree of a map between closed Riemannian manifolds $F : (M, g) \to (N, h)$ can be computed as

$$\deg F = \frac{\int_M F^*(d\mathrm{vol}_h)}{\mathrm{vol}(N)}.$$

In case F is locally a Riemannian isometry, we must have that:

$$F^*(d\mathrm{vol}_h) = \pm d\mathrm{vol}_g.$$

Hence,

$$\deg F = \pm \frac{\mathrm{vol} M}{\mathrm{vol} N}.$$

This gives the well-known formula for the relationship between the volumes of Riemannian manifolds that are related by a finite covering map.

On $\mathbb{R}^n - \{0\}$ we have an interesting $(n-1)$-form

$$w = r^{-n} \sum_{i=1}^{n} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

that is closed. If we restrict this to a sphere of radius ε around the origin we see that

$$\begin{aligned}
\int_{S^{n-1}(\varepsilon)} w &= \varepsilon^{-n} \int_{S^{n-1}(\varepsilon)} \sum_{i=1}^{n} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \varepsilon^{-n} \int_{\bar{B}(0,\varepsilon)} d\left(\sum_{i=1}^{n} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \right) \\
&= \varepsilon^{-n} \int_{\bar{B}(0,\varepsilon)} n dx^1 \wedge \cdots \wedge dx^n \\
&= n\varepsilon^{-n} \mathrm{vol} \bar{B}(0,\varepsilon) \\
&= n \mathrm{vol} \bar{B}(0,1) \\
&= \mathrm{vol}_{n-1} S^{n-1}(1).
\end{aligned}$$

More generally if $F : M^{n-1} \to \mathbb{R}^n - \{0\}$ is a smooth map then it is clearly homotopic to the map $F_1 : M^{n-1} \to S^{n-1}(1)$ defined by $F_1 = F/|F|$ so we obtain

$$\begin{aligned}
\frac{1}{\mathrm{vol}_{n-1} S^{n-1}(1)} \int_M F^* w &= \frac{1}{\mathrm{vol}_{n-1} S^{n-1}(1)} \int_M F_1^* w \\
&= \deg F_1
\end{aligned}$$

This is called the winding number of F.

CHAPTER 4

Characteristic Classes

4.1. Intersection Theory

Let $S^k \subset N^n$ be a closed oriented submanifold of an oriented manifold. The codimension is denoted by $m = n - k$. By integrating k-forms on N over S we obtain a linear functional $H^k(N) \to \mathbb{R}$. The Poincaré dual to this functional is an element $\eta_S^N \in H_c^m(N)$ such that

$$\int_S \omega = \int_M \eta_S^N \wedge \omega$$

for all $\omega \in H^k(N)$. We call η_S^N the dual to $S \subset N$. The obvious defect of this definition is that several natural submanifolds might not have nontrivial duals for the simple reason that $H_c^m(N)$ vanishes, e.g., $N = S^n$.

To get a nontrivial dual we observe that $\int_S \omega$ only depends on the values of ω in a neighborhood of S. Thus we can find duals supported in any neighborhood U of S in N, i.e., $\eta_S^U \in H_c^m(U)$. We normally select the neighborhood so that there is a deformation retraction $\pi : U \to S$. In particular

$$\pi^* : H^k(S) \to H^k(U)$$

is an isomorphism. In case S is connected we also know that integration on $H^k(S)$ defines an isomorphism

$$\int : H^k(S) \to \mathbb{R}$$

This means that η_S^U is just the Poincaré dual to $1 \in \mathbb{R}$ modulo these isomorphisms. Specifically, if $\omega \in H^k(S)$ is a volume form that integrates to 1, then

$$\int_U \eta_S^U \wedge \pi^* \omega = 1.$$

Our first important observation is that if we change the orientation of S, then integration changes sign on S and hence η_S^U also changes sign. This will become important below.

The dual gives us an interesting isomorphism called the *Thom isomorphism.*

Lemma 4.1.1. (Thom) *The map*

$$\begin{array}{rcl} H_c^{p-m}(S) & \to & H_c^p(U) \\ \omega & \to & \eta_S^U \wedge \pi^*(\omega) \end{array}$$

is an isomorphism.

PROOF. Using Poincaré duality twice we see that

$$\begin{aligned} H_c^p(U) &\simeq \hom\left(H^{n-p}(U), \mathbb{R}\right) \\ &\simeq \hom\left(H^{n-p}(S), \mathbb{R}\right) \\ &\simeq H_c^{p-m}(S) \end{aligned}$$

Thus it suffices to show that the map

$$\begin{aligned} H_c^{p-m}(S) &\to H_c^p(U) \\ \omega &\to \eta_S^N \wedge \pi^*(\omega) \end{aligned}$$

is injective. When $p = n$ this is clearly the above construction. For $p < n$ select $\tau \in H^{n-p}(S) \simeq H^{n-p}(U)$, then $\omega \wedge \tau \in H^k(S)$ so

$$\begin{aligned} \int_U \eta_S^U \wedge \pi^*(\omega) \wedge \pi^*(\tau) &= \int_U \eta_S^U \wedge \pi^*(\omega \wedge \tau) \\ &= \int_S \omega \wedge \tau \end{aligned}$$

Note that since τ is closed the form $\eta_S^U \wedge \pi^*(\omega) \wedge \pi^*(\tau)$ is exact provided $\eta_S^U \wedge \pi^*(\omega)$ is exact. Therefore the formula shows that the linear map $\tau \to \int_S \omega \wedge \tau$ is trivial if $\eta_S^U \wedge \pi^*(\omega)$ is trivial in $H_c^p(U)$. Poincaré duality then implies that ω itself is trivial in $H_c^{p-m}(S)$. □

The next goal is to find a characterization of η_S^U when we have a deformation retraction submersion $\pi : U \to S$.

PROPOSITION 4.1.2. *The dual is characterized as a closed form with compact support that integrates to 1 along fibers $\pi^{-1}(p)$ for all $p \in S$.*

PROOF. The characterization requires a choice of orientation for the fibers. It is chosen so that $T_p\pi^{-1}(p) \oplus T_pS$ and T_pN have the same orientation (this is consistent with [**Guillemin-Pollack**], but not with several other texts.) For $\omega \in \Omega^k(S)$ we note that $\pi^*\omega$ is constant on $\pi^{-1}(p)$, $p \in S$. Therefore, if η is a closed compactly supported form that integrates to 1 along all fibers, then

$$\int_U \eta_S^U \wedge \pi^*\omega = \int_S \omega$$

as desired.

Conversely we define

$$\begin{aligned} f &: S \to \mathbb{R}, \\ f(p) &= \int_{\pi^{-1}(p)} \eta_S^U \end{aligned}$$

and note that

$$\int_S \omega = \int_U \eta_S^U \wedge \pi^*\omega = \int_S f\omega$$

for all ω. Since the support of ω can be chosen to be in any open subset of S, this shows that $f = 1$ on S. □

In case S is not connected the dual is constructed on each component.

Next we investigate naturality of the dual.

THEOREM 4.1.3. *Let $F : M \to N$ be transverse to S, then for suitable U we have*

$$F^*\left(\eta_S^U\right) = \eta_{F^{-1}(S)}^{F^{-1}(U)}.$$

PROOF. To make sense of $\eta_{F^{-1}(S)}^{F^{-1}(U)}$ we need to choose orientations for $F^{-1}(S)$. This is done as follows. First note that by shrinking U we can assume that $F^{-1}(U)$ deformation retracts onto $F^{-1}(S)$ in such a way that we have a commutative diagram

$$\begin{array}{ccc} F^{-1}(U) & \stackrel{F}{\longrightarrow} & U \\ \downarrow \pi & & \downarrow \pi \\ F^{-1}(S) & \stackrel{F}{\longrightarrow} & S \end{array}$$

Transversality of F then shows that F restricted to the fibers $F : \pi^{-1}(q) \to \pi^{-1}(F(q))$ is a diffeomorphism. We then select the orientation on $\pi^{-1}(q)$ such that F has degree 1 and then on $T_qF^{-1}(S)$ such that $T_q\pi^{-1}(q) \oplus T_qF^{-1}(S)$ has the orientation of T_qM. In case $F^{-1}(S)$ is a finite collection of points we are simply assigning 1 or -1 to each point depending on whether $\pi^{-1}(q)$ got oriented the same way as M or not. With all of these choices it is now clear that if η_S^U integrates to 1 along fibers then so does the pullback $F^*\left(\eta_S^U\right)$, showing that the pullback must represent $\eta_{F^{-1}(S)}^{F^{-1}(U)}$. □

This gives us a new formula for intersection numbers.

COROLLARY 4.1.4. *If* $\dim M + \dim S = \dim N$, *and* $F : M \to N$ *is transverse to* S, *then*

$$I(F, S) = \int_{F^{-1}(U)} F^*\left(\eta_S^U\right).$$

The advantage of this formula is that the right-hand side can be calculated even when F isn't transverse to S. And since both sides are invariant under homotopies of F this gives us a more general way of calculating intersection numbers. We shall see how this works in the next section.

Another interesting special case of naturality occurs for submanifolds.

COROLLARY 4.1.5. *Assume $S_1, S_2 \subset N$ are transverse and oriented, with suitable orientations on $S_1 \cap S_2$ the dual is given by*

$$\eta_{S_1} \wedge \eta_{S_2} = \eta_{S_1 \cap S_2}.$$

Finally we wish to study to what extent η depends only on its values on the fibers. First we note that if the tubular neighborhood $S \subset U$ is a product neighborhood, i.e. there is a diffeomorphism $F : D \times S \to U$ which is a degree 1 diffeomorphism on fibers: $D \times \{p\} \to \pi^{-1}(p)$ for all $p \in S$, then $\eta_S^{D\times S} = F^*\left(\eta_S^U\right)$ can be represented as the volume form on D pulled back to $D \times S$.

To better measure this effect we define the Euler class

$$e_S^U = i^*\left(\eta_S^U\right) \in H^m(S)$$

as the restriction of the dual to S. Since duals are natural we quickly get

PROPOSITION 4.1.6. *Let $F : M \to N$ be transverse to S, then for suitable U we have*

$$F^*\left(e_S^U\right) = e_{F^{-1}(S)}^{F^{-1}(U)}.$$

This shows

COROLLARY 4.1.7. *If U is a trivial tubular neighborhood of S, then $e_S^U = 0$.*

We also see that intersection numbers of maps are carried by the Euler class.

LEMMA 4.1.8. *If $\dim M + \dim S = \dim N$, and $F : M \to N$, then*

$$I(F,S) = \int_{F^{-1}(U)} F^*\left(\pi^*\left(e_S^U\right)\right).$$

PROOF. Assume that $\pi : U \to S$ is a deformation retraction. Then F and $i \circ \pi \circ F$ are homotopy equivalent as maps from $F^{-1}(U)$. This shows that

$$\begin{aligned} I(F,S) &= \int_{F^{-1}(U)} F^*\left(\eta_S^U\right) \\ &= \int_{F^{-1}(U)} (i \circ \pi \circ F)^*\left(\eta_S^U\right) \\ &= \int_{F^{-1}(U)} (\pi \circ F)^*\left(i^*\eta_S^U\right) \\ &= \int_{F^{-1}(U)} (\pi \circ F)^*\left(e_S^U\right) \\ &= \int_{F^{-1}(U)} F^*\left(\pi^*\left(e_S^U\right)\right). \end{aligned}$$

□

This formula makes it clear that this integral really is an intersection number as it must vanish if F doesn't intersect S.

Finally we show that Euler classes vanish if the codimension is odd.

THEOREM 4.1.9. *The Euler class is characterized by*

$$\eta_S^U \wedge \pi^*\left(e_S^U\right) = \eta_S^U \wedge \eta_S^U \in H_c^{2m}(U).$$

In particular $e_S^U = 0$ if m is odd.

PROOF. Since $\pi^*\left(e_S^U\right)$ and η_S^U represent the same class in $H^m(U)$ we have that

$$\pi^*\left(e_S^U\right) - \eta_S^U = d\omega.$$

Then

$$\begin{aligned} \eta_S^U \wedge \pi^*\left(e_S^U\right) - \eta_S^U \wedge \eta_S^U &= \eta_S^U \wedge (d\omega) \\ &= d\left(\eta_S^U \wedge \omega\right) \end{aligned}$$

Since $\eta_S^U \wedge \omega$ is compactly supported this shows that $\eta_S^U \wedge \pi^*\left(e_S^U\right) = \eta_S^U \wedge \eta_S^U$.

Next recall that we have an isomorphism $\eta_S^U \wedge \pi^*(\cdot) : H^m(S) \to H_c^{2m}(U)$. Thus $e_S^U = 0$ if $\eta_S^U \wedge \eta_S^U = 0$. This applies to the case when m is odd as

$$\eta_S^U \wedge \eta_S^U = -\eta_S^U \wedge \eta_S^U.$$

□

4.2. The Künneth-Leray-Hirch Theorem

In this section we shall compute the cohomology of a fibration under certain simplifying assumptions. We assume that we have a submersion-fibration $\pi : N \to S$ where the fibers are diffeomorphic to a manifold M and that S is connected. As an example we might have the product $M \times S \to S$. We shall further assume that the restriction to any fiber is a surjection in cohomology

$$H^* (N) \to N^* (M) \to 0$$

In the case of a product this obviously holds since the projection $M \times S \to M$ is a right inverse to all the inclusions $M \to M \times \{s\} \subset M \times S$. In general such cohomology classes might not exist, e.g., the fibration $S^3 \to S^2$ is a good counter example.

It seems a daunting task to check the condition for all fibers in a general situation. Assuming we know it is true for a specific fiber $M = \pi^{-1} (s)$ we can select a neighborhood U around s such that $\pi^{-1} (U) = M \times U$. As long as U is contractible we see that $\pi^{-1} (U)$ and M are homotopy equivalent and so the restriction to any of the fibers over U will also give a surjection in cohomology. Covering S with contractible sets now shows that the restriction to all of the fibers has to be a submersion since S is connected. In fact this construction gives us a bit more. First note that for a specific fiber M it is possible to select $\tau_i \in H^* (N)$ that form a basis for $H^* (M)$. The construction now shows that τ_i restrict to a basis for the cohomology of all fibers as long as S is connected.

Theorem 4.2.1. (Künneth-Leray-Hirch) *Given the above collection τ_i, a basis for $H^* (N)$ can be found by selecting a basis ω_k for $H^* (S)$ and then constructing $\tau_i \wedge F^* (\omega_k)$.*

Proof. We employ the usual induction trick over open subsets of S. Note that if $U \subset S$ is an open subset we get a submersion-fibration $\pi : \pi^{-1} (U) \to U$ and the restriction of τ_i to $\pi^{-1} (U)$ obviously still have the desired properties.

In case U is diffeomorphic to an open disc and the bundle is trivial over U the result is obvious as $M \times U$ has the same cohomology as M.

Next assume that the result holds for open sets $U, V, U \cap V \subset S$. A restatement of the theorem will now make it clear that the five-lemma in conjunction with the Meyer-Vietoris sequence shows that also $U \cup V$ must satisfy the theorem.

The restatement is as follows: First note that we have a natural map

$$\begin{array}{rcl} \operatorname{span}\{\tau_i\} \otimes H^* (S) & \to & H^* (N) \\ \tau_i \otimes \omega & \to & \tau_i \wedge \pi^* (\omega) \end{array}$$

that can be graded by collecting all terms on the left hand side that have degree p. Defining

$$H^p = \operatorname{span}\{\tau_i \otimes H^q (S) : \deg \tau_i = p - q\}$$

we assert that the map

$$H^p \to H^p (N)$$

is an isomorphism for all p. □

Künneth's theorem or formula is the above result in the case where the fibration is a product, while the Leray-Hirch theorem or formula is for a fibration of the above type.

4.3. The Hopf-Lefschetz Formulas

We are going to relate the Euler characteristic and Lefschetz numbers to the cohomology of the space.

THEOREM 4.3.1. (Hopf-Poincaré) *Let M be a closed oriented n-manifold, then*

$$\chi(M) = I(\Delta, \Delta) = \sum (-1)^p \dim H^p(M).$$

PROOF. If we consider the map

$$\begin{aligned} (id, id) &: M \to \Delta, \\ (id, id)(x) &= (x, x), \end{aligned}$$

then the Euler characteristic can be computed as the intersection number

$$\begin{aligned} \chi(M) &= I(\Delta, \Delta) \\ &= I((id, id), \Delta) \\ &= \int_M (id, id)^* \left(\eta_\Delta^{M\times M}\right). \end{aligned}$$

Thus we need a formula for the Poincaré dual $\eta_\Delta = \eta_\Delta^{M\times M}$. To find this formula we use Künneth's formula for the cohomology of the product. To this end select a basis ω_i for the cohomology theory $H^*(M)$ and a dual basis τ_i, i.e.,

$$\int_M \omega_i \wedge \tau_j = \delta_{ij},$$

where the integral is assumed to be zero if the form $\omega_i \wedge \tau_j$ doesn't have degree n.

By Künneth's theorem $\pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j)$ is a basis for $H^*(M \times M)$. The dual basis is up to a sign given by $\pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l)$ as we can see by calculating

$$\begin{aligned} & \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\ = & (-1)^{\deg \tau_j \deg \tau_k} \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\tau_j) \wedge \pi_2^*(\omega_l) \\ = & (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \wedge \pi_2^*(\tau_j) \\ = & (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \left(\int_M \omega_i \wedge \tau_k\right) \left(\int_M \omega_l \wedge \tau_j\right) \\ = & (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \delta_{ik} \delta_{lj} \end{aligned}$$

Clearly this vanishes unless $i = k$ and $l = j$.

This can be used to compute η_Δ for $\Delta \subset M \times M$. We assume that

$$\eta_\Delta = \sum c_{ij} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j).$$

On one hand

$$
\begin{aligned}
&\int_{M\times M} \eta_\Delta \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\
= & \sum c_{ij} \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\
= & \sum c_{ij} (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \delta_{ki} \delta_{jl} \\
= & \ c_{kl} (-1)^{\deg \tau_l (\deg \tau_k + \deg \omega_l)}
\end{aligned}
$$

On the other hand using that $(id, id) : M \to \Delta$ is a map of degree 1 tells us that

$$
\begin{aligned}
\int_{M\times M} \eta_\Delta \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) &= \int_\Delta \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\
&= \int_M (id, id)^* (\pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l)) \\
&= \int_M \tau_k \wedge \omega_l \\
&= (-1)^{\deg(\tau_k)\deg(\omega_l)} \delta_{kl}.
\end{aligned}
$$

Thus

$$c_{kl} (-1)^{\deg \tau_l (\deg \omega_l + \deg \tau_k)} = (-1)^{\deg \tau_k \deg \omega_l} \delta_{kl}$$

or in other words $c_{kl} = 0$ unless $k = l$ and in that case

$$
\begin{aligned}
c_{kk} &= (-1)^{\deg \tau_k (2 \deg \omega_k + \deg \tau_k)} \\
&= (-1)^{\deg \tau_k \deg \tau_k} \\
&= (-1)^{\deg \tau_k}.
\end{aligned}
$$

This yields the formula

$$\eta_\Delta = \sum (-1)^{\deg \tau_i} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_i).$$

The Euler characteristic can now be computed as follows

$$
\begin{aligned}
\chi(M) &= \int_M (id, id)^* \left(\eta_\Delta^{M\times M}\right) \\
&= \int_M (id, id)^* \left(\sum (-1)^{\deg \tau_i} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_i)\right) \\
&= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge \tau_i \\
&= \sum (-1)^{\deg \tau_i} \\
&= \sum (-1)^p \dim H^p(M).
\end{aligned}
$$

□

A generalization of this leads us to a similar formula for the Lefschetz number of a map $F : M \to M$.

Theorem 4.3.2. (Hopf-Lefschetz) *Let $F : M \to M$, then*

$$L(F) = I(\text{graph}(F), \Delta) = \sum (-1)^p \operatorname{tr}(F^* : H^p(M) \to H^p(M)).$$

PROOF. This time we use the map $(id, F) : M \to \text{graph}(F)$ sending x to $(x, F(x))$ to compute the Lefschetz number

$$\begin{aligned} I(\text{graph}(F), \Delta) &= \int_M (id, F)^* \eta_\Delta \\ &= \int_M (id, F)^* \left(\sum (-1)^{\deg \tau_i} \pi_1^* (\omega_i) \wedge \pi_2^* (\tau_i)\right) \\ &= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F^* \tau_i \\ &= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F_{ij} \tau_j \\ &= \sum (-1)^{\deg \tau_i} F_{ij} \delta_{ij} \\ &= \sum (-1)^{\deg \tau_i} F_{ii} \\ &= \sum (-1)^p \operatorname{tr}(F^* : H^p(M) \to H^p(M)). \end{aligned}$$

□

The definition $I(\text{graph}(F), \Delta)$ for the Lefschetz number is not consistent with [**Guillemin-Pollack**]. But if we use their definition then the formula we just established would have a sign $(-1)^{\dim M}$ on it. This is a very common confusion in the general literature.

4.4. Examples of Lefschetz Numbers

It is in fact true that $\operatorname{tr}(F^* : H^p(M) \to H^p(M))$ is always an integer, but to see this requires that we know more algebraic topology. In the cases we study here this will be established directly. Two cases where we do know this to be true are when $p = 0$ or $p = \dim M$, in those cases

$$\begin{aligned} \operatorname{tr}\left(F^* : H^0(M) \to H^0(M)\right) &= \# \text{ of components of } M, \\ \operatorname{tr}(F^* : H^n(M) \to H^n(M)) &= \deg F. \end{aligned}$$

4.4.1. Spheres and Real Projective Spaces. The simplicity of the cohomology of spheres and odd dimensional projective spaces now immediately give us the Lefschetz number in terms of the degree.

When $F : S^n \to S^n$ we have $L(F) = 1 + (-1)^n \deg F$. This conforms with our knowledge that any map without fixed points must be homotopic to the antipodal map and therefore have degree $(-1)^{n+1}$.

When $F : \mathbb{RP}^{2n+1} \to \mathbb{RP}^{2n+1}$ we have $L(F) = 1 - \deg(F)$. This also conforms with our feeling for what happens with orthogonal transformations. Namely, if $F \in Gl^+_{2n+2}(\mathbb{R})$ then it doesn't have to have a fixed point as it doesn't have to have an eigenvector, while if $F \in Gl^-_{2n+2}(\mathbb{R})$ there should be at least two fixed points.

The even dimensional version $F : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ is a bit trickier as the manifold isn't orientable and thus our above approach doesn't work. However, as the only nontrivial cohomology group is when $p = 0$ we would expect the mod 2 Lefschetz number to be 1 for all F. When $F \in Gl_{2n+1}(\mathbb{R})$, this is indeed true as such maps have an odd number of real eigenvalues. For general F we can lift it to a map $\tilde{F} : S^{2n} \to S^{2n}$ satisfying the symmetry condition

$$\tilde{F}(-x) = \pm \tilde{F}(x).$$

The sign $\pm$ must be consistent on the entire sphere. If it is $+$ then we have that $\tilde{F}\circ A=\tilde{F}$, where A is the antipodal map. This shows that $\deg\tilde{F}\cdot(-1)^{2n+1}=\deg\tilde{F}$, and hence that $\deg\tilde{F}=0$. In particular, $\tilde{F}$ and also F must have a fixed point. If the sign is $-$ and we assume that $\tilde{F}$ doesn't have a fixed point, then the homotopy to the antipodal map

$$H(x,t)=\frac{(1-t)\tilde{F}(x)-tx}{\left|(1-t)\tilde{F}(x)-tx\right|}$$

must also be odd

$$\begin{aligned}H(-x,t) &= \frac{(1-t)\tilde{F}(-x)-t(-x)}{\left|(1-t)\tilde{F}(-x)-t(-x)\right|}\\ &= -\frac{(1-t)\tilde{F}(x)-t(x)}{\left|(1-t)\tilde{F}(x)-t(x)\right|}\\ &= -H(x,t).\end{aligned}$$

This implies that F is homotopic to the identity on $\mathbb{RP}^{2n}$ and thus $L(F)=L(id)=1$.

4.4.2. Tori. Next let us consider $M=T^n$. The torus is a product of n circles. If we let θ be a generator for $H^1(S^1)$ and $\theta_i=\pi_i^*(\theta)$, where $\pi_i:T^n\to S^1$ is the projection onto the i^{th} factor, then Künneth formula tells us that $H^p(T^n)$ has a basis of the form $\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}$, $i_1<\cdots<i_p$. Thus F^* is entirely determined by knowing what F^* does to θ_i. We write $F^*(\theta_i)=\alpha_{ij}\theta_j$. The action of F^* on the basis $\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}$, $i_1<\cdots<i_p$ is

$$\begin{aligned}F^*\left(\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}\right) &= F^*(\theta_{i_1})\wedge\cdots\wedge F^*\left(\theta_{i_p}\right)\\ &= \alpha_{i_1j_1}\theta_{j_1}\wedge\cdots\wedge\alpha_{i_pj_p}\theta_{j_p}\\ &= \left(\alpha_{i_1j_1}\cdots\alpha_{i_pj_p}\right)\theta_{j_1}\wedge\cdots\wedge\theta_{j_p}\end{aligned}$$

this is zero unless $j_1,...,j_p$ are distinct. Even then, these indices have to be reordered thus introducing a sign. Note also that there are $p!$ ordered $j_1,...,j_p$ that when reordered to be increasing are the same. To find the trace we are looking for the "diagonal" entries, i.e., those $j_1,...,j_p$ that when reordered become $i_1,...,i_p$. If $S(i_1,...,i_p)$ denotes the set of permutations of $i_1,...,i_p$ then we have shown that

$$\operatorname{tr}F^*|_{H^p(T^n)}=\sum_{i_1<\cdots<i_p}\sum_{\sigma\in S(i_1,\ldots,i_p)}\operatorname{sign}(\sigma)\,\alpha_{i_1\sigma(i_1)}\cdots\alpha_{i_p\sigma(i_p)}.$$

This leads us to the formula

$$L(F)=\sum_{p=0}^{n}(-1)^p\sum_{i_1<\cdots<i_p}\sum_{\sigma\in S(i_1,\ldots,i_p)}\operatorname{sign}(\sigma)\,\alpha_{i_1\sigma(i_1)}\cdots\alpha_{i_p\sigma(i_p)}$$

We claim that this can be simplified considerably by making the observation

$$\begin{aligned}\det(\delta_{ij}-\alpha_{ij}) &= \sum_{\sigma\in S(1,\ldots,n)}\operatorname{sign}(\sigma)\left(\delta_{1\sigma(1)}-\alpha_{1\sigma(1)}\right)\cdots\left(\delta_{n\sigma(n)}-\alpha_{n\sigma(n)}\right)\\ &= \sum_{\sigma\in S(1,\ldots,n)}\operatorname{sign}(\sigma)(-1)^p\,\alpha_{i_1\sigma(i_1)}\cdots\alpha_{i_p\sigma(i_p)}\delta_{i_{p+1}\sigma(i_{p+1})}\cdots\delta_{i_n\sigma(i_n)}\end{aligned}$$

where in the last sum $\{i_1, ..., i_p, i_{p+1}, ..., i_n\} = \{1, ..., n\}$. Since the terms vanish unless the permutation fixes $i_{p+1}, ..., i_n$ we have shown that

$$L(F) = \det(\delta_{ij} - \alpha_{ij}).$$

Finally we claim that the $n \times n$ matrix $[\alpha_{ij}]$ has integer entries. To see this first lift F to $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ and think of $T^n = \mathbb{R}^n / \mathbb{Z}^n$ where $\mathbb{Z}^n$ is the usual integer lattice. Let e_i be the canonical basis for $\mathbb{R}^n$ and observe that $e_i \in \mathbb{Z}^n$. The fact that $\tilde{F}$ is a lift of a map in T^n means that $\tilde{F}(x + e_i) - \tilde{F}(x) \in \mathbb{Z}^n$ for all x and $i = 1, ..., n$. Since $\tilde{F}$ is continuous we see that

$$\tilde{F}(x + e_i) - \tilde{F}(x) = \tilde{F}(e_i) - \tilde{F}(0) = Ae_i \in \mathbb{Z}^n$$

For some $A = [a_{ij}] \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$. We can then construct a linear homotopy

$$H(x, t) = (1 - t)\tilde{F}(x) + t(Ax).$$

Since

$$\begin{aligned} H(x + e_i, t) &= (1 - t)\tilde{F}(x + e_i) + tA(x + e_i) \\ &= (1 - t)\left(\tilde{F}(x) + Ae_i\right) + t(Ax + Ae_i) \\ &= (1 - t)\left(\tilde{F}(x)\right) + t(Ax) + Ae_i \\ &= H(x, t) + Ae_i \end{aligned}$$

we see that this defines a homotopy on T^n as well. Thus showing that F is homotopic to the linear map A on T^n. This means that $F^* = A^*$. Since $A^*(\theta_i) = a_{ji}\theta_j$, we have shown that $[\alpha_{ij}]$ is an integer valued matrix.

4.4.3. Complex Projective Space. The cohomology groups of $\mathbb{P}^n = \mathbb{CP}^n$ vanish in odd dimensions and are one dimensional in even dimensions. The trace formula for the Lefschetz number therefore can't be too complicated. It turns out to be even simpler and completely determined by the action of the map on $H^2(\mathbb{P}^n)$, analogously with what happened on tori. To show this we need to find $\omega \in H^2(\mathbb{P}^n)$ such that $\omega^k \in H^{2k}(\mathbb{P}^n)$ always generates the cohomology. We give a concrete description below. This description combined with the fact that $\tau(\mathbb{P}^n)$ and $\mathbb{P}^{n+1} - \{p\}$ are isomorphic bundles over $\mathbb{P}^n$ with conjugate structures, i.e., they have opposite orientations but are isomorphic over $\mathbb{R}$, shows that the Euler class $e_{\mathbb{P}^n}^{\tau(\mathbb{P}^n)} \in H^2(\mathbb{P}^n)$ also generates the cohomology of $\mathbb{P}^n$.

Using the submersion $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ that sends $\left(z^0, ..., z^n\right)$ to $\left[z^0 : \cdots : z^n\right]$ we should be able to construct ω on $\mathbb{C}^{n+1}$. To make the form as nice as possible we want it to be $U(n + 1)$ invariant. This is extremely useful as it will force $\int_{\mathbb{P}^1} \omega$ to be the same for all $\mathbb{P}^1 \subset \mathbb{P}^n$. Since ω is closed it will also be exact on $\mathbb{C}^{n+1}$. We use a bit of auxiliary notation to define the desired 2-form ω on $\mathbb{C}^{n+1} - \{0\}$ as well as some complex differentiation notation

$$\begin{aligned}
dz^i &= dx^i + \sqrt{-1}dy^i, \\
d\bar{z}^i &= dx^i - \sqrt{-1}dy^i, \\
\frac{\partial f}{\partial z^i} &= \frac{1}{2}\left(\frac{\partial f}{\partial x^i} - \sqrt{-1}\frac{\partial f}{\partial y^i}\right), \\
\frac{\partial f}{\partial \bar{z}^i} &= \frac{1}{2}\left(\frac{\partial f}{\partial x^i} + \sqrt{-1}\frac{\partial f}{\partial y^i}\right) \\
\partial f &= \frac{\partial f}{\partial z^i}dz^i, \\
\bar{\partial} f &= \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i.
\end{aligned}$$

The factor $\frac{1}{2}$ and strange signs ensure that the complex differentials work as one would think

$$\begin{aligned}
dz^j\left(\frac{\partial}{\partial z^i}\right) &= \frac{\partial z_j}{\partial z^i} = \delta_i^j = \frac{\partial \bar{z}_j}{\partial \bar{z}^i} = d\bar{z}^j\left(\frac{\partial}{\partial \bar{z}^i}\right), \\
dz^j\left(\frac{\partial}{\partial \bar{z}^i}\right) &= 0 = d\bar{z}^j\left(\frac{\partial}{\partial z^i}\right)
\end{aligned}$$

More generally we can define $\partial\omega$ and $\bar{\partial}\omega$ for complex valued forms by simply computing ∂ and $\bar{\partial}$ of the coefficient functions just as the local coordinate definition of d, specifically

$$\begin{aligned}
\partial\left(f dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}\right) &= \partial f\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}, \\
\bar{\partial}\left(f dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}\right) &= \bar{\partial} f\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}.
\end{aligned}$$

With this definition we see that

$$\begin{aligned}
d &= \partial + \bar{\partial}, \\
\partial^2 &= \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0
\end{aligned}$$

and the Cauchy-Riemann equations for holomorphic functions can be stated as

$$\bar{\partial} f = 0.$$

Working on $\mathbb{C}^{n+1} - \{0\}$ define

$$\begin{aligned}
\Phi(z) &= \log|z|^2 \\
&= \log\left(z^0\bar{z}^0 + \cdots + z^n\bar{z}^n\right)
\end{aligned}$$

and

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi.$$

As r^2 is invariant under $U(n+1)$ the form ω will also be invariant. If we multiply $z \in \mathbb{C}^{n+1} - \{0\}$ by a nonzero scalar λ then

$$\begin{aligned}
\Phi(\lambda z) &= \log\left(|\lambda z|^2\right) = \log|\lambda|^2 + \log|z|^2 \\
&= \log|\lambda|^2 + \Phi(z)
\end{aligned}$$

so when taking derivatives the constant $\log|\lambda|^2$ disappears. This shows that the form ω becomes invariant under multiplication by scalars. That said, it is not possible to define Φ on $\mathbb{P}^n$ as it is essentially forced to be constant and hence have zero derivative. But it is defined in any given coordinate system as we shall see. It is called the potential, or Kähler potential of ω. Finally the form is exact on $\mathbb{C}^{n+1} - \{0\}$ since

$$\partial\bar{\partial} = \left(\partial + \bar{\partial}\right)\bar{\partial} = d\bar{\partial}$$

To show that ω is a nontrivial element of $H^2\left(\mathbb{P}^n\right)$ it suffices to show that $\int_{\mathbb{P}^1}\omega \neq 0$. By deleting a point from $\mathbb{P}^1$ we can coordinatize it by $\mathbb{C}$. Specifically we consider

$$\mathbb{P}^1 = \left[z^0 : z^1 : 0 : \cdots : 0\right],$$

and coordinatize $\mathbb{P}^1 - \{[0 : 1 : 0 : \cdots : 0]\}$ by $z \to [1 : z : 0 : \cdots : 0]$. Then

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1 + z\bar{z}\right) \\
&= \frac{\sqrt{-1}}{2\pi}\left(\partial\left(\frac{zd\bar{z}}{1 + |z|^2}\right)\right) \\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{\partial\left(zd\bar{z}\right)}{1 + |z|^2} - \left(\partial\left(1 + |z|^2\right)\right) \wedge \frac{zd\bar{z}}{\left(1 + |z|^2\right)^2}\right) \\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{dz \wedge d\bar{z}}{1 + |z|^2} - \left(\bar{z}dz\right) \wedge \frac{zd\bar{z}}{\left(1 + |z|^2\right)^2}\right) \\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{dz \wedge d\bar{z}}{1 + |z|^2} - \frac{|z|^2\, dz \wedge d\bar{z}}{\left(1 + |z|^2\right)^2}\right) \\
&= \frac{\sqrt{-1}}{2\pi}\frac{dz \wedge d\bar{z}}{\left(1 + |z|^2\right)^2} \\
&= \frac{\sqrt{-1}}{2\pi}\frac{d\left(x + \sqrt{-1}y\right) \wedge d\left(x - \sqrt{-1}y\right)}{\left(1 + x^2 + y^2\right)^2} \\
&= \frac{\sqrt{-1}}{2\pi}\frac{2\sqrt{-1}dy \wedge dx}{\left(1 + x^2 + y^2\right)^2} \\
&= \frac{1}{\pi}\frac{dx \wedge dy}{\left(1 + x^2 + y^2\right)^2} \\
&= \frac{1}{\pi}\frac{rdr \wedge d\theta}{\left(1 + r^2\right)^2}
\end{aligned}$$

If we delete the π in the formula this is the volume form for the sphere of radius $\frac{1}{2}$ in stereographic coordinates, or the volume form for that sphere in Riemann's conformally flat model.

$$\begin{aligned}
\int_{\mathbb{P}^1} \omega &= \int_{\mathbb{P}^1 - \{[0:1:0:\cdots:0]\}} \omega \\
&= \int_{\mathbb{C}} \frac{1}{2\pi\sqrt{-1}} \frac{d\bar{z} \wedge dz}{\left(1+|z|^2\right)^2} \\
&= \int_{\mathbb{R}^2} \frac{1}{\pi} \frac{dx \wedge dy}{\left(1+x^2+y^2\right)^2} \\
&= \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} \frac{rdr \wedge d\theta}{\left(1+r^2\right)^2} \\
&= \int_0^\infty \frac{2rdr}{\left(1+r^2\right)^2} \\
&= 1.
\end{aligned}$$

The fact that this integral is 1 also tells us that ω is the dual to any $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. To see this let

$$\begin{aligned}
p &= [1:0:0:\cdots:0], \\
\mathbb{P}^{n-1} &= \left\{ \left[0:z^1:\cdots:z^n\right] : \left(z^1,...,z^n\right) \in \mathbb{C}^n - \{0\} \right\}.
\end{aligned}$$

Then we know that

$$\mathbb{P}^n - p = \left\{ \left[z:z^1:\cdots:z^n\right] : \left(z^1,...,z^n\right) \in \mathbb{C}^n - \{0\} \text{ and } z \in \mathbb{C} \right\}.$$

and there is a retract $r_0 : \mathbb{P}^n - p \to \mathbb{P}^{n-1}$, whose fibers consist of the $\mathbb{P}^1$s that pass through p and a point in $\mathbb{P}^{n-1}$. More precisely

$$(r_0)^{-1}\left(\left[0:z^1:\cdots:z^n\right]\right) - \{p\} = \left\{ \left[z:z^1:\cdots:z^n\right] : z \in \mathbb{C} \right\}.$$

Since ω is closed and integrates to 1 over these fibers it must be the dual to $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. This shows that

$$\begin{aligned}
\int_{\mathbb{P}^n} \omega^n &= \int_{\mathbb{P}^n - p} \omega^n \\
&= \int_{\mathbb{P}^n - p} \omega \wedge \omega^{n-1} \\
&= \int_{\mathbb{P}^{n-1}} \omega^{n-1}.
\end{aligned}$$

Next we note that the restriction of ω to $\mathbb{P}^{n-1}$ is simply our natural choice for ω on that space so we have proven that

$$\begin{aligned}
\int_{\mathbb{P}^n} \omega^n &= \int_{\mathbb{P}^{n-1}} \omega^{n-1} \\
&= \int_{\mathbb{P}^k} \omega^k \\
&= \int_{\mathbb{P}^1} \omega \\
&= 1.
\end{aligned}$$

This means that $\omega^k \in H^{2k}\left(\mathbb{P}^n\right)$ is a generator for the cohomology.

Now let $F:\mathbb{P}^n\to\mathbb{P}^n$ and define λ by $F^*(\omega)=\lambda\omega$. Then $F^*\left(\omega^k\right)=\lambda^k\omega^k$ and

$$L(F)=1+\lambda+\cdots+\lambda^n$$

If $\lambda=1$ this gives us $L(F)=n+1$, which was the answer we got for maps from $Gl_{n+1}(\mathbb{C})$. In particular the Euler characteristic $\chi(\mathbb{P}^n)=n+1$. When $\lambda\neq 1$, the formula simplifies to

$$L(F)=\frac{1-\lambda^{n+1}}{1-\lambda}.$$

Since λ is real we note that this can't vanish unless $\lambda=-1$ and $n+1$ is even. Thus all maps on $\mathbb{P}^{2n}$ have fixed points, just as on $\mathbb{RP}^{2n}$. On the other hand $\mathbb{P}^{2n+1}$ does admit a map without fixed points, it just can't come from a complex linear map. Instead we just select a real linear map without fixed points that still yields a map on $\mathbb{P}^{2n+1}$

$$I\left(\left[z^0:z^1:\cdots\right]\right)=\left[-\bar{z}^1:\bar{z}^0:\cdots\right].$$

If I fixes a point then

$$\begin{aligned}-\lambda\bar{z}^1&=z^0,\\ \lambda\bar{z}^0&=z^1\end{aligned}$$

which implies

$$-|\lambda|^2z^i=z^i$$

for all i. Since this is impossible the map does not have any fixed points.

Finally we should justify why λ is an integer. Let $F_1=F|_{\mathbb{P}^1}:\mathbb{P}^1\to\mathbb{P}^n$ and observe that

$$\begin{aligned}\lambda&=\int_{\mathbb{P}^1}F^*(\omega)\\&=\int_{\mathbb{P}^1}F_1^*(\omega)\end{aligned}$$

We now claim that F_1 is homotopic to a map $\mathbb{P}^1\to\mathbb{P}^1$. To see this note that $F_1\left(\mathbb{P}^1\right)\subset\mathbb{P}^n$ is compact and has measure 0 by Sard's theorem. Thus we can find $p\notin\operatorname{im}(F_1)\cup\mathbb{P}^1$. This allows us to deformation retract $\mathbb{P}^n-p$ to a $\mathbb{P}^{n-1}\supset\mathbb{P}^1$. This $\mathbb{P}^{n-1}$ might not be perpendicular to p in the usual metric, but one can always select a metric where p and $\mathbb{P}^1$ are perpendicular and then use the $\mathbb{P}^{n-1}$ that is perpendicular to p. Thus $F_1:\mathbb{P}^1\to\mathbb{P}^n$ is homotopic to a map $F_2:\mathbb{P}^1\to\mathbb{P}^{n-1}$. We can repeat this argument until we get a map $F_n:\mathbb{P}^1\to\mathbb{P}^1$ homotopic to the original F_1. This shows that

$$\begin{aligned}\lambda&=\int_{\mathbb{P}^1}F_1^*(\omega)\\&=\int_{\mathbb{P}^1}F_n^*(\omega)\\&=\deg(F_n)\int_{\mathbb{P}^1}\omega\\&=\deg(F_n).\end{aligned}$$

4.5. The Euler Class

We are interested in studying duals and in particular Euler classes in the special case where we have a vector bundle $\pi : E \to M$ and M is thought of a submanifold of E by embedding it into E via the zero section. The total space E is assumed oriented in such a way that a positive orientation for the fibers together with a positive orientation of M gives us the orientation for E. The dimensions are set up so that the fibers of $E \to M$ have dimension m.

The dual $\eta_M^E \in H_c^m(E)$ is in this case usually called the Thom class of the bundle $E \to M$. The embedding $M \subset E$ is proper so by restriction this dual defines a class $e(E) \in H^m(M)$ called the Euler class (note that we only defined duals to closed submanifolds so $H_c(M) = H(M)$.) Since all sections $s : M \to E$ are homotopy equivalent we see that $e(E) = s^*\eta_M$. This immediately proves a very interesting theorem generalizing our earlier result for trivial tubular neighborhoods.

Theorem 4.5.1. *If a bundle $\pi : E \to M$ has a nowhere vanishing section then* $e(E) = 0$.

Proof. Let $s : M \to E$ be a section and consider $C \cdot s$ for a large constant C. Then the image of $C \cdot s$ must be disjoint from the compact support of η_M and hence $s^*(\eta_M) = 0$. □

This Euler class is also natural

Proposition 4.5.2. *Let $F : N \to M$ be a map that is covered by a vector bundle map $\bar{F} : E' \to E$, i.e., $\bar{F}$ is a linear orientation preserving isomorphism on fibers. Then*

$$e(E') = F^*(e(E)).$$

An example is the pull-back vector bundle is defined by

$$F^*(E) = \{(p, v) \in N \times E : \pi(v) = F(q)\}.$$

Reversing orientation of fibers changes the sign of η_M^E and hence also of $e(E)$. Using $F = id$ and $\bar{F}(v) = -v$ yields an orientation reversing bundle map when k is odd, showing that $e(E) = 0$. Thus we usually only consider Euler classes for even dimensional bundles.

The Euler class can also be used to detect intersection numbers as we have see before. In case M and the fibers have the same dimension, we can define the intersection number $I(s, M)$ of a section $s : M \to E$ with the zero section or simply M. The formula is

$$\begin{aligned} I(s, M) &= \int_M s^*(e(E)) \\ &= \int_M e(E) \end{aligned}$$

since all sections are homotopy equivalent to the zero section.

In the special case of the tangent bundle to an oriented manifold M we already know that the intersection number of a vector field X with the zero section is the Euler characteristic. Thus

$$\chi(M) = I(X, M) = \int_M e(TM)$$

This result was first proven by Hopf and can be used to compute χ using a triangulation. This is explained in [**Guillemin-Pollack**] and [**Spivak**].

The Euler class has other natural properties when we do constructions with vector bundles.

THEOREM 4.5.3. *Given two vector bundles $E \to M$ and $E' \to M$, the Whitney sum has Euler class*

$$e(E \oplus E') = e(E) \wedge e(E').$$

PROOF. As we have a better characterization of duals we start with a more general calculation.

Let $\pi : E \to M$ and $\pi' : E' \to M'$ be bundles and consider the product bundle $\pi \times \pi' : E \times E' \to M \times M'$. With this we have the projections $\pi_1 : E \times E' \to E$ and $\pi_2 : E \times E' \to E'$. Restricting to the zero sections gives the projections $\pi_1 : M \times M' \to M$ and $\pi_2 : M \times M' \to M'$. We claim that

$$\eta_{M \times M'} = (-1)^{n \cdot m'} \pi_1^* (\eta_M) \wedge \pi_2^* (\eta_{M'}) \in H_c^{m+m'} (E \times E').$$

Note that since the projections are not proper it is not clear that $\pi_1^* (\eta_M) \wedge \pi_2^* (\eta_{M'})$ has compact support. However, the support must be compact when projected to E and E' and thus be compact in $E \times E'$. To see the equality we select volume forms $\omega \in H^n (M)$ and $\omega' \in H^{n'} (M')$ that integrate to 1. Then $\pi_1^* (\omega) \wedge \pi_2^* (\omega')$ is a volume form on $M \times M'$ that integrates to 1. Thus it suffices to compute

$$\begin{aligned}
& \int_{E \times E'} \pi_1^* (\eta_M) \wedge \pi_2^* (\eta_{M'}) \wedge (\pi \times \pi')^* (\pi_1^* (\omega) \wedge \pi_2^* (\omega')) \\
= & \int_{E \times E'} \pi_1^* (\eta_M) \wedge \pi_2^* (\eta_{M'}) \wedge \pi_1^* (\pi^* (\omega)) \wedge \pi_2^* \left((\pi')^* (\omega') \right) \\
= & (-1)^{n \cdot m'} \int_{E \times E'} \pi_1^* (\eta_M) \wedge \pi_1^* (\pi^* (\omega)) \wedge \pi_2^* (\eta_{M'}) \wedge \pi_2^* \left((\pi')^* (\omega') \right) \\
= & (-1)^{n \cdot m'} \left(\int_E \eta_M \wedge \pi^* (\omega) \right) \left(\int_{E'} \eta_{M'} \wedge (\pi')^* (\omega') \right) \\
= & (-1)^{n \cdot m'}.
\end{aligned}$$

When we consider Euler classes this gives us

$$e(E \times E') = \pi_1^* (e(E)) \wedge \pi_2^* (e(M')) \in H_c^{m+m'} (M \times M').$$

The sign is now irrelevant since $e(M') = 0$ if m' is odd.

The Whitney sum $E \oplus E' \to M$ of two bundles over the same space is gotten by taking direct sums of the vector space fibers over points in M. This means that $E \oplus E' = (id, id)^* (E \times E')$ where $(id, id) : M \to M \times M$ since

$$(id, id)^* (E \times E') = \{ (p, v, v') \in M \times E \times E' : \pi(v) = p = \pi'(v') \} = E \oplus E'.$$

Thus we get the formula

$$e(E \oplus E') = e(E) \wedge e(E').$$

□

This implies

COROLLARY 4.5.4. *If a bundle $\pi : E \to M$ admits an orientable odd dimensional sub-bundle $F \subset E$, then $e(E) = 0$.*

Proof. We have that $E = F \oplus E/F$ or if E carries an inner product structure $E = F \oplus F^{\perp}$. Now orient F and then E/F so that $F \oplus E/F$ and E have compatible orientations. Then $e(E) = e(F) \wedge e(E/F) = 0$. □

Note that if there is a nowhere vanishing section, then there is a 1 dimensional orientable subbundle. So this recaptures our earlier vanishing theorem. Conversely any orientable 1 dimensional bundle is trivial and thus yields a nowhere vanishing section.

A meaningful theory of invariants for vector bundles using forms should try to avoid odd dimensional bundles altogether. The simplest way of doing this is to consider vector bundles where the vector spaces are complex and then insist on using only complex and Hermitian constructions. This will be investigated further below.

The trivial bundles $\mathbb{R}^m \oplus M$ all have $e(\mathbb{R}^m \oplus M) = 0$. This is because these bundles are all pull-backs of the bundle $\mathbb{R}^m \oplus \{0\}$, where $\{0\}$ is the 1 point space.

To compute $e(\tau(\mathbb{P}^n))$ recall that $\tau(\mathbb{P}^n)$ is the conjugate of $\mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n$ which has dual $\eta_{\mathbb{P}^n} = \omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e(\tau(\mathbb{P}^n)) = -\omega$.

Since $\chi(\mathbb{P}^n) = n+1$ we know that $e(T\mathbb{P}^n) = (n+1)\,\omega^n$.

We go on to describe how the dual and Euler class can be calculated locally. Assume that M is covered by sets U_k such that $E|_{U_k}$ is trivial and that there is a partition of unit λ_k relative to this covering.

First we analyze what the dual restricted to the fibers might look like. For that purpose we assume that the fiber is isometric to $\mathbb{R}^m$. We select a volume form $\psi \in \Omega^{m-1}\left(S^{m-1}\right)$ that integrates to 1 and a bump function $\rho : [0,\infty) \to [-1,0]$ that is -1 on a neighborhood of 0 and has compact support. Then extend ψ to $\mathbb{R}^m - \{0\}$ and consider

$$d(\rho\psi) = d\rho \wedge \psi.$$

Since $d\rho$ vanishes near the origin this is a globally defined form with total integral

$$\begin{aligned} \int_{\mathbb{R}^m} d\rho \wedge \psi &= \int_0^\infty d\rho \int_{S^{m-1}} \psi \\ &= (\rho(\infty) - \rho(0)) \\ &= 1. \end{aligned}$$

Each fiber of E carries such a form. The bump function ρ is defined on all of E by $\rho(v) = \rho(|v|)$, but the "angular" form ψ is not globally defined. As we shall see, the Euler class is the obstruction for ψ to be defined on E. Over each U_k the bundle is trivial so we do get a closed form $\psi_k \in \Omega^{m-1}(S(E|_{U_k}))$ that restricts to the angular form on fibers. As these forms agree on the fibers the difference depends only on the footpoints:

$$\psi_k - \psi_l = \pi^* \phi_{kl},$$

where $\phi_{kl} \in \Omega^{m-1}(U_k \cap U_l)$ are closed. These forms satisfy the cocycle conditions

$$\begin{aligned} \phi_{kl} &= -\phi_{lk}, \\ \phi_{ki} + \phi_{il} &= \phi_{kl}. \end{aligned}$$

Now define

$$\varepsilon_k = \sum_i \lambda_i \phi_{ki} \in \Omega^{m-1}(U_k)$$

and note that the cocycle conditions show that

$$\begin{aligned}\varepsilon_k - \varepsilon_l &= \sum_i \lambda_i \phi_{ki} - \sum_i \lambda_i \phi_{li} \\ &= \sum_i \lambda_i (\phi_{ki} - \phi_{li}) \\ &= \sum_i \lambda_i \phi_{kl} \\ &= \phi_{kl}.\end{aligned}$$

Thus we have a globally defined form $e = d\varepsilon_k$ on M since $d(\varepsilon_k - \varepsilon_l) = d\phi_{kl} = 0$. This will turn out to be the Euler form

$$e = d\left(\sum_i \lambda_i \phi_{ki}\right) = \sum_i d\lambda_i \wedge \phi_{ki}.$$

Next we observe that

$$\pi^* \varepsilon_k - \pi^* \varepsilon_l = \psi_k - \psi_l$$

so

$$\psi = \psi_k - \pi^* \varepsilon_k$$

defines a form on E. This is our global angular form. We now claim that

$$\begin{aligned}\eta &= d(\rho\psi) \\ &= d\rho \wedge \psi + \rho d\psi \\ &= d\rho \wedge \psi - \rho \pi^* d\varepsilon_k \\ &= d\rho \wedge \psi - \rho \pi^* e\end{aligned}$$

is the dual. First we note that it is defined on all of E, is closed, and has compact support. It yields e when restricted to the zero section as $\rho(0) = -1$. Finally when restricted to a fiber we can localize the expression

$$\eta = d\rho \wedge \psi_k - d\rho \wedge \pi^* \varepsilon_k - \rho \pi^* e.$$

But both $\pi^* \varepsilon_k$ and $\pi^* e$ vanish on fibers so η, when restricted to a fiber, is simply the form we constructed above whose integral was 1. This shows that η is the dual to M in E and that e is the Euler class.

We are now going to specialize to complex line bundles with a Hermitian structure on each fiber. Since an oriented Euclidean plane has a canonical complex structure this is the same as studying oriented 2-plane bundles. The complex structure just helps in setting up the formulas.

The angular form is usually denoted $d\theta$ as it is the differential of the locally defined angle. To make sense of this we select a unit length section $s_k : U_k \to S(E|_{U_k})$. For $v \in S(E|_{U_k})$ the angle can be defined by

$$v = h_k(v) s_k = e^{\sqrt{-1}\theta_k} s_k.$$

This shows that the angular form is given by

$$\begin{aligned}d\theta_k &= -\sqrt{-1} \frac{dh_k}{h_k} \\ &= -\sqrt{-1} d \log h_k.\end{aligned}$$

Since we want the unit circles to have unit length we normalize this and define

$$\psi_k = -\frac{\sqrt{-1}}{2\pi} d\log h_k.$$

On $U_k \cap U_l$ we have that

$$h_l s_l = v = h_k s_k$$

So

$$(h_l)^{-1} h_k s_k = s_l.$$

But $(h_l)^{-1} h_k$ now only depends on the base point in $U_k \cap U_l$ and not on where v might be in the unit circle. Thus

$$\pi^* g_{kl} = g_{kl} \circ \pi = h_k (h_l)^{-1}$$

where $g_{kl} : U_k \cap U_l \to S^1$ satisfy the cocycle conditions

$$\begin{aligned} (g_{kl})^{-1} &= g_{lk} \\ g_{ki} g_{il} &= g_{kl}. \end{aligned}$$

Taking logarithmic differentials then gives us

$$\begin{aligned} -\frac{\sqrt{-1}}{2\pi}\pi^* \frac{dg_{kl}}{g_{kl}} &= -\frac{\sqrt{-1}}{2\pi}\pi^* d\log(g_{kl}) \\ &= \left(-\frac{\sqrt{-1}}{2\pi} d\log(h_k)\right) - \left(-\frac{\sqrt{-1}}{2\pi} d\log(h_l)\right) \\ &= \left(-\frac{\sqrt{-1}}{2\pi}\frac{dh_k}{h_k}\right) - \left(-\frac{\sqrt{-1}}{2\pi}\frac{dh_l}{h_l}\right). \end{aligned}$$

Thus

$$\begin{aligned} \varepsilon_k &= -\frac{\sqrt{-1}}{2\pi}\sum_i \lambda_i d\log(g_{ki}), \\ \psi &= \left(-\frac{\sqrt{-1}}{2\pi}\frac{dh_k}{h_k}\right) - \pi^*\varepsilon_k \end{aligned}$$

$$\begin{aligned} e &= d\varepsilon_k \\ &= d\left(\frac{\sqrt{-1}}{2\pi}\sum_i \lambda_i d\log(g_{ki})\right) \\ &= \frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i \wedge d\log(g_{ki}) \end{aligned}$$

This can be used to prove an important result.

Lemma 4.5.5. *Let $E \to M$ and $E' \to M$ be complex line bundles, then*

$$\begin{aligned} e(\hom(E, E')) &= -e(E) + e(E'), \\ e(E \otimes E') &= e(E) + e(E'). \end{aligned}$$

PROOF. Note that the sign ensures that the Euler class vanishes when $E = E'$.

Select a covering U_k such that E and E' have unit length sections s_k respectively t_k on U_k. If we define $L_k \in \hom(E, E')$ such that $L_k(s_k) = t_k$, then h_k is a unit length section of $\hom(E, E')$ over U_k. The transitions functions are

$$\begin{aligned} g_{kl}s_k &= s_l, \\ \bar{g}_{kl}t_k &= t_l. \end{aligned}$$

For $\hom(E, E')$ we see that

$$\begin{aligned} L_l(s_k) &= h_k(g_{lk}s_l) \\ &= g_{lk}L_l(s_l) \\ &= g_{lk}t_l \\ &= g_{lk}\bar{g}_{kl}t_k \\ &= (g_{kl})^{-1}\bar{g}_{kl}t_k \end{aligned}$$

Thus

$$L_l = (g_{kl})^{-1}\bar{g}_{kl}L_k.$$

This shows that

$$\begin{aligned} e(\hom(E, E')) &= -\frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i \wedge d\log\left((g_{ki})^{-1}\bar{g}_{ki}\right) \\ &= \frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i \wedge d\log(g_{ki}) - \frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i \wedge d\log(\bar{g}_{ki}) \\ &= -e(E) + e(E'). \end{aligned}$$

The proof is similar for tensor products using

$$\begin{aligned} s_l \otimes t_l &= (g_{kl}s_k) \otimes (\bar{g}_{kl}t_k) \\ &= g_{kl}\bar{g}_{kl}(s_k \otimes t_k). \end{aligned}$$

□

4.6. Characteristic Classes

All vector bundles will be complex and for convenience also have Hermitian structures. Dimensions etc will be complex so a little bit of adjustment is sometimes necessary when we check where classes live. Note that complex bundles are always oriented since $Gl_m(\mathbb{C}) \subset Gl^+_{2m}(\mathbb{R})$.

We are looking for a characteristic class $c(E) \in H^*(M)$ that can be written as

$$\begin{aligned} c(E) &= c_0(E) + c_1(E) + c_2(E) + \cdots, \\ c_0(E) &= 1 \in H^0(M), \\ c_1(E) &\in H^2(M), \\ c_2(E) &\in H^4(M), \\ &\vdots \\ c_m(E) &= e(E) \in H^{2m}(M), \\ c_l(E) &= 0, l > m \end{aligned}$$

For a 1 dimensional or line bundle we simply define $c(E) = 1 + c_1(E) = 1 + e(E)$. There are two more general properties that these classes should satisfy. First they should be natural in the sense that

$$c(E) = F^*(c(E'))$$

where $F : M \to M'$ is covered by a complex bundle map $E \to E'$ that is an isomorphism on fibers. Second, they should satisfy the product formula

$$\begin{aligned} c(E \oplus E') &= c(E) \wedge c(E') \\ &= \sum_{p=0}^{m+m'} \sum_{i=0}^{p} c_i(E) \wedge c_{p-i}(E') \end{aligned}$$

for Whitney sums.

There are two approaches to defining $c(E)$. In [**Milnor-Stasheff**] an inductive method is used in conjunction with the Gysin sequence for the unit sphere bundle. As this approach doesn't seem to have any advantage over the one we shall give here we will not present it. The other method is more abstract, clean and does not use the Hermitian structure. It is analogous to the construction of splitting fields in Galois theory and is due to Grothendieck.

First we need to understand the cohomology of $H^*(\mathbb{P}(E))$. Note that we have a natural fibration $\pi : \mathbb{P}(E) \to M$ and a canonical line bundle $\tau(\mathbb{P}(E))$. The Euler class of the line bugle is for simplicity denoted

$$e = e(\tau(\mathbb{P}(E))) \in H^2(\mathbb{P}(E)).$$

The fibers of $\mathbb{P}(E) \to M$ are $\mathbb{P}^{m-1}$ and we note that the natural inclusion $i : \mathbb{P}^{m-1} \to \mathbb{P}(E)$ is also natural for the tautological bundles

$$i^*(\tau(\mathbb{P}(E))) = \tau\left(\mathbb{P}^{m-1}\right)$$

thus showing that

$$i^*(e) = e\left(\tau\left(\mathbb{P}^{m-1}\right)\right).$$

As $e\left(\tau\left(\mathbb{P}^{m-1}\right)\right)$ generates the cohomology of the fiber we have shown that the Leray-Hirch formula for the cohomology of the fibration $\mathbb{P}(E) \to M$ can be applied. Thus any element $\omega \in H^*(\mathbb{P}(E))$ has an expression of the form

$$\omega = \sum_{i=1}^{m} \pi^*(\omega_i) \wedge e^{m-i}$$

where $\omega_i \in H^*(M)$ are unique. In particular,

$$\begin{aligned} 0 &= (-e)^m + \pi^*(c_1(E)) \wedge (-e)^{m-1} + \cdots + \pi^*(c_{k-1}(E)) \wedge (-e) + \pi^*(c_k(E)) \\ &= \sum_{i=0}^{m} \pi^*(c_i(E)) \wedge (-e)^{m-i} \end{aligned}$$

This means that $H^*(\mathbb{P}(E))$ is an extension of $H^*(M)$ where the polynomial

$$t^m + c_1(E)\, t^{m-1} + \cdots + c_{m-1}(E)\, t + c_m(E)$$

has $-e$ as a root. The reason for using $-e$ rather than e is that $-e$ restricts to the form ω on the fibers of $\mathbb{P}(E)$.

THEOREM 4.6.1. *Assume that we have vector bundles $E \to M$ and $E' \to M'$ both of rank m, and a smooth map $F : M \to M'$ that is covered by a bundle map that is fiberwise an isomorphism. Then*

$$c(E) = F^*(c(E')).$$

PROOF. We start by selecting a Hermitian structure on E' and then transfer it to E by the bundle map. In that way the bundle map preserves the unit sphere bundles. Better yet, we get a bundle map

$$\pi^*(E) \to (\pi')^*(E')$$

that also yields a bundle map

$$\tau(\mathbb{P}(E)) \to \tau(\mathbb{P}(E')).$$

Since the Euler classes for these bundles is natural we have

$$F^*(e') = e$$

and therefore

$$\begin{aligned} 0 &= F^*\left(\sum_{i=0}^{m} c_i(E') \wedge (-e')^{m-i}\right) \\ &= \sum_{i=0}^{m} F^* c_i(E') \wedge (-e)^{m-i} \end{aligned}$$

Since $c_i(E)$ are uniquely defined by

$$0 = \sum_{i=0}^{m} c_i(E) \wedge (-e)^{m-i}$$

we have shown that

$$c_i(E) = F^* c_i(E').$$

□

The trivial bundles $\mathbb{C}^m \oplus M$ all have $c(\mathbb{C}^m \oplus M) = 1$. This is because these bundles are all pull-backs of the bundle $\mathbb{C}^m \oplus \{0\}$, where $\{0\}$ is the 1 point space.

To compute $e(\tau(\mathbb{P}^n))$ recall that $\tau(\mathbb{P}^n)$ is the conjugate of $\mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n$ which has dual $\eta_{\mathbb{P}^n} = \omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e(\tau(\mathbb{P}^n)) = -\omega$.

The Whitney sum formula is established by proving the splitting principle.

THEOREM 4.6.2. *If a bundle $\pi : E \to M$ splits $E = L_1 \oplus \cdots \oplus L_m$ as a direct sum of line bundles, then*

$$c(E) = \prod_{i=1}^{m} (1 + e(L_i)).$$

PROOF. We pull everything back to $\mathbb{P}(E)$ but without changing notation and note that it then suffices to prove that

$$0 = \prod_{i=1}^{m} (-e + e(L_i)).$$

This is because $\prod_{i=1}^{m}(-e+e(L_i))$ is a polynomial in e of degree m whose coefficients are forced to be the characteristic classes of E. The theorem then follows if we consider how

$$\prod_{i=1}^{m}(1+e(L_i))$$

and

$$\prod_{i=1}^{m}(-e+e(L_i))$$

are multiplied out.

To see that

$$\prod_{i=1}^{m}(-e+e(L_i))=0$$

we identify $-e+e(L_i)$ with the Euler class of $\hom(\tau,L_i)$. This shows that

$$\begin{aligned}\prod_{i=1}^{m}(-e+e(L_i)) &= e\left(\bigoplus_{i=1}^{m}\hom(\tau,L_i)\right)\\ &= e(\hom(\tau,L_1\oplus\cdots\oplus L_m))\\ &= e(\hom(\tau,E))\\ &= e\left(\hom\left(\tau,\tau\oplus\tau^{\perp}\right)\right)\\ &= e(\hom(\tau,\tau))\wedge e\left(\hom\left(\tau,\tau^{\perp}\right)\right)\\ &= 0\end{aligned}$$

since $\hom(\tau,\tau)$ has the identity map as a nowhere vanishing section. □

The splitting principle can be used to compute $c(T\mathbb{P}^n)$. First note that $T\mathbb{P}^n\simeq\hom\left(\tau(\mathbb{P}^n),\tau(\mathbb{P}^n)^{\perp}\right)$. Thus

$$\begin{aligned}T\mathbb{P}^n\oplus\mathbb{C} &= \hom\left(\tau(\mathbb{P}^n),\tau(\mathbb{P}^n)^{\perp}\right)\oplus\mathbb{C}\\ &= \hom\left(\tau(\mathbb{P}^n),\tau(\mathbb{P}^n)^{\perp}\right)\oplus\hom(\tau(\mathbb{P}^n),\tau(\mathbb{P}^n))\\ &= \hom\left(\tau(\mathbb{P}^n),\tau(\mathbb{P}^n)^{\perp}\oplus\tau(\mathbb{P}^n)\right)\\ &= \hom\left(\tau(\mathbb{P}^n),\mathbb{C}^{n+1}\right)\\ &= \hom(\tau(\mathbb{P}^n),\mathbb{C})\oplus\cdots\oplus\hom(\tau(\mathbb{P}^n),\mathbb{C}).\end{aligned}$$

Thus

$$\begin{aligned}c(T\mathbb{P}^n) &= c(T\mathbb{P}^n\oplus\mathbb{C})\\ &= (1+\omega)^{n+1}.\end{aligned}$$

This shows that

$$c_i(T\mathbb{P}^n)=\binom{n+1}{i}\omega^i$$

which conforms with

$$e(T\mathbb{P}^n)=c_n(T\mathbb{P}^n)=(n+1)\,\omega^n.$$

We can now finally establish the Whitney sum formula.

THEOREM 4.6.3. *For two vector bundles $E \to M$ and $E' \to M$ we have*

$$c(E \oplus E') = c(E) \wedge c(E').$$

PROOF. First we repeatedly projectivize so as to create a map $\tilde{N} \to M$ with the property that it is an injection on cohomology and the pull-back of E to $\tilde{N}$ splits as a direct sum of line bundles. Then repeat this procedure on the pull-back of E' to $\tilde{N}$ until we finally get a map $F : N \to M$ such that F^* is an injection on cohomology and both of the bundles split

$$\begin{aligned} F^*(E) &= L_1 \oplus \cdots \oplus L_m, \\ F^*(E') &= K_1 \oplus \cdots \oplus K_{m'} \end{aligned}$$

The splitting principle together with naturality then implies that

$$\begin{aligned} F^*(c(E \oplus E')) &= c(F^*(E \oplus E')) \\ &= c(L_1) \wedge \cdots \wedge c(L_m) \wedge c(K_1) \wedge \cdots \wedge c(K_{m'}) \\ &= c(F^*(E)) \wedge c(F^*(E')) \\ &= F^*c(E) \wedge F^*c(E') \\ &= F^*(c(E) \wedge c(E')). \end{aligned}$$

Since F^* is an injection this shows that

$$c(E \oplus E') = c(E) \wedge c(E').$$

□

4.7. Generalized Cohomology

In this section we are going to explain how one can define relative cohomology and also indicate how it can be used to calculate some of the cohomology groups we have seen earlier.

We start with the simplest and most important situation where $S \subset M$ is a closed submanifold of a closed manifold.

PROPOSITION 4.7.1. *If $S \subset M$ is a closed submanifold of a closed manifold, then*

1. *The restriction map $i^* : \Omega^p(M) \to \Omega^p(S)$ is surjective.*
2. *If $\omega \in \Omega^p(S)$ is closed, then $\omega = i^*\bar{\omega}, \bar{\omega} \in \Omega^p(M)$, where $d\bar{\omega} \in \Omega_c^{p+1}(M - S)$.*
3. *If $\bar{\omega} \in \Omega^p(M)$ with $d\bar{\omega} \in \Omega_c^{p+1}(M - S)$ and $\omega = i^*(\bar{\omega}) \in \Omega^p(S)$ is exact, then $\bar{\omega} - d\bar{\theta} \in \Omega_c^p(M - S)$ for some $\bar{\theta} \in \Omega^{p-1}(M)$.*

PROOF. Select a neighborhood $S \subset U \subset M$ that deformation retracts $\pi : U \to S$. Then $i^* : H^p(U) \to H^p(S)$ is an isomorphism. We also need a function $\lambda : M \to [0, 1]$ that is compactly supported in U and is 1 on a neighborhood of S.

1. Given $\omega \in \Omega^p(S)$ let $\bar{\omega} = \lambda\pi^*(\omega)$.

2. With that choice $d\bar{\omega} = d\lambda \wedge \pi^*(\omega) + \lambda\pi^*(d\omega)$ so the second property is also verified.

3. Conversely assume that $\bar{\omega} \in \Omega^p(M)$ has $d\bar{\omega} \in \Omega_c^{p+1}(M - S)$. By possibly shrinking the support for λ to make it disjoint from the support of $d\bar{\omega}$ we can assume that $\lambda\bar{\omega}$ is closed. If we assume that $i^*(\lambda\bar{\omega}) = \omega$ is exact, then $\lambda\bar{\omega} = d\theta$ for

$\theta \in \Omega^p(U)$. We can then use $\bar{\theta} = \lambda\theta$ and note that

$$\begin{aligned} \bar{\omega} - d\bar{\theta} &= \bar{\omega} - d\lambda \wedge \theta - \lambda d\bar{\theta} \\ &= \bar{\omega} - d\lambda \wedge \theta - \lambda^2 \bar{\omega} \\ &\in \Omega_c^p(M - S). \end{aligned}$$

□

This shows that we have a short exact sequence

$$\begin{aligned} 0 &\to \Omega^p(M,S) \to \Omega^p(M) \to \Omega^p(S) \to 0, \\ \Omega^p(M,S) &= \ker\left(i^* : \Omega^p(M) \to \Omega^p(S)\right) \end{aligned}$$

as well as a natural inclusion

$$\Omega_c^p(M - S) \to \Omega^p(M,S)$$

that is an isomorphism on cohomology.

Corollary 4.7.2. *Assume $S \subset M$ is a closed submanifold of a closed manifold, then*

$$\to H_c^p(M - S) \to H^p(M) \to H^p(S) \to H_c^{p+1}(M - S) \to$$

is a long exact sequence of cohomology groups.

Good examples are $S^{n-1} \subset S^n$ with $S^n - S^{n-1}$ being two copies of $\mathbb{R}^n$ and $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ where $\mathbb{P}^n - \mathbb{P}^{n-1} \simeq \mathbb{F}^n$. This gives us a different inductive method for computing the cohomology of the spaces. Conversely, given the cohomology groups of those spaces, it computes the compactly supported cohomology of $\mathbb{R}^n$. It can also be used on manifolds with boundary:

$$\to H_c^p(\mathrm{int}M) \to H^p(M) \to H^p(\partial M) \to H_c^{p+1}(\mathrm{int}M) \to$$

where we can specialize to $M = D^n \subset \mathbb{R}^n$, the closed unit ball. The Poincaré lemma computes the cohomology of D^n so we get that

$$H_c^{p+1}(B^n) \simeq H^p\left(S^{n-1}\right).$$

For general connected compact manifolds with boundary we also get some interesting information.

Theorem 4.7.3. *If M is an oriented compact n-manifold with boundary, then*

$$H^n(M) = 0.$$

Proof. If M is oriented, then we know that ∂M is also oriented and that

$$\begin{aligned} H^n(M,\partial M) &= H_c^n(\mathrm{int}M) \simeq \mathbb{R} \\ H^n(\partial M) &= \{0\}, \\ H^{n-1}(\partial M) &\simeq \mathbb{R}^k, \end{aligned}$$

where k is the number of components of ∂M. The connecting homomorphism $H^{n-1}(\partial M) \to H_c^n(\mathrm{int}M)$ can be analyzed from the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^n(M,\partial M) & \to & \Omega^n(M) & \to & \Omega^n(\partial M) & \to & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \to & \Omega^{n-1}(M,\partial M) & \to & \Omega^{n-1}(M) & \to & \Omega^{n-1}(\partial M) & \to & 0 \end{array}$$

Evidently any $\omega \in \Omega^{n-1}(\partial M)$ is the restriction of some $\bar{\omega} \in \Omega^{n-1}(M)$, where we can further assume that $d\bar{\omega} \in \Omega_c^n(M)$. Stokes' theorem then tells us that

$$\int_M d\bar{\omega} = \int_{\partial M} \bar{\omega} = \int_{\partial M} \omega.$$

This shows that the map $H^{n-1}(\partial M) \to H_c^n(\text{int}M)$ is nontrivial and hence surjective, which in turn implies that $H^n(M) = \{0\}$. □

It is possible to extend the above long exact sequence to the case where M is noncompact by using compactly supported cohomology on M. This gives us the long exact sequence

$$\to H_c^p(M-S) \to H_c^p(M) \to H^p(S) \to H_c^{p+1}(M-S) \to$$

It is even possible to also have S be noncompact if we assume that the embedding is proper and then also use compactly supported cohomology on S.

We can generalize further to a situation where S is simply a compact subset of M. In that case we define the deRham-Cech cohomology groups $\bar{H}^p(S)$ using

$$\begin{aligned} \bar{\Omega}^p(S) &= \frac{\{\omega\}}{\omega_1 \sim \omega_2 \text{ iff } \omega_1 = \omega_2 \text{ on a ngbd of } S}, \\ \omega &\in \Omega^p(M) \end{aligned}$$

and the short exact sequence

$$0 \to \Omega_c^p(M-S) \to \Omega_c^p(M) \to \bar{\Omega}^p(S) \to 0.$$

This in turn gives us a long exact sequence

$$\to H_c^p(M-S) \to H_c^p(M) \to \bar{H}^p(S) \to H_c^{p+1}(M-S) \to$$

Finally we can define a more general relative cohomology group. We take a differentiable map $F : S \to M$ between manifolds. It could, e.g., be an embedding of $S \subset M$, but S need not be closed. Define

$$\Omega^p(F) = \Omega^p(M) \oplus \Omega^{p-1}(S)$$

and the differential

$$\begin{aligned} d &: \Omega^p(F) \to \Omega^{p+1}(F) \\ d(\omega, \psi) &= (d\omega, F^*\omega - d\psi) \end{aligned}$$

Note that $d^2 = 0$ so we get a complex and cohomology groups $H^p(F)$. These "forms" fit into a sort exact sequence

$$0 \to \Omega^{p-1}(S) \to \Omega^p(F) \to \Omega^p(M) \to 0,$$

where the maps are just the natural inclusion and projection. When we include the differential we get a large diagram where the left square is anticommutative and the right one commutative

$$\begin{array}{ccccccccc} 0 & \to & \Omega^p(S) & \to & \Omega^{p+1}(M) \oplus \Omega^p(S) & \to & \Omega^{p+1}(M) & \to & 0 \\ & & \uparrow d & & \uparrow (d, F^* - d) & & \uparrow d & & \\ 0 & \to & \Omega^{p-1}(S) & \to & \Omega^p(M) \oplus \Omega^{p-1}(S) & \to & \Omega^p(M) & \to & 0 \end{array}$$

This still leads us to a long exact sequence

$$\to H^{p-1}(S) \to H^p(F) \to H^p(M) \to H^p(S) \to$$

The connecting homomorphism $H^p(M) \to H^p(S)$ is in fact the pull-back map F^* as can be seen by a simple diagram chase.

In case $i : S \subset M$ is an embedding we also use the notation $H^p(M, S) = H^p(i)$. In this case it'd seem that the connecting homomorphism is more naturally defined to be $H^{p-1}(S) \to H^p(M, S)$, but we don't have a short exact sequence

$$0 \to \Omega^p(M) \oplus \Omega^{p-1}(S) \to \Omega^p(M) \to \Omega^p(S) \to 0$$

hence the tricky shift in the groups.

We can easily relate the new relative cohomology to the one defined above. This shows that the relative cohomology, while trickier to define, is ultimately more general and useful.

Proposition 4.7.4. *If $i : S \subset M$ is a closed submanifold of a closed manifold then the natural map*

$$\begin{aligned} \Omega_c^p(M - S) &\to \Omega^p(M) \oplus \Omega^{p-1}(S) \\ \omega &\to (\omega, 0) \end{aligned}$$

defines an isomorphism

$$H_c^p(M - S) \simeq H^p(i).$$

Proof. Simply observe that we have two long exact sequences

$$\to H^p(i) \to H^p(M) \to H^p(S) \to H^{p+1}(i) \to$$

$$\to H_c^p(M - S) \to H^p(M) \to H^p(S) \to H_c^{p+1}(M - S) \to$$

where two out of three terms are equal. □

Finally, now that we have a fairly general relative cohomology theory we can establish the well-known excision property. This property is actually a bit delicate to establish in general algebraic topology and is also requires a bit of work here.

Theorem 4.7.5. *Assume that a manifold $M = U \cup V$, where U and V are open, then the restriction map*

$$H^p(M, U) \to H^p(V, U \cap V)$$

is an isomorphism.

Proof. First select a partition of unity λ_U, λ_V relative to U, V. Then λ_U, λ_V are constant on $M - U \cap V$ and hence $d\lambda_U = 0 = d\lambda_V$ on $M - U \cap V$.

We start with injectivity. Take a class $[(\omega, \psi)] \in H^p(M, U)$, then

$$\begin{aligned} d\omega &= 0, \\ \omega|_U &= d\psi. \end{aligned}$$

If the restriction to $(V, U \cap V)$ is exact then we can find $(\bar{\omega}, \bar{\psi}) \in \Omega^{p-1}(V) \oplus \Omega^{p-2}(U \cap V)$ such that

$$\begin{aligned} \omega|_V &= d\bar{\omega}, \\ \psi|_{U\cap V} &= \bar{\omega}|_{U\cap V} - d\bar{\psi}. \end{aligned}$$

This shows that

$$\begin{aligned} \left(\psi + d\left(\lambda_V \bar{\psi}\right)\right)|_{U\cap V} &= \left(\bar{\omega} - d\left(\lambda_U \bar{\psi}\right)\right)|_{U\cap V}, \\ \psi + d\left(\lambda_V \bar{\psi}\right) &\in \Omega^{p-1}(U), \\ \bar{\omega} - d\left(\lambda_U \bar{\psi}\right) &\in \Omega^{p-1}(V). \end{aligned}$$

Thus we have a form $\tilde{\omega} \in \Omega^{p-1}(M)$ defined by $\psi + d\left(\lambda_V \bar{\psi}\right)$ on U and $\bar{\omega} - d\left(\lambda_U \bar{\psi}\right)$ on V. Clearly $d\tilde{\omega} = \omega$ and $\psi = \tilde{\omega}|_U - d\left(\lambda_V \bar{\psi}\right)$ so we have shown that (ω, ψ) is exact.

For surjectivity select $\left(\bar{\omega}, \bar{\psi}\right) \in \Omega^{p-1}(V) \oplus \Omega^{p-2}(U \cap V)$ that is closed

$$\begin{aligned} d\bar{\omega} &= 0, \\ \bar{\omega}|_{U\cap V} &= d\bar{\psi}. \end{aligned}$$

Using

$$\begin{aligned} \bar{\omega}|_{U\cap V} - d\left(\lambda_U \bar{\psi}\right) &= d\left(\lambda_V \bar{\psi}\right), \\ \bar{\omega} - d\left(\lambda_U \bar{\psi}\right) &\in \Omega^p(V), \\ d\left(\lambda_V \bar{\psi}\right) &\in \Omega^p(U) \end{aligned}$$

we can define ω as $\bar{\omega} - d\left(\lambda_U \bar{\psi}\right)$ on V and $d\left(\lambda_V \bar{\psi}\right)$ on U. Clearly ω is closed and $\omega|_U = d\left(\lambda_V \bar{\psi}\right)$. Thus we define $\psi = \lambda_V \bar{\psi}$ in order to get a closed form $(\omega, \psi) \in \Omega^p(M) \oplus \Omega^{p-1}(U)$. Restricting this form to $\Omega^p(V) \oplus \Omega^{p-1}(U \cap V)$ yields $\left(\bar{\omega} - d\left(\lambda_U \bar{\psi}\right), \lambda_V \bar{\psi}\right)$ which is not $\left(\bar{\omega}, \bar{\psi}\right)$. However, the difference

$$\begin{aligned} \left(\bar{\omega}, \bar{\psi}\right) - \left(\bar{\omega} - d\left(\lambda_U \bar{\psi}\right), \lambda_V \bar{\psi}\right) &= \left(d\left(\lambda_U \bar{\psi}\right), \lambda_U \bar{\psi}\right) \\ &= d\left(\lambda_U \bar{\psi}, 0\right) \end{aligned}$$

is exact. Thus $[(\omega, \psi)] \in H^p(M, U)$ is mapped to $\left[\left(\bar{\omega}, \bar{\psi}\right)\right] \in H^p(V, U \cap V)$. □

4.8. The Gysin Sequence

This sequence allows us to compute the cohomology of certain fibrations where the fibers are spheres. As we saw above, these fibrations are not necessarily among the ones where we can use the Hirch-Leray formula. This sequence uses the Euler class and will recapture the dual, or Thom class, from the Euler class.

We start with an oriented vector bundle $\pi : E \to M$. It is possible to put a smoothly varying inner product structure on the vector spaces of the fibration, using that such bundles are locally trivial and gluing inner products together with a partition of unity on M. The function $E \to \mathbb{R}$ that takes v to $|v|^2$ is then smooth and the only critical value is 0. As such we get a smooth manifold with boundary

$$D(E) = \{v \in E : |v| \leq 1\}$$

called the disc bundle with boundary

$$S(E) = \partial D(E) = \{v \in E : |v| = 1\}$$

being the unit sphere bundle and interior

$$\mathrm{int}D(E) = \{v \in E : |v| < 1\}.$$

Two different inner product structures will yield different disc bundles, but it is easy to see that they are all diffeomorphic to each other. We also note that $\mathrm{int}D(E)$ is diffeomorphic to E, while $D(E)$ is homotopy equivalent to E. This gives us a diagram

$$\begin{array}{ccccccccc} \to & H_c^p(\mathrm{int}D(E)) & \to & H^p(D(E)) & \to & H^p(S(E)) & \to & H_c^{p+1}(\mathrm{int}D(E)) & \to \\ & \downarrow & & \uparrow & & \updownarrow & & \uparrow & \\ \to & H_c^p(E) & \to & H^p(E) & \to & H^p(S(E)) & \dashrightarrow & H_c^{p+1}(E) & \to \end{array}$$

where the vertical arrows are simply pull-backs and all are isomorphims. The connecting homomorphism

$$H^p(S(E)) \to H_c^{p+1}(\mathrm{int}D(E))$$

then yields a map

$$H^p(S(E)) \dashrightarrow H_c^{p+1}(E)$$

that makes the bottom sequence a long exact sequence. Using the Thom isomorphism

$$H^{p-m}(M) \to H_c^p(E)$$

then gives us a new diagram

$$\begin{array}{ccccccccc} \to & H^{p-m}(M) & \xrightarrow{e\wedge} & H^p(M) & \to & H^p(S(E)) & \dashrightarrow & H^{p+1-m}(M) & \to \\ & \downarrow \eta_M \wedge \pi^*(\cdot) & & \updownarrow & & \updownarrow & & \downarrow & \\ \to & H_c^p(E) & \to & H^p(E) & \to & H^p(S(E)) & \to & H_c^{p+1}(E) & \to \end{array}$$

Most of the arrows are pull-backs and the vertical arrows are isomorphisms. The first square is commutative since $\pi^* i^*(\eta_M) = \pi^*(e)$ is represented by η_M in $H^m(E)$. This is simply because the zero section $I : M \to E$ and projection $\pi : E \to M$ are homotopy equivalences. The second square is obviously commutative. Thus we get a map

$$H^p(S(E)) \dashrightarrow H^{p+1-m}(M)$$

making the top sequence exact. This is the Gysin sequence of the sphere bundle of an oriented vector bundle. The connecting homomorphism which lowers the degree by $m-1$ can be constructed explicitly and geometrically by integrating forms on $S(E)$ along the unit spheres, but we won't need this interpretation.

The Gysin sequence also tells us how the Euler class can be used to compute the cohomology of the sphere bundle from M.

To come full circle with the Leray-Hirch Theorem we now assume that $E \to M$ is a complex bundle of complex dimension m and construct the projectivized bundle

$$\mathbb{P}(E) = \left\{(p, L) : L \subset \pi^{-1}(p) \text{ is a 1 dimensional subspace}\right\}$$

This gives us projections

$$S(E) \to \mathbb{P}(E) \to M.$$

There is also a tautological bundle

$$\tau(\mathbb{P}(E)) = \{(p, L, v) : v \in L\}.$$

The unit-sphere bundle for τ is naturally identified with $S(E)$ by

$$\begin{array}{rcl} S(E) & \to & S(\tau(\mathbb{P}(E))), \\ (p, v) & \to & (p, \mathrm{span}\{v\}, v). \end{array}$$

This means that $S(E)$ is part of two Gysin sequences. One where M is the base and one where $\mathbb{P}(E)$ is the base. These two sequences can be connected in a very interesting manner.

If we pull back E to $\mathbb{P}(E)$ and let

$$\tau^\perp = \left\{(p, L, w) : w \in L^\perp\right\}$$

be the orthogonal complement then we have that

$$\pi^*(e(E)) = e(\pi^*(E)) = e(\tau(\mathbb{P}(E))) \wedge e\left(\tau^\perp\right) \in H^*(\mathbb{P}(E)).$$

Thus we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & H^{p-2}\left(\mathbb{P}\left(E\right)\right) & \overset{e(\tau)\wedge\cdot}{\longrightarrow} & H^{p}\left(\mathbb{P}\left(E\right)\right) & & & \\
\searrow & & \nearrow & & & & \searrow & & \nearrow \\
 & H^{p-1}\left(S\left(E\right)\right) & & \uparrow e\left(\tau^{\perp}\right)\wedge\pi^{*}\left(\cdot\right) & & \uparrow\pi^{*} & & H^{p}\left(S\left(E\right)\right) & \\
\nearrow & & \searrow & & & & \nearrow & & \searrow \\
 & & & H^{p-2m}\left(M\right) & \overset{e(E)\wedge\cdot}{\longrightarrow} & H^{p}\left(M\right) & & &
\end{array}
$$

What is more we can now show in two ways that

$$\mathrm{span}\left\{1,e,...,e^{m-1}\right\}\otimes H^{*}\left(M\right)\to H^{*}\left(\mathbb{P}\left(E\right)\right)$$

is an isomorphism. First we can simply use the Leray-Hirch result by noting that the classes $1,e,...,e^{m-1}$ when restricted to the fibers are the usual cohomology classes of the fiber $\mathbb{P}^{m}$. Or we can use diagram chases on the above diagram.

4.9. Further Study

There are several texts that expand on the material covered here. The book by [**Guillemin-Pollack**] is the basic prerequisite for the material covered here. What we cover here corresponds to a simplified version of [**Bott-Tu**]. Another text is the well constructed [**Madsen-Tornehave**], which in addition explains how characteristic classes can be computed using curvature. The comprehensive text [**Spivak**, vol. V] is also worth consulting for many aspects of the theory discussed here. For a more topological approach we recommend [**Milnor-Stasheff**]. Other useful texts are listed in the references.

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