## Manifold Theory

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## CHAPTER 1

## Manifolds

### 1.1. Smooth Manifolds

A manifold is a topological space, $M$, with a maximal atlas or a maximal smooth structure.

The standard definition of an atlas is as follows:
Definition 1.1.1. An atlas $\mathscr{A}$ consists of maps $x_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}^{n_{\alpha}}$ such that
(1) $U_{\alpha}$ is an open covering of $M$.
(2) $x_{\alpha}$ is a homeomorphism onto its image.
(3) The transition functions $x_{\alpha} \circ x_{\beta}^{-1}: x_{\beta}\left(U_{\alpha} \cap U_{\beta}\right) \rightarrow x_{\alpha}\left(U_{\alpha} \cap U_{\beta}\right)$ are diffeomorphisms.

In condition (3) it suffices to show that the transition functions are smooth since $x_{\beta}$ 。 $x_{\alpha}^{-1}: x_{\alpha}\left(U_{\alpha} \cap U_{\beta}\right) \rightarrow x_{\beta}\left(U_{\alpha} \cap U_{\beta}\right)$ is an inverse. The atlas is maximal provided we cannot add a map to it so as to create a larger atlas. The maps $x_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}^{n_{\alpha}}$ are called coordinates or charts or even coordinate charts.

The second definition is a compromise between the first and a more sheaf theoretic approach. It is, however, essentially the definition of a submanifold of Euclidean space where parametrizations are given as local graphs.

DEFINITION 1.1.2. A smooth structure is a collection $\mathscr{D}$ consisting of continuous functions whose domains are open subsets of $M$ with the property that: For each $p \in M$, there is an open neighborhood $U \ni p$ and functions $x^{i} \in \mathscr{D}, i=1, \ldots, n$ such that
(1) The domains of $x^{i}$ contain $U$.
(2) The map $x=\left(x^{1}, \ldots, x^{n}\right): U \rightarrow \mathbb{R}^{n}$ is a homeomorphism onto its image $V \subset \mathbb{R}^{n}$.
(3) For each $f: O \rightarrow \mathbb{R}$ in $\mathscr{D}$ there is a smooth function $F: x(U \cap O) \rightarrow \mathbb{R}$ such that $f=F \circ x$ on $U \cap O$.

The map in (2) in both definitions is called a chart or coordinate system on $U$. The topology of $M$ is recovered by these maps. Observe that in condition (3), $F=f \circ x^{-1}$, but it is usually possible to find $F$ without having to invert $x . F$ is called the coordinate representation of $f$ and is normally also denoted by $f$. The smooth structure is maximal provided we cannot add a function to it and still have a smooth structure.

Note that it is very easy to see that these two definitions are equivalent. Both have advantages. The first in certain proofs. The latter is generally easier to work with when showing that a concrete space is a manifold and is also often easier to work with when it comes to defining foundational concepts.

DEFINITION 1.1.3. A continuous function $f: O \rightarrow \mathbb{R}$ is said to be smooth with respect to $\mathscr{D}$ if $\mathscr{D} \cup\{f\}$ is also a smooth structure. In other words we only need to check that condition (3) still holds when we add $f$ to our collection $\mathscr{D}$. We can more generally define
what it means for $f$ to be $C^{k}$ for any $k$ with smooth being $C^{\infty}$ and continuous $C^{0}$. We shall generally only use smooth or continuous functions.

The space of all smooth functions is a maximal smooth structure. We use the notation $C^{k}(M)$ for the space of $C^{k}$ functions defined on all of $M$ and $\mathfrak{C}^{k}(M)$ for the space of $f: O \rightarrow \mathbb{R}$ where $O \subset M$ is open and $f$ is $C^{k}$.

It is often the case that all the functions in a $\mathscr{D}$ have domain $M$. In fact, it is possible to always select the smooth structure such that this is the case. We shall also show that it is possible to always use a finite collection $\mathscr{D}$.

A manifold of dimension $n$ or an $n$-manifold is a manifold such that coordinate charts always use $n$ functions.

PROPOSITION 1.1.4. If $U \subset \mathbb{R}^{m}$ and $V \subset \mathbb{R}^{n}$ are open sets that are diffeomorphic, then $m=n$.

Proof. The differential of the diffeomorphism is forced to be a linear isomorphism. This shows that $m=n$.

COROLLARY 1.1.5. A connected manifold is an n-manifold for some integer $n$.
Proof. It is not possible to have coordinates around a point into Euclidean spaces of different dimensions. Let $A^{n} \subset M$ be the set of points that have coordinates using $n$ functions. This is clearly an open set. Moreover if $p_{i} \rightarrow p$ and $p_{i} \in A^{n}$ then we see that if $p$ has a chart that uses $m$ functions then $p_{i}$ will also have this property showing that $m=n$.

### 1.2. Examples

If we start with $M \subset \mathbb{R}^{k}$ as a subset of Euclidean space, then we should obviously use the induced topology and the ambient coordinate functions $\left.x^{i}\right|_{M}: M \rightarrow \mathbb{R}$ as the potential differentiable structure $\mathscr{D}$. Depending on what subset we start with this might or might not work. Even when it doesn't there might be other obvious ways that could make it work. For example, we could start with a subset which has corners, such as a triangle. While the obvious choice of a differentiable structure will not work we note that the subset is homeomorphic to a circle, which does have a valid differentiable structure. This structure will be carried over to the triangle via the homeomorphism. This is a rather subtle point and begs the very difficult question: Does every topological manifold carry a smooth structure? The answer is yes in dimensions 1,2 , and 3 , but no in dimension 4 and higher. There are also subsets where the induced topology won't make the space even locally homeomorphic to Euclidean space. A figure eight 8 is a good example. But again there is an interesting bijective continuous map $\mathbb{R} \rightarrow 8$. It "starts" at the crossing, wraps around in the figure 8 and then ends at the crossing on the opposite side. As the interval was open every point on 8 only gets covered once in this process. This map is clearly also continuous. However, it is not a homeomorphism onto its image. Thus we see again that an even more subtle game can be played where we can refine the topology of a given subset and to make it a manifold.
1.2.1. Spheres. The $n$-sphere is defined as

$$
S^{n}=\left\{x \in \mathbb{R}^{n+1}| | x \mid=1\right\}
$$

Thus we have $n+1$ natural coordinate functions. On any open hemisphere $O_{i}^{ \pm}=\left\{x \in S^{n} \mid \pm x^{i}>0\right\}$ we use the coordinate system that comes from using the $n$ functions $x^{j}$ where $j \neq i$ and the
remaining coordinate function is obtained as a smooth expression:

$$
\pm x^{i}=\sqrt{1-\sum_{j \neq i}\left(x^{j}\right)^{2}}
$$

A somewhat different atlas of charts can be constructed via stereographic projection from the points $\pm e_{i}$, where $e_{i}$ are the usual basis vectors. The map is geometrically given by drawing a line through a point $z \in e_{i}^{\perp}=\left\{z \in \mathbb{R}^{n+1} \mid z \perp e_{i}\right\}$ and $\pm e_{i}$ and then checking where it intersects the sphere. The equator where $x^{i}=0$ stays fixed, while the hemisphere closest to $\pm e_{i}$ is mapped outside this equatorial band, and the hemisphere farthest from $\pm e_{i}$ is mapped inside the band, finally the map is not defined at $\pm e_{i}$. The map from the sphere to the subspace is given by the formula:

$$
z=\frac{1}{1 \mp x^{i}}\left(x \mp e_{i}\right) \pm e_{i}
$$

and the inverse

$$
x=\frac{ \pm 2}{1+|z|^{2}}\left(z \mp e_{i}\right) \pm e_{i} .
$$

Any two of these maps suffice to create an atlas. But we must check that the transition functions are also smooth. To be specific we consider the ones coming from opposite points, say $e_{n+1}$ and $-e_{n+1}$. In this case the transition is an inversion in the equatorial band and is given by

$$
z \mapsto \frac{z}{|z|^{2}}
$$

1.2.2. Basic Geometry of Projective Spaces. Given a vector space $V$ we define $\mathbb{P}(V)$ as the space of 1-dimensional subspaces or lines through the origin. It is called the projective space of $V$. In the concrete case were $V=\mathbb{F}^{n+1}$ we use the notation $\mathbb{P}\left(\mathbb{F}^{n+1}\right)=\mathbb{F} \mathbb{P}^{n}=$ $P^{n}$.

One can similarly develop a theory of the space of subspaces of any given dimension. The space of $k$-dimensional subspaces is denoted $G_{k}(V)$ and is called the Grassmannian.

The space of operators or endomorphisms on $V$ is denoted $\operatorname{End}(V)$ and the invertible operators or automorphisms by $\operatorname{Aut}(V)$. When $V=\mathbb{F}^{n}$ these are represented by matrices $\operatorname{End}\left(\mathbb{F}^{n}\right)=\operatorname{Mat}(\mathbb{F})$ and $\operatorname{Aut}\left(\mathbb{F}^{n}\right)=G l_{n}(\mathbb{F})$. Since invertible operators map lines to lines we see that $\operatorname{Aut}(V)$ acts in a natural way on $\mathbb{P}(V)$. In fact this action is transitive, i.e., if we have $p, q \in \mathbb{P}(V)$, then there is an operator $A \in \operatorname{Aut}(V)$ such that $A(p)=q$. Moreover, as any two bases in $V$ can be mapped to each other by invertible operators it follows that any collection of $k$ independent lines $p_{1}, \ldots, p_{k}$, i.e., $p_{1}+\cdots+p_{k}=p_{1} \oplus \cdots \oplus p_{k}$ can be mapped to any other collection of $k$ independent lines $q_{1}, \ldots, q_{k}$. This means that the action of $\operatorname{Aut}(V)$ on $\mathbb{P}(V)$ is $k$-point homogeneous for all $k \leq \operatorname{dim}(V)+1$. Note that this action is not effective as all homotheties $A=\lambda 1_{V}$ act trivially on $\mathbb{P}(V)$.

Since an endomorphism might have a kernel it is not true that it maps lines to lines, however, if we have $A \in \operatorname{End}(V)$, then we do get a map $A: \mathbb{P}(V)-\mathbb{P}(\operatorname{ker} A) \rightarrow \mathbb{P}(V)$ defined on lines that are not in the kernel of $A$.

Let us now assume that $V$ is an inner product space with an inner product $\langle v, w\rangle$ that can be real or complex. The key observation in relation to subspaces is that they are completely characterized by the orthogonal projections onto the subspaces. Thus the space of $k$-dimensional subspaces is the same as the space of orthogonal projections of rank $k$. It is convenient to know that an endomorphism $E \in \operatorname{End}(V)$ is an orthogonal projection
iff it is a projection, $E^{2}=E$ that is self-adjoint, $E^{*}=E$. In the case of a one-dimensional subspace $p \in \mathbb{P}(V)$ spanned by a unit vector $v \in V$, the orthogonal projection is given by

$$
\operatorname{proj}_{p}(x)=\langle x, v\rangle v .
$$

Clearly we get the same formula for all unit vectors in $p$. Note that the formula is quadratic in $v$. This yields an inclusion $\mathbb{P}(V) \rightarrow \operatorname{End}(V)$ and endows $\mathbb{P}(V)$ with a natural topology. One can also easily see that $\mathbb{P}(V)$ is compact.

The angle between lines in $V$ gives a natural metric on $\mathbb{P}(V)$. Automorphisms clearly do not preserve angles between lines and so are not necessarily isometries. However if we restrict attention to unitary or orthogonal transformations $U \subset$ Aut $(V)$, then we know that they preserve inner products of vectors. Therefore, they must also preserve angles between lines. Thus $U$ acts by isometries on $\mathbb{P}(V)$. This action is again homogeneous so $\mathbb{P}(V)$ looks geometrically the same everywhere.

One way of finding coordinates around $p \in \mathbb{P}(V)$ is to consider the set of 1-dimensional subspaces $\mathbb{P}(V)-\mathbb{P}\left(p^{\perp}\right)$ that are not perpendicular to $p$. This is clearly an open set in $\mathbb{P}(V)$ and we claim that there is a coordinate map $G_{p}: \operatorname{Hom}\left(p, p^{\perp}\right) \rightarrow \mathbb{P}(V)-\mathbb{P}\left(p^{\perp}\right)$. To construct this map decompose $V \simeq p \oplus p^{\perp}$ and note that any 1-dimensional subspace not in $p^{\perp}$ is a graph over $p$ given by a unique homomorphism in $\operatorname{Hom}\left(p, p^{\perp}\right)$. The next thing to check is that $G_{p}$ is a homeomorphism onto its image and is differentiable as a map into End $(V)$. Neither fact is hard to verify. Finally observe that $\operatorname{Hom}\left(p, p^{\perp}\right)$ is a vector space of dimension $\operatorname{dim} V-1$. In this way $\mathbb{P}(V)$ becomes a manifold of dimension $\operatorname{dim} V-1$.
1.2.3. Projective Coordinates. We saw that the $n$-dimensional (real) projective space $\mathbb{R P}^{n}$ can be identified with the space of orthogonal projections of rank 1 . More concretely, if

$$
x=\left[\begin{array}{c}
x^{0} \\
x^{1} \\
\vdots \\
x^{n}
\end{array}\right] \in \mathbb{R}^{n+1}-\{0\}
$$

then the matrix that describes the orthogonal projection onto span $\{x\}$ is given by

$$
\begin{aligned}
E_{x} & =\frac{1}{|x|^{2}}\left[\begin{array}{cccc}
x^{0} x^{0} & x^{0} x^{1} & \cdots & x^{0} x^{n} \\
x^{1} x^{0} & x^{1} x^{1} & \cdots & x^{1} x^{n} \\
\vdots & \vdots & \ddots & \vdots \\
x^{n} x^{0} & x^{n} x^{1} & \cdots & x^{n} x^{n}
\end{array}\right] \\
& =\frac{1}{|x|^{2}} x x^{*} .
\end{aligned}
$$

Clearly $E_{x}^{*}=E_{x}$ and as $x^{*} x=|x|^{2}$ we have $E_{x}^{2}=E_{x}$ and $E_{x} x=x$. Thus $E_{x}$ is the orthogonal projection onto span $\{x\}$. Finally note that $E_{x}=E_{y}$ if and only if $x=\lambda y, \lambda \neq 0$. With that in mind we obtain a natural differentiable system by using the coordinate functions

$$
f^{i j}\left(E_{x}\right)=\frac{x^{i} x^{j}}{|x|^{2}}
$$

If we fix $j$ and consider the $n+1$ functions $f^{i j}$, then we have the relationship

$$
f^{j j}=\left(f^{j j}\right)^{2}+\sum_{i \neq j}\left(f^{i j}\right)^{2}
$$

This describes a sphere of radius $\frac{1}{2}$ centered at the point where $f^{i j}=0$ for $i \neq j$ and $f^{j j}=\frac{1}{2}$. The origin on this sphere corresponds to all points where $x^{j}=0$. But any other point on the sphere corresponds to a unique element of $O_{j}=\left\{E_{x} \mid x_{j} \neq 0\right\}$. This means that around any given point in $O_{j}$ we can use $n$ of the functions $f^{i j}$ as a coordinate chart. The remaining function is then expressed smoothly in terms of the other coordinate functions. This still leaves us with the other functions $f^{k l}$, but they satisfy

$$
f^{k l}=\frac{f^{k j} f^{l j}}{f^{j j}}
$$

and so on the given neighborhood in $O_{j}$ they are also smoothly expressed in terms of our chosen coordinate functions. The more efficient collection of functions $f^{i j}, i \leq j$ yields the Veronese map

$$
\mathbb{R P}^{n} \rightarrow \mathbb{R}^{\frac{(n+2)(n+1)}{2}}
$$

A more convenient differentiable system can be constructed using homogeneous coordinates on $\mathbb{R P}^{p}$. These are written $\left[x^{0}: x^{1}: \cdots: x^{n}\right]$ and represent the equivalence class of non-zero vectors that are multiples of $x$. The notation is suggestive of the fact that all elements in the equivalence class have the same ratios $x^{i}: x^{j}=\frac{x^{i}}{x^{j}}$ on $O_{j}$. We can now define a differentiable system by using the functions

$$
f_{j}^{i}\left(\left[x^{0}: x^{1}: \cdots: x^{n}\right]\right)=\frac{x^{i}}{x^{j}}=\frac{f^{i j}}{f^{j j}}
$$

These have domain $O_{j}$ and are smoothly expressed in terms of the coordinate functions we already considered. Conversely note that on $O_{i} \cap O_{j}$ the old coordinates are also expressed smoothly in terms of the new functions:

$$
f^{i j}=\left(\sum_{k} f_{i}^{k} f_{j}^{k}\right)^{-1}
$$

On $O_{j}$ we can use $f_{j}^{i}, i \neq j$ as a coordinate chart. The other coordinate functions $f_{l}^{k}$ can easily be expressed as smooth combinations by noting that on $O_{l} \cap O_{j}$ we have

$$
f_{l}^{k}=\frac{f_{j}^{k}}{f_{j}^{l}}
$$

Thus using the obvious coordinate functions works, but it is often desirable to use a different collection of functions for a differentiable system.

Homogeneous coordinates also work over $\mathbb{C}$. We offer a few extra formulas of these coordinates and how they tie in with the geometry of projective space.

For $z=\left(z^{0}, \ldots, z^{n}\right) \in \mathbb{F}^{n+1}-\{0\}$ denote the 1 -dimensional subspace generated by $z$ as $\left[z^{0}: \cdots: z^{n}\right]$. Thus $\left[z^{0}: \cdots: z^{n}\right]=\left[w^{0}: \cdots: w^{n}\right]$ iff and only if $z$ and $w$ are proportional. If we let $p=[1: 0: \cdots: 0]$, then the coordinate map is simply $G_{p}\left(z^{1}, \ldots, z^{n}\right)=$ $\left[1: z^{1}: \cdots: z^{n}\right]$.

Keeping in mind that $p$ is the only line perpendicular to all lines in $p^{\perp}$ we see that $\mathbb{P}^{n}-p$ can be represented by

$$
\mathbb{P}^{n}-p=\left\{\left[z: z^{1}: \cdots: z^{n}\right] \mid\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{F}^{n}-\{0\} \text { and } z \in \mathbb{F}\right\} .
$$

Here the subset

$$
\mathbb{P}\left(p^{\perp}\right)=\left\{\left[0: z^{1}: \cdots: z^{n}\right] \mid\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{F}^{n}-\{0\}\right\}
$$

can be identified with $\mathbb{P}^{n-1}$. Using the projection

$$
\begin{aligned}
R_{0} & =\left[\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 1 & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array}\right], \\
\operatorname{ker}\left(R_{0}\right) & =p
\end{aligned}
$$

we obtain a retract $R_{0}: \mathbb{P}^{n}-p \rightarrow \mathbb{P}^{n-1}$, whose preimages are diffeomorphic to $\mathbb{F}$. Using the family of transformations

$$
R_{t}=\left[\begin{array}{cccc}
t & 0 & \cdots & 0 \\
0 & 1 & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array}\right]
$$

we see that $R_{0}$ is in fact a deformation retraction.
Finally we check the projective spaces in low dimensions. When $\operatorname{dim} V=1, \mathbb{P}(V)$ is just a point and that point is in fact $V$ it self. Thus $\mathbb{P}(V)=\{V\}$.

When $\operatorname{dim} V=2$, we note that for each $p \in \mathbb{P}(V)$ the orthogonal complement $p^{\perp}$ is again a one-dimensional subspace and therefore an element of $\mathbb{P}(V)$. This gives us an involution $p \rightarrow p^{\perp}$ on $\mathbb{P}(V)$ just like the antipodal map on the sphere. In fact

$$
\begin{aligned}
\mathbb{P}(V) & =(\mathbb{P}(V)-\{p\}) \cup\left(\mathbb{P}(V)-\left\{p^{\perp}\right\}\right), \\
\mathbb{P}(V)-\{p\} & \simeq \mathbb{F} \simeq \mathbb{P}(V)-\left\{p^{\perp}\right\}, \\
\mathbb{F}-\{0\} & \simeq(\mathbb{P}(V)-\{p\}) \cap\left(\mathbb{P}(V)-\left\{p^{\perp}\right\}\right) .
\end{aligned}
$$

Thus $\mathbb{P}(V)$ is simply a one point compactification of $\mathbb{F}$. In particular, we have that $\mathbb{R} \mathbb{P}^{1} \simeq$ $S^{1}$ and $\mathbb{C P}^{1} \simeq S^{2}$, (you need to convince yourself that these maps are diffeomorphisms). Since the geometry doesn't allow for distances larger than $\frac{\pi}{2}$ it is natural to identify these projective "lines" with spheres of radius $\frac{1}{2}$ in $\mathbb{F}^{2}$.
1.2.4. Matrix Spaces. Define Mat ${ }_{n \times m}^{k}$ as the matrices with $n$ rows, $m$ columns, and rank $k$. We will focus on real matrices but everything carries over to the complex case with the modification that all dimensions will be complex dimensions.

The special case where $k=n=m$ is denoted $G l_{n}$ and is known as the general linear group. It evidently consists of the nonsingular $n \times n$ matrices and is an open subset of all the $n \times n$ matrices. As such it is obviously a manifold of dimension $n^{2}$.

In the general case $\mathrm{Mat}_{n \times m}^{k}$ is still a subspace of a Euclidean space so it is natural to suspect that the entries will suffice as a differentiable system. The trick is to discover how many of them are needed to create a coordinate system. To that end, assume that we look at the matrices of rank $k$ where the first $k$ rows and the first $k$ columns are linearly independent. If such a matrix is written in block form

$$
\left[\begin{array}{ll}
A & C \\
B & D
\end{array}\right]
$$

then we know that $B=Y A, Y \in \operatorname{Mat}_{(n-k) \times k}, C=A X, X \in \operatorname{Mat}_{k \times(m-k)}$, and $D=Y A X$. Thus those matrices are uniquely represented by the invertible matrix $A$ and the two general matrices $X, Y$. Next observe that $Y=B A^{-1}, X=A^{-1} C$. Thus we can use the $n m-$
$(n-k)(m-k)$ entries that correspond to $A, B, C$ as a coordinate chart on this set. The remaining entries corresponding to $D$ are then smooth functions of these coordinates as $D=B A^{-1} C$.

More generally we define the sets $O_{i_{1}, \ldots, i_{k}, j_{1}, \ldots, j_{k}} \subset \operatorname{Mat}_{n \times m}^{k}$ as the rank $k$ matrices where the rows indexed by $i_{1}, \ldots, i_{k}$ and columns by $j_{1}, \ldots, j_{k}$ are linearly independent. On these sets all entries that lie in the corresponding rows and columns are used as coordinates and the remaining entries are smoothly expressed in terms of these using the above expression with the necessary index modifications.

When $m=n$ we can add other conditions such as having constant determinant, being skew- or self-adjoint, orthogonal, unitary and much more.

A particularly intricate situation is the Grassmannian of $k$-planes in $\mathbb{R}^{n}$ (or $\mathbb{C}^{n}$ ). These are, as indicated, the $k$-dimensional subspaces of an $n$-dimensional vector space. When $k=1$ they are simply the projective spaces. As such, they are represented as the subset of orthogonal projections:

$$
\operatorname{Gr}_{k}=\operatorname{Gr}_{k}\left(\mathbb{F}^{n}\right)=\left\{E \in \operatorname{Mat}_{n \times n}^{k} \mid E^{2}=E \text { and } E^{*}=E\right\} .
$$

If $X \in \operatorname{Mat}_{n \times k}^{k}$, then

$$
E_{X}=X\left(X^{*} X\right)^{-1} X^{*} \in \mathrm{Gr}_{k} .
$$

Moreover, $E_{X}=E_{Y}$ if and only if $X=Y A$ where $A \in G l_{k}$. Instead of analyzing the entries of $E_{X}$ as our differentiable system, we will imitate the construction of homogeneous coordinates to create an efficient way of parametrizing suitable open sets in $\mathrm{Gr}_{k}$. Let $O_{i_{1}, \ldots, i_{k}} \subset \mathrm{Gr}_{k}$ be the open set with the property that the rows of $E$ corresponding to the indices $i_{1}, \ldots, i_{k}$ are linearly independent. As $E$ is self-adjoint the corresponding columns are also linearly independent. If $E=E_{X}$, then $O_{i_{1}, \ldots, i_{k}}$ corresponds to the $X \in \operatorname{Mat}{ }_{n \times k}^{k}$ where the rows indexed by $i_{1}, \ldots, i_{k}$ are linearly independent. We can then consider the matrix $A_{X} \in G l_{k}$ which consists of those rows from $X$. Then the remaining rows in $X A_{X}^{-1}$ parametrize $E_{X}=E_{X A_{X}^{-1}}$. To see this more explicitly assume that the first $k$ rows are linearly independent. Then we can use

$$
X=\left[\begin{array}{c}
I_{k} \\
Z
\end{array}\right], Z \in \operatorname{Mat}_{(n-k) \times k}
$$

and

$$
E_{X}=\left[\begin{array}{cc}
A & C \\
B & D
\end{array}\right]=\left[\begin{array}{cc}
I_{k}+Z^{*} Z & \left(I_{k}+Z^{*} Z\right) Z^{*} \\
Z\left(I_{k}+Z^{*} Z\right) & Z\left(I_{k}+Z^{*} Z\right) Z^{*}
\end{array}\right]
$$

Note that $Z=B A^{-1}$ depends smoothly on the entries in $E$ regardless of how $E \in O_{1, \ldots, k}$ is expressed as a matrix. In this way we have created smooth bijections

$$
\operatorname{Mat}_{(n-k) \times k} \rightarrow O_{i_{1}, \ldots, i_{k}} \subset \operatorname{Gr}_{k} \subset \operatorname{Mat}_{n \times n}^{k}
$$

The inverse maps will now yield the differentiable system or equivalently atlas for $\operatorname{Gr}_{k}$. The formula $Z=B A^{-1}$ makes it clear that these coordinates are smooth on an overlap $O_{i_{1}, \ldots, i_{k}} \cap O_{j_{1}, \ldots, j_{k}}$.
1.2.5. Tangent Spaces to Spheres. The last example for now is somewhat different in nature and can easily be generalized to manifolds that come from subsets of Euclidean space where standard coordinate functions give a differentiable system.

We consider the set of vectors tangent to a sphere. By tangent to the sphere we mean that they are velocity vectors for curves in the sphere. If $c: I \rightarrow S^{n}$, then $|c|^{2}=1$ and consequently $\dot{c} \cdot c=(\dot{c} \mid c)=1$. Thus the velocity is always perpendicular to the base vector. This means that we are considering the set

$$
T S^{n} \simeq\left\{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}| | x \mid=1 \text { and }(x \mid v)=0\right\}
$$

Conversely we see that for $(x, v) \in T S^{n}$ the curve

$$
c(t)=x \cos t+v \sin t
$$

is a curve on the sphere that has velocity $v$ at the base point $x$. Now suppose that we are considering the points $x \in O_{j}^{ \pm}$with $\pm x^{j}>0$. We know that on this set we can use $x^{i}$, $i \neq j$ as coordinates. It seems plausible that we could similarly use $v^{i}, i \neq j$ for the vector component. We already know that we can write $x^{j}$ as a smooth function of $x^{i}, i \neq j$. So we now have to write $v^{j}$ as a smooth function of $v^{i}$ and $x^{i}$. The equation $(x \mid v)=0$ tells us that

$$
v^{j}=-\frac{\sum_{i \neq j} x^{i} v^{i}}{x^{j}}
$$

so this is certainly possible.
This also helps us in the general case where we might be considering tangent vectors to a general $M$. For simplicity assume that $x^{n+1}=F\left(x^{1}, \ldots, x^{n}\right)$. If $c$ is a curve, then we also have $c^{n+1}(t)=F\left(c^{1}(t), \ldots, c^{n}(t)\right)$. Thus

$$
\dot{c}^{n+1}(t)=\frac{\partial F}{\partial x^{i}} \dot{c}^{i}(t)
$$

This means that for the tangent vectors

$$
v^{n+1}=\frac{\partial F}{\partial x^{i}} v^{i}
$$

Thus we have again written $v^{n+1}$ as a smooth function of our chosen coordinates given that $x^{n+1}$ is already written as a smooth function of $x^{1}, \ldots, x^{n}$.

This argument is general enough that we can use it to create a differentiable structure for similarly defined tangent spaces $T M$ for $M^{m} \subset \mathbb{R}^{n}$ where we used the $n$-coordinate functions from $\mathbb{R}^{n}$ to generate the differentiable structure on $M$. The only difference is that we now need $n-m$ functions to describe $n-m$ of the coordinates on any given set where we've used a specific set of $m$ coordinates as a chart. For instance

$$
x^{j}=F^{j}\left(x^{1}, \ldots, x^{m}\right), j>m
$$

yields

$$
v^{j}=\sum_{i=1}^{m} \frac{\partial F^{j}}{\partial x^{i}} v^{i}, j>m
$$

### 1.3. Topological Properties of Manifolds

The goal is to show that we can construct partitions of unity on smooth manifolds. This means that we have to start by showing that the space is paracompact. The simplest topological assumptions for this to work is that the space is second countable (there is a countable basis for the topology) and Hausdorff (points can be separated by disjoint open sets). For a manifold, as defined above, this means that the topology will henceforth be assumed to be second countable and Hausdorff. The Hausdorff property is essential for many obvious properties, but it will also seem as if it is rarely used explicitly. Two essential properties come from the Hausdorff axiom. First, that limits of sequences are uniquely defined. Second, that compact subsets are closed sets and thus have complements that are open.

Checking that the topology is second countable generally follows by checking that the space can be covered by countably many coordinate charts. Clearly open subsets of $\mathbb{R}^{n}$ are
second countable. So this means that the space is a countable union of open sets that are all second countable and thus itself second countable.

Checking that it is Hausdorff is generally also easy. Either two points will lie the same chart in which case they can easily be separated. Otherwise they'll never lie in the same chart and one must then check that there are small charts around the points whose domains don't intersect.
1.3.1. Bump Functions. The goal is to prove a smooth version of Urysohn's lemma and construct partitions of unity.

THEOREM 1.3.1. A smooth manifold has a compact exhaustion, i.e., is $\sigma$-compact, and is in addition paracompact.

Proof. A compact exhaustion is an increasing countable collection of compact sets $K_{1} \subset K_{2} \subset \cdots$ such that $M=\cup K_{i}$ and $K_{i} \subset \operatorname{int} K_{i+1}$ for all $i$. The crucial ingredients for finding such an exhaustion is second countability and local compactness.

First we show that open sets $O$ in $\mathbb{R}^{n}$ have this property. Around each $p \in O$ select an open neighborhood $U_{p}$ such that the closure is compact and $\bar{U}_{p} \subset O$. Since $O$ is second countable (or just Lindelöf) we can select a countable collection $U_{p_{i}}$ that covers $O$. Define $K_{1}=\bar{U}_{p_{1}}$ and given $K_{i}$ let $K_{i+1}=\bar{U}_{p_{1}} \cup \cdots \cup \bar{U}_{p_{k}}$ where $p_{1}, \ldots, p_{k}$ are chosen so that $k \geq i$ and $K_{i} \subset U_{p_{1}} \cup \cdots \cup U_{p_{k}}$.

By definition $M$ is a countable union of open sets that have exhaustions, i.e., there are compact sets $K_{i, j}$ where for fixed $j, K_{i, j}, i=1,2,3 \ldots$ form an exhaustion of $O_{j}$, and $O_{j}$ is an open covering of $M$. The desired exhaustion is then given by $K_{i}=\cup_{j \leq i} K_{i, j}$.

To show that the space is paracompact consider the compact "annuli" $C_{i}=K_{i}-\operatorname{int} K_{i-1}$ and note that $C_{i} \cap C_{j}=\emptyset$ when $|i-j|>1$. Extend this to a covering of open sets $U_{i}=$ $\operatorname{int} K_{i+1}-K_{i-1} \supset C_{i}$ and note that $U_{i} \cap U_{j}=\emptyset$ when $|i-j|>4$. In other words these are locally finite covers. Given an open cover $B_{\alpha}$ we can consider the refinement $B_{\alpha} \cap U_{i}$. For fixed $i$ we can then extract a finite collection of $B_{\alpha} \cap U_{i}$ that cover the compact set $V_{i}$. This leads to a locally finite refinement of the original cover $B_{\alpha}$.

Another fundamental lemma we need is a smooth version of Urysohn's lemma.
LEMMA 1.3.2. (Smooth Urysohn Lemma) If $M$ is a smooth manifold and $C_{0}, C_{1} \subset M$ are disjoint closed sets, then there exists a smooth function $f: M \rightarrow[0,1]$ such that $C_{0}=$ $f^{-1}(0)$ and $C_{1}=f^{-1}(1)$.

Proof. First we claim that for each open set $O \subset M$ there is a smooth function $f$ : $M \rightarrow[0, \infty)$ such that $M-O=f^{-1}(0)$.

We start by proving this in Euclidean space. First note that for any open cube

$$
O=\left(a_{1}, b_{1}\right) \times \cdots \times\left(a_{n}, b_{n}\right)
$$

there is a bump function $\mathbb{R}^{n} \rightarrow[0, \infty)$ that is positive on $O$ and vanishes on the complement. Simply select such bump functions for each interval $\left(a_{i}, b_{i}\right)$ and multiply them. Then write a general open set $O$ as a union of open cubes such that for all $p \in O$ there is a neighborhood $U$ that intersects only finitely many open cubes. Using bump functions on each of the cubes we can then add them up to get a function that is positive only on $O$.

Next note that if $U \subset M$ is open and the closure is contained in a chart $\bar{U} \subset V$, where $x: V \rightarrow O \subset \mathbb{R}^{n}$, then this construction gives us a function that is positive on $U$ and vanishes on $V-\bar{U}$. If we extend this function to vanish on $M-V$ we obtain a smooth function.

More generally, we can find a locally finite cover of $M$ consisting of open $U_{\alpha}$, where $\bar{U}_{\alpha} \subset V_{\alpha}$ and $V_{\alpha}$ is the domain for a chart (convince yourself that this is indeed possible).

For a fixed open set $O \subset M$ consider the nonempty intersections $U_{\alpha} \cap O$ and construct a function as just explained on each of them. Then add all of these functions to obtain a smooth function on $M$ that is positive on $O$ and vanishes on $M-O$.

Finally, the Urysohn function is constructed by selecting $f_{i}: M \rightarrow[0, \infty)$ such that $f_{i}^{-1}(0)=C_{i}$ and defining

$$
f(x)=\frac{f_{0}(x)}{f_{0}(x)+f_{1}(x)}
$$

This function is well-defined as $C_{0} \cap C_{1}=\emptyset$ and is the desired Urysohn function.
We can now easily construct the partitions of unity we need.
LEMMA 1.3.3. Let $M$ be a smooth manifold. Any countable locally finite covering $U_{\alpha}$ of open sets has partition of unity subordinate to this covering, i.e., there are smooth functions $\phi_{\alpha}: M \rightarrow[0,1]$ such that $\phi_{\alpha}^{-1}(0)=M-U_{\alpha}$ and $1=\sum_{\alpha} \phi_{\alpha}$.

Proof. The previous result gives us functions $\lambda_{\alpha}: M \rightarrow[0,1]$ such that $\lambda_{\alpha}^{-1}(0)=$ $M-U_{\alpha}$. As the cover is locally finite the sum $\sum_{\alpha} \lambda_{\alpha}$ is well-defined. Moreover, it is always positive as $U_{\alpha}$ cover $M$. We can then define

$$
\phi_{\alpha}=\frac{\lambda_{\alpha}}{\sum_{\alpha} \lambda_{\alpha}}
$$

Proposition 1.3.4. If $U \subset M$ is an open set in a smooth manifold and $f: U \rightarrow \mathbb{R}^{n}$ is smooth, then $\lambda f$ defines a smooth function on $M$ provided $\lambda: M \rightarrow \mathbb{R}$ is smooth and vanishes on $M-U$.

Proof. Clearly $\lambda f$ is smooth away from the boundary of $U$. On the boundary, $\lambda$ and all it derivatives vanish so the product rule shows that $\lambda f$ is also smooth there.

Finally, we can use this to show
Proposition 1.3.5. A smooth manifold admits a proper smooth function.
Proof. Select a compact exhaustion $K_{1} \subset K_{2} \subset \cdots$, where each $K_{i}$ is compact, $K_{i} \subset$ $\operatorname{int} K_{i+1}$, and $M=\bigcup K_{i}$. Choose Urysohn functions $\phi_{i}: M \rightarrow[0,1]$ such that $\phi_{i}\left(K_{i-1}\right)=0$ and $\phi_{i}\left(M-\operatorname{int} K_{i}\right)=1$. Then use $\rho=\sum \phi_{i}$.
1.3.2. Metrizability. We next mention several interesting results that help us understand the topological properties that are crucial for manifolds. It should also be mentioned that if we use the topology on $\mathbb{R}$ generated by the half open intervals $[a, b)$ then we obtain a paracompact space that is separable but not second countable and not locally compact (51 in [Steen \& Seebach]).

The Urysohn metrization theorem asserts that a second countable normal Hausdorff space is metrizable. In particular, manifolds are always metrizable. The proof of this result is remarkably simple.

THEOREM 1.3.6. A second countable normal Hausdorff space is metrizable. Moreover, if the space admits a compact exhaustion, then it is metrizable with a complete metric.

Proof. We shall only use that the space is completely regular. In fact Tychonoff's Lemma shows that a regular Lindelöf space is normal. So it suffices to assume that the space is second countable and regular. There are second countable Hausdorff spaces that are not regular (79 in [Steen \& Seebach]). Note that such spaces can't be locally compact.

The key is to use that the Hilbert cube: $\times_{i=1}^{\infty} I_{i}$ where $I_{i}=[0,1]$ is a metric space with distance

$$
d\left(\left(x_{i}\right),\left(y_{i}\right)\right)=\sum_{i} 2^{-i}\left|x_{i}-y_{i}\right|
$$

The goal is then to show that our space is homeomorphic to a subset in the Hilbert cube.
Choose a countable collection of closed sets $\mathscr{C}$ such that their complements generate the topology of $M$. Enumerate the all pairs $\left(C_{i}, F_{i}\right) \in \mathscr{C} \times \mathscr{C}$ with $C_{i} \subset \operatorname{int} F_{i}$, and for each such pair select a function $\phi_{i}: M \rightarrow[0,1]$ such that $\phi_{i}\left(C_{i}\right)=0$ and $\phi_{i}\left(M-\operatorname{int} F_{i}\right)=1$. This results in a map $\Phi: M \rightarrow \times_{i=1}^{\infty} I_{i}$ by defining $\Phi(x)=\times_{i=1}^{\infty} \phi_{i}(x)$.

This map is injective since distinct points can be separated by open sets whose complements are in $\mathscr{C}$. Next we show that for each $C \in \mathscr{C}$ the image $\Phi(C)$ is closed. Consider a sequence $c_{n} \in C$ such that $\Phi\left(c_{n}\right) \rightarrow \Phi(x)$. Note that for any fixed index $i$ we have $\phi_{i}\left(c_{n}\right) \rightarrow \phi_{i}(x)$. If $x \notin C$, then we can find a pair $\left(C_{i}, F_{i}\right)$ where $x \in M-\operatorname{int} F_{i}$. Therefore, $\phi_{i}\left(c_{n}\right)=0$ and $\phi_{i}(x)=1$, which is impossible. Thus $x \in C$ and $\Phi(x) \in \Phi(C)$. This shows that the map is a homeomorphism onto its image.

An explicit metric on $M$ can given by

$$
d(x, y)=\sum_{i} 2^{-i}\left|\phi_{i}(x)-\phi_{i}(y)\right|
$$

In case the space also has a compact exhaustion we can find a proper function $\rho: M \rightarrow$ $[0, \infty)$ and use map: $(\rho, \Phi): M \rightarrow[0, \infty) \times{ }_{i=1}^{\infty} I_{i}$ which is also proper. In this way the metric has the property that bounded closed sets are compact. In particular, Cauchy sequences have accumulations points and are consequently convergent.

THEOREM 1.3.7. A connected locally compact metric space has a compact exhaustion.

Proof. Assume $(M, d)$ is the metric space. For each $x \in M$ let

$$
r(x)=\sup \{r \mid \overline{B(x, r)} \text { is compact }\} .
$$

If $r(x)=\infty$ for some $x$ we are finished. Otherwise $r(x)$ is a continuous function, in fact

$$
|r(x)-r(y)| \leq d(x, y)
$$

since

$$
r(y) \leq d(x, y)+r(x)
$$

and

$$
r(x) \leq d(x, y)+r(y)
$$

We now claim that for a fixed compact set $C$ the set $C^{\#}=\left\{x \in M \mid \exists z \in C: d(x, z) \leq \frac{1}{2} r(z)\right\}$ is also compact and contains $C$ in its interior. The latter statement is obvious since $B\left(x, \frac{1}{2} r(x)\right) \subset$ $C^{\#}$ for all $x \in C$. Next select a sequence $x_{i} \in C^{\#}$ and select $z_{i} \in C$ such that $d\left(x_{i}, z_{i}\right) \leq \frac{1}{2} r\left(z_{i}\right)$. Since $C$ is compact we can after passing to a subsequence assume that $z_{i} \rightarrow z \in C$ and that $d\left(z, z_{i}\right)<\frac{1}{4} r(z)$ for all $i$. Then $d\left(z, x_{i}\right) \leq d\left(z, z_{i}\right)+d\left(z_{i}, x_{i}\right)<\frac{1}{4} r(z)+\frac{1}{2} r\left(z_{i}\right)$. Continuity of $r\left(z_{i}\right)$ then shows that $x_{i} \in B\left(z, \frac{3}{4} r(z)\right)$ for large $i$. As $\overline{B\left(z, \frac{3}{4} r(z)\right)}$ is compact we can then extract a convergent subsequence of $x_{i}$.

Finally consider the compact sets $K_{i+1}=K_{i}^{\#}$ where $K_{1}$ is any non-empty compact set. We claim that $\cup_{i} K_{i}$ is both open and closed. The set is open since $B\left(x, \frac{1}{2} r(x)\right) \subset K_{i}^{\#}=K_{i+1}$ for any $x \in K_{i}$. To see that the set is closed select a convergent sequence $x_{n} \in \cup_{i} K_{i}$ and let $x$ be the limit point. We have $r\left(x_{n}\right) \rightarrow r(x)$ and $d\left(x_{i}, x\right) \rightarrow 0$. So it follows that for large $n$ we have $x \in B\left(x_{n}, \frac{1}{2} r\left(x_{n}\right)\right)$ showing that $x \in K_{i}^{\#}$ if $x_{n} \in K_{i}$. So the fact that $M$ is connected shows that it has a compact exhaustion.

COROLLARY 1.3.8. A second countable locally compact metric space has a compact exhaustion and is paracompact.

Proof. There are at most countably many connected components and each of these has a compact exhaustion. We can then proceed as above.

THEOREM 1.3.9 (Baire Category Theorem). A Hausdorff space that is locally compact satisfies: A countable union of closed sets without interiors has no interior.

Proof. Let $C_{i} \subset M$ be a countable collection of closed sets with no interior points. Select an open set $V_{0} \subset X$. Then $V_{0}-C_{1}$ is a nonempty open set as $C_{1}$ has no interior points. As $M$ is locally compact we can find an open set $V_{1}$ such that $\bar{V}_{1} \subset V_{0}-C_{1}$ is compact. Similarly we can find open sets $V_{i}$ such that $\bar{V}_{i} \subset V_{i-1}-C_{i} \subset V_{i-1}$ is compact. By compactness $\bigcap_{i=1}^{\infty} \bar{V}_{i}$ is nonempty and we also have $\bigcap_{i=1}^{\infty} \bar{V}_{i} \subset V_{0}-\bigcup_{i=1}^{\infty} C_{i}$. In particular, $V_{0}-\bigcup_{i=1}^{\infty} C_{i}$ is nonempty for any open set $V_{0}$. This shows that $\bigcup_{i=1}^{\infty} C_{i}$ has no interior points.

Example 1.3.10. The set of rationals $\mathbb{Q} \subset \mathbb{R}$ is a metric space that does not admit a complete metric nor is it locally compact.
1.3.3. The Meta Theorem. The above topological properties of manifolds lead to a very general principle that offers a very abstract general condition for when a statement holds for manifolds.

Consider a class $\mathscr{M}^{n}$ manifolds with the following properties:
(1) Every $M \in \mathscr{M}$ is $\sigma$-compact and has dimension $n$.
(2) $\mathbb{R}^{n} \in \mathscr{M}^{n}$.
(3) If $M \in \mathscr{M}^{n}$ and $U \subset M$ is open, then $U \in \mathscr{M}^{n}$.
(4) If $M \in \mathscr{M}^{n}$ and $M$ is diffeomorphic to $N$, then $N \in \mathscr{M}^{n}$.

This can for example be the class of all $n$-manifolds or all oriented $n$-manifolds or simply all open subsets of a manifold. The key property to be extracted from $\sigma$-compactness is that each manifold has a proper function $\rho: M \rightarrow[0, \infty)$.

The goal is to consider the validity of a statement $P(M)$ for all $M \in \mathscr{M}^{n}$. We will assume that the statement only depends on the diffeomorphism type of the manifold.

THEOREM 1.3.11. The statement $P(M)$ is true for all manifolds in $\mathscr{M}^{n}$ provided the following conditions hold:
(1) $P\left(\mathbb{R}^{n}\right)$ is true.
(2) If $A, B \subset M \in \mathscr{M}^{n}$ are open and $P(A), P(B), P(A \cap B)$ are true, then $P(A \cup B)$ is true.
(3) If $A_{i} \subset M \in \mathscr{M}^{n}$ form a countable collection of pairwise disjoint open sets such that $P\left(A_{i}\right)$ are true, then $P\left(\cup A_{i}\right)$ is true.

Proof. We start by showing that $P(U)$ is true for all open sets $U \subset \mathbb{R}^{n}$. Observe first that any open box $\left(a_{1}, b_{1}\right) \times \cdots \times\left(a_{n}, b_{n}\right)$ is diffeomorphic to $\mathbb{R}^{n}$ and that the intersection of two boxes is either empty or a box. Consider next an open subset of $\mathbb{R}^{n}$ that is a finite union of open boxes. The claim follows for such sets by induction on the number of boxes. To see this, assume it holds for any union of $k$ or fewer open boxes and consider $k+1$ open boxes $B_{i}$. Then the statement holds for $B_{1} \cup \cdots \cup B_{k}, B_{k+1}$, and the intersection as it is a union of $k$ or fewer boxes:

$$
\left(B_{1} \cup \cdots \cup B_{k}\right) \cap B_{k+1}=\left(B_{1} \cap B_{k+1}\right) \cup \cdots \cup\left(B_{k} \cap B_{k+1}\right) .
$$

This in turn shows that we can prove the theorem for all open sets in $\mathbb{R}^{n}$. Fix an open set $U \subset \mathbb{R}^{n}$ and a proper function $\rho: U \rightarrow[0, \infty)$. Now cover each compact set $\rho^{-1}[i, i+1] \subset$ $U_{i}$ by an open set $U_{i}$ that is a finite union of open boxes, where $U_{i} \cap U_{j}=\emptyset$ when $|i-j| \geq 2$. Thus the theorem holds for $\cup U_{2 i}, \cup U_{2 i+1}$. It also holds for the intersection $\left(\cup U_{2 i}\right) \cap$ $\left(\bigcup U_{2 i+1}\right)=\bigcup\left(U_{j} \cap U_{j+1}\right)$ as $U_{i} \cap U_{i+1} \cap U_{j} \cap U_{j+1}=\emptyset$ when $i \neq j$. Consequently, the statement holds for the entire union.

Having come this far we use the exact same strategy to prove the statement for an $M \in$ $\mathscr{M}^{n}$ by considering the class of all open subsets $U \subset M$ and replacing the first statement with:
(1) $P(U)$ is true for all open $U \subset M$ that are diffeomorphic to an open subset of $\mathbb{R}^{n}$, i.e., all charts $U \subset M$.

Using induction this shows that the statement is true for any open subset of $M$ that is a finite union of charts. Next write $M=\bigcup U_{i}$ where each $U_{i}$ is a finite union of charts and $U_{i} \cap U_{j}=\emptyset$ when $|i-j| \geq 2$. This means the theorem holds for $\cup U_{2 i}, \cup U_{2 i+1}$, and $\left(\cup U_{2 i}\right) \cap\left(\cup U_{2 i+1}\right)$ and consequently for the entire union.

Based on the proof of this theorem we also obtain the following less abstract version. We consider a statement $P$ about all open subsets of a fixed manifold. Thus we don't necessarily assume that the statement is invariant under diffeomorphisms.

Corollary 1.3.12. The statement $P(M)$ is true for a manifold $M$ provided
(1) $M$ has a cover of open sets $O_{\alpha}$ that are diffeomorphic to $\mathbb{R}^{n}$ such that for all $\alpha$ the statement $P(B)$ is true for any box $B \subset O_{\alpha}$.
(2) If $A, B \subset M \in \mathscr{M}^{n}$ are open and $P(A), P(B), P(A \cap B)$ are true, then $P(A \cup B)$ is true.
(3) If $A_{i} \subset M \in \mathscr{M}^{n}$ form a countable collection of pairwise disjoint open sets such that $P\left(A_{i}\right)$ are true, then $P\left(\cup A_{i}\right)$ is true.

### 1.4. Smooth Maps

1.4.1. Smooth Maps. A map $F: M \rightarrow N$ between spaces has a natural dual or pull back that takes functions defined on subsets of $N$ to functions defined on subsets of $M$. Specifically if $f: A \subset N \rightarrow \mathbb{R}$ then $F^{*}(f)=f \circ F: F^{-1}(A) \subset M \rightarrow \mathbb{R}$. Here it could happen that $F^{-1}(A)=\emptyset$. Note that if $F$ is continuous then its pull back will map continuous functions on open subsets of $N$ to continuous functions on open subsets of $M$. Conversely, if $N$ is normal, and the pull back takes continuous functions to continuous functions, then it will be continuous. To see this fix $O \subset N$ that is open and select a continuous function $\lambda$ : $N \rightarrow[0, \infty)$ such that $\lambda^{-1}(0, \infty)=O$. Then $(\lambda \circ F)^{-1}(0, \infty)=F^{-1}(O)$ and is in particular open as we assumed that $\lambda \circ F$ was continuous.

DEFINITION 1.4.1. A map $F: M \rightarrow N$ is said to be smooth if $F^{*}$ takes smooth functions to smooth functions, i.e., $F^{*}\left(\mathfrak{C}^{\infty}(N)\right) \subset \mathfrak{C}^{\infty}(M)$.

Proposition 1.4.2. Let $F: M \rightarrow N$ be continuous. The following conditions are equivalent:
(1) $F$ is smooth.
(2) If $\mathscr{D}$ is a differentiable structure on $N$, then $F^{*}(\mathscr{D}) \subset \mathfrak{C}^{\infty}(M)$.
(3) $F^{*}\left(C^{\infty}(N)\right) \subset C^{\infty}(M)$.
(4) If $x_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}^{m}$ is an atlas for $M$ and $y_{\beta}: V_{\beta} \rightarrow \mathbb{R}^{n}$ an atlas for $N$, then the coordinate representations $y_{\alpha} \circ F \circ x_{\beta}^{-1}$ are smooth when- and where-ever they are defined.

### 1.4.2. The Rank of a Map.

DEFINITION 1.4.3. The rank of a smooth map at $p \in M$ is denoted $\operatorname{rank}_{p} F$ and is defined as the rank of the differential $D\left(y \circ F \circ x^{-1}\right)$ at $x(p)$. This definition is independent of the coordinate systems we choose due to the chain rule and the fact that the transition functions have nonsingular differentials at all points.

Proposition 1.4.4. If $F: M \rightarrow N$ and $G: N \rightarrow O$ are smooth maps, then

$$
\operatorname{rank}_{p}(G \circ F) \leq \min \left\{\operatorname{rank}_{p} F, \operatorname{rank}_{F(p)} G\right\}
$$

Proof. Using coordinates $x$ around $p \in M, y$ around $F(p) \in N$, and $z$ around $G(F(p)) \in$ $O$ we can consider the composition

$$
z \circ G \circ F \circ x^{-1}=\left(z \circ G \circ y^{-1}\right) \circ\left(y \circ F \circ x^{-1}\right)
$$

The chain rule then implies

$$
\left.D\left(z \circ G \circ F \circ x^{-1}\right)\right|_{p}=\left.\left.D\left(z \circ G \circ y^{-1}\right)\right|_{y \circ F(p)} \circ D\left(y \circ F \circ x^{-1}\right)\right|_{x(p)}
$$

This reduces the claim to the corresponding result for linear maps.

### 1.4.3. Coordinates.

Definition 1.4.5. We say that $F$ is a diffeomorphism if it is a bijection and both $F$ and $F^{-1}$ are smooth.

Proposition 1.4.6. Let $y: U \rightarrow \mathbb{R}^{m}$ be smooth where $U \subset M$ is an open subset. If $\operatorname{rank}_{p} y=\operatorname{dim} M=m$, then $y$ is a chart on a neighborhood of $p$. Moreover, if $\operatorname{rank}_{p} y=m<$ $\operatorname{dim} M$, then it is possible to select coordinate functions $y^{m+1}, \ldots, y^{n}$ such that $y^{1}, \ldots, y^{n}$ form coordinates around $p$.

Proof. This follows from the inverse function theorem. Select a chart $x: V \rightarrow \mathbb{R}^{m}$ on a neighborhood of $p$ and consider the smooth map $y \circ x^{-1}: x(U \cap V) \rightarrow \mathbb{R}^{m}$. By the definition of rank the map has nonsingular differential at $x(p)$ and must therefore be a diffeomorphism from a neighborhood around $x(p)$ to its image. This shows in turn that $y$ is a diffeomorphism on some neighborhood of $p$ onto its image.

For the second claim select an arbitrary coordinate system $z^{1}, \ldots, z^{n}$ around $p$. Then the map $\left(y \circ z^{-1}, z^{1}, \ldots, z^{n}\right)$ has a differential at $z(p)$ that looks like

$$
\left[\begin{array}{c}
D\left(y \circ z^{-1}\right) \\
I_{n}
\end{array}\right]
$$

where $I_{n}$ is the identity matrix and $D\left(y \circ z^{-1}\right)$ has linearly independent rows. We can then use the replacement procedure to eliminate $m$ of the bottom $n$ rows so as to get a nonsingular $n \times n$ matrix. Assuming after possibly rearranging indices that the remaining rows are the last $n-m$ rows we see that $\left(y \circ z^{-1}, z^{m+1}, \ldots, z^{n}\right)$ has rank $n$ at $p$ and thus forms a coordinate system around $p$.

### 1.4.4. Immersions.

DEFINITION 1.4.7. We say that $F: M \rightarrow N$ is an immersion if $\operatorname{rank}_{p} F=\operatorname{dim} M$ for every $p \in M$.

Proposition 1.4.8. For a smooth map $F: M \rightarrow N$ the following conditions are equivalent:
(1) $F$ is an immersion.
(2) For each $p \in M$ there are charts $x: U \rightarrow \mathbb{R}^{m}$ and $y: V \rightarrow \mathbb{R}^{n}$ with $p \in U$ and $F(p) \in V$ such that

$$
y \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{m}, 0, \ldots, 0\right)
$$

(3) If $\mathscr{D}$ is a differentiable structure on $N$ then $F^{*}(\mathscr{D})$ is a differentiable structure on $M$.

Proof. It is obvious that 2 implies 1. For 1 implies 2. Select coordinates $u: U \rightarrow \mathbb{R}^{m}$ around $p$ and $v: V \rightarrow \mathbb{R}^{n}$ around $F(p) \in N$. The composition $v \circ F \circ u^{-1}$ has rank $m$ at $u(p)$. After possibly reordering the indices for the $v$-coordinates we can assume that $\left(v^{1}, \ldots, v^{m}\right) \circ F \circ u^{-1}$ also has rank $m$ at $u(p)$. But this means that it is a diffeomorphism on some neighborhood around $u(p)$. Consequently $x=\left(v^{1}, \ldots, \nu^{m}\right) \circ F$ is a chart around $p$. Consider the functions

$$
\begin{aligned}
y^{i} & =v^{i}, i=1, \ldots, m \\
y^{i} & =v^{i}-v^{i} \circ F \circ x^{-1}\left(v^{1}, \ldots, v^{m}\right), i>m
\end{aligned}
$$

These are defined on a neighborhood of $F(p)$ and when $i>m$ we have

$$
y^{i} \circ F=v^{i} \circ F-v^{i} \circ F \circ x^{-1}\left(v^{1} \circ F, \ldots, v^{m} \circ F\right)=0
$$

So it remains to check that $y=\left(y^{1}, \ldots, y^{n}\right)$ are coordinates at $F(p)$. The composition $y \circ v^{-1}$ has a differential that is in lower triangular block form

$$
\left[\begin{array}{cc}
I_{m} & 0 \\
* & I_{n-m}
\end{array}\right]
$$

where the diagonal entries are the identity matrices on first $m$ and last $n-m$ coordinate subspaces. This shows that they will form coordinates on some neighborhood of $F(p)$.

As 1 and 2 are equivalent we can now use the proof that 1 implies 2 to show that if 1 or 2 hold, then 3 also holds.

Conversely assume that 3 holds. Select a coordinate chart $z^{i}=y^{i} \circ F$ around $p$ where $y^{i} \in \mathscr{D}, i=1, \ldots, m$. The chart $z$ has rank $m$ at $p$, so it follows that the corresponding smooth map $y$ must have rank at least $m$ at $F(p)$. However, the rank can't be greater than $m$ as it maps into $\mathbb{R}^{m}$. We can now add $n-m$ coordinate functions $z^{i}$ from some other coordinate system around $F(p)$ so as to obtain a map $\left(y^{1}, \ldots, y^{m}, z^{m+1}, \ldots, z^{n}\right)$ that has rank $n$ at $F(p)$. These coordinate choices show that 1 holds.

COROLLARY 1.4.9. A smooth map $F: M \rightarrow N$ is an immersion iff for any smooth map $G: L \rightarrow M$ and $o \in L$ we have

$$
\operatorname{rank}_{o}(F \circ G)=\operatorname{rank}_{o} G
$$

DEFINITION 1.4.10. We say that $F$ is an embedding if it is an immersion, injective, and $F: M \rightarrow F(M)$ is a homeomorphism, where the image is endowed with the induced topology.

Proposition 1.4.11. For a smooth map $F: M \rightarrow N$ the following conditions are equivalent:
(1) $F$ is an embedding.
(2) $F^{*}\left(\mathfrak{C}^{\infty}(N)\right)=\mathfrak{C}^{\infty}(M)$, i.e., $F^{*}$ is surjective on smooth functions.

Proof. Start by assuming that 2 holds. Given $p, q \in M$ select $f \in \mathfrak{C}^{\infty}(M)$ such that $f(p) \neq f(q)$. Then find $g \in \mathfrak{C}^{\infty}(N)$ such that $f=g \circ F$. Thus $g(F(p)) \neq g(F(q))$ showing that $F$ is injective. To see that the topology of $M$ agrees with the induced topology on
$F(M)$ select an open set $O \in M$ and $\lambda: M \rightarrow[0, \infty)$ such that $\lambda^{-1}(0, \infty)=O$. Select $\mu: U \subset N \rightarrow \mathbb{R}$ such that $\lambda=\mu \circ F$. Note that $F(M) \subset U$ as $\lambda$ is defined on all of $M$. Thus

$$
\mu^{-1}(0, \infty) \cap F(M)=F\left(\lambda^{-1}(0, \infty)\right)=F(O)
$$

and $F(O)$ is open in $F(M)$. Finally select coordinates $x$ around $p \in M$ and write $x^{i}=y^{i} \circ F$ for smooth functions on some neighborhood of $F(p)$. The composition $y \circ F \circ x^{-1}$ has rank $m$ at $x(p)$. So the map $F \circ x^{-1}$ must have rank at least $m$ at $x(p)$. However, the rank can't exceed $m$ so this shows that $\operatorname{rank}_{p} F=m$ and in turn that $F$ is an immersion.

Conversely assume that $F$ is an embedding and $f: O \subset M \rightarrow \mathbb{R}$ a smooth function. Using that $F$ is an immersion we can for each $p \in M$ select charts $x_{p}: O_{p} \rightarrow \mathbb{R}^{m}$ around $p$ and $y_{p}: U_{p} \rightarrow \mathbb{R}^{n}$ around $F(p)$ such that $\left.y_{p}^{j}\right|_{F\left(o_{p}\right) \cap U_{p}}=0$ for $j>m$. Since $F$ is an embedding $U_{p} \cap F\left(O_{p}\right) \subset F(M)$ is open. This means that we can assume that $U_{p}$ is chosen so that $F\left(O_{p}\right)=U_{p} \cap F(M)$. Now select a locally finite subcover $U_{\alpha}$ of $F(M)$ from the cover $U_{p}$ and let $O_{\alpha}=F^{-1}\left(U_{\alpha}\right)$. On each $U_{\alpha}$ define $g_{\alpha}$ such that $g_{\alpha} \circ y_{\alpha}^{-1}\left(a^{1}, . ., a^{n}\right)=$ $f \circ x_{\alpha}^{-1}\left(a^{1}, \ldots, a^{m}\right)$. We can then define $g=\sum_{\alpha} \mu_{\alpha} g_{\alpha}$, where $\mu_{\alpha}$ is a partition of unity for $U_{\alpha}$. This gives us a function on the open set $\cup U_{\alpha}$. Since $F$ is injective it follows that $g \circ F=f$.

Corollary 1.4.12. If $F: M \rightarrow N$ is an embedding such that $F(M) \subset N$ is closed, then $F^{*}\left(C^{\infty}(N)\right)=C^{\infty}(M)$.

Proof. The only additional item to worry about is whether the function $g$ just constructed can be extended to $N$ and remain fixed on $F(M)$. When the image is a closed subset this is easily done by finding a smooth Urysohn function $v$ that is 1 on $F(M)$ and vanishes on $N-U$. The function $v g$ is then a smooth function on $N$ that can be used instead of $g$.

REMARK 1.4.13. We note that when an injective immersion is also a proper map, then it becomes an embedding whose image is a closed subset. Such maps are called proper embeddings or proper submanifolds. Calling them "closed submanifolds" might cause confusion as closed manifolds are generally compact manifolds without boundary.

DEFINITION 1.4.14. A subset $S \subset M$ is a submanifold if it admits a topology such that the restriction of the differentiable structure on $M$ to $S$ is a differentiable structure. The dimension of the structure on $S$ will generally be less than that of $M$ unless $S$ is an open subset with the induced topology. Note that the topology on $S$ can be different from the induced topology, but it has to be finer as we require all smooth functions on $M$ to be smooth on $S$. In this way we see that a submanifold is in fact the image of an injective immersion.

Suppose $F: M \rightarrow N$ is a smooth map whose image lies in a submanifold $S \subset N$. When is $F: M \rightarrow S$ smooth? This is definitely the case when $S$ is embedded and also the case when $S$ is immersed provided $F: M \rightarrow S$ is continuous.

Proposition 1.4.15. Let $F: M \rightarrow N$ be a smooth map whose image lies in a submanifold $S \subset N$. If $F: M \rightarrow S$ is continuous, then $F: M \rightarrow S$ is smooth

Proof. We fix $p \in M$ and with it $q=F(p) \in S$. There are coordinates $y^{1}, \ldots, y^{n}$ on a neighborhood of $q \in N$, such that $y^{1}, \ldots, y^{k}$ restrict to coordinates on a neighborhood $q \in V \subset S$. Since $F$ is continuous, the preimage $U=F^{-1}(V)$ is open. Smoothness of $\left.F\right|_{U}: U \rightarrow N$, then shows that that $\left.y^{i} \circ F\right|_{U}$ is smooth for $i=1, \ldots, k$. This shows that also $\left.F\right|_{U}: U \rightarrow S$ is smooth.

We finish with a useful lemma about when a map that becomes an embedding when restricted to a submanifold can be extended to an embedding on a neighborhood of the submanifold.

LEMMA 1.4.16. If $F: M \rightarrow N$ is a proper immersion that is an embedding when restricted to a properly embedded submanifold $S \subset M$, then $F$ is an embedding on an open set containing $S$.

Proof. It suffices to show that $F$ is injective on a neighborhood of $S$. If it is not injective on any neighborhood, then we can find sequences $x_{i}$ and $y_{i}$ that approach $S$ with $F\left(x_{i}\right)=F\left(y_{i}\right)$. Note that if a sequence approaches $S$, then all but finitely many elements of the sequence must lie in any open set that contains $S$. In particular, if a sequence does not have any accumulation points in $S$, then it cannot approach $S$ (any closed set in $M$ that is disjoint from the closed set $S$ lies in the complement of an open set that contains $S$ ). Thus both sequences have accumulation points that lie in $S$. By passing to subsequences, assume that they converge to points $x$ and $y$ in $S$. Then $F(x)=F(y)$ so $x=y$ and $x_{i}=y_{i}$ for large $i$ as they lie in a neighborhood of $x=y$ where $F$ is injective.

### 1.4.5. Submersions.

DEFInition 1.4.17. We say that $F: M \rightarrow N$ is a submersion if $\operatorname{rank}_{p} F=\operatorname{dim} N$ for all $p \in M$.

Proposition 1.4.18. For a smooth map $F: M \rightarrow N$ the following conditions are equivalent:
(1) $F$ is a submersion.
(2) For each $p \in M$ there are charts $x: U \rightarrow \mathbb{R}^{m}$ and $y: V \rightarrow \mathbb{R}^{n}$ with $p \in U$ and $F(p) \in V$ such that

$$
y \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{n}\right) .
$$

(3) For each $f \in \mathfrak{C}^{\infty}(N)$ and $p \in M$ we have that $\operatorname{rank}_{p}(f \circ F)=\operatorname{rank}_{F(p)}(f)$.

Proof. Assume that 1 holds and select a chart $y$ around $F(p)$. Then $y \circ F$ has rank $n$ at $p$. We can now supplement with $m-n$ coordinate functions $x^{i}$ from any coordinate system around $p$ such that $x^{1}=y^{1} \circ F, \ldots, x^{n}=y^{n} \circ F, x^{n+1}, \ldots, x^{m}$ are coordinates around $p$. This yields the desired coordinates.

Clearly 2 implies 3 .
If we assume that 3 holds and that we have a chart $y$ around $F(p)$. Then we can consider smooth functions $f=\sum \alpha_{i} y^{i}$, where $\alpha_{i} \in \mathbb{R}$. These have rank 1 at $F(p)$ unless $\alpha^{1}=\cdots=\alpha^{n}=0$. If we choose coordinates $x$ around $p$, then $\left.\left.D\left(f \circ F \circ x^{-1}\right)\right|_{x^{-1}(p)}\right)=$ $\left.\sum \alpha_{i} D\left(y^{i} \circ F \circ x^{-1}\right)\right|_{x^{-1}(p)}$. So it follows that $\left.D\left(y^{i} \circ F \circ x^{-1}\right)\right|_{x^{-1}(p)}$ are linearly independent, which in turn implies that $y \circ F \circ x^{-1}$ has rank $n$ at $x^{-1}(p)$.

COROLLARY 1.4.19. A smooth map $F: M \rightarrow N$ is a submersion iff for any smooth map $G: N \rightarrow O$ and $p \in M$ we have

$$
\operatorname{rank}_{p}(G \circ F)=\operatorname{rank}_{F(p)} G
$$

Finally we mention a few useful properties.
PROPOSITION 1.4.20. Let $F: M^{m} \rightarrow N^{n}$ be a smooth map.
(1) If $F$ is proper, then it is closed.
(2) IfF is proper, $K \subset N$ is compact, and $O \supset F^{-1}(K)$ is open, then there exists an open set $V \supset K$ such that $F^{-1}(V) \subset O$.
(3) If $F$ is a submersion, then it is open.
(4) If $F$ is a proper submersion and $N$ is connected then it is surjective.

Proof. 1. Let $C \subset M$ be a closed set and assume $F\left(x_{i}\right) \rightarrow y$, where $x_{i} \in C$. The set $\left\{y, F\left(x_{i}\right)\right\}$ is compact. Thus the preimage is also compact. This implies that $\left\{x_{i}\right\}$ has an accumulation point. If we assume that $x_{i_{j}} \rightarrow x \in C$, then continuity shows that $F\left(x_{i_{j}}\right) \rightarrow F(x)$. Thus $y=F(x) \in F(C)$.
2. The set $M-O$ is closed, so by 1 we obtain an open neighborhood $V=N-$ $F(M-O)$ around C. If $F(x) \in V$, then $x \notin M-O$ and consequently $F^{-1}(V) \subset O$.
3. Consequence of local coordinate representation of $F$.
4. Follows directly from properties 1 and 3.

COROLLARY 1.4.21. Let $F: M \rightarrow N$ be a submersion. If $f: O \subset F(N) \rightarrow \mathbb{R}$ is a function on an open set such that $f \circ F$ is smooth, then $f$ is smooth.

Proof. Smoothness is clearly a local property so we can confine ourselves to functions that are defined on the coordinate systems guaranteed from 2 in proposition 1.4.18 But then the claim is obvious.
1.4.6. Constant Rank. The canonical forms for immersions and submersions can be combined into a more general result for maps that have constant rank on all of the manifold.

REMARK 1.4.22. First note that $\operatorname{rank} A \geq k$ is an open condition on Mat ${ }_{n \times m}$. Thus if $\operatorname{rank}_{p} F=k$, then $\operatorname{rank}_{x} F \geq k$ for all $x$ in a neighborhood of $p$. Similarly, when $V \subset \mathbb{R}^{n}$ is a subspace, then the condition that $\operatorname{im} A+V=\mathbb{R}^{n}$ is also an open condition for $A \in \mathrm{Mat}_{n \times m}$.

THEOREM 1.4.23 (Rank Theorem). Let $F: M \rightarrow N$ be a map of constant rank $k$ on all of $M$. For each $p \in M$ there are charts $x: U \rightarrow \mathbb{R}^{m}$ and $y: V \rightarrow \mathbb{R}^{n}$ with $p \in U$ and $F(p) \in V$ such that $x(p)=0, y(F(p))=0$, and

$$
y \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{k}, 0, \ldots, 0\right) .
$$

Proof. We start with general charts around $p$ and $F(p)$ such that $u(p)=0, v(F(p))=$ 0 , and

$$
v \circ F \circ u^{-1}\left(u^{1}, \ldots, u^{m}\right)=(A(u), B(u))
$$

where $A$ takes up the first $k$ coordinates and $B$ the remaining $n-k$. After possibly reordering these two coordinate systems we can assume that $u \mapsto A(u)$ has rank $k$ in a neighborhood of 0 . Now consider the map $u \mapsto x(u)=\left(A(u), u^{k+1}, \ldots, u^{m}\right)$. This map has rank $m$ at 0 and is consequently a local diffeomorphism. This gives us a new chart $x$ around $p$ where

$$
v \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{k}, B(x)\right) .
$$

Since this map has constant rank $k$ it must follow that

$$
\frac{\partial B}{\partial x^{i}}=0, i=k+1, \ldots, m
$$

After possibly shrinking the domain of the chart we have that

$$
v \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{k}, B\left(x^{1}, \ldots, x^{k}\right)\right) .
$$

We can now define $y=\left(v^{1}, \ldots, v^{k}, v^{k+1}-B^{k+1}, \ldots, v^{n}-B^{n}\right)$. This map is nonsingular at 0 and

$$
y \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{k}, 0, \ldots, 0\right) .
$$

1.4.7. Regular and Critical Points. We say that $F: M \rightarrow N$ is non-singular on $M$ if it is both a submersion and an immersion. This is evidently equivalent to saying that it is locally a diffeomorphism.

A point $p \in M$ is called a regular point if $\operatorname{rank}_{p} F=\operatorname{dim} N$, otherwise it is a critical point. A point $q \in N$ is called a regular value if $F^{-1}(q)$ is empty or only contains regular points, otherwise it is a critical value.

REMARK 1.4.24. Note that if $p \in M$ is a regular point for $F: M \rightarrow N$, then there is a neighborhood $p \in U \subset M$ such that $q$ is a regular value for $\left.F\right|_{U}: U \rightarrow N$. Thus the set of regular points is open. This however does not tell us that the set of regular values is open. In case $F$ is proper we can use proposition 1.4 .20 to conclude that the set of regular values is open.

THEOREM 1.4.25 (The Preimage Theorem). If $q \in N$ is a regular value for a smooth function $F: M^{m} \rightarrow N^{n}$, then $F^{-1}(p)$ is empty or a properly embedded submanifold of $M$ of dimension $m-n$.

Proof. Note that the preimage is closed so it follows that its intersections with compact sets is compact. We shall also use the induced topology and show that it is a submanifold with respect to that topology. We claim that $\mathfrak{C}^{\infty}(M)$ restricts to a differential system on the preimage.

If we select coordinates $y^{i}, i=1, \ldots, n$ around $q \in N$, then the functions $y^{i} \circ F$ are part of a coordinate system $x^{i}$ around any point $p \in F^{-1}(q)$. This means that we can find a neighborhood $p \in U$ such that $U \cap F^{-1}(q)=\left\{x \in U \mid y^{i}(F(x))=y^{i}(F(q))\right\}$, i.e., $x^{i}=y^{i} \circ F$ are constant on the preimage. Given $f \in \mathfrak{C}^{\infty}(M)$ defined around $p$ we have that $f=F\left(x^{1}, \ldots, x^{m}\right)$. Now on $U \cap F^{-1}(q)$ the first $n$ coordinates are constant so it follows that $\left.f\right|_{U \cap F^{-1}(q)}=F\left(x^{1}(p), \ldots, x^{n}(p), x^{n+1}, \ldots, x^{m}\right)$. Thus the restriction can be written as a smooth function of the last $m-n$ coordinates. Finally we note that these last $m-n$ coordinates also define the desired chart on $U \cap F^{-1}(q)$ as they are injective and yield a homeomorphism onto the image.

REMARK 1.4.26. The constant rank theorem implies that the preimage theorem remains true as long as the map has constant rank $k$ on $M$. In this case the preimages have dimension $m-k$.

To complement the preimage theorem we next prove.
THEOREM 1.4.27 (Brown, 1935, A.P. Morse, 1939 and Sard, 1942). The set of regular values for a smooth function $F: M^{m} \rightarrow N^{n}$ is a countable intersection of open dense sets and in particular dense. Moreover, the set of critical values has measure 0.

Proof. We prove Brown's original statement: the set of critical values has no interior points. The proof we give is fairly standard and is very close to Brown's original proof. The same proof is easily adapted to prove Sard's measure zero version, but this particular statement is in fact rarely used. A.P. Morse proved the measure theoretic result when the target space is $\mathbb{R}$.

Note that the set of critical points is closed but its image need not be closed. However, the set of critical points is a countable union of compact sets and thus the image is also a countable union of compact sets. This means that we rely on the Baire category theorem: a set that is the countable union of closed sets with empty interiors also has empty interior. Thus we only need to show that there are no interior points in the set of critical values
that come from critical points in a compact set. Further note that it suffices to prove the theorem for the restriction of $F$ to any open covering of $M$.

To clarify the meaning of measure 0 and prove Sard's theorem in the case where it is most used, we make some simple observations.

Consider a map $F: O \subset \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. When $F$ is locally Lipschitz, then it maps sets of measure zero to sets of measure zero. Moreover, any differentiable map that has bounded derivative on compact sets is locally Lipschitz. Thus $C^{1}$ diffeomorphisms preserve sets of measure zero. This shows that the notion of sets of measure zero is well-defined in a smooth manifold. Now consider $F: M^{m} \rightarrow N^{n}$, where $m<n$ and construct $\bar{F}: M \times \mathbb{R}^{n-m} \rightarrow$ $N$, by $\bar{F}(x, z)=F(x)$. Then $F(M)=\bar{F}(M \times\{0\})$ has measure zero as $M \times\{0\} \subset M \times$ $\mathbb{R}^{n-m}$ has measure zero.

In the general case the proof uses induction on $m$. For $m=0$ the claim is trivial as $M$ is forced to be a countable set with the discrete topology. As mentioned above, it suffices to prove it for maps $F: U \subset \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$, where $U$ is open. For such a map let $C_{0}$ be the set of critical points and define $C_{k} \subset C_{0}$ as the set of critical points where all derivatives of order $\leq k$ vanish. Note that all of these sets are closed.

First we show that $F\left(C_{k}\right)$ has no interior points when $k \geq m / n$ : Fix a compact set $K$. Taylor's theorem shows that we can select $r>0$ and $C>0$ such that for any $x \in B(p, r)$ with $p \in C_{k} \cap K$ we have

$$
|F(p)-F(x)| \leq C|p-x|^{k+1}
$$

Now cover $C_{k} \cap K$ by finitely many cubes $I_{i}^{\varepsilon}$ of side length $\varepsilon<r$, then $F\left(I_{i}^{\varepsilon}\right)$ lies in a cube $J_{i}^{\varepsilon}$ of side length $\leq C(m, n) \varepsilon^{k+1}$ for a constant $C(m, n)$ that depends on $C, m$, and $n$. Thus

$$
\begin{aligned}
\left|J_{i}^{\varepsilon}\right| & \leq(C(m, n))^{n} \varepsilon^{n(k+1)} \\
& =(C(m, n))^{n} \varepsilon^{n(k+1)-m}\left|I_{i}^{\varepsilon}\right|
\end{aligned}
$$

Since $C_{k} \cap K$ is compact we can assume that $\sum\left|I_{i}^{\varepsilon}\right|$ remains bounded as $\varepsilon \rightarrow 0$. Thus $\sum\left|J_{i}^{\varepsilon}\right|$ will converge to 0 since $n(k+1)>m$. This shows that $F\left(C_{k} \cap K\right)$ does not contain any interior points as it could otherwise not be covered by cubes whose total volume is arbitrarily small.

Next we show that $F\left(C_{k}-C_{k+1}\right)$ has no interior points for $k>0$ : Denote by $\partial^{k}$ some specific partial derivative of order $k$. Thus $\left(\partial^{k} F\right)(p)=0$ for $p \in C_{k}-C_{k+1}$ but some partial derivative $\frac{\partial \partial^{k} F}{\partial x^{j}}(p) \neq 0$. Without loss of generality we can assume that $\frac{\partial \partial^{k} F^{1}}{\partial x^{j}}(p) \neq 0$. This means that near $p$ the set where $\partial^{k} F^{1}=0$ will be a submanifold of dimension $m-1$. Since $p$ is critical for $F$ it'll also be a critical point for the restriction of $F$ to any submanifold. By induction hypothesis the image of such a set has no interior points. Thus for any fixed compact set $K$ the set $K \cap\left(C_{k}-C_{k+1}\right)$ can be divided into a finite collection of sets whose images have no interior points.

Finally we show that $F\left(C_{0}-C_{1}\right)$ has no interior points: Note that when $n=1$ it follows that $C_{0}=C_{1}$ so there is nothing to prove in this case. Assume that $p \in C_{0}-C_{1}$ is a point where $\frac{\partial F^{i}}{\partial x^{j}} \neq 0$. After rearranging the coordinates in $\mathbb{R}^{m}$ and $\mathbb{R}^{n}$ we can assume that $\frac{\partial F^{1}}{\partial x^{1}} \neq 0$. In particular, the set $L=\left\{x \mid F^{1}(x)=F^{1}(p)\right\}$ is a submanifold of dimension $m-1$ in a neighborhood of $p$. Let $G=\left(F^{2}, \ldots, F^{n}\right): L \rightarrow \mathbb{R}^{n-1}$. Now observe that if $F(p)$ is an interior point in $F\left(C_{0}-C_{1}\right)$, then $G(p)$ is an interior point for $G\left(L \cap\left(C_{0}-C_{1}\right)\right)$. This, however, contradicts our induction hypothesis since all the points in $L \cap\left(C_{0}-C_{1}\right)$ are critical for $G$. (For the measure zero statement, this last part requires a precursor to the Tonelli/Fubini theorem or Cavalieri's principle: A set has measure zero if its intersection with all parallel hyperplanes has measure zero in the hyperplanes.)

Putting these three statements together implies that the set of critical values has no interior points.
1.4.8. Covering Maps. We start with a more general result about proper maps.

LEMMA 1.4.28. Let $F: M^{m} \rightarrow N^{m}$ be a smooth proper map. If $y \in N$ is a regular value, then there exists a neighborhood $V$ around $y$ such that $F^{-1}(V)=\bigcup_{k=1}^{n} U_{k}$ where $U_{k}$ are mutually disjoint and $F: U_{k} \rightarrow V$ is a diffeomorphism.

Proof. First use that $F$ is proper to show that $F^{-1}(y)=\left\{x_{1}, \ldots, x_{n}\right\}$ is a finite set. Next use that $y$ is regular to find mutually disjoint neighborhoods $W_{k}$ around $x_{k}$ such that $F: W_{k} \rightarrow F\left(W_{k}\right)$ is a diffeomorphism. Finally, use proposition 1.4 .20 to find an open neighborhood $V$ of $y$ such that $F^{-1}(V) \subset \bigcup_{k=1}^{n} W_{k}$. We can the use $U_{i}=W_{i} \cap F^{-1}(V)$.

DEFINITION 1.4.29. A smooth map $\pi: \bar{N} \rightarrow N$ is called a covering map if each point in $N$ is evenly covered, i.e., for every $y \in N$ there is a neighborhood $V$ around $y$ such that $\pi^{-1}(V)=\bigcup U_{i}$ where $\pi: U_{i} \rightarrow V$ is a diffeomorphism and the sets $U_{i}$ are pairwise disjoint. In other words: $\pi^{-1}(V)$ is diffeomorphic to $\pi^{-1}(y) \times V$ :

$$
\begin{array}{rll}
\pi^{-1}(V) & \longrightarrow & \pi^{-1}(y) \times V \\
\searrow & & \swarrow
\end{array}
$$

and $F^{-1}(y) \subset \bar{N}$ is a 0 -dimensional submanifold.
Corollary 1.4.30. If $\pi: \bar{N} \rightarrow N$ is a proper non-singular map with $N$ connected, then $\pi$ is a covering map.

The key property for covering maps is the unique path lifting property. A lift of a continuous map $F: M \rightarrow N$ into the base of a covering map $\pi: \bar{N} \rightarrow N$ is a continuous map $\bar{F}: M \rightarrow \bar{N}$ such that $\pi \circ \bar{F}=F$. If $\bar{F}\left(x_{0}\right)=\pi\left(y_{0}\right)$, then we say that the lift goes through $y_{0}$.


When $F$ is smooth then the lift is also forced to be smooth. Moreover, when the covering is trivial:

then $F$ has a lift through any $y_{0} \in \pi^{-1}\left(F\left(x_{0}\right)\right)$
Proposition 1.4.31. If $M$ is connected, $x_{0} \in M$, and $y_{0} \in \bar{N}$ such that $F\left(x_{0}\right)=\pi\left(y_{0}\right)$, then there is at most one lift $\bar{F}$ such that $\bar{F}\left(x_{0}\right)=y_{0}$.

Proof. Assume that we have two lifts $F_{1}$ and $F_{2}$ with this property and let $A=$ $\left\{x \in M \mid F_{1}(x)=F_{2}(x)\right\}$. Clearly $A$ is non-empty and closed. The covering maps property shows that $A$ is open. So when $M$ is connected $A=M$.

DEFINITION 1.4.32. We say that two maps $F_{0}, F_{1}: M \rightarrow N$ are homotopic if there is a smooth map $H:[0,1] \times M \rightarrow N$ such that $F_{0}(x)=H(0, x)$ and $F_{1}(x)=H(1, x)$. Smoothness of such a homotopy near the boundary points hasn't been defined yet. However, using a function $\lambda:[0,1] \rightarrow[0,1]$ we can alter any homotopy to a new homotopy $H(\lambda(t), x)$. If $\lambda \equiv 0$ for $t<\varepsilon$ and $\lambda \equiv 1$ for $t>1-\varepsilon$, then the new homotopy becomes
stationary at the ends. This also allows us to smoothly concatenate homotopies provided $H_{1}(1, x)=H_{2}(0, x)$. Thus maps being homotopic is an equivalence relation.

DEFINITION 1.4.33. Curves are very simple homotopies between maps from a one point space. Thus curves can easy be concatenated to smooth curves if we don't care about how they are parametrized. The equivalences of points created by curves are the path connected components of a manifold. We say that a manifold is path connected if any two points can be joined by a curve. A manifold is simply connected if it is path connected and any closed curve is homotopic to a constant map.

THEOREM 1.4.34. Let $\pi: \bar{N} \rightarrow N$ be a covering map. If $M$ is connected and simply connected, then any continuous $F: M \rightarrow N$ has lift through each point in $\pi^{-1}\left(F\left(x_{0}\right)\right)$.

Proof. Cover $N$ by connected open sets $V_{\alpha}$ such that $\pi^{-1}\left(V_{\alpha}\right) \simeq \pi^{-1}\left(F\left(x_{0}\right)\right) \times V_{\alpha}$.
Next suppose that $M$ is covered by a string of connected sets $U_{i}, i=0,1,2 \ldots$ such that $F\left(U_{i}\right) \subset V_{\alpha_{i}}$. We can then lift $F$ on each of the sets $U_{i}$ to go through a given point in $\pi^{-1}\left(F\left(U_{i}\right)\right)$. If we further have the property that $U_{k} \cap\left(\bigcup_{i=0}^{k-1} U_{i}\right)$ is non-empty and connected, then we can use the uniqueness of liftings to successively define $\left.F\right|_{U_{k}}$ given that it is defined on $\bigcup_{i=0}^{k-1} U_{i}$. Note that the sets $U_{i}$ need not be open.

Unfortunately not a lot of manifolds admit such covers. Clearly $\mathbb{R}^{k}$ does as it can be covered by coordinate cubes. Also any interval, disc, and square has this property. However, the circle $S^{1}$ cannot be covered by such a string of sets. On the other hand spheres $S^{n}, n>1$ do have this property. We will use the property for the interval and square.

We can now show that if we have a map $G: M_{0} \rightarrow M$, where $M_{0}$ has the desired covering property, then $F \circ G$ can be lifted. Given two curves $c_{i}:[0,1] \rightarrow M$ where $c_{i}(0)=x_{0}$ and $c_{i}(1)=x \in M$, where $i=0,1$, we invoke simple connectivity of $M$ to find a homotopy $H:[0,1]^{2} \rightarrow M$ where $H(s, 0)=x_{0}, H(s, 1)=x$, and $H(i, t)=c_{i}(t)$. We can then find a lift of $F \circ H$ such that $\overline{F \circ H}(s, 0)=y_{0}$. The unique path lifting property then guarantees that $\overline{F \circ H}(s, 1)$ is constant, and, in particular, that the lift of $F$ at $x \in M$ does not depend on the path connecting it to $x_{0}$. This gives us a well-defined lift of $F$ that is smooth when composed with any curve that starts at $x_{0}$.

COROLLARY 1.4.35. If $\pi: \bar{N} \rightarrow N$ is a covering map and $F: M \rightarrow N$ is a map such that for every closed curve $c: S^{1} \rightarrow M$ the map $F \circ c$ has a lift that passes through each point in $\pi^{-1}\left(F \circ c\left(t_{0}\right)\right)$ for a fixed $t_{0} \in S^{1}$, then $F$ has a lift through each point in $\pi^{-1}\left(F\left(x_{0}\right)\right)$.

Proof. This proof is almost identical to the above proof. The one difference is that the curves are no longer necessarily homotopic to each other. However, the fact that lifts of closed curves in $M$ are assumed to become closed shows that the construction is independent of the paths we choose.

COROLLARY 1.4.36. If $F_{0}: M_{0} \rightarrow N$ and $F_{1}: M_{1} \rightarrow N$ are coverings where all manifolds are connected and $M_{0,1}$ are both simply connected, then $M_{0}$ and $M_{1}$ are diffeomorphic.

Proof. This is an immediate consequence of the lifting property of each of the covering maps to the other covering space.

COROLLARY 1.4.37. (Hadamard) Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a proper non-singular map, then $F$ is a diffeomorphism.

### 1.5. Exersises

(1) Let $F: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ be a smooth map such that $F(\lambda x)=\lambda F(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^{m}$. Show that $F$ is linear.
(2) Let $F: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ be a smooth map such that $F(0)=0$ and define

$$
H(t, x)= \begin{cases}t^{-1} F(t x) & t \neq 0 \\ \left.D F\right|_{0}(x) & t=0\end{cases}
$$

Show that $H(t, x)$ is smooth.
(3) Show that $\mathbb{R} \mathbb{P}^{1}$ and $S^{1}$ are diffeomorphic. Hint: Read subsection 1.2.3
(4) Show that $\mathbb{C P} \mathbb{P}^{1}$ and $S^{2}$ are diffeomorphic. $\mathbb{C P}{ }^{1}$ is also called the Riemann sphere. Hint: Read subsection 1.2.3
(5) Let $p: \mathbb{C} \rightarrow \mathbb{C}$ be a nontrivial polynomial and define $P: \mathbb{C P}^{1} \rightarrow \mathbb{C P}^{1}$ by

$$
P([z: 1])=[p(z): 1] \text { and } P([1: 0])=[1: 0] .
$$

(a) Show that $P$ is smooth.
(b) Show that a similar definition works for any smooth proper map $f: \mathbb{C} \rightarrow$ $\mathbb{C}$ and will always define a continuous extension $F: \mathbb{C P}^{1} \rightarrow \mathbb{C P}^{1}$. Will it always be smooth?
(6) Let $\frac{p}{q}$ be a rational function, where $p$ and $q$ are complex polynomials without common roots. Show that

$$
R([z: 1])=[p(z): q(z)]
$$

can be extended to a smooth map $R: \mathbb{C P}^{1} \rightarrow \mathbb{C P}^{1}$ (the extension depends on the degrees of the polynomials).
(7) Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a smooth proper map with finitely many critical values.
(a) Show that if $n \geq 2$, then $F$ is surjective (Hint: The set of regular values is connected).
(b) Give a counter example when $n=1$.
(8) Show that

$$
\left\{\left[z_{0}: \cdots: z_{n}\right] \in \mathbb{F P}^{n} \mid \lambda_{0} z_{0}+\cdots+\lambda_{n} z_{n}=0\right\}
$$

defines a submanifold diffeomorphic to $\mathbb{F P}^{n-1}$ as long as $\left(\lambda_{0}, \ldots, \lambda_{n}\right) \neq 0$.
(9) Consider the immersion $\mathbb{R} \rightarrow \mathbb{R}^{2}$ that looks like a figure 8 with loops that are in the first and third quadrants and is invariant under the involution $A(x, y)=(y, x)$. Show that the restriction of $A$ to this immersed submanifold is not continuous.
(10) Show that the space of symmetric matrices of rank $k \leq n$ in $\operatorname{Mat}_{n \times n}(\mathbb{R})$ is a manifold of dimension $\binom{k+1}{2}+k(n-k)$.
(11) Let

$$
V_{k}\left(\mathbb{R}^{n}\right)=\left\{\left(v_{1}, \ldots, v_{k}\right) \in \mathbb{R}^{n} \times \cdots \times \mathbb{R}^{n} \mid v_{i} \cdot v_{j}=\delta_{i j}\right\} \subset \operatorname{Mat}_{n \times k}^{k}(\mathbb{R})
$$

be the Stiefel manifold of $k$ ordered orthonormal vectors. Show that this is a manifold of dimension $k n-\binom{k+1}{2}$.
(12) Let $G_{k}\left(\mathbb{R}^{n}\right)$ be the Grassmannian of $k$-dimensional subspaces in $\mathbb{R}^{n}$.
(a) Show that the map

$$
\begin{aligned}
\operatorname{Mat}_{n \times k}^{k}(\mathbb{R}) & \rightarrow G_{k}\left(\mathbb{R}^{n}\right) \\
X & \mapsto E_{X}=X\left(X^{*} X\right)^{-1} X^{*}
\end{aligned}
$$

is a submersion whose preimages are diffeomorphic to $G l_{k}$.
(b) Show that there is a submersion $V_{k}\left(\mathbb{R}^{n}\right) \rightarrow G_{k}\left(\mathbb{R}^{n}\right)$ whose preimages can be identified with $O(k)$.
(13) Show that the complex exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}^{\times}=\mathbb{C}-\{0\}$ is a covering map.
(14) Show that $\mathbb{R}^{n} \rightarrow T^{n}=S^{1} \times \cdots \times S^{1}$ is a covering map and use this to show that any map $M \rightarrow T^{n}$ is homotopic to a constant provided $M$ is simply connected.
(15) Let $M, N$ be manifolds. If $S \subset M$ is a closed subset and $q \in N$, then there is a smooth map $F: M \rightarrow N$ such that $S=F^{-1}(q)$.
(16) Show that if $F: M \rightarrow N$ admits a section $s: N \rightarrow M$, i.e., $F \circ s=i d_{N}$, then $s$ is an embedding. Is $F$ necessarily a submersion?
(17) Show that the map

$$
\begin{aligned}
\mathbb{F P}^{m} \times \mathbb{F P}^{m} & \rightarrow \mathbb{F P}^{m n+m+n} \\
\left(\left[\cdots: x_{i}: \cdots\right],\left[\cdots: y_{j}: \cdots\right]\right) & \mapsto
\end{aligned}
$$

gotten by multiplying all of the homogeneous coordinates is well-defined and an embedding.
(18) Show that

$$
\left\{\left[z_{1}: z_{2}: z_{3}\right] \in \mathbb{C P}^{2} \mid z_{1}^{n}+z_{2}^{n}+z_{3}^{n}=0\right\}
$$

is a compact submanifold.
(19) More generally, show that

$$
\left\{\left[z_{1}: z_{2}: z_{3}\right] \in \mathbb{C P}^{2} \mid p\left(z_{1}, z_{2}, z_{3}\right)=0\right\}
$$

is a compact submanifold when $p$ is homogeneous and irreducible. What happens in the real case?
(20) Show that

$$
\left\{\left(z_{0}, \ldots, z_{n}\right) \in \mathbb{C}^{n+1} \mid z_{0}^{2}+\cdots+z_{n}^{2}=1\right\}
$$

defines a submanifold and that it is diffeomorphic to $T S^{n}$.
(21) Show that the Brieskorn variety $W^{2 n-1}(d) \subset \mathbb{C}^{n+1}$ defined by the equations

$$
\begin{aligned}
z_{0}^{2}+\cdots+z_{n}^{2} & =0 \\
z_{0} \bar{z}_{0}+\cdots+z_{n} \bar{z}_{n} & =2
\end{aligned}
$$

is a manifold of dimension $2 n-1$.
(22) Show that the Milnor manifold with $m \leq n$ given by
$H(m, n)=\left\{\left(\left[z_{0}: \cdots: z_{m}\right],\left[w_{0}: \cdots: w_{n}\right]\right) \in \mathbb{F P}^{m} \times \mathbb{F P}^{n} \mid z_{0} w_{0}+\cdots+z_{m} w_{n}=0\right\}$
is a manifold of dimension $\operatorname{dim}_{\mathbb{R}} \mathbb{F} \cdot(m+n-1)$.
(23) Let $F: M \rightarrow N$ be a submersion. Show that if $S \subset N$ is an embedded submanifold, then $F^{-1}(S) \subset M$ is an embedded submanifold that, when nonempty, satisfies

$$
\operatorname{dim} N-\operatorname{dim} S=\operatorname{dim} M-\operatorname{dim} F^{-1}(S)
$$

Hint: Start with the case where $S=G^{-1}(z)$ and $z$ is a regular value. Then localize to prove the result.
(24) Note that the torus $S^{1} \times S^{1}$ can be embedded in $\mathbb{R}^{3}$.
(a) Show that the $n$ torus $S^{1} \times \cdots \times S^{1}$ can be embedded in $\mathbb{R}^{n+1}$.
(b) Show that $S^{p} \times S^{q}$ can be embedded in $\mathbb{R}^{p+q+1}$.
(c) Show that $S^{p_{1}} \times \cdots \times S^{p_{k}}$ can be embedded in $\mathbb{R}^{p_{1}+\cdots+p_{k}+1}$.
(25) Let $F: S^{1} \rightarrow \mathbb{R}$.
(a) Show that if $y \in \mathbb{R}$ is a regular value, then it has an even number of preimages.
(b) Show that there are at least as many critical points as there are preimages of a regular value.
(26) Show that $\mathbb{R}^{n} \subset \mathbb{R}^{n+1}$ is not the preimage of a regular value of a function $\mathbb{R} \mathbb{P}^{n+1} \rightarrow \mathbb{R}$.
(27) A classical way of embedding $\mathbb{R}^{\mathbb{P}^{n}}$ into $S^{n+k}$ uses a symmetric bilinear map $b: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+k+1}$ with the property that if $b(x, y)=0$, then $x=y=0$. Define $F: S^{n} \rightarrow S^{n+k}$, by $F(x)=\frac{b(x, x)}{\mid b(x, x)}$.
(a) Show that $F(x)=F(y)$ if and only if $x= \pm y$. Hint: Consider $b(x+\lambda y, x-\lambda y)$ when $b(x, x)=\lambda^{2} b(y, y)$.
(b) Show that $F$ induces an embedding $\mathbb{R P}^{n} \rightarrow S^{n+k}$.
(c) Use $b(x, y)=\left(z^{0}, \ldots, z^{2 n}\right)$, where $x=\left(x^{0}, \ldots, x^{n}\right), y=\left(y^{0}, \ldots, y^{n}\right)$ and $z^{k}=$ $\sum_{i+j=k} x^{i} y^{j}$, to obtain an embedding $\mathbb{R}^{P^{n}} \rightarrow S^{2 n}$.
(d) Use the multiplicative structure on $\mathbb{R}^{2} \simeq \mathbb{C}$ to obtain a diffeomorphism $\mathbb{R} \mathbb{P}^{1} \rightarrow S^{1}$. Can this be adapted to obtain a diffeomorphism $\mathbb{C P}^{1} \rightarrow S^{2} ?$
(28) A regular closed curve $c: S^{1} \rightarrow \mathbb{R}^{2}$ is said to have a crossing at $q \in \mathbb{R}^{2}$ provided $F^{-1}(q)=\left\{p_{1}, p_{2}\right\}$ consists of exactly two points and the derivatives of $c$ at these two points are linearly independent. For each positive integer $k$ there exists a curve with $k$ crossings (draw it). It can even be realized as the level set $F^{-1}(0)$ of a polynomial $F(x, y): \mathbb{R}^{2} \rightarrow \mathbb{R}$.
(a) Check this in these three cases: $F(x, y)=x^{2}+y^{2}-1$ for $p=0, F(x, y)=$ $\left(x^{2}+y^{2}-4\right)^{2}$ for $p=1, F(x, y)=\left(4 x^{2}\left(1-x^{2}\right)-y^{2}\right)^{2}$ for $p=2$.
(b) Show that if $F^{-1}(0)$ is the image of a curve with $p$ crossings, then the zero level set of $F(x, y)+z^{2}-\varepsilon$ is a compact surface of genus $p$ ( $p$ holes) for sufficiently small $\varepsilon$. Essentially the curve has been fattened so that each of $p+1$ enclosed regions of the curve correspond to a hole in the surface and the crossings become necks.

## CHAPTER 2

# Tangent Spaces and Differentials of Maps 

### 2.1. The Tangent Bundle

2.1.1. Motivation. To motivate let us start by selecting a countable differentiable system $\left\{f^{i}\right\}, i=1,2, \ldots$ of functions $f^{i}: M \rightarrow \mathbb{R}$. To find such a system we invoke paracompactness, partitions of unity, extensions of smooth function etc from the last chapter.

Tangent vectors are supposed to be tangents or velocities to curves on the manifold. These vectors have, as such, no place to live unless we know that the manifold is in Euclidean space. In the general case we can use the countable collection $f^{i}$ of smooth functions coming from a differential structure to measure the coordinates of the velocities by calculating the derivatives

$$
\frac{d\left(f^{i} \circ c\right)}{d t}
$$

for a smooth curve $c: I \rightarrow M$. Thus a tangent vector $v \in T M$ looks like a countable collection $v^{i}$ of its coordinates. However, around any given point we know that there will be $n$ coordinate functions, say $f^{1}, \ldots, f^{n}$, that yield a chart and then other smooth functions $F^{j}$, $j>n$ such that $f^{j}=F^{j}\left(f^{1}, \ldots, f^{n}\right)$. Thus we also have the relations

$$
v^{j}=\sum_{i=1}^{n} \frac{\partial F^{j}}{\partial x^{i}} v^{i}
$$

In other words the $n$ coordinates $v^{1}, \ldots, v^{n}$ determine the rest of the coordinate components of $v$. Note that at a fixed point $p$, the tangent vectors $v \in T_{p} M$ form an $n$-dimensional vector space, which is an $n$-dimensional subspace of a fixed infinite dimensional vector space. Moreover, this tangent space is well-defined as the set of vectors tangent to curves going through $p$ and is thus not dependent on the chosen coordinates. However, the coordinates help us select a basis for this vector space and thus to create suitable coordinates that yield a differentiable structure on $T M$.

As it stands, the definition does depend on our initial choice of a differentiable system. To get around this we could simply use the entire space of smooth functions $C^{\infty}(M)$ to get around this. This is more or less what we shall do below.
2.1.2. Abstract Derivations. The space of all smooth functions $\mathfrak{C}^{\infty}(M)$ is not a vector space as we can't add functions that have different domains especially if these domains do not even intersect. If we fix $p \in M$, then we consider the subset $\mathfrak{C}_{p}(M) \subset \mathfrak{C}^{\infty}(M)$ of smooth functions whose domain contains $p$. Thus any two functions in $\mathfrak{C}_{p}(M)$ can now be added in a meaningful way by adding them on the intersection of their domains and then noting that this is again an open set containing $p$. Thus we get a nice and very large vector space of smooth functions defined on neighborhoods of $p$. To get a logically meaningful theory this space is often modified by considering instead equivalence classes of function in $\mathfrak{C}_{p}(M)$, the relation being that two functions are equivalent if they are equal on some neighborhood of $p$. This quotient space is denoted $\mathfrak{F}_{p}(M)$ and the elements are called
germs of functions at $p$. This is not unlike the idea that the space of $L^{2}$ functions is really supposed to be a quotient space where we divide out by the subspace of functions that vanish almost everywhere.

Now consider a curve $c: I \rightarrow M$ with $c\left(t_{0}\right)=p$. The goal is to make sense of the velocity of $c$ at $t_{0}$. If $f \in \mathfrak{C}_{p}(M)$, then $f \circ c$ measures how $c$ changes with respect to $f$. If $f$ had been a coordinate function this would be the corresponding coordinate component of $c$ in a chart. Similarly the derivative $\frac{d}{d t}(f \circ c)$ measures the change in velocity with respect to $f$, i.e., what should be the $f$-component of the velocity.

DEFINITION 2.1.1. The velocity $\dot{c}\left(t_{0}\right)$ of $c$ at $t_{0}$ is the map

$$
\begin{aligned}
\mathfrak{C}_{p}(M) & \rightarrow \mathbb{R} \\
f & \mapsto \frac{d}{d t}(f \circ c)\left(t_{0}\right) .
\end{aligned}
$$

Thus $\dot{c}\left(t_{0}\right)$ is implicitly defined by specifying how it creates directional derivatives

$$
D_{\dot{c}\left(t_{0}\right)} f=\frac{d}{d t}(f \circ c)\left(t_{0}\right)
$$

for all smooth functions defined on a neighborhood of $p=c\left(t_{0}\right)$.

DEFINITION 2.1.2. A derivation at $p$ or on $\mathfrak{C}_{p}(M)$ is a linear map $D: \mathfrak{C}_{p}(M) \rightarrow \mathbb{R}$ that also satisfies the product rule for differentiation at $p$ :

$$
D(f g)=D(f) g(p)+f(p) D(g)
$$

There is an alternate way of defining derivations as linear functions on $\mathfrak{C}_{p}(M)$. Let $\mathfrak{C}_{p}^{0}(M) \subset \mathfrak{C}_{p}(M)$ be the maximal ideal of functions that vanish at $p$ and $\left(\mathfrak{C}_{p}^{0}(M)\right)^{2} \subset$ $\mathfrak{C}_{p}^{0}(M)$ the ideal generated by products of elements in $\mathfrak{C}_{p}^{0}(M)$.

LEMMA 2.1.3. The derivations at p are isomorphic to the subspace of linear maps on $\mathfrak{C}_{p}^{0}(M)$ that vanish on $\left(\mathfrak{C}_{p}^{0}(M)\right)^{2}$.

Proof. If $D$ is a derivation, then the derivation property shows that it vanishes on $\left(\mathfrak{C}_{p}^{0}(M)\right)^{2}$. Furthermore, it also vanishes on constant functions as linearity and the derivation property implies

$$
D(a)=a D(1)=a D(1 \cdot 1)=a(D(1)+D(1)) .
$$

Conversely, any linear map $D$ on $\mathfrak{C}_{p}^{0}(M)$ that vanishes on $\left(\mathfrak{C}_{p}^{0}(M)\right)^{2}$ defines a unique linear map on $\mathfrak{C}_{p}(M)$ by also defining it to vanish on constant functions. If $f, g \in \mathfrak{C}_{p}(M)$, then we have

$$
\begin{aligned}
0 & =D((f-f(p))(g-f(p))) \\
& =D(f g)-f(p) D g-g(p) D f+D(f(p) g(p)) \\
& =D(f g)-f(p) D g-g(p) D f
\end{aligned}
$$

showing that it is a derivation.
Next we show that derivations exist.
PROPOSITION 2.1.4. The map $f \mapsto \frac{d}{d t}(f \circ c)\left(t_{0}\right)$ is a derivation on $\mathfrak{C}_{p}(M)$.

Proof. That it is linear in $f$ is obvious from the fact that differentiation is linear. The derivation property follows from the product rule for differentiation:

$$
\frac{d}{d t}((f g) \circ c)\left(t_{0}\right)=\left(\frac{d}{d t}(f \circ c)\left(t_{0}\right)\right)(g \circ c)\left(t_{0}\right)+(f \circ c)\left(t_{0}\right) \frac{d}{d t}(g \circ c)\left(t_{0}\right) .
$$

DEFINITION 2.1.5. The tangent space $T_{p} M$ for $M$ at $p$ is the vector space of derivations on $\mathfrak{C}_{p}(M)$.

PROPOSITION 2.1.6. If $p \in U \subset M$, where $U$ is open, then $T_{p} U=T_{p} M$.
Proof. We already saw that derivations must vanish on constant function. Next consider a function $f$ that vanishes on a neighborhood of $p$. We can then find $\lambda: M \rightarrow \mathbb{R}$ that is 1 on a neighborhood of $p$ and $\lambda=0$ on the complement of the region where $f$ vanishes. Thus $\lambda f=0$ on $M$ and

$$
0=D(\lambda f)=D(\lambda) f(p)+\lambda(p) D(f)=D(f)
$$

This in turns shows that if two functions $f, g$ agree on a neighborhood of $p$, then $D(f)=D(g)$. This means that a derivation $D$ on $\mathfrak{C}_{p}(M)$ restricts to a derivation on $\mathfrak{C}_{p}(U)$ and conversely that any derivation on $\mathfrak{C}_{p}(U)$ also defines a derivation on $\mathfrak{C}_{p}(M)$. This proves the claim.

We are now ready to prove that there are no more derivations than one would expect.
LEMMA 2.1.7. The natural map $\mathbb{R}^{n} \rightarrow T_{0} \mathbb{R}^{n}$ that maps $v$ to $D_{v} f=\left.\left(\frac{d f}{d t}\right)(t v)\right|_{t=0}$ is an isomorphism.

Proof. The map is clearly linear and as

$$
D_{v} x^{i}=v^{i}
$$

it follows that its kernel is trivial. Thus we need to show that it is surjective. This claim depends crucially on the fact that derivations are defined on $C^{\infty}$ functions. The key observation is that we have a Taylor formula

$$
f(x)=f(0)+x^{i} f_{i}(x)
$$

where $f_{i}$ are also smooth and $f_{i}(0)=\frac{\partial f}{\partial x^{i}}(0)$. The functions are defined by

$$
f_{i}(x)=\int_{0}^{1} \frac{\partial f}{\partial x^{i}}(t x) d t
$$

and the formula follows from the fundamental theorem of calculus applied to the identity:

$$
\frac{d}{d t}(f(t x))=x^{i} \frac{\partial f}{\partial x^{i}}(t x)
$$

Now select an abstract derivation $D \in T_{0} \mathbb{R}^{n}$ and observe that

$$
D(f)=D(f(0))+D\left(x^{i}\right) f_{i}(0)+0 D\left(f_{i}\right)=\frac{\partial f}{\partial x^{i}}(0) D\left(x^{i}\right)
$$

So if we define a vector $v=\left(D\left(x^{1}\right), \ldots, D\left(x^{n}\right)\right)$, then in fact

$$
D(f)=D_{v}(f)
$$

REMARK 2.1.8. The space of linear maps on $C^{k}\left(\mathbb{R}^{n}\right), 1 \leq k<\infty$ that satisfy the product rule

$$
D(f g)=D(f) g(0)+f(0) D(g)
$$

is infinite dimensional! It clearly suffices to show this for $n=1$. Observe that if $Z \subset C^{k}(\mathbb{R})$ is the subset of functions that vanish at 0 , then we merely need to show that $Z / Z^{2}$ is infinite dimensional. To see this first note that if $f$ is $C^{0}$ and $g \in Z$, then $f g$ is differentiable with derivative $f(0) g^{\prime}(0)$ at 0 . This in turn implies that functions in $Z^{2}$ are not only $C^{k}$ but also have derivatives of order $k+1$ at 0 . However, there is a vast class of functions in $Z$ that do not have derivatives of order $k+1$ at 0 .
2.1.3. Concrete Derivations. To avoid the issue of crucially using $C^{\infty}$ functions we give an alternate definition of the tangent space that obviously gives the above definition.

DEFINITION 2.1.9. $T_{p} M$ is the space of derivations that are constructed from the derivations coming from curves that pass through $p$.

Without the above result it is not obvious that this is a vector space so a little more work is needed.

Proposition 2.1.10. If $x^{1}, \ldots, x^{n}$ are coordinates on a neighborhood of $p$, then two curves $c_{i}$ passing through $p$ at $t=0$ define the same derivations if and only if for all $i=1, \ldots, n$

$$
\frac{d\left(x^{i} \circ c_{1}\right)}{d t}(0)=\frac{d\left(x^{i} \circ c_{2}\right)}{d t}(0) .
$$

Proof. The necessity is obvious. Conversely note that any $f \in \mathfrak{C}_{p}(M)$ can be expressed smoothly as $f=F\left(x^{1}, \ldots, x^{n}\right)$ on some neighborhood of $p$. Thus

$$
\begin{aligned}
\frac{d\left(f \circ c_{1}\right)}{d t}(0) & =\frac{d\left(F\left(x^{1} \circ c_{1}, \ldots, x^{n} \circ c_{1}\right)\right)}{d t}(0) \\
& =\frac{\partial F}{\partial x^{i}} \frac{d\left(x^{i} \circ c_{1}\right)}{d t}(0) \\
& =\frac{\partial F}{\partial x^{i}} \frac{d\left(x^{i} \circ c_{2}\right)}{d t}(0) \\
& =\frac{d\left(f \circ c_{2}\right)}{d t}(0)
\end{aligned}
$$

PROPOSITION 2.1.11. The subset of derivations on $\mathfrak{C}_{p}(M)$ that come from curves through p form a subspace.

Proof. First note that for a curve $c$ through $p$ we have

$$
\alpha \frac{d(f \circ c)}{d t}(0)=\frac{d(f \circ c)(\alpha t)}{d t}(0) .
$$

So scalar multiplication preserves this subset.
Next assume that we have two curves $c_{i}$ and select a coordinate system $x^{i}$ around $p$. Define

$$
c=x^{-1}\left(x^{1} \circ c_{1}+x^{1} \circ c_{2}, \ldots, x^{n} \circ c_{1}+x^{n} \circ c_{2}\right)
$$

where $x^{-1}$ is the inverse of the chart map $x: U \rightarrow V \subset \mathbb{R}^{n}$. Then

$$
x^{i} \circ c=x^{i} \circ c_{1}+x^{i} \circ c_{2}
$$

and

$$
\frac{d(f \circ c)}{d t}(0)=\frac{d\left(f \circ c_{1}\right)}{d t}(0)+\frac{d\left(f \circ c_{2}\right)}{d t}(0)
$$

This shows that addition of such derivations also remain in this subset.
DEFINITION 2.1.12. The velocity of a curve $c: I \rightarrow M$ at $t_{0}$ is denoted by $\dot{c}\left(t_{0}\right) \in$ $T_{c\left(t_{0}\right)} M$ and is the derivation corresponding to the map:

$$
f \mapsto \frac{d(f \circ c)}{d t}\left(t_{0}\right) .
$$

As any vector $v \in T_{p} M$ can be written as $v=\dot{c}\left(t_{0}\right)$ we can also define the directional derivative of $f$ by

$$
D_{v} f=\frac{d(f \circ c)}{d t}\left(t_{0}\right) .
$$

2.1.4. Local Coordinate Formulas, Differentials, and the Tangent Bundle. Finally let us use coordinates to specify a basis for the tangent space. Fix $p \in M$ and a coordinate system $x^{i}$ around $p$. For any $f \in \mathfrak{C}_{p}(M)$ write $f=F\left(x^{1}, \ldots, x^{n}\right)$ and define

$$
\frac{\partial f}{\partial x^{i}}=\frac{\partial F}{\partial x^{i}}
$$

The map $f \mapsto \frac{\partial f}{\partial x^{i}}(p)$ is a derivation on $\mathfrak{C}_{p}(M)$. We denote it by $\left.\frac{\partial}{\partial x^{i}}\right|_{p}$. These tangent vectors in fact form a basis as we saw that

$$
D(f)=\left.D\left(x^{i}\right) \frac{\partial f}{\partial x^{i}}\right|_{p}
$$

i.e.,

$$
D=\left.v^{i} \frac{\partial}{\partial x^{i}}\right|_{p}
$$

where the components $v^{i}$ are uniquely determined. Moreover, as

$$
\frac{d(f \circ c)}{d t}(0)=\left.\frac{\partial f}{\partial x^{i}}\right|_{p} \frac{d\left(x^{i} \circ c\right)}{d t}(0)
$$

we also get this as a natural basis if we use curves to define the tangent space.
Definition 2.1.13. The cotangent space $T_{p}^{*} M$ to $M$ at $p \in M$ is the vector space of linear functions on $T_{p} M$. Alternately this can also be defined as the quotient space $\mathfrak{C}_{p}^{0}(M) /\left(\mathfrak{C}_{p}^{0}(M)\right)^{2}$ without even referring to tangent vectors.

Using coordinates we obtain a natural dual basis $d x^{i}$ satisfying

$$
d x^{i}\left(\frac{\partial}{\partial x^{j}}\right)=\frac{\partial x^{i}}{\partial x^{j}}=\delta_{j}^{i}
$$

In particular,

$$
d x^{i}(v)=d x^{i}\left(v^{j} \frac{\partial}{\partial x^{j}}\right)=v^{i}
$$

calculates the $i^{\text {th }}$ coordinate of a vector.
We also obtain a natural set of transformation laws when we have another coordinate system $y^{i}$ around $p$ :

$$
d y^{i}=\frac{\partial y^{i}}{\partial x^{j}} d x^{j}
$$

and

$$
\frac{\partial}{\partial y^{i}}=\frac{\partial x^{j}}{\partial y^{i}} \frac{\partial}{\partial x^{j}}
$$

Here the matrices $\left[\frac{\partial y^{i}}{\partial x^{j}}\right]$ and $\left[\frac{\partial x^{j}}{\partial y^{i}}\right]$ have entries that are functions on the common domain of the charts and are inverses of each other. These are also the natural transformation laws for a change of basis as well as the change of the dual basis.

The differential $d$ also has a coordinate free definition. Let $f \in \mathfrak{C}_{p}(M)$, then we can define $d f \in T_{p}^{*} M$ by

$$
d f(v)=D_{v} f=\frac{d(f \circ c)}{d t}(0)
$$

if $c$ is a curve with $\dot{c}(0)=v$. In coordinates we already know that

$$
d f(v)=\frac{\partial f}{\partial x^{i}} v^{i}
$$

so in fact

$$
d f=\frac{\partial f}{\partial x^{i}} d x^{i}
$$

This shows that our definition of $d x^{i}$ is consistent with the more abstract definition and that the transformation law for switching coordinates is simply just the law of how to write a vector or co-vector out in components with respect to a basis.

It now becomes very simple to define a differentiable structure on the tangent bundle $T M$. This space is the disjoint union of the tangent spaces $T_{p} M$ where $p \in M$. There is also a natural base point projection $p: T M \rightarrow M$ that takes a vector in $T_{p} M$ to its base point $p$. Starting with a differential system $\left\{f^{i}\right\}$ for $M$, we obtain a differentiable system $\left\{f^{i} \circ p, d f^{i}\right\}$ for $T M$. Moreover when $f^{1}, . ., f^{n}$ form a chart on $U \subset M$, then $f^{1} \circ p, \ldots, f^{n} \circ p, d f^{1}, \ldots, d f^{n}$ form a chart on $T U$. This takes us full circle back to our preliminary definition of tangent vectors.

IMPORTANT: The isomorphism between $T_{p} M$ and $\mathbb{R}^{n}$ depends on a choice of coordinates and is not canonically defined. We just saw that in a coordinate system we have a natural identification

$$
T U \rightarrow U \times \mathbb{R}^{n}
$$

which for fixed $p \in U$ yields a linear isomorphism

$$
T_{p} U \rightarrow\{p\} \times \mathbb{R}^{n} \simeq \mathbb{R}^{n}
$$

However, this does not mean that $T M$ has a natural map to $M \times \mathbb{R}^{n}$, that is a linear isomorphism when restricted to tangent spaces. Manifolds that admit such maps are called parallelizable. Euclidean space is parallelizable as are all matrix groups. But, as we shall see, $S^{2}$ is not parallelizable.

### 2.2. Derivatives of Maps and Vector Fields

2.2.1. Derivatives of Maps. Given a smooth function $F: M \rightarrow N$ we obtain a derivative or differential $\left.D F\right|_{p}: T_{p} M \rightarrow T_{F(p)} N$. If we let $D=v=\dot{c}(0) \in T_{p} M$ represent a tangent vector, then

$$
\begin{gathered}
\left.D F\right|_{p}(D)=D \circ F^{*}, \\
D_{\left.D F\right|_{p}(v)} f=D_{v}(f \circ F), \\
\left.D F\right|_{p}(v)=\left.\frac{d(F(c(t)))}{d t}\right|_{t=0} .
\end{gathered}
$$

When using coordinates around $p \in M$ we can also create the partial derivatives

$$
\frac{\partial F}{\partial x^{i}} \in T N
$$

as the velocities of the $x^{i}$-curves for $F \circ x^{-1}$ where the other coordinates are kept constant, in fact

$$
\left.\frac{\partial F}{\partial x^{i}}\right|_{p}=D F\left(\left.\frac{\partial}{\partial x^{i}}\right|_{p}\right)
$$

Note that $\frac{\partial F}{\partial x^{i}}$ is a function from (a subset of) $M$ to $T N$ which at $p \in M$ is mapped to $T_{F(p)} N$.
These partial derivatives represent the columns in a matrix representation for $D F$ since

$$
D F(v)=D F\left(\frac{\partial}{\partial x^{i}} v^{i}\right)=D F\left(\frac{\partial}{\partial x^{i}}\right) v^{i}=\frac{\partial F}{\partial x^{i}} v^{i} .
$$

If we also have coordinates at $F(p)$ in $N$, then

$$
D F(v)=\frac{\partial F}{\partial x^{i}} v^{i}=\frac{\partial\left(y^{j} \circ F\right)}{\partial x^{i}} v^{i} \frac{\partial}{\partial y^{j}}
$$

So the matrix representation for $D F$ is precisely the matrix of partial derivatives

$$
[D F]=\left[\frac{\partial\left(y^{j} \circ F\right)}{\partial x^{i}}\right]=\left[\frac{\partial\left(y^{j} \circ F \circ x^{-1}\right)}{\partial x^{i}}\right] .
$$

We can now reformulate what it means for a smooth function to be an immersion or submersion.

Definition 2.2.1. The smooth function $F: M \rightarrow N$ is an immersion if $\left.D F\right|_{p}$ is injective for all $p \in M$. It is a submersion if $\left.D F\right|_{p}$ is surjective for all $p \in M$.

REMARK 2.2.2. When we consider a map $F: M \rightarrow \mathbb{R}^{k}$, then we also have a differential

$$
d F=\left[\begin{array}{c}
d F^{1} \\
\vdots \\
d F^{k}
\end{array}\right]: T M \rightarrow \mathbb{R}^{k}
$$

The identification $I: \mathbb{R}^{k} \times \mathbb{R}^{k} \rightarrow T \mathbb{R}^{k}$ defined by $I(p, v)=\left.\frac{d}{d t}(p+t v)\right|_{t=0}$ shows that $D F=$ $I(F, d F)$.

We can use differentials together with some elementary linear algebra to prove an interesting result for retracts. A retract $F: M \rightarrow M$ is an idempotent map, $F \circ F=F$. Linear projections are examples of retracts.

THEOREM 2.2.3. If $F: M \rightarrow M$ satisfies $F \circ F=F$, then the image $F(M) \subset M$ is $a$ submanifold.

Proof. First note that the image is a closed subset. By the rank theorem 1.4.23) it suffices to show that $F$ has constant rank on a open set that contains the image. First fix $p \in F(M)$. This is clearly a fixed point of $F$ and $\left.D F\right|_{p}: T_{p} M \rightarrow T_{p} M$ also satisfies $\left(\left.D F\right|_{p}\right)^{2}=\left.D F\right|_{p}$. In particular, $\left.D F\right|_{p}$ can only have eigenvalues 0 or 1 and $\left.\operatorname{ker} D F\right|_{p} \oplus$ $\left.\operatorname{im} D F\right|_{p}=T_{p} M$. Moreover $\left.\operatorname{ker} D F\right|_{p}=\operatorname{im}\left(i d_{T_{p} M}-\left.D F\right|_{p}\right)$ so it follows that

$$
\operatorname{rank}\left(i d_{T_{p} M}-\left.D F\right|_{p}\right)+\left.\operatorname{rank} D F\right|_{p}=n=\operatorname{dim} M
$$

As both ranks can only be the same or larger (see remark 1.4.22 for points near $p$ it follows that they are constant on $F(M)$. Let $k=\left.\operatorname{rank} D F\right|_{p}$. For a general $x \in M$ with $F(x)=p$ we have

$$
\left.\left.D F\right|_{p} \circ D F\right|_{x}=\left.D F\right|_{x}
$$

implying that $\operatorname{rank} D F \leq k$ on $M$. However, as $\operatorname{rank} D F \geq k$ is an open condition there must be an open set containing $F(M)$ where $\operatorname{rank} D F=k$.
2.2.2. Vector Fields. A vector field is a smooth map (called a section) $X: M \rightarrow T M$ such that $\left.X\right|_{p} \in T_{p} M$. We use $\left.X\right|_{p}$ instead of $X(p)$ as $\left.X\right|_{p}$ is a map that can also be evaluated on function. In fact we note that we obtain a derivation

$$
D_{X}: C^{\infty}(M) \rightarrow C^{\infty}(M)
$$

by defining

$$
\left(D_{X} f\right)(p)=D_{X \mid p} f
$$

Conversely any such derivation corresponds to a vector field in the same way that tangent vectors correspond to derivations at a point.

In local coordinates we obtain

$$
X=D_{X}\left(x^{i}\right) \frac{\partial}{\partial x^{i}}
$$

Given two vector fields $X$ and $Y$ we can construct their Lie bracket $[X, Y]$ implicitly as a derivation

$$
D_{[X, Y]}=D_{X} D_{Y}-D_{Y} D_{X}=\left[D_{X}, D_{Y}\right] .
$$

This clearly defines a linear map and is a derivation as

$$
\begin{aligned}
D_{[X, Y]}(f g)= & D_{X}\left(g D_{Y} f+f D_{Y} g\right)-D_{Y}\left(g D_{X} f+f D_{X} g\right) \\
= & D_{X} g D_{Y} f+D_{X} f D_{Y} g+g D_{X} D_{Y} f+f D_{X} D_{Y} g \\
& -D_{Y} g D_{X} f+-D_{Y} f D_{X} g-g D_{Y} D_{X} f-f D_{Y} D_{X} g \\
= & g\left[D_{X}, D_{Y}\right] f+f\left[D_{X}, D_{Y}\right] g .
\end{aligned}
$$

In local coordinates this is conveniently calculated by ignoring second order partial derivatives:

$$
\begin{aligned}
{\left[X^{i} \frac{\partial}{\partial x^{i}}, Y^{j} \frac{\partial}{\partial x^{j}}\right]=} & X^{i} \frac{\partial Y^{j}}{\partial x^{i}} \frac{\partial}{\partial x^{j}}-Y^{j} \frac{\partial X^{i}}{\partial x^{j}} \frac{\partial}{\partial x^{i}} \\
& +X^{i} Y^{j} \frac{\partial^{2}}{\partial x^{i} \partial x^{j}}-Y^{j} X^{i} \frac{\partial^{2}}{\partial x^{j} \partial x^{i}} \\
= & X^{i} \frac{\partial Y^{j}}{\partial x^{i}} \frac{\partial}{\partial x^{j}}-Y^{j} \frac{\partial X^{i}}{\partial x^{j}} \frac{\partial}{\partial x^{i}} \\
= & \left(X^{j} \frac{\partial Y^{i}}{\partial x^{j}}-Y^{j} \frac{\partial X^{i}}{\partial x^{j}}\right) \frac{\partial}{\partial x^{i}} .
\end{aligned}
$$

Since tangent vectors are also velocities to curves it would be convenient if vector fields had a similar interpretation. A curve $c(t)$ such that

$$
\dot{c}(t)=\left.X\right|_{c(t)}
$$

is called an integral curve for $X$. Given an initial value $p \in M$, there is in fact a unique integral curve $c(t)$ such that $c(0)=p$ and it is defined on some maximal interval $I$ that contains 0 as an interior point.

In local coordinates we can write $x^{i} \circ c(t)=x^{i}(t)$ and $X=v^{i} \frac{\partial}{\partial x^{i}}$. The condition that $c$ is an integral curve then comes down to

$$
\dot{c}(t)=\frac{d x^{i}}{d t} \frac{\partial}{\partial x^{i}}=v^{i} \frac{\partial}{\partial x^{i}}
$$

or more precisely

$$
\frac{d x^{i}}{d t}(t)=v^{i}(c(t))
$$

This is a first order ODE and as such will have a unique solution given an initial value.
To get a maximal interval for an integral curve we have to use the local uniqueness of solutions and patch them together through a covering of coordinate charts.

We state the main theorem on integral curves that will be used again and again.
THEOREM 2.2.4. Let $X$ be a vector field on a manifold $M$. For each $p \in M$ there is a unique integral curve $c_{p}(t): I_{p} \rightarrow M$ where $c_{p}(0)=p, \dot{c}_{p}(t)=X_{c_{p}(t)}$ for all $t \in I_{p}$, and $I_{p}$ is the maximal open interval for any curve satisfying these two properties. Moreover, the map $(t, p) \mapsto c_{p}(t)$ is defined on an open subset of $\mathbb{R} \times M$ and is smooth. Finally, for given $p \in M$ the interval $I_{p}$ either contains $[0, \infty)$ or $c_{p}(t)$ is not contained in a compact set as $t \rightarrow \infty$.

Proof. The first part is simply existence and uniqueness of solutions to ODEs. The second part is that such solutions depend smoothly on initial data. This is far more subtle to prove (see e.g. Michel and Miller). The last statement is a basic compactness argument.

We use the general notation that $\Phi_{X}^{t}(p)=c_{p}(t)$ is the flow corresponding to a vector field $X$, i.e.

$$
\frac{d}{d t} \Phi_{X}^{t}=\left.X\right|_{\Phi_{X}^{t}}=X \circ \Phi_{X}^{t}
$$

REMARK 2.2.5. If we have a smooth family of vector fields $X_{\lambda}: L \times M \rightarrow T M, \lambda \in L$, then the corresponding flows $\Phi_{X_{\lambda}}^{t}$ as also smooth with respect to $\lambda$ (see e.g. Michel and Miller).

Let $F: M^{m} \rightarrow N^{n}$ be a smooth map between manifolds. If $X$ is a vector field on $M$ and $Y$ a vector field on $N$, then we say that $X$ and $Y$ are $F$-related provided $D F\left(\left.X\right|_{p}\right)=\left.Y\right|_{F(p)}$, or in other words $D F(X)=Y \circ F$. Given that tangent vectors are defined as derivations we note that it is equivalent to say that for all $f \in C^{\infty}(N)$ we have $\left(D_{Y} f\right) \circ F=D_{X}(f \circ F)$. In particular, when $X_{i}$ are $F$-related to $Y_{i}$ for $i=1,2$, it follows that $\left[X_{1}, X_{2}\right]$ is $F$-related to $\left[Y_{1}, Y_{2}\right]$.

We can also relate this concept to the integral curves for the vector fields.
Proposition 2.2.6. $X$ and $Y$ are $F$-related iff $F \circ \Phi_{X}^{t}=\Phi_{Y}^{t} \circ F$ whenever both sides are defined.

Proof. Assuming that $F \circ \Phi_{X}^{t}=\Phi_{Y}^{t} \circ F$ we have

$$
\begin{aligned}
D F(X) & =D F\left(\left.\frac{d}{d t}\right|_{t=0} \Phi_{X}^{t}\right) \\
& =\left.\frac{d}{d t}\right|_{t=0}\left(F \circ \Phi_{X}^{t}\right) \\
& =\left.\frac{d}{d t}\right|_{t=0}\left(\Phi_{Y}^{t} \circ F\right) \\
& =Y \circ \Phi_{Y}^{0} \circ F \\
& =Y \circ F .
\end{aligned}
$$

Conversely $D F(X)=Y \circ F$ implies that

$$
\begin{aligned}
\frac{d}{d t}\left(F \circ \Phi_{X}^{t}\right) & =D F\left(\frac{d}{d t} \Phi_{X}^{t}\right) \\
& =D F\left(\left.X\right|_{\Phi_{X}^{t}}\right) \\
& =\left.Y\right|_{F \circ \Phi_{X}^{t}}
\end{aligned}
$$

This shows that $t \mapsto F \circ \Phi_{X}^{t}$ is an integral curve for $Y$. At $t=0$ it agrees with the integral curve $t \mapsto \Phi_{Y}^{t} \circ F$ so by uniqueness we obtain $F \circ \Phi_{X}^{t}=\Phi_{Y}^{t} \circ F$.

### 2.3. Vector Bundles

Define vector bundles and give examples in the form of tangent, cotangent, normal, and trivial bundles. Pull-backs. Normal bundles to preimages are trivial. Use the section from Spivak vol 1 ed 3.

Good check for being a vb is that one can locally find sections that form a bases for the fibers. Subbundle is equivalent to having a bundle map that is a projection onto the subbundle.
2.3.1. Bundles over Projective Spaces. In this short final section we put together several of the concepts from this section to discuss projective spaces and their associated bundles.

The tautological or canonical line bundle is defined as

$$
\tau\left(\mathbb{P}^{n}\right)=\left\{(p, v) \in \mathbb{P}^{n} \times \mathbb{F}^{n+1} \mid v \in p\right\}
$$

This is a natural subbundle of the trivial vector bundle $\mathbb{P}^{n} \times \mathbb{F}^{n+1}$ and consequently has a natural orthogonal complement

$$
\tau^{\perp}\left(\mathbb{P}^{n}\right) \simeq\left\{(p, v) \in \mathbb{P}^{n} \times \mathbb{F}^{n+1} \mid p \perp v\right\}
$$

Note that in the complex case we are using Hermitian orthogonality. These are related to the tangent bundle in an interesting fashion. From our coordinatization around a point $p \in \mathbb{P}^{n}$ as in subsection 1.2.3 where we think of $p \subset \mathbb{F}^{n+1}$ as a 1-dimensional subspace we see that

$$
T_{p} \mathbb{P}^{n} \simeq \operatorname{Hom}\left(p, p^{\perp}\right)
$$

and globally

$$
T \mathbb{P}^{n} \simeq \operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \tau^{\perp}\left(\mathbb{P}^{n}\right)\right)
$$

For each $p \in \mathbb{P}^{n}$ these bundles are trivial over the coordinate neighborhood $\mathbb{P}^{n}-\mathbb{P}\left(p^{\perp}\right)$.

Note that the fibrations $\tau\left(\mathbb{P}^{n}\right) \rightarrow \mathbb{P}^{n}$ and $\mathbb{F}^{n+1}-\{0\} \rightarrow \mathbb{P}^{n}$ are suspiciously similar. The latter has fibers $p-\{0\}$ where the former has $p$. Thus the latter fibration can be identified with the nonzero vectors in $\tau\left(\mathbb{P}^{n}\right)$. In other words the missing 0 in $\mathbb{F}^{n+1}-\{0\}$ is replaced by the zero section in $\tau\left(\mathbb{P}^{n}\right)$ in order to create a larger bundle. This process is called a blow up of the origin in $\mathbb{F}^{n+1}$. Essentially we have a map $\tau\left(\mathbb{P}^{n}\right) \rightarrow \mathbb{F}^{n+1}$ that maps the zero section to 0 and is otherwise a bijection. We can use $\mathbb{F}^{n+1}-\{0\} \rightarrow \mathbb{P}^{n}$ to create a new fibration by restricting it to the unit sphere $S \subset \mathbb{F}^{n+1}-\{0\}$.

The conjugate to the tautological bundle can also be seen internally in $\mathbb{P}^{n+1}$ as the map

$$
\mathbb{P}^{n+1}-\{p\} \rightarrow \mathbb{P}^{n}
$$

where $\mathbb{P}^{n+1}-\{p\}$ is a tubular neighborhood of $\mathbb{P}^{n} \subset \mathbb{P}^{n+1}$. When $p=[1: 0: \cdots: 0]$ this fibration is given by

$$
\left[z: z^{0}: \cdots: z^{n}\right] \rightarrow\left[z^{0}: \cdots: z^{n}\right]
$$

and looks like a vector bundle if we use fiberwise addition and scalar multiplication in the $z$-variable.

The equivalence is obtained by mapping

$$
\begin{gathered}
\mathbb{P}^{n+1}-\{[1: 0: \cdots: 0]\} \rightarrow \tau\left(\mathbb{P}^{n}\right) \\
{\left[z: z^{0}: \cdots: z^{n}\right] \rightarrow\left(\left[z^{0}: \cdots: z^{n}\right], \bar{z} \frac{\left(z^{0}, \ldots, z^{n}\right)}{\left|\left(z^{0}, \ldots, z^{n}\right)\right|^{2}}\right)}
\end{gathered}
$$

It is necessary to conjugate $z$ to get a well-defined map. This is why the identification is only conjugate linear. The conjugate to the tautological bundle can also be identified with the dual bundle $\operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \mathbb{C}\right)$ via the natural inner product structure coming from $\tau\left(\mathbb{P}^{n}\right) \subset \mathbb{P}^{n} \times \mathbb{F}^{n+1}$. The relevant linear functional corresponding to $\left[z: z^{0}: \cdots: z^{n}\right]$ is given by

$$
v \rightarrow\left\langle v, \bar{z} \frac{\left(z^{0}, \ldots, z^{n}\right)}{\left|\left(z^{0}, \ldots, z^{n}\right)\right|^{2}}\right\rangle
$$

This functional appears to be defined on all of $\mathbb{F}^{n+1}$, but as it vanishes on the orthogonal complement to $\left(z^{0}, \ldots, z^{n}\right)$ we only need to consider the restriction to span $\left\{\left(z^{0}, \ldots, z^{n}\right)\right\}=$ $\left[z^{0}: \cdots: z^{n}\right]$.

### 2.4. Frobenius

Involutive, integrable,
If $F: N \rightarrow M$ is smooth and the image lies in a maximal integral submanifold $L \subset M$, then $F: N \rightarrow L$ is also smooth.

### 2.5. Exercises

(1) A manifold $M^{n}$ is said to be parallelizable if $T M \simeq M \times \mathbb{R}^{n}$. Show that $M$ is parallelizable if and only if there are $n$ vector fields that span the tangent space at every point.
(2) Show that $\mathbb{R} \times S^{n}$ and $S^{1} \times S^{n}$ are parallelizable.
(3) Show that $S^{p} \times S^{q}$ is parallelizable if $p$ or $q$ is odd.
(4) Show that $S^{3}$ is parallelizable.

## CHAPTER 3

## Submersions and Immersions

### 3.1. Submersions

In this section we present a number of results about the deeper structure of submersions.
3.1.1. Submersion-Fibrations. We study the relationship of the topologies of the manifolds related to a submersion.

In case $F$ is a submersion it is possible to construct vector fields in $M$ that are $F$-related to a given vector field in $N$.

Proposition 3.1.1. Assume that $F$ is a submersion. Given a vector field $Y$ in $N$, there are vector fields $X$ in $M$ that are $F$-related to $Y$.

Proof. First we do a local construction of $X$. Since $F$ is a submersion proposition 1.4.18 shows that for each $p \in M$ there are charts $x: U \rightarrow \mathbb{R}^{m}$ and $y: V \rightarrow \mathbb{R}^{n}$ with $p \in U$ and $F(p) \in V$ such that

$$
y \circ F \circ x^{-1}\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{n}\right) .
$$

This relationship evidently implies that $\frac{\partial}{\partial y^{i}}$ and $\frac{\partial}{\partial x^{i}}$ are $F$-related for $i=1, \ldots, n$. Thus, if we write $Y=Y^{i} \frac{\partial}{\partial y^{i}}$, then we can simply define $X=\sum_{i=1}^{n} Y^{i} \circ F \frac{\partial}{\partial x^{i}}$. This gives the local construction.

For the global construction assume that we have a covering $U_{\alpha}$, vector fields $X_{\alpha}$ on $U_{\alpha}$ that are $F$-related to $Y$, and a partition of unity $\lambda_{\alpha}$ subordinate to $U_{\alpha}$. Then simply define $X=\sum \lambda_{\alpha} X_{\alpha}$ and note that

$$
\begin{aligned}
D F(X) & =D F\left(\sum \lambda_{\alpha} X_{\alpha}\right) \\
& =\sum \lambda_{\alpha} D F\left(X_{\alpha}\right) \\
& =\sum \lambda_{\alpha} Y \circ F \\
& =Y \circ F .
\end{aligned}
$$

Finally we can say something about the maximal domains of definition for the flows of $F$-related vector fields given $F$ is proper.

Proposition 3.1.2. Assume that $F$ is proper and that $X$ and $Y$ are $F$-related vector fields. If $F(p)=q$ and $\Phi_{Y}^{t}(q)$ is defined on $[0, b)$, then $\Phi_{X}^{t}(p)$ is also defined on $[0, b)$. In other words the relation $F \circ \Phi_{X}^{t}=\Phi_{Y}^{t} \circ F$ holds for as long as the RHS is defined.

Proof. Assume $\Phi_{X}^{t}(p)$ is defined on $[0, a)$. If $a<b$, then the set

$$
\begin{aligned}
K & =\left\{x \in M \mid F(x)=\Phi_{Y}^{t}(p) \text { for some } t \in[0, a]\right\} \\
& =F^{-1}\left(\left\{\Phi_{Y}^{t}(p) \mid t \in[0, a]\right\}\right)
\end{aligned}
$$

is compact in $M$ since $F$ is proper. The integral curve $t \mapsto \Phi_{X}^{t}(q)$ lies in $K$ since $F\left(\Phi_{X}^{t}(p)\right)=$ $\Phi_{Y}^{t}(q)$. From theorem 2.2.4 we know that a maximally defined integral curves are either defined for all time or leave every compact set. In particular, $[0, a)$ is not the maximal interval on which $t \mapsto \Phi_{X}^{t}(p)$ is defined.

These relatively simple properties lead to some very general and tricky results.
DEFINITION 3.1.3. A fibration $F: M \rightarrow N$ is a smooth map which is locally trivial in the sense that for every $p \in N$ there is a neighborhood $U$ of $p$ such that $F^{-1}(U)$ is diffeomorphic to $U \times F^{-1}(p)$. This diffeomorphism must commute with the natural maps of these sets on to $U$. In other words $(x, y) \in U \times F^{-1}(p)$ must be mapped to a point in $F^{-1}(x)$. Note that it is easy to destroy the fibration property by simply deleting a point in $M$. Note also that in this context fibrations are necessarily submersions.

Special cases of fibrations are covering maps and vector bundles. The Hopf fibration $S^{3} \rightarrow S^{2}=\mathbb{P}^{1}$ is a more non-trivial example of a fibration, which we shall study further below. Tubular neighborhoods are also examples of fibrations.

THEOREM 3.1.4 (Ehresman). If $F: M \rightarrow N$ is a proper submersion, then it is a fibration.

Proof. As far as $N$ is concerned this is a local result. In $N$ we simply select a set $U$ that is diffeomorphic to $\mathbb{R}^{n}$ and claim that $F^{-1}(U) \approx U \times F^{-1}(0)$. Thus we just need to prove the theorem in case $N=\mathbb{R}^{n}$, or more generally a coordinate box around the origin.

Next select vector fields $X_{1}, \ldots, X_{n}$ in $M$ that are $F$-related to the coordinate vector fields $\partial_{1}, \ldots, \partial_{n}$. Our smooth map $G: \mathbb{R}^{n} \times F^{-1}(0) \rightarrow M$ is then defined by $G\left(t^{1}, \ldots, t^{n}, x\right)=\Phi_{X_{1}}^{t^{1}} \circ$ $\cdots \circ \Phi_{X_{n}}^{t^{n}}(x)$. The inverse to this map is $G^{-1}(z)=\left(F(z), \Phi_{X_{n}}^{-t^{n}} \circ \cdots \circ \Phi_{X_{1}}^{-t^{1}}(z)\right)$, where $F(z)=\left(t^{1}, \ldots, t^{n}\right)$.

REMARK 3.1.5. Note that proposition 1.4 .20 shows in analogy with lemma 1.4 .28 that if $F: M \rightarrow N$ is a proper map and $y \in F(M)$ a regular value, then there is an open neighborhood $V \ni y$ such that $F^{-1}(V) \simeq V \times F^{-1}(y)$.

The theorem also unifies several different results.
Corollary 3.1.6 (Basic Lemma in Morse Theory). Let $F: M \rightarrow \mathbb{R}$ be a proper map. If $F$ is regular on $(a, b) \subset \mathbb{R}$, then $F^{-1}(a, b) \simeq F^{-1}(c) \times(a, b)$ where $c \in(a, b)$.

Corollary 3.1.7 (Reeb). Let $M$ be a closed manifold that admits a map with two critical points, then $M$ is homeomorphic to a sphere. (This is a bit easier to show if we also assume that the critical points are nondegenerate.)

THEOREM 3.1.8. Let $\pi: S \rightarrow B$ be a fibration, where $S$ is a sphere. If the fibration admits a section, then $B$ is either simply connected or $\operatorname{dim} B<\operatorname{dim} S$. In particular, the fibrations $S \rightarrow \mathbb{P}^{n}$ are nontrivial.

Proof. The proof uses that the identity map on is not homotopically trivial (see proposition 5.4.5.

When $\operatorname{dim} B=\operatorname{dim} S$ the fibration is a covering map and so must be a diffeomorphism when $B$ is simply connected.

In general a section $s: B \rightarrow S$ is a lift of the identity map on $B$. By Sard's theorem it follows that it must be homotopically trivial when $\operatorname{dim} B<\operatorname{dim} S$ as it can't be surjective. This in turn implies that the identity map on $B$ is homotopically trivial.

Finally we can extend the fibration theorem to the case when $M$ has boundary.
THEOREM 3.1.9. Assume that $M$ is a manifold with boundary and that $N$ is a manifold without boundary, if $F: M \rightarrow N$ is proper and a submersion on $M$ as well as on $\partial M$, then it is a fibration.

Proof. The proof is identical and reduced to the case when $N=\mathbb{R}^{n}$. The assumptions allow us to construct the lifted vector fields so that they are tangent to $\partial M$. The flows will then stay in $\partial M$ or int $M$ for all time if they start there.

REMARK 3.1.10. This theorem is sometimes useful when we have a submersion whose fibers are not compact. It is then occasionally possible to add a boundary to $M$ so as to make the map proper. A good example is a tubular neighborhood around a closed submanifold $S \subset U$. By possibly making $U$ smaller we can assume that it is a compact manifold with boundary such that the fibers of $U \rightarrow S$ are closed discs rather than open discs.

EXAMPLE 3.1.11. Consider the the projection $\mathbb{R}^{2} \rightarrow \mathbb{R}$ onto the first axis. This is clearly a submersion and a trivial bundle. The standard vector field $\partial_{x}$ on $\mathbb{R}$ can be lifted to the related field $\partial_{x}+y^{2} \partial_{y}$ on $\mathbb{R}^{2}$. However, the integral curves for this lifted field are not complete as they are given by $\left(t+t_{0}, \frac{x_{0}}{1-x_{0}\left(t+t_{0}\right)}\right)$ and diverge as $t$ approaches $\frac{1}{x_{0}}-t_{0}$. In particular, neither the above proportion or theorem 3.1.4 can be made to work when the submersion isn't proper even though the submersion is a trivial fibration.
3.1.2. Quotient Manifolds. If $M$ is a manifold and $\sim$ an equivalence relation on $M$ : when is $M / \sim$ a manifold and $\pi: M \rightarrow M / \sim$ a submersion? Clearly the equivalence classes must form a foliation and the leaves/equivalence classes be closed subsets of $M$. Also their normal bundles have to be trivial as preimages of regular values have trivial normal bundle.

The most basic and still very nontrivial case is that of a Lie group $G$ and a subgroup $H$. The equivalence classes are the cosets $g H$ in $G$ and the quotient space is $G / H$. When $H$ is dense in $G$ the quotient topology is not even Hausdorff. However one can prove that if $H$ is closed in $G$, so that the equivalence classes are all closed embedded submanifolds, then the quotient is a manifold and the quotient map a submersion.

A nasty example is $\mathbb{R}^{2}-\{0\}$ with the equivalence relation being that two points are equivalent if they have the same $x$-coordinate and lie in the same component of the corresponding vertical line. This means that the above general assumptions are not sufficient as all equivalence classes are closed embedded submanifolds with trivial normal bundles. The quotient space is the line with double origin and so is not Hausdorff!

REMARK 3.1.12. The key to getting a Hausdorff quotient is to assume that the graph of the equivalence relation

$$
R=\{(x, y) \mid x \sim y\} \subset M \times M
$$

is a proper submanifold. We can in fact find necessary and sufficient conditions that guarantee that the quotient space becomes a manifold and $\pi: M \rightarrow M / \sim$ a submersion. We let $\pi_{1,2}: M \times M \rightarrow M$ denote that projections onto the first and second factor. The equivalence class $\pi(p)$ that contains $p \in M$ is both a subset in $M$ and a point in the quotient. Note that

$$
\begin{aligned}
& R \cap(M \times\{p\})=\pi(p) \times\{p\}, \\
& R \cap(\{p\} \times M)=\{p\} \times \pi(p) .
\end{aligned}
$$

Proposition 3.1.13. If $M / \sim$ has a manifold structure such that $\pi: M \rightarrow M / \sim$ becomes a submersion, then $R \subset M \times M$ is a properly embedded submanifold and the restrictions of the projection maps $\left.\pi_{1,2}\right|_{R}: R \subset M \times M \rightarrow M$ are submersions.

Proof. Note that a submanifold is properly embedded exactly when it is a closed subset of the ambient manifold.

Consider the graph

$$
G(\pi)=\{(p, \pi(p)) \in M \times(M / \sim) \mid p \in M\}
$$

We have that $\mathrm{id} \times \pi: M \times M \rightarrow M \times(M / \sim)$ is a submersion and $R=(\mathrm{id} \times \pi)^{-1}(G(\pi))$. Since $G(\pi)$ is a properly embedded submanifold $R$ also becomes a properly embedded submanifold. This also tells us that $\left.(\mathrm{id} \times \pi)\right|_{R}: R \rightarrow G(\pi)$ becomes a submersion. Composing this map with the diffeomorphism $\left.\pi_{1}\right|_{G(\pi)}: G(\pi) \rightarrow M$, then implies that $\left.\pi_{1}\right|_{R}: R \rightarrow M$ becomes a submersion. Since $R$ is invariant under the involution $(p, q) \mapsto(q, p)$ we have also shown that $\left.\pi_{2}\right|_{R}: R \rightarrow M$ is a submersion.

The converse is also true and offers a particularly nice characterization of quotient manifolds that rarely makes it into text books.

THEOREM 3.1.14 (Godement). If $\sim$ is an equivalence relation on a smooth manifold $M$, then $M / \sim$ has a manifold structure such that $\pi: M \rightarrow M / \sim$ becomes a submersion provided $R \subset M \times M$ is a properly embedded submanifold and the restriction $\left.\pi_{1}\right|_{R}: R \rightarrow M$ is a submersion.

Proof. We first settle the topological aspects of the quotient by showing that $\pi: M \rightarrow$ $M / \sim$ is open and that $M / \sim$ is Hausdorff. Let $O \subset M$ be open and note that by definition of the quotient topology that $\pi(O)$ is open precisely when $\pi^{-1}(\pi(O))$ is open. The latter set is open since,

$$
\pi^{-1}(\pi(O))=\pi_{1}((M \times O) \cap R)=\{q \in M \mid \exists q \in O: p \sim q\}
$$

and $\left.\pi_{1}\right|_{R}$ is a submersion and thus an open map. For the Hausdorff property fix two equivalence classes $\pi(p), \pi(q)$. Select shrinking open neighborhoods $U_{i} \ni p$ and $V_{i} \ni q$ with $\cap_{i} U_{i}=\{p\}$ and $\cap_{i} V_{i}=\{q\}$. If $\pi\left(U_{i}\right) \cap \pi\left(V_{i}\right) \neq \emptyset$ for all $i$, then there exists $x_{i} \in U_{i}$ and $y_{i} \in V_{i}$ such that $\pi\left(x_{i}\right)=\pi\left(y_{i}\right) \in \pi\left(U_{i}\right) \cap \pi\left(V_{i}\right)$. But then $x_{i} \rightarrow p, y_{i} \rightarrow q$, and $x_{i} \sim y_{i}$. Since $R$ is a closed set this implies that $p \sim q$ and consequently $\pi(p)=\pi(q)$.

Note that $\Delta$ and $M \times\{p\}$ are transverse at $(p, p)=\Delta \cap M \times\{p\}$, i.e.,

$$
T_{(p, p)}(M \times M)=T_{(p, p)} \Delta+T_{(p, p)}(M \times\{p\})
$$

In particular, $R \supset \Delta$ and $M \times\{p\}$ are transverse at $(p, p)$. The intersection of the tangent spaces, $T_{(p, p)} R \cap T_{(p, p)}(M \times\{p\})$, has dimension $k$ if $\operatorname{dim} R=n+k$. This intersection is naturally isomorphic to the $k$-dimensional space $E_{p}=\left\{v \in T_{p} M \mid(v, 0) \in T_{(p, p)} R\right\}$. In this way we obtain a subbundle of $T M$. Select a submanifold $W \subset M$ whose tangent space $T_{p} W$ is a complement to $E_{p} \subset T_{p} M$. Since also $\left.\pi_{2}\right|_{R}$ is a submersion a simple generalization of the preimage theorem 1.4 .25 shows that

$$
V=R \cap(M \times W)=\left(\left.\pi_{2}\right|_{R}\right)^{-1}(W)=\{(p, q) \mid q \in W, p \sim q\}
$$

is a submanifold of dimension $n$ as $\operatorname{codim} W=k$ and $\operatorname{dim} R=n+k$. We claim that the restriction $\left.\pi_{1}\right|_{V}: V \rightarrow M$ is nonsingular at $(p, p)$. Note that $(v, w) \in T_{(p, p)} V$ iff $(v, w) \in$ $T_{(p, p)} R$ and $w \in T_{p} W$. So if $\left.D \pi_{1}\right|_{V}(v, w)=v=0$, then $w \in E_{p} \cap T_{p} W=\{0\}$. This shows that we can find neighborhoods $p \in U_{1}, U_{2} \subset O$ such that $\left.\pi_{1}\right|_{R}: V \cap\left(U_{1} \cap U_{1}\right) \rightarrow U_{2}$ becomes
a diffeomorphism. The inverse has the form $f(x)=(x, r(x))$, in particular, $U_{2} \subset U_{1}$ and $r: U_{2} \rightarrow W$. Now define

$$
U=\left\{x \in U_{2} \mid r(x) \in U_{2} \cap W\right\} .
$$

Since $(r(x), r(x)) \in \Delta \subset R \cap(M \times W)$ it follows that $r^{2}=r$ and that $U$ is invariant under $r$ (as in theorem 2.2.3. Finally consider

$$
(x, y) \in R \cap(U \times(U \cap W)) \subset V
$$

As $\pi_{1}(x, y)=x$ and $f(x)=(x, r(x))=(x, y)$ we see that $r(x)$ is the only point in $U \cap W$ that is equivalent to $x$. In other words, if $x, y \in U$ are equivalent then $r(x)=r(y)$.

Now suppose that $W$ was chosen so as to have global coordinates $\phi=\left(x^{1}, \ldots, x^{n-k}\right)$. These descend to coordinates $\bar{\phi}=\left(\bar{x}^{1}, \ldots, \bar{x}^{n-k}\right)$ on the open set $\pi(U)=\pi(U \cap W)$. To obtain a differentiable structure we note that any function $f$ whose domain intersects $\pi(U \cap W)$ has the property that $f \circ \bar{\phi}^{-1}=f \circ \pi \circ \phi^{-1}$. Thus $f \circ \bar{\phi}^{-1}$ is smooth when $f \circ \pi$ is smooth. This shows if we can define a differentiable structure by declaring that $f \in \mathfrak{C}^{\infty}(M / \sim)$ if and only if $f \circ \pi \in \mathfrak{C}^{\infty}(M)$.

### 3.2. Embeddings

3.2.1. Embeddings into Euclidean Space. The goal is to show that any manifold is a proper submanifold of Euclidean space. This requires most importantly that we can find a way to reduce the dimension of the ambient Euclidean space into which the manifold can be embedded.

THEOREM 3.2.1 (Whitney Embedding, Dimension Reduction). If $F: M^{m} \rightarrow \mathbb{R}^{n}$ is an injective immersion, then there is also an injective immersion $M^{m} \rightarrow \mathbb{R}^{2 m+1}$. Moreover, if one of the coordinate functions of $F$ is proper, then we can keep this property. In particular, when $M$ is compact we obtain an embedding.

Proof. For each $v \in \mathbb{R}^{n}-\{0\}$ consider the orthogonal projection onto the orthogonal complement of $v$ :

$$
f_{v}(x)=x-\frac{(x \mid v) v}{|v|^{2}}
$$

The image is an $(n-1)$-dimensional subspace. So if we can show that $f_{v} \circ F$ is an injective immersion, then the ambient dimension has been reduced by 1 .

Note that $f_{v} \circ F(x)=f_{v} \circ F(y)$ iff $F(x)-F(y)$ is proportional to $v$. Similarly $d\left(f_{v} \circ F\right)(w)=$ 0 iff $d F(w)$ is proportional to $v$. With that in mind consider the images of the following two maps:

$$
\begin{aligned}
H: M \times M \times \mathbb{R} & \rightarrow \mathbb{R}^{n} \\
h(x, y, t) & =t(F(x)-F(y)) \\
G: T M & \rightarrow \mathbb{R}^{n} \\
G(w) & =d F(w)
\end{aligned}
$$

As long as $2 m+1<n$ Sard's theorem implies that the union of the two images has dense complement. Therefore, we can select $v \in \mathbb{R}^{n}-(H(M \times M \times \mathbb{R}) \cup G(T M))$. Clearly $v \neq 0$ as 0 lies in the image of both maps. So if $F(x)-F(y)=s v$, then either $s=0$ and $x=y$ or $v=\frac{1}{s}(F(x)-F(y))$ which is impossible. Similarly, if $d F(w)=s v$, then either $s=0$ showing that $w=0$ or $v=\frac{1}{s}(d F(w))$ which is impossible.

Note that the $v$ we selected could be taken from $O-(H(M \times M \times \mathbb{R}) \cup G(T M))$, where $O \subset \mathbb{R}^{n}$ is any open subset. This gives us a bit of extra information. While we can't get the ultimate map $M^{m} \rightarrow \mathbb{R}^{2 m+1}$ to target a specific $(2 m+1)$-dimensional subspace of $\mathbb{R}^{n}$, we can map it into a subspace arbitrarily close to a fixed subspace of dimension $2 m+1$. To be specific simply assume that $\mathbb{R}^{2 m+1} \subset \mathbb{R}^{n}$ consists of the first $2 m+1$ coordinates. By selecting $v \in(-\varepsilon, \varepsilon)^{2 m+1} \times(1-\varepsilon, 1+\varepsilon)^{n-2 m-1}$ we see that $f_{v}$ changes the first coordinates with an error that is small.

This can be used to obtain proper maps $f_{v} \circ F$. When the first coordinate for $F$ is proper, then $f_{v} \circ F$ is also proper provided $v$ is not proportional to $e_{1}$. This means that we merely have to select $v \in\left\{|v|<2 \mid\left(v \mid e_{1}\right)<\varepsilon\right\}$ to obtain a proper injective submersion.

REMARK 3.2.2. Note also that if $F$ starts out only being an immersion, then we can find an immersion into $\mathbb{R}^{2 m}$. This is because $G(T M) \subset \mathbb{R}^{n}$ has measure zero as long as $n>2 m$.

Lemma 3.2.3. If $A, B \subset M^{m}$ are open sets that both admit embeddings into $\mathbb{R}^{2 m+1}$, then the union $A \cup B$ also admits an embedding into $\mathbb{R}^{2 m+1}$.

Proof. Select a partition of unity $\lambda_{A}, \lambda_{B}: A \cup B \rightarrow[0,1]$, i.e., supp $\lambda_{A} \subset A$, supp $\lambda_{B} \subset$ $B$, and $\lambda_{A}+\lambda_{B}=1$. Further, choose embeddings $F_{A}: A \rightarrow \mathbb{R}^{2 m+1}$ and $F_{B}: B \rightarrow \mathbb{R}^{2 m+1}$. Note multiplying these embeddings with our bump functions we obtain well-defined maps $\lambda_{A} F_{A}, \lambda_{B} F_{B}: A \cup B \rightarrow \mathbb{R}^{2 m+1}$. This gives us a map

$$
\begin{aligned}
F: A \cup B & \rightarrow \mathbb{R}^{2 m+1} \times \mathbb{R}^{2 m+1} \times \mathbb{R} \times \mathbb{R}, \\
F(x) & =\left(\lambda_{A}(x) F_{A}(x), \lambda_{B}(x) F_{B}(x), \lambda_{A}(x), \lambda_{B}(x)\right),
\end{aligned}
$$

which we claim is an injective immersion.
If $F(x)=F(y)$, then $\lambda_{A, B}(x)=\lambda_{A, B}(y)$. If, e.g., $\lambda_{B}(x)>0$ then $F_{B}(x)=F_{B}(y)$. This shows that $x=y$ as $F_{B}$ is an injection.

If $d F(v)=0$ for $v \in T_{p} M$, then $d \lambda_{A, B}(v)=0$. So if, e.g., $\lambda_{A}(p)>0$, then by the product rule:

$$
\left.d\left(\lambda_{A} F_{A}\right)\right|_{p}=\left.\left(d \lambda_{A}\right)\right|_{p} F_{A}(p)+\left.\lambda_{A}(p) d F_{A}\right|_{p}=\left.\lambda_{A}(p) d F_{A}\right|_{p}
$$

and consequently

$$
\left.d F_{A}\right|_{p}(v)=0
$$

showing that $v=0$.
If, in addition, we select a proper function $\rho: A \cup B \rightarrow[0, \infty)$, then we obtain a proper injective immersion

$$
(\rho, F): A \cup B \rightarrow \mathbb{R} \times \mathbb{R}^{2 m+1} \times \mathbb{R}^{2 m+1} \times \mathbb{R} \times \mathbb{R}
$$

and consequently an embedding. The dimension reduction result above then gives us a (proper) embedding into $\mathbb{R}^{2 m+1}$.

THEOREM 3.2.4 (Whitney Embedding, Final Version). An m-dimensional manifold $M$ admits a proper embedding into $\mathbb{R}^{2 m+1}$.

Proof. We only need to check the hypotheses in theorem 1.3.11 Clearly the statement is invariant under diffeomorphisms and holds for $\mathbb{R}^{m}$. Condition (2) was established in the previous lemma. Condition (3) is almost trivial. Given embeddings $F_{i}: A_{i} \rightarrow \mathbb{R}^{2 m+1}$, where $A_{i} \subset M$ are open and pairwise disjoint we can construct new embeddings $G_{i}: A_{i} \rightarrow$ $\left(i, i+\frac{1}{2}\right)^{2 m+1}$ with disjoint images. This yields an embedding $G: \bigcup_{i} A_{i} \rightarrow \mathbb{R}^{2 m+1}$.

This shows that any $m$-manifold has an embedding into $\mathbb{R}^{2 m+1}$. To obtain a proper embedding we select a proper function $\rho: M \rightarrow[0, \infty)$ and use the dimension reduction result on the proper embedding $(\rho, F): M \rightarrow \mathbb{R} \times \mathbb{R}^{2 m+1}$.

### 3.2.2. Tubular Neighborhoods.

LEMMA 3.2.5. If $M \subset \mathbb{R}^{n}$ is a properly embedded submanifold, then some neighborhood of the normal bundle of $M$ in $\mathbb{R}^{n}$ is diffeomorphic to a neighborhood of $M$ in $\mathbb{R}^{n}$.

Proof. The normal bundle is defined as

$$
v\left(M \subset \mathbb{R}^{n}\right)=\left\{(v, p) \in T_{p} \mathbb{R}^{n} \times M \mid v \perp T_{p} M\right\}
$$

There is a natural map

$$
\begin{aligned}
v\left(M \subset \mathbb{R}^{n}\right) & \rightarrow \mathbb{R}^{n} \\
(v, p) & \mapsto v+p
\end{aligned}
$$

This map is proper provided $M \subset \mathbb{R}^{n}$ is properly embedded. It is also clearly an embedding when restricted to the zero section. Note that the image of the differential at a point $(0, p)$ contains $T_{p} M$ and $\left\{v \in T_{p} \mathbb{R}^{n} \mid v \perp T_{p} M\right\}$. Consequently the differential is nonsingular. This shows that it is also a local diffeomorphism on some neighborhood of the zero section $M$. Lemma 1.4 .16 then shows that it is a diffeomorphism on a neighborhood of the zero section.

THEOREM 3.2.6. If $M \subset N$ is a properly embedded submanifold, then some neighborhood of the normal bundle of $M$ in $N$ is diffeomorphic to a neighborhood of $M$ in $N$.

Proof. Any subbundle of $\left.T N\right|_{M}$ that is transverse to $T M$ is a normal bundle. It is easy to see that all such bundles are isomorphic. One specific choice comes from a proper embedding $N \subset \mathbb{R}^{n}$ and then defining

$$
v(M \subset N)=\left\{(v, p) \in T_{p} N \times M \mid v \perp T_{p} M\right\}
$$

We don't immediately obtain a map $v(M \subset N) \rightarrow N$. First we select a neighborhood $N \subset$ $U \subset \mathbb{R}^{n}$ as in the previous lemma. The projection $\pi: U \rightarrow N$ that takes $w+q \in U$ to $q \in N$ is a fibration. This gives us a neighborhood $V=\{(v, p) \in v(M \subset N) \mid v+p \in U\}$ of $M$. This allows us to define a map

$$
\begin{aligned}
V & \rightarrow N \\
(v, p) & \mapsto \pi(v+p)
\end{aligned}
$$

that is a local diffeomorphism near the zero section and an embedding on the zero section.

### 3.3. Exercises

(1) Show that if an equivalence relation $R \subset M \times M$ is a submanifold, then $\left.\pi_{2}\right|_{R}$ : $R \rightarrow M$ is a submersion.
(2) Use theorem 3.1.14 on $\operatorname{Mat}_{n \times k}^{k}(\mathbb{R})$ to show that the Grassmannian $G_{k}\left(\mathbb{R}^{n}\right)$ is a manifold.
(3) Use theorem 3.1.14 on $V_{k}\left(\mathbb{R}^{n}\right)$ to show that the Grassmannian $G_{k}\left(\mathbb{R}^{n}\right)$ is a manifold.
(4) Use theorem 3.1.14 on $O(n)$ to show that the Grassmannian $G_{k}\left(\mathbb{R}^{n}\right)$ is a manifold.
(5) Show that the submersions $V_{k}\left(\mathbb{R}^{n}\right) \rightarrow G_{k}\left(\mathbb{R}^{n}\right)$ and $O(n) \rightarrow G_{k}\left(\mathbb{R}^{n}\right)$ are fibrations.
(6) Show that $\operatorname{Mat}_{n \times k}^{k}(\mathbb{R}) \rightarrow G_{k}\left(\mathbb{R}^{n}\right)$ is a fibration (hint: this can be proven directly from the definition or by using that $V_{k}\left(\mathbb{R}^{n}\right) \rightarrow G_{k}\left(\mathbb{R}^{n}\right)$ is a fibration).
(7) Give an example of an injective immersion $\mathbb{R} \rightarrow T^{2}$ whose image is dense, e.g., of the form $\left(e^{i 2 \pi t}, e^{i 2 \pi a t}\right)$. Extend this map to an immersion $\mathbb{R} \times(-\varepsilon, \varepsilon) \rightarrow T^{2}$ and show that it is not injective on any neighborhood of $\mathbb{R} \times\{0\}$.
(8) Let $F: M \rightarrow N$ be a proper immersion. Show that if $F$ is injective when restricted to a closed subset $C \subset M$, then $F$ is also injective on an open neighborhood of $C$.

## CHAPTER 4

## Lie Groups

### 4.1. General Properties

A Lie group is a smooth manifold with a group structure that is also smooth, i.e., a manifold $G$ with an associative multiplication

$$
\begin{array}{rlll}
G \times G & \rightarrow & G \\
(g, h) & \mapsto & g h
\end{array}
$$

that is smooth and inverse operation

$$
\begin{array}{rll}
G & \rightarrow & G \\
g & \mapsto & g^{-1}
\end{array}
$$

that is smooth. The identity is generally denoted $e$. The most obvious example of a Lie group is is simply a vector space with addition as the product structure. A more interesting example is the space of invertible matrices, $\operatorname{Gl}(n, \mathbb{F})$ with matrix multiplication as the product structure.

A Lie group homomorphism is a homomorphism between Lie groups that is also smooth. A Lie subgroup $H \subset G$ is a subgroup that is also an immersed submanifold, i.e., it is the image of a Lie group under an injective immersion that is also a homomorphism.

A Lie group is homogeneous in a canonical way as left translation by group elements: $l_{g}(x)=g \cdot x$ maps the identity element $e$ to $g$. Consequently, $l_{g h^{-1}}$ maps $h$ to $g$. Since left translation is a diffeomorphism it can be used to calculate the differential of a Lie group homomorphism from the differential at the identity. For a smooth homomorphism $\phi: G_{1} \rightarrow G_{2}$ between Lie groups the homomorphism property can be expressed as

$$
\phi \circ l_{g}=l_{\phi(g)} \circ \phi
$$

or

$$
\phi=l_{\phi(g)} \circ \phi \circ l_{g^{-1}}
$$

This shows that

$$
\left.D \phi\right|_{g}=\left.D l_{\phi(g)} \circ D \phi\right|_{e} \circ D l_{g^{-1}}
$$

In particular, $\phi$ has constant rank and it's kernel must be a properly embedded submanifold Lie subgroup.

We could equally well have used right translation $r_{g}(x)=x g$ for these observations.
A vector field is left invariant if it is $l_{g}$-related to itself for all $g$, i.e., $D l_{g}\left(\left.X\right|_{h}\right)=\left.X\right|_{g h}$. This shows that $\left.X\right|_{e}$ determines $X$. Conversely, given $\left.X\right|_{e} \in T_{e} G$ it is easy to see that $\left.X\right|_{g}=D l_{g}\left(\left.X\right|_{e}\right)$ defines a smooth left invariant vector field. The space of left invariant vector fields is denoted by $\mathfrak{g}$ and is identified with $T_{e} G$ as a vector space. However, there is an extra structure on $\mathfrak{g}$ as the Lie bracket of left invariant fields is again left invariant (see section 2.2.2. This makes $\mathfrak{g}$ in to a Lie algebra, i.e., an algebra with a bracket operation $[X, Y]$ that is bilinear, skew symmetric, and satisfies the Jacobi identity. Any associative
algebra $(A,+, \cdot)$ has such a bracket structure defined by commutation $[a, b]=a b-b a$. The space of square matrices $\operatorname{Mat}_{n \times n}(\mathbb{F})$ with this commutator bracket is denoted by $\mathfrak{g l}(n, \mathbb{F})$ when we think of it as a Lie algebra.

When $H \subset G$ is a Lie subgroup it follows that $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra as left multiplication $l_{g}$ on $G$ preserves $H$ when $g \in H$. Similarly, for a smooth homomorphism $\phi: H \rightarrow G$ we see that any $X \in \mathfrak{h}$ is $\phi$-related to the left invariant field $Y \in \mathfrak{g}$ that is determined by $\left.Y\right|_{e}=D \phi\left(\left.X\right|_{e}\right)$ showing that we obtain a Lie algebra homomorphism $\phi_{*}: \mathfrak{h} \rightarrow \mathfrak{g}$, i.e., $\phi_{*}$ is linear and preserves Lie brackets, $\phi_{*}[X, Y]=\left[\phi_{*}(X), \phi_{*}(Y)\right]$.

### 4.2. Matrix Groups

We explain the various basic matrix groups that come from the general linear groups.
4.2.1. The General Linear Groups. The most obvious examples of Lie groups are matrix groups starting with the general linear groups

$$
\begin{aligned}
& G l(n, \mathbb{R}) \subset \operatorname{Mat}_{n \times n}(\mathbb{R})=\mathfrak{g l}(n, \mathbb{R}), \\
& G l(n, \mathbb{C}) \subset \operatorname{Mat}_{n \times n}(\mathbb{C})=\mathfrak{g l}(n, \mathbb{C}) .
\end{aligned}
$$

These are open subsets of the vector space of $n \times n$ matrices and and the group operations are explicitly given in terms of multiplication and division of numbers. The identity is usually denoted $e=I$ for matrix groups. As such we have right and left translation on $\operatorname{Mat}_{n \times n}(\mathbb{F})$ for any $A \in \operatorname{Mat}_{n \times n}(\mathbb{F})$ defined by $l_{A}(X)=A X$ and $r_{A}(X)=X A$. These are linear maps but not invertible unless $A \in G l(n, \mathbb{F})$. With that in mind we note that the equation for left invariant fields $\left.X\right|_{g}=D L_{g}\left(\left.X\right|_{I}\right)$ becomes $\left.X\right|_{g}=\left.g X\right|_{I}=r_{\left.X\right|_{I}}(g)$. This allows us to show that the Lie bracket of left invariant fields is the same as the Lie algebra $\mathfrak{g l}(n, \mathbb{F})$. Let $X=r_{A}$ and $Y=r_{B}$ be two left invariant fields and $f: \operatorname{Mat}_{n \times n}(\mathbb{F}) \rightarrow \mathbb{R}$ a linear function. For any tangent vector $v \in \operatorname{Mat}_{n \times n}(\mathbb{F})$ we have $D_{v} f=f(v)$. This shows that

$$
\left.\left(D_{X} f\right)\right|_{g}=D_{g A} f=f(g A)=f \circ r_{A}(g)
$$

and as $r_{A}$ is linear

$$
\left.\left(D_{Y}\left(D_{X} f\right)\right)\right|_{I}=\left.\left(D_{Y}\left(f \circ r_{A}\right)\right)\right|_{I}=f \circ r_{A} \circ r_{B}(I)=f(B A)
$$

Similarly,

$$
\left.\left(D_{X}\left(D_{Y} f\right)\right)\right|_{I}=f(A B)
$$

and we can conclude that

$$
\left.D_{[X, Y]} f\right|_{I}=f([A, B])=f(A B-B A) .
$$

Thus $\left.[X, Y]\right|_{I}=[A, B]=A B-B A$.
4.2.2. The Special Linear Groups. The determinant map det : $\operatorname{Mat}_{n \times n}(\mathbb{F}) \rightarrow \mathbb{F}$ is multiplicative and smooth, and the general linear group is in fact the open subset of matrices with non-zero determinant.

The derivative of the determinant is important to calculate. The determinant function is multi-linear in the columns of the matrix. So if we denote the identity matrix by $I$, then it follows that

$$
\operatorname{det}(I+t X)=1+t(\operatorname{tr} X)+o(t)
$$

and for $A \in G l$ that

$$
\operatorname{det}(A+t X)=\operatorname{det} A\left(1+t\left(\operatorname{tr}\left(A^{-1} X\right)\right)+o(t)\right)
$$

In particular, all non-zero values in $\mathbb{F}-\{0\}$ are regular values of det. This gives us the special linear groups $S l(n, \mathbb{F})$ of matrices with det $=1$. The tangent space $T_{I} S l$ is given as the kernel of the differential and is thus the space of traceless matrices:

$$
T_{I} S l=\left\{X \in \text { Mat }_{n \times n} \mid \operatorname{tr} X=0\right\}=\mathfrak{s l}(n, \mathbb{F}) .
$$

4.2.3. The Polar Decomposition. Using that the operation of taking adjoints $A \rightarrow A^{*}$ is smooth we obtain a smooth map $F: \operatorname{Mat}_{n \times n}(\mathbb{F}) \rightarrow \operatorname{Sym}_{n}(\mathbb{F})$ defined by $A \rightarrow A A^{*}$ where $\operatorname{Sym}_{n}(\mathbb{F})$ denotes the real vector space of self-adjoint operators (symmetric or Hermitian depending on the field.) Note that the image of this map consists of the self-adjoint matrices that are nonnegative definite, i.e., have nonnegative eigenvalues. The differential of this map at the identity can be found from the expansion

$$
(I+t X)\left(I+t X^{*}\right)=I+t\left(X+X^{*}\right)+o(t)
$$

and is

$$
X+X^{*}
$$

This is clearly surjective since it is simply multiplication by 2 when restricted to $\operatorname{Sym}_{n}(\mathbb{F})$. More generally the differential at an invertible $A \in G l$ is given by

$$
X A^{*}+A X^{*}
$$

which is also surjective as it is a bijection when restricted to the real subspace $\left\{X\left(A^{-1}\right)^{*} \mid X \in \operatorname{Sym}_{n}(\mathbb{F})\right\}$. Thus we obtain a submersion to the space of positive definite self-adjoint matrices:

$$
F: G l(n, \mathbb{F}) \rightarrow \operatorname{Sym}_{n}^{+}(\mathbb{F})
$$

Note that $\operatorname{Sym}_{n}^{+}(\mathbb{F}) \subset \operatorname{Sym}_{n}(\mathbb{F})$ is an open convex cone of a real vector space and diffeomorphic to a Euclidean space. Finally we observe that this submersion is also proper as $A_{k} A_{k}^{*} \rightarrow \infty$ when $A_{k} \rightarrow \infty$. In particular, we can use Ehresmann's theorem to conclude that $G l(n, \mathbb{F})$ is diffeomorphic to $\operatorname{Sym}_{n}^{+}(\mathbb{F}) \times F^{-1}(I)$. The fiber over the identity is identified with the orthogonal group:

$$
O(n)=\left\{O \in G l(n, \mathbb{R}) \mid O O^{*}=I\right\}
$$

or the unitary group

$$
U(n)=\left\{U \in G l(n, \mathbb{C}) \mid U U^{*}=I\right\}
$$

and both are compact Lie groups. We note that left translates $l_{A} F^{-1}(I)=A \cdot F^{-1}(I)$ are diffeomorphic to each other and $A \cdot F^{-1}(I) \subset F^{-1}\left(A A^{*}\right)$. Thus fibers are precisely the left translates of the orthogonal or unitary groups. This is the content of the polar decomposition for invertible matrices.

The tangent spaces to the orthogonal and unitary groups are given as the kernel of the differential of the map $A \rightarrow A A^{*}$ and are thus given by the skew-adjoint matrices

$$
\begin{aligned}
& T_{I} O(n)=\left\{X \in \operatorname{Mat}_{n \times n}(\mathbb{R}) \mid X^{*}=-X\right\}=\mathfrak{s o}(n), \\
& T_{I} U(n)=\left\{X \in \operatorname{Mat}_{n \times n}(\mathbb{C}) \mid X^{*}=-X\right\}=\mathfrak{u}(n) .
\end{aligned}
$$

These two families of groups can be intersected with the special linear groups to obtain the special orthogonal groups $S O(n)=O(n) \cap S l(n, \mathbb{R})$ and the special unitary groups $S U(n)=U(n) \cap S l(n, \mathbb{C})$. It is not immediately clear that these new groups have welldefined smooth structures. However, it follows from the canonical forms of orthogonal matrices that $S O(n)$ is the connected component of $O(n)$ that contains $I$. The other component consists of the orthogonal matrices with det $=-1$. For the unitary group we obtain a Lie group homomorphism det : $U(n) \rightarrow S^{1} \subset \mathbb{C}$ where all values are regular values.

The tangent spaces are the traceless skew-adjoint matrices. In the real case skewadjoint matrices are skew-symmetric and thus automatically traceless, this conforms with $S O(n) \subset O(n)$ being open. In the complex case, the skew-adjoint matrices have purely imaginary entries on the diagonal so the additional assumption that they be traceless reduces the real dimension by 1 , this conforms with 1 being a regular value of det : $U(n) \rightarrow$ $S^{1}$.
4.2.4. The Matrix Exponential. The matrix exponential map exp : $\operatorname{Mat}_{n \times n}(\mathbb{F}) \rightarrow$ $G l(n, \mathbb{F})$ is defined using the usual power series expansion. The relationship

$$
\operatorname{det} \exp (A)=\exp (\operatorname{tr} A)
$$

shows that it image is in the general linear group and in case $\mathbb{F}=\mathbb{R}$ that it maps into the matrices with positive determinant.

It also commutes with the operation of taking adjoints $\exp A^{*}=(\exp A)^{*}$. This shows that we obtain the following restrictions

$$
\begin{aligned}
\exp : T_{I} O(n) & \rightarrow S O(n), \\
\exp : T_{I} U(n) & \rightarrow U(n), \\
\exp : T_{I} S U(n) & \rightarrow S U(n),
\end{aligned}
$$

as well as

$$
\exp : \operatorname{Sym}_{n}(\mathbb{F})=T_{I} \operatorname{Sym}_{n}^{+}(\mathbb{F}) \rightarrow \operatorname{Sym}_{n}^{+}(\mathbb{F})
$$

These maps are all surjective. In all cases this uses that a matrix in the target can be conjugated to a nice canonical form, $O^{*} C O$, where $C$ is diagonal in the last three cases and has a block diagonal form in the first case that consists of $2 \times 2$ rotations and diagonal entries that are $\pm 1$. In the unitary case the diagonal entries are of the form $e^{i \theta}$. Thus $C=\exp (i D)$, where $D$ is a real diagonal matrix, and $O^{*} C O=O^{*} \exp (i D) O$. Similarly, in the last case $C$ is a diagonal matrix with positive entries and $C=\exp (D)$ for a unique diagonal matrix $D$ with real entries. The first case is the most intricate. First observe that rotations do come from skew-symmetric matrices:

$$
\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right]=\exp \left[\begin{array}{cc}
0 & -\theta \\
\theta & 0
\end{array}\right] .
$$

This also takes care of pairs of real eigenvalues of the same sign as they correspond to rotations where $\theta=0$ or $\pi$. Since elements in $S O(n)$ have determinant 1 we can always ensure that the real eigenvalues get paired up except when $n$ is odd, in which case the remaining eigenvalue is 1 .

The polar decomposition diffeomorphism $G l(n, \mathbb{C}) \cong \operatorname{Sym}_{n}^{+}(\mathbb{R}) \times U(n)$ now tells us that $G l(n, \mathbb{C})$ is connected. Similarly, $G l^{+}(n, \mathbb{R}) \simeq \operatorname{Sym}_{n}^{+}(\mathbb{R}) \times S O(n)$ is connected. As the elements of $O(n)$ with determinant -1 are diffeomorphic to $S O(n)$ via multiplication by any reflection in a coordinate hyperplane it follows that $G l(n, \mathbb{R})$ has precisely two connected components.
4.2.5. Low Dimensional Groups and Spheres. There are several interesting connections between low dimensional Lie groups and low dimensional spheres.

First we note that rotations in the plane are also complex multiplication by numbers on the unit circle $S^{1} \subset \mathbb{C}$ so:

$$
S O(2)=U(1)=S^{1}
$$

The 3-sphere can be thought of as the unit sphere $S^{3} \subset \mathbb{C}^{2}$ and thus

$$
S^{3}=\left\{\left.(z, w) \in \mathbb{C}^{2}| | z\right|^{2}+|w|^{2}=1\right\}
$$

On the other hand:

$$
S U(2)=U(2) \cap S l(2, \mathbb{C})=\left\{\left.\left[\begin{array}{cc}
z & -\bar{w} \\
w & \bar{z}
\end{array}\right] \in U(2) \right\rvert\, z \bar{z}+w \bar{w}=1\right\}
$$

so we have:

$$
S U(2)=S^{3}
$$

Next we note that

$$
\begin{aligned}
S O(3) & =\left\{\left.\left[\begin{array}{lll}
e_{1} & e_{2} & e_{3}
\end{array}\right] \right\rvert\, e_{i} \cdot e_{j}=\delta_{i j}, \operatorname{det}\left[\begin{array}{lll}
e_{1} & e_{2} & e_{3}
\end{array}\right]=1\right\} \\
& =\left\{\left[\begin{array}{lll}
e_{1} & e_{2} & e_{1} \times e_{2}
\end{array}\right]\left|e_{1} \cdot e_{2}=0,\left|e_{1}\right|=\left|e_{2}\right|=1\right\}\right. \\
& =U S^{2}
\end{aligned}
$$

where $U S^{2}=\{(p, v)| | p|=|v|=1, p \cdot v=0\}$ is the set of unit tangent vectors.
There is a another important identification for this space

$$
S O(3)=\mathbb{R} \mathbb{P}^{3}
$$

This comes from exhibiting a homomorphism $S U(2) \rightarrow S O(3)$ whose kernel is $\{ \pm I\}$. This shows that via the identification $S U(2)=S^{3}$ the preimages are precisely antipodal points. The specifics of the construction take a bit of work and will also lead us to quaternions. First make the identification

$$
\mathbb{C}^{2}=\left\{\left.\left[\begin{array}{cc}
z & -\bar{w} \\
w & \bar{z}
\end{array}\right] \right\rvert\,(z, w) \in \mathbb{C}^{2}\right\} .
$$

On the right hand side we obtain a collection of matrices that is closed under addition and multiplication by real scalars. Since $\mathbb{C}$ is a commutative algebra the right hand side is also closed under multiplication. Thus it forms an algebra over $\mathbb{R}$. It is also a division algebra as non-zero elements have $\operatorname{det}=|z|^{2}+|w|^{2}>0$ and thus have inverses. This is the algebra of quaternions also denoted $\mathbb{H}$. Note that $X \in \mathbb{C}^{2}$ has Euclidean length

$$
|X|^{2}=|z|^{2}+|w|^{2}=\operatorname{det} X
$$

Any $A \in S U(2)$ acts by conjugation on this algebra by

$$
A \cdot X=A X A^{*}
$$

The map $X \mapsto A X A^{*}$ is an orthogonal transformation as it doesn't alter the Euclidean length of $X$ :

$$
\left|A X A^{*}\right|^{2}=\operatorname{det}\left(A X A^{*}\right)=\operatorname{det} A \operatorname{det} X \operatorname{det} A^{*}=\operatorname{det} X=\left|X^{2}\right|
$$

A natural orthonormal basis is given by

$$
1=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right], i=\left[\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right], j=\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right], k=\left[\begin{array}{cc}
0 & -i \\
-i & 0
\end{array}\right]
$$

Note that these matrices each have Euclidean norm $\sqrt{2}$. So the inner product is scaled to make them have norm 1. The last matrix is defined so that we obtain

$$
\begin{gathered}
i j=k=-j i, \\
j k=i=-k j, \\
k i=j=-i k, \\
i^{2}=j^{2}=k^{2}=-1 .
\end{gathered}
$$

In fact conjugation fixes 1 so it also fixes the orthogonal complement spanned by $i, j, k$. Thus we obtain a homomorphism $S U(2) \rightarrow S O(3)$ by letting $A \in S U(2)$ act by conjugation on $\operatorname{span}_{\mathbb{R}}\{i, j, k\}$. The kernel of this map consists of matrices $A \in S U(2)$ that commute with all elements in $\mathbb{H}$ since $A X=X A$. This shows that such $A$ must be homotheties and consequently the only possibilities are $\pm I= \pm 1$. It is also not hard to check that $S U(2) \rightarrow S O(3)$ is a submersion by calculating the differential at the identity. Thus the image is both open and closed and all of $S O(3)$. This shows that $S O(3)=\mathbb{R} \mathbb{P}^{3}$.

From all of this we obtain a special proof of the "Hairy Ball Theorem" for $S^{2}$.
THEOREM 4.2.1. Every vector field on $S^{2}$ vanishes somewhere.
PROOF. The proof is by contradiction. If we have a non-zero vector field, then we also have a unit vector field $p \mapsto(p, v(p)) \in U S^{2}$. This gives us a diffeomorphism $S O(3) \rightarrow$ $S^{2} \times S^{1} \subset S^{2} \times \mathbb{R}^{2}$ by mapping each $\left[p, e_{2}, p \times e_{2}\right] \in S O(3)$ to

$$
\left(p, e_{2} \cdot v(p),\left(p \times e_{2}\right) \cdot v(p)\right)
$$

This contradicts that $S O(3)=\mathbb{R P}^{3}$ as $S^{2} \times S^{1}$ has a non-compact simply connected cover $S^{2} \times \mathbb{R}$.

### 4.3. The Exponential Map

Note that exponential map for matrices satisfies the law of exponents $\exp (A+B)=$ $\exp A \exp B$ when $A, B$ commute. In particular, the map $t \mapsto \exp (t A)$ is a homomorphism from the Abelian Lie group $(\mathbb{R},+)$. This one-parameter group is also the integral curve for the left-invariant vector field defined by $\left.X\right|_{g}=g A$ since

$$
\left.\frac{d \exp (t A)}{d t}\right|_{t=t_{0}}=\left.\exp \left(t_{0} A\right) \frac{d \exp (s A)}{d s}\right|_{s=0}=\left(\exp \left(t_{0} A\right)\right) A
$$

As such, we can define the abstract exponential map on a general Lie group, exp : $T_{e} G \rightarrow G$, by declaring $t \mapsto \exp (t X)$ to be the integral curve through $e$ for the left-invariant field defined by $\left.X\right|_{g}=D l_{g}(X)$, i.e., $\exp (t X)=\Phi_{X}^{t}(e)$. Smoothness of exp then follows from remark 2.2.5 Note that

$$
l_{g} \exp (t X)=g \exp (t X)=\Phi_{X}^{t}(g)
$$

as they agree at $t=0$ and are both integral curves for $X$ :

$$
\frac{d\left(l_{g} \exp (t X)\right)}{d t}=D l_{g}\left(\frac{d(\exp (t X))}{d t}\right)=D l_{g}\left(\left.X\right|_{\exp (t X)}\right)=\left.X\right|_{l_{g} \exp (t X)}
$$

If $Y$ is the left invariant field defined by $\left.D \phi(X)\right|_{e}$ then $X$ and $Y$ are $\phi$-related and hence by proposition 2.2.6

$$
\phi\left(\Phi_{X}^{t}(e)\right)=\Phi_{Y}^{t}(e) .
$$

In other words

$$
\phi(\exp (t X))=\exp \left(\left.t D \phi\right|_{e}(X)\right)
$$

and the diagram

$$
\begin{array}{ccc}
T_{e} G & \xrightarrow{D \phi} & T_{e} H \\
\exp \downarrow & & \exp \downarrow \\
G & \xrightarrow{\phi} & H
\end{array}
$$

is commutative.
Proposition 4.3.1. The exponential map has the following properties.
(1) $D \exp : T_{0} T_{e} G \rightarrow T_{e} G$ is an isomorphism. In particular, there is a neighborhood $U$ around $0 \in T_{e} M$ such that $\exp : U \rightarrow \exp (U)$ is a diffeomorphism.
(2) $\exp \left(s X+t Y+O\left(s^{2}+t^{2}\right)\right)=\exp (s X) \exp (t Y)$. In particular, for integers $m$ we have

$$
\exp (X+Y)=\lim _{m \rightarrow \infty}\left(\exp \left(\frac{1}{m} X\right) \exp \left(\frac{1}{m} Y\right)\right)^{m}
$$

(3) If $T_{e} G=V \oplus W$, then the map $(X, Y) \mapsto \exp (X) \exp (Y), X \in V, Y \in W$ is a diffeomorphism near the origin onto a neighborhood of $e \in G$.
Proof. Recall that we can identify $T_{0} T_{e} G \simeq T_{e} G$ by sending $X \in T_{e} G$ to $\left.\frac{d}{d t}(t X)\right|_{0} \in$ $T_{0} T_{e} G$. As $\left.\frac{d \exp (t x)}{d t}\right|_{t=0}=X$ we see that $\left.D \exp \right|_{0}=i d_{T_{e} G}$. This proves (1).

For (2) let log be the inverse of exp on a neighborhood of $e \in G$ and consider the two maps

$$
\begin{aligned}
(s, t) & \mapsto \log (\exp (s X) \exp (t Y)) \\
(s, t) & \mapsto s X+t Y
\end{aligned}
$$

From (1) it follows that the derivatives $\left.\frac{\partial}{\partial s}\right|_{(0,0)}=X$ and $\left.\frac{\partial}{\partial t}\right|_{(0,0)}=Y$ are the same for both maps. This proves the first part of the claim. For the second claim let $s=t=\frac{1}{m}$ and note that

$$
\exp \left(\frac{1}{m} X\right) \exp \left(\frac{1}{m} Y\right)=\exp \frac{1}{m}\left(X+Y+O\left(\frac{1}{m}\right)\right)
$$

Thus

$$
\left(\exp \left(\frac{1}{m} X\right) \exp \left(\frac{1}{m} Y\right)\right)^{m}=\exp \left(X+Y+O\left(\frac{1}{m}\right)\right)
$$

when $m$ is an integer and the claim follows by letting $m \rightarrow \infty$.
For (3) we again use (1) and the identification $T_{0} T_{e} G \simeq T_{e} G=V \oplus W$. As in (2) we note that

$$
(s, t) \mapsto \log (\exp (s X) \exp (t Y)), X \in V, Y \in W
$$

again has partial derivatives at $(0,0)$ that respect the splitting $T_{e} G=V \oplus W$. This shows that

$$
\begin{aligned}
V \oplus W & \rightarrow T_{e} G \\
(X, Y) & \mapsto \log (\exp (X) \exp (Y))
\end{aligned}
$$

is nonsingular at the origin which proves the claim.
THEOREM 4.3.2. Let $G$ and $H$ be Lie groups with $G$ connected. A homomorphism $\phi: G \rightarrow H$ is uniquely determined by its differential $\left.D \phi\right|_{e}: T_{e} G \rightarrow T_{e} H$. In particular, $a$ connected Lie subgroup $G \subset H$ is determined by $T_{e} G \subset T_{e} H$.

Proof. Part (1) of the previous proposition together with $\phi(\exp X)=\exp \left(\left.D \phi\right|_{e} X\right)$ shows that the $\left.D \phi\right|_{e}$ determines $\phi$ in a neighborhood of the identity. We also have $\phi(g \exp X)=$ $\phi(g) \exp \left(\left.D \phi\right|_{e} X\right)$ so in a neighborhood of any $g \in G$ the map $\phi$ is determined by $\phi(g)$ and $\left.D \phi\right|_{e}$. Thus any two homomorphisms with the same differential at $e$ must agree on a set that is clearly closed and by what we just saw also open. This establishes the claim.

With the use of the exponential map we can also offer a very simple topological criterion for when a subgroup is an embedded Lie group. However, most embedded subgroups are also preimages of submersions so we can generally apply the preimage theorem 1.4.25 or constant rank theorem 1.4.25]1.4.23.

THEOREM 4.3.3 (Cartan). A closed subgroup $H \subset G$ of a Lie group, is an embedded submanifold and hence also a Lie group.

Proof. Define the tangent space to $H$ inside $T_{e} G$ as

$$
V=\left\{X \in T_{e} G \mid \exp (t X) \in H \text { for all } t \in \mathbb{R}\right\}
$$

and let $W$ be a complement such that $T_{e} G=V \oplus W$.
Clearly $\alpha X \in V$ if $X \in V$ for any $\alpha \in \mathbb{R}$. If $X, Y \in V$, then the formula

$$
\exp t(X+Y)=\lim _{m \rightarrow \infty}\left(\exp \left(\frac{1}{m} t X\right) \exp \left(\frac{1}{m} t Y\right)\right)^{m}
$$

shows that $X+Y \in H$ as the right-hand side is the limit of elements in $H$ and $H$ is closed. This shows that $V$ is a vector space.

Consider the restriction $\exp : V \rightarrow H$. We claim that this is a bijection near the origin. If not, then we can find $h_{m} \in H$, with $h_{m} \rightarrow e$ such that $h_{m}=\exp X_{m} \exp Y_{m}$ where $X_{m} \in V$, $Y_{m} \in W-\{0\}$. Here $X_{m}, Y_{m} \rightarrow 0$ and we can assume that $\frac{Y_{m}}{\left|Y_{m}\right|} \rightarrow Y \in W$. Note that $\exp Y_{m} \in$ $H$ as $h_{m}, \exp X_{m} \in H$. For fixed $t \in \mathbb{R}-\{0\}$ let $k_{m}$ be the integer closest to $\frac{t}{\left|Y_{m}\right|}$ so that $k_{m} Y_{m} \rightarrow t Y$. This shows that

$$
\exp k_{m} Y_{m}=\left(\exp Y_{m}\right)^{k_{m}} \in H
$$

Since the limit is $\exp t Y$ and $H$ is closed we have shown that $Y \in V$ which is a contradiction.
This creates a chart on a neighborhood of $e$ and by left translation around every point in $H$. This makes $H$ an embedded submanifold as $(X, Y) \mapsto \exp X \exp Y,(X, Y) \in V \oplus W$ restricts to $\exp : V \rightarrow H$ when $Y=0$ and is a local diffeomorphism.

### 4.4. Coverings and Quotients of Lie Groups

THEOREM 4.4.1. A surjective Lie group homomorphism $\phi: G \rightarrow H$ with a differential that is bijective is a covering map. Moreover, when $G$ is connected the kernel is central and in particular Abelian.

Proof. Consider a surjective Lie group homomorphism $\phi: G \rightarrow H$ whose differential is bijective. The kernel $\operatorname{ker} \phi$ is by definition the pre-image of the identity and by the regular value theorem a closed 0 -dimensional submanifold of $G$. Thus we can select a neighborhood $U$ around $e \in G$ that has compact closure and $\bar{U} \cap \operatorname{ker} \phi=\{e\}$ and that is mapped diffeomorphically to $\phi(U)$. It follows from continuity of the group multiplication and that inversion is a diffeomorphism that there is neighborhood around $e \in G$ such that $V^{2} \subset U$ and $V^{-1}=V$ i.e., if $a, b \in V$ then $a \cdot b \in U$ and $a^{-1} \in V$. We claim that if $g, h \in$ $\operatorname{ker} \phi$ and $g \cdot V \cap h \cdot V \neq \emptyset$, then $g=h$. In fact, if $g \cdot v_{1}=h \cdot v_{2}$, then $g^{-1} \cdot h=v_{2} \cdot v_{1}^{-1} \in$ $U \cap \operatorname{ker} \phi$, which implies that $g^{-1} \cdot h=e$. In this way we have found disjoint open sets $g \cdot V$ for $g \in \operatorname{ker} \phi$ that are mapped diffeomorphically to $\phi(V)$. We claim that additionally $\phi^{-1}(\phi(V))=\bigcup_{g \in \operatorname{ker} \phi} g \cdot V$. To see this let $\phi(x)=\phi(y)$ with $y \in V$. Then $g=x y^{-1} \in \operatorname{ker} \phi$ and $x \in g V$.

This shows that a neighborhood of $e \in H$ is evenly covered. Using left translations we can then show that all points in $H$ are evenly covered.

Finally assume that $G$ is connected. For a fixed $g \in G$ consider conjugation $x \rightarrow g x g^{-1}$. We say that $x$ is central if it commutes with all elements in $G$ and this comes down to checking that $x$ is fixed by all conjugations. Now $\operatorname{ker} \phi \subset G$ is already a normal and thus preserved by all conjugations. Consider a path $g(t)$ from $e \in G$ to $g \in G$, then for fixed $x$ we obtain a path $g(t) \cdot x \cdot(g(t))^{-1}$. When $x \in \operatorname{ker} \phi$ this path is necessarily in $\operatorname{ker} \phi$ and
4.5. THE LIE GROUP LIE ALGEBRA CORRESPONDENCE
starts at $x$. However, $\operatorname{ker} \phi$ is discrete and so the path must be constant. This shows that any $x \in \operatorname{ker} \phi$ commutes with all elements in $G$.

There is also a converse to the above theorem.
THEOREM 4.4.2. Let $f: \bar{G} \rightarrow G$ be a covering map, with $\bar{G}$ connected. If $G$ is a Lie group, then $\bar{G}$ has a Lie group structure that makes $f$ a homomorphism. Moreover, the fundamental group of a connected Lie group is Abelian.

Proof. The most important and simplest case is when $\bar{G}=\tilde{G}$ is simply connected. In that case we can simply use the unique lifting property to lift the composite map $\tilde{G} \times \tilde{G} \rightarrow$ $G \times G \rightarrow G$ to a product structure on $\bar{G}$. The inverse structure is created in a similar way. We then have to use the uniqueness of lifts to establish associativity as we would otherwise obtain to different lifts for multiplying three elements $\tilde{G} \times \tilde{G} \times \tilde{G} \rightarrow G$.

Covering space theory shows that the fundamental group is also a group of deck transformations on the universal cover. These deck transformations are all lifts of the projection $\tilde{G} \rightarrow G$. Composition and inverses of these lifts are simply new lifts and so they form a group. This is the fundamental group as a lift is identified by an element in the preimage of $e \in G$. As the preimage of $e \in G$ is also the kernel of $\tilde{G} \rightarrow G$ the lifts can then be more precisely identified with left translations by elements in the kernel of $\tilde{G} \rightarrow G$. The group structure on the kernel is also the same as composition since $l_{g_{1}} \circ l_{g_{2}}=l_{g_{1} g_{2}}$. Thus the deck transformations form an Abelian group.

In general, covering space theory shows that any connected cover $\bar{G} \rightarrow G$ is determined by its fundamental group $\pi_{1}(\bar{G}) \subset \pi_{1}(G)$. As $\pi_{1}(G)$ is a central subgroup of $\tilde{G}$ we can identify $\bar{G}$ with the group $\tilde{G} / \pi_{1}(\bar{G})$. This induces a group structure on $\bar{G}$. This group structure is also a lift of $\bar{G} \times \bar{G} \rightarrow G \times G \rightarrow G$ and is consequently smooth.

We can now also address the question of when the coset space of a subgroup becomes a manifold.

THEOREM 4.4.3. If $H \subset G$ is a closed subgroup of a Lie group, then the quotient space $G / H$ is a manifold and $\pi: G \rightarrow G / H$ is a submersion.

Proof. We have to check that the corresponding equivalence relation

$$
R=\left\{(x, y) \in G \times G \mid x y^{-1} \in H\right\}
$$

is a properly embedded submanifold such that the restriction $\left.\pi_{1}\right|_{R}: R \rightarrow G$ is a submersion. Consider the smooth map $p: G \times G \rightarrow G$ defined by $p(x, y)=x y^{-1}$. Clearly $R=p^{-1}(H)$, so the goal is to show that $p$ is a submersion. To see that, fix $x, y \in G$ and define $\phi_{y}: G \rightarrow$ $G \times G$ by $\phi_{y}(z)=(z y, y)$ such that $\phi_{y}\left(x y^{-1}\right)=(x, y)$ and $p\left(\phi_{y}(z)\right)=z$, i.e., $p \circ \phi_{y}=\operatorname{id}_{G}$. This shows that $R$ is a properly embedded submanifold. Next observe that if we use $\psi$ : $G \times H \rightarrow G \times G$ defined by $\psi(x, y)=\left(x, h^{-1} x\right)$, then the image of $\psi$ is precisely $H$ and the composition with $\pi_{1}$ is the projection $G \times H \rightarrow G$ which we know is a submersion. Thus also $\left.\pi_{1}\right|_{R}$ becomes a submersion.

### 4.5. The Lie group Lie algebra Correspondence

We saw at the very end of section 4.1 that a Lie subgroup $H \subset G$ defines a subalgebra of left invariant fields $\mathfrak{h} \subset \mathfrak{g}$. The left translates of $H$ form the coset space $G / H$. As subsets of $G$ they are all submanifolds that create a foliation of $G$. The corresponding distribution consists of the left translates $D l_{g}\left(T_{e} H\right)$. As we shall see this construction sets up a bijective correspondence between subalgebras and connected Lie subgroups.

THEOREM 4.5.1. Let $G$ be a Lie group. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra for a unique connected Lie subgroup $H \subset G$.

Proof. The Lie algebra $\mathfrak{h}$ consists of left invariant vector fields on $G$ whose Lie brackets also lie in $\mathfrak{h}$. As such they define an involutive distribution. By Frobenius' theorem there is a unique maximal integral submanifold through $e \in G$. This submanifold $H \subset G$ is by definition connected and the left translates $l_{g} H$ are all maximal integral submanifolds for the distribution. Note that $g \in H \cap l_{g} H$ provided $g \in H$ and consequently $H=l_{g} H$. This shows that group multiplication on $G$ defines a multiplication $H \times H \rightarrow H$. However, this does not necessarily guarantee that multiplication is smooth (or even continuous) as a map into $H$ in case $H$ is not embedded. Fortunately we have more structure. Note that $\{\exp (t X) \mid t \in \mathbb{R}\}$ is a maximal integral submanifold for $\operatorname{span}\{X\}$. If $X \in \mathfrak{h}$, this shows that $\left.\exp \right|_{\mathfrak{h}}: \mathfrak{h} \rightarrow H$. Moreover, this map is smooth as a map into $H$ since $t \mapsto \exp (t X)$ is also an integral curve for $X$ as a vector field on $H$ (see remark 2.2.5). Smoothness of multiplication around $e \in H$ can now be checked by selecting an open neighborhood $0 \in B \subset \mathfrak{g}$ such that $\exp : B \rightarrow U=\exp (B)$ is a diffeomorphism with inverse log. Next choose $e \in V \subset U$ such that $V \cdot V \subset U$. Smoothness of multiplication in $G$ is now equivalent to smoothness of

$$
(X, Y) \mapsto \log (\exp X \exp Y), X, Y \in \log (V)
$$

Restricting this to $X, Y \in \mathfrak{h} \cap \log (V)$ then shows that multiplication is also smooth on $H$ as $\exp : \mathfrak{h} \cap B \rightarrow H$ is a diffeomorphisms onto a neighborhood of $e \in H$. Smoothness of the inverse operation near $e \in H$ is even easier as it corresponds to $X \mapsto-X$ in the Lie algebra.

It follows from theorem 4.3 .2 that $H$ is the only connected Lie subgroup with $T_{e} H \simeq$ $\mathfrak{h}$.

This theorem can be used to construct homomorphisms from Lie algebra homomorphisms.

THEOREM 4.5.2. Let $G$ and $H$ be connected Lie groups. Any Lie algebra homomorphism $L: \mathfrak{g} \rightarrow \mathfrak{h}$ corresponds to a unique homomorphism $\phi: \tilde{G} \rightarrow H$, where $\tilde{G} \rightarrow G$ is the universal cover of $G$.

Proof. It suffices to prove this in case $G$ is itself simply connected as all covers of a Lie group have isomorphic Lie algebras. We start by observing that the graph of a smooth homomorphism $\phi: G \rightarrow H$

$$
\operatorname{Graph}(\phi)=\{(g, \phi(g)) \mid g \in G\} \subset G \times H
$$

is a Lie subgroup that is isomorphic to $G$, via inclusion $g \mapsto(g, \phi(g))$ and projection $(g, h) \mapsto g$. Similarly the graph of a Lie algebra homomorphism $L: \mathfrak{g} \mapsto \mathfrak{h}$

$$
\operatorname{Graph}(L)=\{(X, L(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{h}
$$

is a subalgebra isomorphic to $\mathfrak{g}$.
The graph Graph $(L)$ will by the previous theorem correspond to a unique maximal connected subalgebra $G^{\prime} \subset G \times H$. The projection onto $G$ restricts to a homomorphism $\left.\pi_{1}\right|_{G^{\prime}}=\pi: G^{\prime} \rightarrow G$. By construction the tangent space $T_{e} G^{\prime}$ is mapped isomorphically on to $T_{e} G$. Theorem 4.4.1 then tells us that $\pi$ is a covering map. If we assume that $G$ is simply connected, then $\pi$ becomes a smooth isomorphism and the inverse followed by projection onto $H$ defines the homomorphism whose differential corresponds to $L$.

These results lead to a Lie group-Lie algebra correspondence. One missing piece is Ado's theorem which we will not prove.

THEOREM 4.5.3 (Ado). Each (complex) finite dimensional Lie algebra is a subalgebra of $\mathfrak{g l}(n, \mathbb{R})$ or $(\mathfrak{g l}(n, \mathbb{C}))$ for some $n$.

The simplest case of the theorem is when the Lie algebra has no center. The center

$$
\mathfrak{z}=\left\{X \in \mathfrak{g} \mid[X, Y]=\operatorname{ad}_{X} Y=0, \text { for all } Y \in \mathfrak{g}\right\}
$$

is an ideal and is the kernel of the homomorphism:

$$
\begin{array}{rll}
\mathrm{ad}: \mathfrak{g} & \rightarrow \mathfrak{g l}(\mathfrak{g}) \\
X & \mapsto \operatorname{ad}_{X} .
\end{array}
$$

An Abelian Lie algebra is clearly also a subalgebra as it can be identified with the space of diagonal matrixes. However, it is not so easy to piece these two observations together as the quotient algebra $\mathfrak{g} / \mathfrak{z}$ can also have a center. A good example is the Lie algebra of upper triangular matrices.

Assuming Ado's theorem we obtain
THEOREM 4.5.4. Each Lie group corresponds to a unique Lie algebra and each finite dimensional Lie algebra corresponds to a unique simply connected Lie group.

### 4.6. Actions and Exercises

Let $G$ be a Lie group and $M$ a connected manifold. An action of $G$ on $M$ is a generalization of left translation on a Lie group. It is a smooth map

$$
\begin{aligned}
G \times M & \rightarrow M \\
(g, x) & \mapsto g x
\end{aligned}
$$

such that $g_{1}\left(g_{2} x\right)=\left(g_{1} g_{2}\right) x$. It'll be convenient to introduce the action map

$$
\begin{aligned}
A: G \times M & \rightarrow M \times M \\
(g, x) & \mapsto(g x, x)
\end{aligned}
$$

Note that $\pi_{2} \circ A(g, x)=x$ is a submersion.
The orbits are denoted by $G \cdot x=\{g x \mid g \in M\}$ and generate an equivalence relation. The corresponding relation $R \subset M \times M$ is in fact the image of the action map $R=A(G \times M)$. The quotient or orbit space is denoted by $G \backslash M$ as we are acting on the left. An action is transitive if $R=M \times M$ or equivalently $G \backslash M$ is one point. An action is proper if $A$ is proper, in particular, actions by compact groups are always proper.

The isotropy or stabilizer group at $x \in M$ is $G_{x}=\{g \in G \mid g x=x\}$. As such, $G_{x}$, is a closed subgroup and consequently also a Lie group by theorem4.3.3. An action is effective if $\bigcap_{x \in M} G_{x}=\{e\}$, i.e., only the identity acts trivially on $M$. An action is free if $G_{x}=\{e\}$ for all $x \in M$.
(1) Show that if an embedded submanifold $H \subset G$ of a Lie group is a subgroup, then it is a closed subset of $G$. Recall that embedded submanifolds are in general not closed subsets of the ambient space.
(2) Show that if a manifold has a group structure such that multiplication is smooth, then the inverse operation is also smooth. Hint: Consider the smooth map $(x, y) \mapsto(x, x y)$ and show that it is a bijection with non-singular differential at $(e, e)$.
(3) Show that if a subgroup $H \subset G$ of a Lie group is an open subset, then it is also a closed subset.
(4) Show that if a Lie group, $G$, is not connected, then the connected component, $G_{0}$, containing $e$ is an open and closed subgroup.
(5) Let $G$ be a connected Lie group. Show that $G$ is generated by any neighborhood $U \ni e$. Hint A: Find a smaller neighborhood $e \in V \subset U$ such that $V^{-1}=V$ and consider $\bigcup_{m=1}^{\infty} V^{m}$, where $V^{m}=V^{m-1} \cdot V$. Hint B: Select a path $c:[0,1] \rightarrow G$ with $c(0)=e$ and find a subdivision $0=t_{0}<\cdots<t_{k}=1$ such that $\left(c\left(t_{i-1}\right)\right)^{-1} c\left(t_{i}\right) \in$ $U$.
(6) Show that a continuous homomorphism between Lie groups is necessarily smooth. (Hint: use the graph).
(7) Show that $U(n)$ and $S U(n) \times S^{1}$ are diffeomorphic as manifolds, but not isomorphic as Lie groups when $n>1$. Hint: Find a section $s: S^{1} \rightarrow U(n)$ for $\operatorname{det}: U(n) \rightarrow S^{1}$, i.e., $\operatorname{det} \circ s=i d_{S^{1}}$.
(8) Let $\bar{G} \rightarrow G$ be a covering of Lie groups. Show that there is a natural isomorphism between the Lie algebras of these Lie groups.
(9) This is a generalization of theorem 4.4.1Consider a smooth homomorphism $\phi$ : $G \rightarrow H$, where $H$ is connected. Show that this is a fibration when $\left.D \phi\right|_{e}$ is surjective and in particular induces a smooth structure on the group $G / \operatorname{ker} \phi$ via $H$. Hint A: Lift left invariant fields to left invariant fields and argue as in theorem 3.1.4 Hint B: Use exponential maps to find $U \subset T_{e} G$ that exp $\circ D \phi$ maps diffeomorphically on to a neighborhood of $e \in H$ and observe that the preimage is $\{g \exp U \mid g \in \operatorname{ker} \phi\} \simeq \exp (D \phi(U)) \times \operatorname{ker} \phi$.
(10) The group $G l_{n+1}$ acts on $\mathbb{F P} \mathbb{P}^{n}$.
(a) Show that the action is not proper.
(b) Show that the action is transitive.
(c) Show that the isotropy groups are isomorphic to $G l_{n} \times G l_{1}$.
(d) Show that any element that acts trivially is a homothety $\lambda I, \lambda \in \mathbb{F}-\{0\}$.
(e) Show that the homotheties $C=\{\lambda I \mid \lambda \in \mathbb{F}-\{0\}\}$ are the center of $G l_{n+1}$ and that $P l_{n+1}=G l_{n+1} / C$ is a Lie group that acts effectively on $\mathbb{F} \mathbb{P}^{n}$. $P l$ stands for projective linear, the abbreviation $P S l$ is also used as $S l \rightarrow P l$ is a proper submersion.
(11) Let $M=S^{1}$ and $G$ the group with two elements that acts as a reflection in the $x$-axis. Is $R \subset S^{1} \times S^{1}$ smoothly embedded?
(12) Show that if $G \backslash M$ is a smooth manifold such that $M \rightarrow G \backslash M$ is a submersion, then $R \subset M \times M$ is properly embedded.
(13) Assume that the action is proper.
(a) Show that $G_{x}$ is compact.
(b) Show that there is a proper embedding $G / G_{x} \rightarrow M$ whose image is $G \cdot x$.
(c) Show that $G \backslash M$ is Hausdorff and second countable.
(14) Assume that the action map $A$ is proper and has constant rank. Show that $G \backslash M$ is a smooth manifold such that $M \rightarrow G \backslash M$ is a submersion.
(15) Give an example of a free action of $\mathbb{R}$ on $S^{1} \times S^{1}$ that is not proper.
(16) Show that $S U(2) / S O(2)$ is diffeomorphic to $S^{2}=\mathbb{C P}^{1}$.
(17) Assume that the action is proper and free.
(a) Show that $G \backslash M$ is a smooth manifold such that $M \rightarrow G \backslash M$ is a submersion.
(b) Show that $M \rightarrow G \backslash M$ is a fibration. Hint: Theorem 3.1.4 cannot be applied directly, but the proof can be adapted by showing that vector fields can be lifted to vector fields that are invariant under the action.
(18) Conjugation on a Lie group $G$ defines what is called the adjoint action

$$
\begin{aligned}
\operatorname{Ad}: G \times G & \rightarrow G \\
(g, x) & \mapsto \operatorname{Ad}_{g} x=g x g^{-1}
\end{aligned}
$$

(a) Show that its differential with respect to $x, \operatorname{Ad}_{g}=\left.D \operatorname{Ad}_{g}\right|_{e}$, defines an action on $T_{e} G=\mathfrak{g}$.
(b) Taking next the differential with respect to $g, \operatorname{ad}_{X}=\left.D \operatorname{Ad}_{g}\right|_{e}$, defines a Lie algebra action on $\mathfrak{g}$, i.e., $\operatorname{ad}_{X} \circ \operatorname{ad}_{Y}-\operatorname{ad}_{Y} \circ \operatorname{ad}_{X}=\operatorname{ad}_{[X, Y]}$.
(c) Show that $\operatorname{ad}_{X}(Y)=[X, Y]$.

## CHAPTER 5

## Transversality and Incidence Theory

The goal of this chapter is to introduce transversality and use it to define several important invariants such as degree, winding number, Lefschetz number, and Euler characteristic. In chapter 7 and 8 we will show that these invariants can also be calculated using de Rham cohomology. We prove several profound results that have been used widely in the literature: Brouwer's fixed point theorem, the Jordan-Brouwer separation theorem, the Borsuk-Ulam theorem, the Poincaré-Hopf and Lefschetz theorems, and finally the Hopf degree theorem. Each section contains one or more of these results.

### 5.1. Preimages

We say that a map $F: M \rightarrow N$ is transverse to a submanifold $S \subset N$ provided

$$
T_{F(p)} S+D F\left(T_{p} M\right)=T_{F(p)} N
$$

for all $p \in M$ with the property that $F(p) \in S$. When $M$ is itself a submanifold of $N$, then $F$ is the inclusion map. With this definition we obtain a very useful generalization of the preimage theorem.

THEOREM 5.1.1. If $F: M \rightarrow N$ is transverse to a (properly) embedded submanifold $S \subset N$, then $F^{-1}(S) \subset M$ is a (properly) embedded submanifold. When $F^{-1}(S) \neq \emptyset$ its dimension satisfies:

$$
\operatorname{codim} F^{-1}(S)=\operatorname{dim} M-\operatorname{dim} F^{-1}(S)=\operatorname{dim} N-\operatorname{dim} S=\operatorname{codim} S
$$

Proof. The preimage of $S$ will be a closed subset of $M$ provided $S$ is a closed subset of $N$. To show the preimage is a submanifold fix $p \in F^{-1}(S)$ and let $q=F(p)$. Around $q$ we can select coordinates $\left(y^{1}, \ldots, y^{n}\right): U \rightarrow \mathbb{R}^{n}$ such that $S \cap U=\left\{y^{1}=0, \ldots, y^{k}=0\right\}$, i.e., $0 \in \mathbb{R}^{k}$ is a regular value for $\left(y^{1}, \ldots, y^{k}\right): U \rightarrow \mathbb{R}^{k}$. Thus we have an open set $F^{-1}(U)$ around $p$ such that $F^{-1}(U) \cap F^{-1}(S)=F^{-1}(U \cap S)$ is the preimage of $0 \in \mathbb{R}^{k}$ for the map $G=\left(y^{1} \circ F, \ldots, y^{k} \circ F\right)$. If $G(x)=0$, then the kernel of $\left.D G\right|_{x}$ consists of $\left(\left.D F\right|_{x}\right)^{-1}\left(T_{F(x)} S\right)$. Let $E_{x}$ be a complement to $\left(\left.D F\right|_{x}\right)^{-1}\left(T_{F(x)} S\right) \subset T_{x} M$. The fact that $F$ is transverse to $S$ implies that $\left.D F\right|_{x}\left(E_{x}\right) \oplus T_{F(x)} S=T_{F(x)} N$. The differential of $\left(y^{1}, \ldots, y^{k}\right): U \rightarrow \mathbb{R}^{k}$ is also surjective on $\left.D F\right|_{x}\left(E_{x}\right)$ so it follows that $\left.D G\right|_{x}: E_{x} \rightarrow T_{0} \mathbb{R}^{k}$ is surjective (in fact an isomorphism). This shows that $0 \in \mathbb{R}^{k}$ is a regular value for $G$ and consequently that $F^{-1}(S)$ is a submanifold of codimension $k=\operatorname{codim} S$.

COROLLARY 5.1.2. Let $G: M \rightarrow N$ be transverse to an embedded submanifold $S \subset N$. $A$ map $F: L \rightarrow M$ is transverse to $G^{-1}(S) \subset M$ if and only if $G \circ F$ is transverse to $S$.

Proof. This is essentially the second part of the above proof. Select $x \in L$ with $F(x) \in G^{-1}(S)$ and let $E_{x} \subset T_{x} L$ be a complement to

$$
\begin{aligned}
\left(\left.D F\right|_{x}\right)^{-1}\left(T_{F(x)} G^{-1}(S)\right) & =\left(\left.D F\right|_{x}\right)^{-1}\left(\left(\left.D G\right|_{F(x)}\right)^{-1} T_{G \circ F(x)} S\right) \\
& =\left(\left.D(G \circ F)\right|_{x}\right)^{-1}\left(T_{G \circ F(x)} S\right) .
\end{aligned}
$$

As $G$ is transverse to $S$ it follows that $\left.D F\right|_{x}\left(E_{x}\right)$ is a complement to $T_{F(x)} G^{-1}(S)$ if and only if $\left.D(G \circ F)\right|_{x}\left(E_{x}\right)$ is a complement to $T_{G \circ F(x)} S$.

Definition 5.1.3. Manifolds with boundary are defined like manifolds, but modeled on open sets in $L^{n}=\left\{x \in \mathbb{R}^{n} \mid x^{1} \leq 0\right\}$. The boundary $\partial M$ is the set of points that correspond to elements in $\partial L^{n}=\left\{x \in \mathbb{R}^{n} \mid x^{1}=0\right\}$.

It is not hard to prove that if $F: M \rightarrow \mathbb{R}$ has $a \in \mathbb{R}$ as a regular value then $F^{-1}(-\infty, a]$ is a manifold with boundary. If $M$ is oriented, then the boundary is oriented in such a way that if we add the outward pointing normal to the boundary as the first basis vector then we get a positively oriented basis for $M$. Thus $\partial_{2}, \ldots, \partial_{n}$ is the positive orientation for $\partial L^{n}$ since $\partial_{1}$ points away from $L^{n}$ and $\partial_{1}, \partial_{2}, \ldots, \partial_{n}$ is the usual positive orientation for $L^{n}$.

When $F: M \rightarrow N$, then we denote the restriction to the boundary as $\partial F=\left.F\right|_{\partial M}$.
THEOREM 5.1.4. Let $F: M \rightarrow N$, where $M$ has boundary. If $S \subset N$ has no boundary and both $F$ and $\partial F$ are transverse to $S$, then $F^{-1}(S)$ is a submanifold with $\partial\left(F^{-1}(S)\right)=$ $F^{-1}(S) \cap \partial M$.

Proof. The transversality assumptions for $F$ and $\partial F$ at $x \in(\partial F)^{-1}(S)$ imply that we can find a subspace $E_{x} \subset T_{x} \partial M$ such that $\left.E_{x} \oplus \operatorname{ker} D \partial F\right|_{x}=T_{x} \partial M$ and $\left.E_{x} \oplus \operatorname{ker} D F\right|_{x}=T_{x} M$. In particular, ker $\left.D F\right|_{x}$ contains vectors that are not tangent to $\partial M$.

To see how this helps us we select coordinates around $q=F(p) \in S, p \in \partial M$, such that $S$ is the preimage of $0 \in \mathbb{R}^{k}$. By also choosing coordinates around $p$ we are reduced to a situation where $F: L^{m} \rightarrow \mathbb{R}^{k}$ and $0 \in \mathbb{R}^{k}$ is a regular value for both $F$ and $\partial F$. By further restricting around $p \in \partial L$ we can assume that $F$ extends to $\bar{F}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{k}$ where $0 \in \mathbb{R}^{k}$ is a regular value. The preimage $\bar{F}^{-1}(0) \subset \mathbb{R}^{n}$ is a submanifold and

$$
F^{-1}(0)=\left\{x \in \bar{F}^{-1}(0) \mid x^{1}(x) \leq 0\right\} .
$$

Thus $F^{-1}(0)$ becomes a manifold with boundary $\partial F^{-1}(0)=\bar{F}^{-1}(0) \cap \partial L$ provided 0 is a regular value for $\left.x^{1}\right|_{\bar{F}^{-1}}$. This is equivalent to $\bar{F}^{-1}(0)$ being transverse to $\partial L$. If $x \in \bar{F}^{-1}(0) \cap \partial L$, then we saw at the beginning of the proof that

$$
\left.\operatorname{ker} D F\right|_{p}=T_{x} \bar{F}^{-1}(0)=T_{x} F^{-1}(0)
$$

contains vectors that are not tangent to $T_{x} \partial M$. This shows that $\bar{F}^{-1}(0)$ is transverse to $\partial L$.

Before we can apply this to our first interesting result we need to classify one-dimensional manifolds.

THEOREM 5.1.5. A connected one-dimensional manifold is diffeomorphic to either $S^{1}$ or $\mathbb{R}$ when it has no boundary and either $[0,1]$ or $[0, \infty)$ when it has boundary.

Proof. If $M$ is orientable, then it has a vector field that never vanishes. It can be constructed locally to be nonzero and point in the positive direction and then using a partition of unity to create a global nonvanishing vector field. Any maximal integral curve will cover the manifold and thus parametrize it. To see this, assume that $c:(a, b) \rightarrow M$ is a maximal integral curve, where $0 \in(a, b)$. Any $x \in M$ is connected to $c(0)$ by a continuous
path which is also a compact set $C \subset M$. The integral curve must either lie in $C$ as $t \rightarrow b$ (or $t \rightarrow a$ ) or leave $C$. In the latter case it will hit $x$. In the former, the integral curve will have an accumulation point $z$ as $t \rightarrow b$ (or $t \rightarrow a$ ). However, there is also a nonstationary integral curve through $z$ which must overlap with $c$ and thus up to translation coincide with $c$. This shows that $z$ lies in the image of $c$ and thus that it was not a maximal integral curve.

If the manifold is not orientable, then the orientation covering has an involution that is orientation reversing. However, any diffeomorphism on $[0, \infty)$ clearly fixes the boundary. On $[0,1]$ it either fixes the boundary points or reverses them, in the later case the intermediate value theorem guarantees an interior fixed point as it must cross the diagonal $y=x$. When the manifold has no boundary we must in addition use that it is orientation reversing to get a fixed point. For $\mathbb{R}$ the map is strictly decreasing and so is also forced to cross $y=x$. On $S^{1}$ we can lift the map to $\mathbb{R}$ where it will have a fixed point.

COROLLARY 5.1.6. A compact manifold with boundary admits no retracts onto the boundary.

Proof. Consider a map $F: M \rightarrow \partial M$, such that $\left.F\right|_{\partial M}=i d_{\partial M}$. If $p \in \partial M$ is a regular value, then $F^{-1}(p) \subset M$ is a one-dimensional manifold with $\partial\left(F^{-1}(p)\right)=F^{-1}(p) \cap$ $\partial M=\{p\}$. Thus $F^{-1}(p)$ is noncompact and consequently $M$ must also be noncompact.

Corollary 5.1.7 (Brouwer's Fixed Point Theorem). Any map on the closed unit ball in Euclidean space has a fixed point.

Proof. Consider a map $F: \bar{B} \rightarrow \bar{B}$, where $\bar{B} \subset \mathbb{R}^{n}$ is the closed unit ball. If $F$ has no fixed points, then there is a unique line through $p$ and $F(p)$ for all $p \in \bar{B}$. Let $G(p) \in \partial \bar{B}$ be the point on this line closest to $p$. We offer an explicit formula by solving

$$
|t p+(1-t) F(p)|^{2}=1
$$

This quadratic equation has no solutions on $(0,1)$ as that corresponds to the point between $p$ and $F(p)$ and there is a solution $t_{0} \leq 1$ and another $t_{1} \geq 1$. The latter corresponds to $G(p)=t_{1} p+\left(1-t_{1}\right) F(p)$. When $p \in \partial \bar{B}$ we have $G(p)=p$ so we can use the previous corollary to obtain a contradiction.

REMARK 5.1.8. This corollary uses that the function is smooth. As any continuous function can be approximated by smooth functions we also obtain the more general results for continuous functions.

### 5.2. Thom's Transversality Theorem

Throughout the section we will consider maps from $M^{n}$ (possibly with boundary) into $N$ (without boundary). We are interested in finding maps that are transverse to a specific (properly) embedded submanifold $S \subset N$.

Lemma 5.2.1. Let L be a manifold without boundary and $F: M \times L \rightarrow N$. If $F$ and $\partial F$ are transverse to $S \subset N$, then $F_{l}: M \rightarrow N$ and $\partial F_{l}: \partial M \rightarrow N$ are transverse to $S$ for almost all $l \in L$, where $F_{l}(x)=F(x, l)$.

Proof. Our assumptions allow us to conclude that $S^{*}=F^{-1}(S) \subset M \times L$ is a (properly) embedded submanifold with boundary $\partial S^{*}=S^{*} \cap \partial M \times L$. Consider the restriction of the projection onto $L, \pi: S^{*} \rightarrow L$. We claim that if $l \in L$ is a regular value for $\pi$ and $\partial \pi$, then $F_{l}$ and $\partial F_{l}$ are transverse to $S$. For simplicity we focus on $F_{l}$ as the argument is identical for $\partial F_{l}$.

For the given $l$ consider $(x, l) \in S^{*}$ and $y=F(x, l)=F_{l}(x) \in S$. By assumption

$$
\left.D F\right|_{(x, l)}\left(T_{x} M \times T_{l} L\right)+T_{y} S=T_{y} N .
$$

For any fixed $a \in T_{y} N$, there exists $(w, e) \in T_{x} M \times T_{l} L$ such that

$$
\left.D F\right|_{(x, l)}(w, e)-a \in T_{y} S .
$$

Since $l$ is a regular value for $\pi: S^{*} \rightarrow L$ we can also find $(u, e) \in T_{(x, l)} S^{*}$ such that $\left.D \pi\right|_{(x, l)}(u, e)=$ $e$. Now $\left.D F\right|_{(x, l)}(u, e) \in T_{y} S$ as $S^{*}=F^{-1}(S)$. So if $v=w-u \in T_{x} M$, then

$$
D F_{l}(v)-a=\left.D F\right|_{(x, l)}(v, 0)=\left.D F\right|_{(x, l)}(w, e)-a-\left.D F\right|_{(x, l)}(u, e) \in T_{y} S
$$

This shows that $F_{l}$ is transverse to $S \subset N$.
This lemma can be used to prove the Borsuk-Ulam theorem. A map $F: O \subset \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ is said to be odd (or even), if $O$ is invariant under $x \mapsto A(x)=-x$ and $F \circ A=A \circ F$ (or $F \circ A=F$ ). Note that all matrices in Mat ${ }_{n \times m}$ induce odd maps.

THEOREM 5.2.2. The following statements are equivalent and true:
(1) If $f: S^{n} \rightarrow \mathbb{R}^{n}$, then there exists $x \in S^{n}$ such that $f(x)=f(-x)$.
(2) If $f: S^{n} \rightarrow \mathbb{R}^{n}$ is odd, then there exists $x \in S^{n}$ such that $f(x)=0$.
(3) There is no odd map $f: S^{n} \rightarrow S^{n-1}$.

Proof. We first establish equivalence and then prove (2).
Clearly (1) implies (2) and (2) implies (3). For (3) implies (1) simply note that if (1) fails for some $f$, then

$$
g(x)=\frac{f(x)-f(-x)}{|f(x)-f(-x)|}
$$

contradicts (3).
Let $L=$ Mat $_{n \times(n+1)}^{n}$ be the open set of rank $n$ matrices. If $B \in$ Mat $_{n \times(n+1)}^{n}$, then it induces an odd map $B: S^{n} \rightarrow \mathbb{R}^{n}$ with exactly two zeros $\left\{ \pm v_{B}\right\}$ both of which span $\operatorname{ker}\left(B: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n}\right)$. Note that the rank $n$ assumption also implies that 0 is a regular value for $B: S^{n} \rightarrow \mathbb{R}^{n}$.

Assume now that $f: S^{n} \rightarrow \mathbb{R}^{n}$ has no zeros and consider the linear homotopies

$$
F(t, x, B)=H_{B}(t, x)=t f(x)+(1-t) B(x):[0,1] \times S^{n} \times \operatorname{Mat}_{n \times(n+1)}^{n} \rightarrow \mathbb{R}^{n}
$$

We claim that $F$ is transverse to $0 \in \mathbb{R}^{n}$. Suppose $F(t, x, B)=0$. As $f$ has no zeros we must have $t<1$ and we can use that

$$
\left.D F\right|_{(t, x, B)}(0,0, H)=(1-t) H(x) .
$$

Since $x \neq 0$ and $t<1$ we can for any $u \in \mathbb{R}^{n}=T_{0} \mathbb{R}^{n}$ find $H \in \operatorname{Mat}_{n \times(n+1)}=T_{B} \operatorname{Mat}_{n \times(n+1)}^{n}$ such that $u=(1-t) H(x)$. Thus $F$ is actually a submersion when $t<1$.

Lemma 5.2.1 then implies that there is a $B \in \operatorname{Mat}_{n \times(n+1)}^{n}$ such that the homotopy $H_{B}$ is transverse to $0 \in \mathbb{R}^{n}$. Let $N=H_{B}^{-1}(0)$. This a compact one-dimensional manifold with $\partial N=N \cap\{0,1\} \times S^{n}$ showing that $\partial N=\left\{\left(0, \pm v_{B}\right)\right\}$. Thus $N$ is a union of circles and one $\operatorname{arc} N_{0}$ that joins the two zeros $\left\{ \pm v_{B}\right\}$ on the boundary. Note that the homotopy is a homotopy of odd maps, $H_{B}(t,-x)=-H_{B}(t, x)$. Thus $A(N)=N$ and $A\left(N_{0}\right)=N_{0}$. Now parametrize the arc $N_{0}$ to be a unit speed curve $c:[0, b] \rightarrow[0,1] \times S^{n} \subset \mathbb{R}^{n+2}$ with $c(0)=\left(0, v_{B}\right)$ and $c(b)=\left(0,-v_{B}\right)$. Since $A$ is an isometry that preserves $N_{0}$ we see that $A \circ c$ is also a unit speed curve with $A \circ c(0)=\left(0,-v_{B}\right)$ and $A \circ c(b)=\left(0, v_{B}\right)$. In other words $A \circ c$ is simply $c$ parametrized backwards:

$$
A(c(s))=c(b-s)
$$

This shows that $A\left(c\left(\frac{b}{2}\right)\right)=c\left(\frac{b}{2}\right) \in[0,1] \times S^{n}$ which is impossible.
Since $L$ is locally path connected lemma 5.2 .1 shows that any $F_{l}$ is homotopic to nearby maps that are transverse to $S \subset N$. This will allow us to show that any map is homotopic to a nearby map that is transverse. We will prove a slightly more complicated version that also works for maps that are sections, e.g., vector fields.

THEOREM 5.2.3 (Thom). Any map $f: M \rightarrow N$ is homotopic to a nearby map that is transverse to $S \subset N$. Moreover, if $f$ is a section for $\pi: N \rightarrow M$, i.e., $\pi \circ f=i d_{M}$, then the homotopy $H:[0,1] \times M \rightarrow N$ can be chosen so that all of the maps $H_{t}: M \rightarrow N$ are sections. Finally, if $f$ is proper, then the homotopy is also proper.

Proof. In case were $f$ is a section note that the property $\pi \circ f=i d_{M}$ implies that $f: M \rightarrow N$ becomes an embedding. Moreover, $\left.D \pi\right|_{f(x)}: T_{f(x)} N \rightarrow T_{x} M$ is a surjection and hence is a submersion on a neighborhood of $f(M)$. This also shows that it is transverse to each of the preimages of $\pi$. For the rest of the proof we will assume that $N$ is a neighborhood of $f(M)$ on which $\pi$ is a submersion.

Using a proper embedding $N \rightarrow \mathbb{R}^{k}$ we can orthogonally project the coordinate vector fields on to $T N$ to obtain vector fields $X_{1}, \ldots, X_{k}$ on $N$ that span the tangent space at every point. We can further orthogonally project on to the tangent spaces of the preimages of $\pi$ to obtain vector fields $Y_{1}, \ldots, Y_{k}$ on $N$ that span the tangent spaces to the preimages of $\pi$. Let $\Phi_{1}^{t_{1}}, \ldots, \Phi_{k}^{t_{k}}$ be the flows of either $X_{1}, \ldots, X_{k}$ or $Y_{1}, \ldots, Y_{k}$ depending of which case we are considering. For each $y \in N$, there exists $0<\varepsilon(y) \ll 1$, such that

$$
\begin{aligned}
B(0, \varepsilon(y)) & \rightarrow N \\
\left(t_{1}, \ldots, t_{k}\right) & \mapsto \Phi_{1}^{t_{1}} \circ \cdots \circ \Phi_{k}^{t_{k}}(y)
\end{aligned}
$$

is a submersion to a neighborhood of $y$ in $N$ or in a preimage of $\pi$. The function $\varepsilon(y)$ can be chosen to be smooth as the existence of flows is locally uniform. We can then scale the parameters $s_{i}=t_{i} / \varepsilon(y)$ to obtain maps

$$
\begin{aligned}
B(0,1) & \rightarrow N \\
\left(s_{1}, \ldots, s_{k}\right) & \mapsto \Phi_{1}^{\varepsilon(y) s_{1}} \circ \cdots \circ \Phi_{k}^{\varepsilon(y) s_{k}}(y)
\end{aligned}
$$

that are submersions into $N$ or preimages of $\pi$.
We claim that

$$
\begin{aligned}
F: M \times B(0,1) & \rightarrow N \\
\left(x, s_{1}, \ldots s_{k}\right) & \mapsto \Phi_{1}^{\varepsilon(f(x)) s_{1}} \circ \cdots \circ \Phi_{k}^{\varepsilon(f(x)) s_{k}}(f(x))
\end{aligned}
$$

is transverse to $S \subset N$. Note that $F_{(0, \ldots, 0)}(x)=f(x)$. Moreover, the maps $F_{\left(s_{1}, \ldots, s_{k}\right)}: M \rightarrow N$ are sections to $\pi$ since the flows preserve preimages of $\pi$.

In case the vector fields span $T N$ the map $\left(s_{1}, \ldots, s_{k}\right) \mapsto F\left(x, s_{1}, \ldots, s_{k}\right)$ is a submersion for each $x$ and in particular transverse to $S$. In case the vector fields only span the tangent spaces to the preimages of $\pi$ the whole map $F$ becomes a submersion since each section $F_{\left(s_{1}, \ldots, s_{k}\right)}: M \rightarrow N$ is transverse to the preimages of $\pi$.

The previous lemma now guarantees a point $\left(s_{1}, \ldots s_{k}\right)$ so that $F_{\left(s_{1}, \ldots, s_{k}\right)}: M \rightarrow N$ is transverse to $S$. The homotopy is then defined by

$$
H(t, x)=F\left(x, t s_{1}, \ldots, t s_{k}\right)
$$

Finally, assume $f$ is proper. First observe that as $N \subset \mathbb{R}^{k}$ is properly embedded it follows that: $y_{i} \rightarrow \infty$ in $N$ if and only if $y_{i} \rightarrow \infty$ in $\mathbb{R}^{k}$. Moreover, as the vector fields $X_{i}$ or $Y_{i}$ all have norm $\leq 1$ in $T N \subset T \mathbb{R}^{k}$ we have that in the Euclidean distance:

$$
\left|\Phi_{1}^{\varepsilon(y) s_{1}} \circ \cdots \circ \Phi_{k}^{\varepsilon(y) s_{k}}(y)-y\right| \leq k \varepsilon(y)
$$

Thus $y_{i} \rightarrow \infty$ if and only if $\Phi_{1}^{\varepsilon\left(y_{i}\right) s_{1}} \circ \cdots \circ \Phi_{k}^{\varepsilon\left(y_{i}\right) s_{k}}\left(y_{i}\right) \rightarrow \infty$. As $f$ is proper we have that $y_{i}=f\left(x_{i}\right) \rightarrow \infty$ provided $x_{i} \rightarrow \infty$ in $M$. This shows that also $\Phi_{1}^{\varepsilon\left(y_{i}\right) s_{1}} \circ \cdots \circ \Phi_{k}^{\varepsilon\left(y_{i}\right) s_{k}}\left(y_{i}\right) \rightarrow \infty$. In particular, each $F_{\left(s_{1}, \ldots, s_{k}\right)}: M \rightarrow N$ is proper. This implies that the homotopy $H(t, x)=$ $F\left(x, t s_{1}, \ldots, t s_{k}\right)$ is proper when $f$ is proper.

We obtain a series of useful consequences. The first is a relative version of the above theorem.

THEOREM 5.2.4. If $f: M \rightarrow N$ is transverse to $S$ for all $x \in C \cap f^{-1}(S)$, where $C \subset M$ is closed, then the homotopy can be chosen so that $H(t, x)=f(x)$ for all $x \in C$ and $x \mapsto$ $H(1, x)$ is transverse to $S \subset N$.

Proof. We use the same notation as in theorem 5.2.3. Select a bump function $\lambda$ : $M \rightarrow[0,1]$ such that $\lambda^{-1}(0)=C$. Define

$$
\begin{aligned}
m: M \times B(0,1) & \rightarrow M \times B(0,1) \\
(x, s) & \mapsto\left(x, \lambda^{2}(x) s\right)
\end{aligned}
$$

and note that

$$
\operatorname{Dm}(v, e)=\left(v, \lambda^{2}(x) e+2 \lambda(x) d \lambda(v) s\right)
$$

We can now define $G=F \circ m$ so that both $G(x, s)=f(x)$ and $\left.D G\right|_{(x, s)}(v, e)=\left.D f\right|_{x}(v)$ for $x \in C$. This shows that $G$ is transverse to $S$ for all $x \in C \cap f^{-1}(S)$. For $x \in M-C$ we have

$$
\left.D G\right|_{(x, s)}(0, e)=\left.D F\right|_{(x, s)}\left(0, \lambda^{2}(x) e\right)
$$

showing that $G_{x}: B(0,1) \rightarrow N$ becomes a submersion.
Corollary 5.2.5. Let $F: M \rightarrow N$. If $\partial F: \partial M \rightarrow N$ is transverse to $S \subset N$, then there is a homotopy $H:[0,1] \times M \rightarrow N$, such that $H(t, x)=\partial F(x)$ for all $x \in \partial M$ and $x \mapsto H(1, x)$ is transverse to $S \subset M$.

REMARK 5.2.6. In particular, if two maps are homotopic and transverse to $S$, there there exists a homotopy between the maps that is also transverse to $S$.

COROLLARY 5.2.7. Any manifold admits a vector field that is transverse to the zero section $p \mapsto 0_{p} \in T_{p} M$.

COROLLARY 5.2.8. Any map $F: M \rightarrow M$ is homotopic to a map $G: M \rightarrow M$ such that

$$
\begin{aligned}
\left(i d_{M}, G\right): M & \rightarrow M \times M \\
x & \mapsto(x, G(x))
\end{aligned}
$$

is transverse to the diagonal $\Delta=\{(p, p) \mid p \in M\}$.
Proof. Just use that $\left(i d_{M}, F\right)$ is a section of the projection $\pi_{1}: M \times M \rightarrow M$ on to the first coordinate.

### 5.3. Mod 2 Intersection Theory

We start with some elementary observations about intersections of subspaces $V^{k}, W^{l} \in$ $\mathbb{R}^{n}$. They will always intersect in the origin and when $k+l>n$ they actually intersect in a nontrivial subspace.

This leads to some observations about spheres and projective spaces. In $S^{2}$ any two great circles intersect in two points or coincide. However, we can always homotope one of these great circles away from the other. This means that the fact that great circles intersect is not a topological property. When we pass to the projective plane $\mathbb{R P}^{2}$, the great circles become projective lines $\mathbb{R} \mathbb{P}^{1} \subset \mathbb{R} \mathbb{P}^{2}$ and they will intersect in one point or coincide (2dimensional subspaces in $\mathbb{R}^{3}$ intersect in a line or coincide). In contrast to the sphere we will show that the fact that projective lines intersect is a topological property and cannot be changed via homotopies.

The general set-up in this section and the next is a map $F: M \rightarrow N$ where $M$ is compact and $N$ is connected. We wish to study how $F$ intersects a closed submanifold $S \subset N$. When $M$ has boundary we further assume that $\partial F$ does not intersect $S$ and, in particular, is transverse to $S$. If we assume that $F$ is transverse to $S$ and that

$$
\operatorname{dim} M+\operatorname{dim} S=\operatorname{dim} N
$$

then $F^{-1}(S) \subset M$ is a finite collection of points none of which lie on the boundary. We define

$$
I_{2}(F, S)=\# F^{-1}(S) \bmod 2= \begin{cases}0 & \text { if } \# F^{-1}(S) \text { is even } \\ 1 & \text { if } \# F^{-1}(S) \text { is odd }\end{cases}
$$

We proceed to show that this intersection number is a homotopy invariant of $F$. Note by contrast that the number of preimages is not a homotopy invariant. In the next section we will define a more subtle integer valued intersection number.

THEOREM 5.3.1. If $F_{0}, F_{1}: M^{m} \rightarrow N^{n}$ are homotopic and transverse to $S^{n-m} \subset N$, then $I_{2}\left(F_{0}, S\right)=I_{2}\left(F_{1}, S\right)$. When $\partial M \neq \emptyset$, we assume that $\partial F_{0}=\partial F_{1}$, does not intersect $S$, and that the homotopy is fixed on $\partial M$.

Proof. When $\partial M \neq \emptyset$ the space $[0,1] \times M$ is not a manifold with boundary. However, we are assuming that any homotopy maps $[0,1] \times \partial M$ to a set that is disjoint from $S$. Theorem 5.2.4 and its corollary can easily be reframed to work in this context. Thus we obtain a homotopy $H:[0,1] \times M \rightarrow N$ that is transverse to $S$ and such that $H([0,1] \times \partial M) \cap$ $S=\emptyset$. The preimage $H^{-1}(S) \subset[0,1] \times \operatorname{int} M$ is a compact one-manifold with boundary

$$
\partial H^{-1}(S)=H^{-1}(S) \cap\{0,1\} \times \operatorname{int} M=\{0\} \times F_{0}^{-1}(S) \cup\{1\} \times F_{1}^{-1}(S) .
$$

As \# $\partial H^{-1}(S)=\# F_{0}^{-1}(S)+\# F_{1}^{-1}(S)$ is even it follows that the two terms on the right have the same parity. This proves the theorem.

This immediately explains why projective lines can't be homotopied away from each other as they intersect in one point. The theorem also allows us to define the intersection number of a map.

DEFINITION 5.3.2. If $F: M^{m} \rightarrow N^{n}$ and $S^{n-m} \subset N$, then $I_{2}(F, S)$ is defined as the mod 2 intersection number of any map that is homotopic to $F$ and transverse to $S$. In case $M$ has boundary, $\partial F$ does not intersect $S$ and the homotopies are all $\partial F$ when restricted to $\partial M$.

When $M^{m} \subset N^{n}$, then we define the intersection number as $I_{2}(M, S)=I_{2}(i, S)$, where $i: M \rightarrow N$ is the inclusion map.

We can now also place corollary 5.1.6 in a more general context. For this to be more clear note that $I_{2}\left(i d_{M},\{x\}\right)=1$.

THEOREM 5.3.3. Let $B^{m+1}$ be a compact manifold with boundary $\partial B=M^{m}$ and $f$ : $M^{m} \rightarrow N^{n}$ with $S^{n-m} \subset N^{n}$ a closed submanifold. If $f=\partial F$, where $F: B \rightarrow N$, then $I_{2}(f, S)=0$.

Proof. We can use theorem 5.2.3 to find a map $G: B \rightarrow N$ that is homotopic to $F$ and such that both $G$ and $\partial G$ are transverse to $S$. The preimage $G^{-1}(S)$ is a compact onemanifold with an even number of boundary components $\partial\left(G^{-1}(S)\right)=(\partial G)^{-1}(S)$. Thus $0=I_{2}(\partial G, S)=I_{2}(f, S)$.

REMARK 5.3.4. All of the above results also work for proper maps in case $M$ is not compact. Theorem 5.2.3 guarantees that proper maps are homotopic to proper maps that are transverse through homotopies that are proper. Likewise theorems 5.3.1 and its more general version 5.3.3 as long as we assume that the homotopy or extension map are proper.

DEFINITION 5.3.5. The mod 2 Euler characteristic of a manifold is defined as $\chi_{2}(M)=$ $I_{2}\left(X, M_{0}\right)$, where $X: M \rightarrow T M$ is a vector field and $M_{0}=\left\{\left(p, 0_{p}\right) \mid p \in M\right\}$ the zero section. Corollary 5.2.7 implies that this is well-defined as all vector fields are homotopy equivalent.

Similarly corollary 5.2 .8 shows that the mod 2 Lefschetz number of a map $F: M \rightarrow M$, $L_{2}(F)=I_{2}\left(\left(i d_{M}, F\right), \Delta\right)$, is a well-defined homotopy invariant of $F$.

PROPOSITION 5.3.6. We have $\chi_{2}(M)=L_{2}\left(i d_{M}\right)$.
Proof. We identify $M$ with the diagonal $\Delta \subset M \times M$ and $T M$ with the normal bundle $N(\Delta)=\{(v,-v) \mid v \in T M\}$ to the diagonal in the product. For a vector field $X$ on $M$ we obtain a section $(X,-X)$ of $N(\Delta)$ that is homotopic to the zero section

$$
\Delta_{0}=\left\{\left((p, p),\left(0_{p},-0_{p}\right)\right) \mid p \in M\right\} .
$$

This tells us that

$$
L_{2}\left(i d_{M}\right)=I_{2}\left(\left(i d_{M}, i d_{M}\right), \Delta\right)=I_{2}(\Delta, \Delta)=I_{2}\left(\Delta_{0}, \Delta_{0}\right)=I_{2}\left((-X, X), \Delta_{0}\right)=I_{2}\left(X, M_{0}\right)
$$

The mod 2 Euler characteristic of a sphere is always 0 . For odd dimensional spheres this is because they admit a nonvanishing vector field. For even dimensional spheres we can select such a nonvanishing field on the equator and extend it to the entire space creating only two zeros. What is more interesting is that $\chi_{2}\left(\mathbb{R} \mathbb{P}^{2 n}\right)=1$ as we can select a vector field on $S^{2 n}$ that is invariant under the antipodal map and has two zeros. In projective space this yields a vector field with one zero. By the same construction we also see that $\chi_{2}\left(\mathbb{R}^{2 n+1}\right)=0$.

We can now prove another difficult result, the Jordan-Brouwer separation theorem.
THEOREM 5.3.7. If $S \subset \mathbb{R}^{n+1}$ is a closed, connected, $n$-dimensional submanifold, then $\mathbb{R}^{n+1}-S$ has two connected components.

Note that $\mathbb{R} \mathbb{P}^{n} \subset \mathbb{R} \mathbb{P}^{n+1}$ has a complement that is a disc and is thus connected. The theorem holds with virtually the same proof when the ambient space $\mathbb{R}^{n+1}$ is replaced with a simply connected manifold. Note also that transversality can be used to show that the complement of a submanifold of codim $\geq 2$ is always connected. Thus the complement of a finite set in a connected manifold is connected when the manifold has dimension $\geq 2$.

Proof. We start with the observation that $I_{2}(f, S)=0$ for any closed curve $f: S^{1} \rightarrow$ $\mathbb{R}^{n+1}$. This is where simple connectivity is used.

This simple observation implies that there exists a unit normal field $X: S \rightarrow \mathbb{R}^{n+1}$, i.e., $X(p) \perp T_{p} S$ for all $p \in S$. Note that at each point there are only two choices for this unit normal and the existence is equivalent to saying that $S$ is orientable.

First note that the space of unit normal vectors

$$
U N\left(S \subset \mathbb{R}^{n+1}\right)=\left\{v \in T_{p} \mathbb{R}^{n+1}\left|p \in S, v \perp T_{p} S,|v|=1\right\}\right.
$$

is a two-fold covering space of $S$. We can now appeal to corollary 1.4.35 and obtain a section $S \rightarrow U N\left(S \subset \mathbb{R}^{n+1}\right)$ provided the unique lift of a closed curve in $S$ becomes a closed curve in $U N\left(S \subset \mathbb{R}^{n+1}\right)$. Let $c:[0,1] \rightarrow S$ be a curve, and $X:[0,1] \rightarrow U N\left(S \subset \mathbb{R}^{n+1}\right)$ the unique lift. In case $c(0)=c(1)$ we need to show that $X(0)=X(1)$. If not, then $X(0)=-X(1)$. Consider the curve $c_{\varepsilon}(t)=c(t)+\varepsilon X(t)$. Since $X$ is nontrivial and transverse to $S$ there must be a small $\varepsilon$ such that $c_{\varepsilon}(t)$ does not intersect $S$. Now join the end points $c(0)+\varepsilon X(0)$ and $c(1)+\varepsilon X(1)=c(0)-\varepsilon X(0)$ by a straight line that intersects $S$ orthogonally and only in $c(0)=c(1)$. This leads to a closed curve that intersects $S$ only once.

We can now create a tubular neighborhood, more like a band neighborhood, by considering $H(s, p)=p+s X(p)$ on $(-\varepsilon, \varepsilon) \times S$. As $S$ is compact and $X$ transverse to $S$ the differential of $H$ is an isomorphism at all points $(0, p)$. Thus it is a diffeomorphism on a neighborhood of $\{0\} \times S$. By decreasing $\varepsilon$ we obtain a diffeomorphism $H:(-\varepsilon, \varepsilon) \times S \rightarrow U$ onto a neighborhood of $S$.

This neighborhood allows us to deform curves $c:[0,1] \rightarrow \mathbb{R}^{n+1}$ between points $p, q \in$ $\mathbb{R}^{n+1}-S$ to curves with a minimal number of intersections with $S$. Note that $I_{2}(c, S)$ is well-defined for curves with $c(0)=p$ and $c(1)=q$ and is invariant under homotopies that fix $p$ and $q$. We claim that given $p, q \in \mathbb{R}^{n+1}-S$ there is a curve that intersects $S$ transversely and intersects $S$ in $I_{2}(c, S)$ points, where $c$ is any curve from $p$ to $q$ that is transverse to $S$. In the tubular neighborhood $U$ we can write $c(t)=p(t)+s(t) X(p(t))$ and note that if $c\left(t_{0}\right) \in S$, i.e., $s\left(t_{0}\right)=0$, then either it crosses from negative $s$ to positive $s$, or the other way around. We say that it has a positive or negative crossing. Now assume that the first crossing $t_{0}$ is positive, the last crossing $t_{k}$ can be negative or positive. If $t_{k}$ is negative, then we can replace $c$ on $\left[-\delta+t_{0}, t_{k}+\boldsymbol{\delta}\right]$ by a curve in $H(\{-\delta\} \times S)$ (this is where connectivity of $S$ is used) to obtain a new curve that does not intersect $S$. This gives a curve that does not intersect $S$ provided $I_{2}(c, S)=0$. While if $t_{k}$ is positive we can replace $c$ on $\left[-\delta+t_{0}, t_{k}-\delta\right]$ by a curve in $H(\{-\delta\} \times S)$ to obtain a new curve that intersects $S$ in only one point (at $t_{k}$ ). This gives a curve that intersects $S$ once when $I_{2}(c, S)=1$.

We can now finish the proof. Fix $p_{0} \in \mathbb{R}^{n+1}-S$ and define

$$
O_{0 \text { or } 1}=\left\{p \in \mathbb{R}^{n+1}-S \mid I_{2}(c, S)=0 \text { or } 1\right\}
$$

We claim that both sets are nonempty, open, and connected. Clearly $p_{0} \in O_{0}$. For $O_{1}$ select a shortest line segment from $p_{0}$ to $S$. It'll intersect $S$ orthogonally and its continuation will yield a slightly longer segment that intersects $S$ orthogonally in exactly one point. Both sets are open as any point $p \in \mathbb{R}^{n+1}-S$ has a neighborhood $B(p, \delta) \subset \mathbb{R}^{n+1}-S$. Thus any point in $B(p, \delta)$ can be joined to $p$ by a segment that doesn't intersect $S$, and hence to $p_{0}$ by a curve with the same intersection number as a curve from $p_{0}$ to $p$. Finally, both sets are connected since any two points $p, q \in O_{0}$ or 1 are joined to $p_{0}$ by curves whose intersection number with $S$ have the same parity. This leads to a concatenated curve from $p$ to $q$ with an even number of intersections with $S$. By the above argument it can be replaced with a curve that doesn't intersect $S$.

### 5.4. Oriented Intersection Theory

We refine the mod 2 intersection numbers from the last section to integer valued intersection numbers provided all of the manifolds involved in our standard set-up

$$
F: M \rightarrow N \supset S
$$

are oriented. We shall further assume that $M$ is closed.
5.4.1. The Oriented Intersection Number. Recall that an orientation for a vector space $V$ is a choice of an equivalence class of ordered bases. It can be denoted $[V]$ or [ $v_{1}, \ldots, v_{n}$ ] if it refers to a specific ordered basis.

Given a subspace $V_{0} \subset V$ that also comes with an orientation we can select a unique orientation on a complement $V_{0} \oplus V_{1}=V$ so that a positively oriented basis on $V_{0}$ followed by a positively oriented on $V_{1}$ gives a positively oriented basis for $V$. We also write $\left[V_{0}\right] \oplus$ $\left[V_{1}\right]=[V]$. Note that could also select an orientation $\left[V_{1}\right]^{\prime}$ such that $\left[V_{1}\right]^{\prime} \oplus\left[V_{0}\right]=[V]$. These orientations agree unless both subspaces are odd dimensional as it takes $\operatorname{dim} V_{0} \cdot \operatorname{dim} V_{1}$ transpositions to switch the ordered bases from $V_{0} \oplus V_{1}$ to $V_{1} \oplus V_{0}$.

When $M$ has boundary we orient the boundary by first declaring that outward pointing vectors are positively oriented: $\left[n_{x}\right] \oplus\left[T_{x} \partial M\right]=\left[T_{x} M\right]$. This is consistent with how $H^{n}=$ $\left\{x \in \mathbb{R}^{n} \mid x^{1} \leq 0\right\}$ and its boundary are oriented when we use $n=\partial_{1}$.

In case $M=[0,1]$ we simply assign numbers $\pm 1$ to the points on the boundary. Thus $\{1\}$ is assigned $\mathrm{a}+1$ while $\{0\}$ gets $\mathrm{a}-1$. Note that these signs cancel. Thus any compact oriented one-manifold has the property that the sum of the signs assigned to the boundary points is 0 . This will be fundamental for homotopy invariance of oriented intersection numbers.

Now suppose that $F: M \rightarrow N$ is transverse to $S$ and that $M, N, S$ are oriented manifolds. We wish to assign an orientation to $S^{*}=F^{-1}(S)$. Select $x \in S^{*}$ and a complement $E_{x}$ to $T_{x} S^{*} \subset T_{x} M, E_{x} \oplus T_{x} S^{*}=T_{x} M$. Since $F$ is transverse we note that $\left.D F\right|_{x}\left(E_{x}\right) \oplus T_{x} S=T_{x} N$. Thus we select the orientation of this complement so that

$$
\left[\left.D F\right|_{x}\left(E_{x}\right)\right] \oplus\left[T_{x} S\right]=\left[T_{x} N\right] .
$$

Since $E_{x}$ and $\left.D F\right|_{x}\left(E_{x}\right)$ are isomorphic this also induces an orientation on $E_{x}$ and thus we can orient $S$ so that

$$
\left[E_{x}\right] \oplus\left[T_{x} S\right]=\left[T_{x} M\right] .
$$

A slight consistency issue now develops when $M$ has boundary and also $\partial F$ is transverse. Here $\partial S^{*}$ obtains two possible orientations, one as the boundary of $S^{*}$ which is oriented by $F$ and one simply via $\partial F$. We need to check what affects this possible difference in orientations. Fix $x \in \partial S^{*}$ and start by noting that an outward pointing $n_{x} \in T_{x} S^{*} \subset T_{x} M$ is also outward pointing for $M$. Next we need a complement $E_{x}$ to $T_{x} S^{*} \subset T_{x} M$. We obtain such a complement by selecting a complement $E_{x}$ for $T_{x} \partial S^{*} \subset T_{x} \partial M$, this will then also be a complement for $T_{x} S^{*} \subset T_{x} M$. We now have from $S^{*}$ that $\partial S^{*}$ gets oriented by via $F$ by:

$$
\begin{aligned}
{\left[E_{x}\right] \oplus\left[T_{x} S^{*}\right] } & =\left[T_{x} M\right] \\
{\left[n_{x}\right] \oplus\left[T_{x} \partial S^{*}\right] } & =\left[T_{x} S^{*}\right] .
\end{aligned}
$$

In other words

$$
\left[E_{x}\right] \oplus\left[n_{x}\right] \oplus\left[T_{x} \partial S^{*}\right]=\left[T_{x} M\right]
$$

On the other hand via $\partial F$ we obtain

$$
\left[E_{x}\right] \oplus\left[T_{x} \partial S^{*}\right]^{\prime}=\left[T_{x} \partial M\right]
$$

where

$$
\left[n_{x}\right] \oplus\left[T_{x} \partial M\right]=\left[T_{x} M\right]
$$

i.e.,

$$
\left[n_{x}\right] \oplus\left[E_{x}\right] \oplus\left[T_{x} \partial S^{*}\right]^{\prime}=\left[T_{x} M\right]
$$

Thus we conclude that

$$
\left[n_{x}\right] \oplus\left[E_{x}\right] \oplus\left[T_{x} \partial S^{*}\right]^{\prime}=\left[E_{x}\right] \oplus\left[n_{x}\right] \oplus\left[T_{x} \partial S^{*}\right]
$$

Here the orientations $\left[n_{x}\right] \oplus\left[E_{x}\right]$ and $\left[E_{x}\right] \oplus\left[n_{x}\right]$ agree if $\operatorname{dim} E_{x}$ is even, and are opposite when $\operatorname{dim} E_{x}$ is odd. This shows that we have the predictable relationship

$$
\left[T_{x} \partial S^{*}\right]^{\prime}=(-1)^{\operatorname{dim} E_{x}}\left[T_{x} \partial S^{*}\right]
$$

We can now define oriented intersection numbers. If $F: M \rightarrow N$ is transverse to $S$ and $M^{m}, N^{n}, S^{n-m}$ are oriented manifolds, then we assign a sign/orientation $[x]= \pm 1$ to each $x \in F^{-1}(S)$ with the understanding that it is +1 precisely when the orientation of $T_{x} M=E_{x}$ is mapped to the positive orientation for $D F\left(T_{x} M\right)$ :

$$
[x]=\left.\operatorname{sign} \operatorname{det} D F\right|_{x}
$$

where $\left.\operatorname{det} D F\right|_{x}$ is calculated with respect to positively oriented bases for $T_{x} M$ and $D F\left(T_{x} M\right)$.
THEOREM 5.4.1. When $F=\partial G$, where $G: B \rightarrow N$ and $B$ is compact and oriented, then

$$
\sum_{x \in F^{-1}(S)}[x]=0
$$

Proof. By theorem 5.2.4 we can assume that $G$ is transverse to $S$. Here $G^{-1}(S)$ is a compact one-manifold with $\partial G^{-1}(S)=F^{-1}(S)$. Orientations assigned to points in $F^{-1}(S)$ differ by the same sign $(-1)^{m}$ depending on whether we use the definition from $F$ or as the boundary of $G^{-1}(S)$. We conclude that they add up to 0 as they come in pairs of opposite signs corresponding to each arc in $G^{-1}(S)$.

REMARK 5.4.2. This shows that two homotopic and transverse maps on a closed manifold must have the same value for the sum $\sum[x]$. Also note that as in remark 5.3.4 we can generalize this theorem the the case where $B$ is not compact provided $G$ is proper.

DEFINITION 5.4.3. The oriented intersection number $I(F, S)$ is defined as

$$
I\left(F_{1}, S\right)=\sum_{x \in F_{1}^{-1}(S)}[x]
$$

for any map $F_{1}$ that is homotopic to $F$ and transverse to $S$. This differs in absolute value from $\# F^{-1}(S)$ by cancelling pairs of opposite signs, in particular

$$
I(F, S)=I_{2}(F, S) \bmod 2
$$

When $M^{m} \subset N^{n}$ we obtain two possible intersection numbers $I(M, S)$ and $I(S, M)$. Since

$$
\left[T_{x} M\right] \oplus\left[T_{x} S\right]=(-1)^{\operatorname{dim} M \operatorname{dim} S}\left[T_{x} S\right] \oplus\left[T_{x} M\right]
$$

this intersection number vanishes when the submanifolds are odd dimensional and homotopic to each other.

Consider the intersection $\mathbb{F P}^{m}, \mathbb{F P}^{n-m} \subset \mathbb{F P}^{n}$. When the subspaces are in generic position they will intersect in a point $\mathbb{F P}^{0}$. When $\mathbb{F}=\mathbb{C}$ this is the oriented intersection number. When $\mathbb{F}=\mathbb{R}$ it is the mod 2 intersection number as at least one of the three spaces is even dimensional and so not orientable.
5.4.2. Degree and Winding Numbers. We can now also define the oriented degree of a map $F: M^{n} \rightarrow N^{n}$ between oriented manifolds where we in addition assume that $N$ is connected. Using remark 5.4.2 the degree for proper maps is also well-defined as long as we modify all extensions and homotopies to be proper.

We start by considering the intersection numbers $I(F,\{q\}), q \in N$. We know from lemma 1.4 .28 that when $q$ is a regular value, i.e., $F$ is transverse to $\{q\}$, then some connected neighborhood $V$ is evenly covered: $F^{-1}(V)=\bigcup U_{i}$, where $F: U_{i} \rightarrow V$ is a diffeomorphism. Thus $D F$ preserves (or reverses) the orientation on $U_{i}$ if it preserves (or reverses) orientations at just one point. Thus $I(F,\{y\})=I(F,\{q\})$ for all $y \in V$. Now for a given $F$ there is always a map homotopic to $F$ that is transverse to $\{q\}$. So we can again conclude that $I(F,\{y\})=I(F,\{q\})$ for all $y$ in a neighborhood of $q$. This means that $y \mapsto I(F,\{y\})$ is locally constant on $N$, and in particular constant when $N$ is connected.

DEFInITION 5.4.4. The oriented degree for $F: M^{n} \rightarrow N^{n}$ is well-defined as

$$
\operatorname{deg} F=I(F,\{q\}), q \in N
$$

when $N$ is connected and $F$ is proper.
We get several nice results using degree theory. The key observation is that the degree of a map is a homotopy invariant as it is simply an intersection number. However, as we can only compute degrees of proper maps it is important that the homotopies are through proper maps. When working on closed manifolds this is not an issue. However, if the manifold is Euclidean space, then all maps are homotopy equivalent, although not necessarily through proper maps.

Proposition 5.4.5. The identity map on a closed manifold is not homotopic to a constant map.

Proof. The constant map has degree 0 while the identity map has degree 1 on an oriented manifold. In case the manifold isn't oriented we can use the mod 2 degree.

THEOREM 5.4.6. Even dimensional spheres do not admit non-vanishing vector fields.
Proof. A nowhere vanishing vector field $X$ on $S^{n}$ can be scaled so that it is a unit vector field. If we consider it as a function $X: S^{n} \rightarrow S^{n} \subset \mathbb{R}^{n+1}$ then it is always perpendicular to its foot point as $T_{p} S^{n} \perp p$ in $\mathbb{R}^{n+1}$. We can then create a homotopy

$$
H(p, t)=p \cos (\pi t)+X_{p} \sin (\pi t)
$$

Since $p \perp X_{p}$ and both are unit vectors the Pythagorean theorem shows that $H(p, t) \in S^{n}$ as well. When $t=0$ the homotopy is the identity, and when $t=1$ it is the antipodal map. Since the antipodal map reverses orientations on even dimensional spheres it is not possible for the identity map to be homotopic to the antipodal map.

Next we offer two interesting results for proper maps. The first is related to corollary 1.4.37

THEOREM 5.4.7. If $F: M \rightarrow N$ is a proper nonsingular map of degree $\pm 1$ between oriented connected manifolds, then $F$ is a diffeomorphism.

Proof. Since $F$ is non-singular everywhere it either reverses or preserves orientations at all points. Moreover by corollary 1.4 .30 it is also a covering map. Thus $|\operatorname{deg} F|=$ $\# F^{-1}(y)$ for all $y \in N$. This shows that it must be a diffeomorphism.

Next we offer an interesting and very broad extension of the Fundamental Theorem of Algebra.

THEOREM 5.4.8. Let $F: M \rightarrow N$ be a proper map between oriented noncompact $n$ manifolds, where $N$ is connected. If $F$ is nonsingular and orientation preserving outside a compact set, then $F$ is surjective.

Proof. We assume that all critical points lie in the compact set $C \subset M$ and consider a value $y \in F(M)-F(C)$. This is a regular value and by assumption $\operatorname{deg} F=\# F^{-1}(y)>0$. In particular, $F$ is surjective.

REMARK 5.4.9. Note that when $n=1$ the function $f(x)=x^{2}$ is proper and nonsingular outside a compact set. When $n \geq 2$, it is often possible to ensure that $M-C$ is connected as long as $M$ is itself connected. Thus it often suffices to assume that the map is proper and nonsingular outside a compact set.

The classical winding number for curves in the plane is the number of times a closed curve goes around a fixed point such as the origin. It can be calculated using degrees and as we shall see later also with integration.

DEFINITION 5.4.10. Let $F: M^{n} \rightarrow \mathbb{R}^{n+1}$, where $M$ is closed and oriented. When $z \notin F(M)$ we define the winding number

$$
W(F, z)=\operatorname{deg}\left(\frac{F(x)-z}{|F(x)-z|}: M \rightarrow S^{n}\right)
$$

While it is simply a degree, and the degree is simply an intersection number, it is convenient to maintain these terminologies.

We note that $W(F, z)=W(F-z, 0)$ and that the winding number is a homotopy invariant under homotopies that map in to $\mathbb{R}^{n+1}-\{z\}$.

The winding number can also be calculated in a different way as an intersection number and in return intersection numbers can be calculated as degrees.

THEOREM 5.4.11. Let $G: B^{n+1} \rightarrow \mathbb{R}^{n+1}$ have $0 \in \mathbb{R}^{n+1}$ as a regular value. If $B$ is compact and oriented with boundary $M=\partial B$ and $F=\partial G$ does not contain 0 in its image, then

$$
W(F, 0)=I(G,\{0\}) .
$$

Proof. We select pairwise disjoint coordinate balls $B_{x} \approx B(0, \varepsilon)$ around each $x \in$ $G^{-1}(0)$ such that $G(h)=\left.D G\right|_{x} h+o(h)$.

Let $N^{n+1}=B-\bigcup_{x \in G^{-1}(0)} B_{x}$. This is a new compact manifold with boundary $M \bigcup_{x \in G^{-1}(0)} \partial B_{x}$. The boundaries $\partial B_{x}$ come with two orientations. One from being the boundary of $B_{x}$ and the opposite from being part of the boundary of $N$. By theorem5.4.1 we conclude that

$$
0=\operatorname{deg}\left(\frac{G}{|G|}: \partial N \rightarrow S^{n}\right)
$$

Here the degree is the sum of the degrees from decomposing the boundary as $\partial N=$ $M \bigcup_{x \in G^{-1}(0)} \partial B_{x}$ but where the degrees from the restrictions to $\partial B_{x}$ come with the opposite sign. More precisely:

$$
\operatorname{deg}\left(\frac{G}{|G|}: M \rightarrow S^{n}\right)=\sum_{x \in G^{-1}(0)} \operatorname{deg}\left(\frac{G}{|G|}: \partial B_{x} \rightarrow S^{n}\right)
$$

where $\partial B_{x}$ is oriented as the boundary of $B_{x}$.
We now have to calculate the terms on the right. Since $x \in G^{-1}(0)$ is regular the differential $\left.D G\right|_{x}$ is nonsingular. In particular, we can assume that $B_{x}$ is so small that
$|o(h)| \ll|D G|_{x} h \mid$ for all $h \in \partial B_{x}$. Consequently, we obtain a homotopy from $\left.D G\right|_{x} h$ to $G$ defined by

$$
H(t, h)=\left.D G\right|_{x} h+\text { to }(h):[0,1] \times \partial B_{x} \rightarrow \mathbb{R}^{n+1}-\{0\} .
$$

This reduces the task to calculating the winding number of the differential. However, the space of nonsingular matrices $G l_{n}(\mathbb{R})$ has two components. The orientation preserving matrices contributing +1 and the orientation reversing matrices -1 . This proves the theorem.

REMARK 5.4.12. In case 0 is not a regular value for $G$ but still has a finite preimage that lies in the interior we instead obtain the formula
$W(F, 0)=\operatorname{deg}\left(\frac{G}{|G|}: M \rightarrow S^{n}\right)=\sum_{x \in G^{-1}(0)} \operatorname{deg}\left(\frac{G}{|G|}: \partial B_{x} \rightarrow S^{n}\right)=\sum_{x \in G^{-1}(0)} W\left(\left.G\right|_{\partial B_{x}}, 0\right)$.
as the condition that each $x \in G^{-1}(0)$ is regular is only used to calculate the local winding number and show that it agrees with the intersection number. Thus the total winding number can be split up into local winding numbers.

The above theorem also gives us a new proof of a stronger version of the Borsuk-Ulam theorem 5.2.2. Note that an even map on a sphere obviously has even degree.

THEOREM 5.4.13. An odd map $F: S^{n} \rightarrow S^{n}$ has odd degree. In particular, there cannot exist an odd map $S^{n} \rightarrow S^{n-1}$.

Proof. We use induction on $n$. For $n=0$ we have $S^{0}=\{ \pm 1\}$. As the map is odd it is a bijection and so has degree $\pm 1$. When $n>0$ select $S^{n-1} \subset S^{n}$ and $y \notin F\left(S^{n-1}\right)$. As $F$ is odd and $S^{n-1}$ is invariant under the antipodal map also $-y \notin F\left(S^{n-1}\right)$. We can additionally assume that $\{ \pm y\}$ are regular values for $F$. Now project along great circles through $\pm y$ onto the orthogonal equator $S_{y}^{n-1}$ to obtain a new odd map

$$
G(x)=\frac{F(x)-(F(x), y) y}{|F(x)-(F(x), y) y|}=\frac{\pi \circ F}{|\pi \circ F|}
$$

where $\pi$ is the orthogonal projection along $y$ in Euclidean space. Using counting and that $F$ is odd we obtain

$$
\begin{aligned}
\operatorname{deg} F & =I(F,\{y\}) \\
& =\frac{1}{2} I(F,\{ \pm y\}) \\
& =I\left(\left.F\right|_{S_{+}^{n}},\{ \pm y\}\right), \text { where } S_{+}^{n} \text { is the hemisphere with pole } y \\
& =I(\pi \circ F, 0) \\
& =\operatorname{deg} G,
\end{aligned}
$$

where the previous theorem was used for the last equality.
5.4.3. Lefschetz numbers and the Euler Characteristic. To define Lefschetz numbers and the Euler characteristic we need to select orientations for $M \times M$ and $T M$. A closer look at how orientations are used tells us that the ambient space only needs to have orientations defined along the submanifolds that the maps are intersecting. For $\Delta \subset M \times M$ we note that $T_{(p, p)} M \times M=T_{p} M \times T_{p} M$ comes with a canonical orientation: any choice of an ordered basis $e_{1}, \ldots, e_{m}$ for $T_{p} M$ gives the same choice of orientation

$$
\left(e_{1}, 0\right), \ldots,\left(e_{m}, 0\right),\left(0, e_{1}\right), \ldots,\left(0, e_{m}\right)
$$

Similarly for $M_{0} \subset T M$, there is a natural identification $T_{0_{p}} T M=T_{p} M \times T_{0_{P}} T_{p} M=T_{p} M \times$ $T_{p} M$ where the first factor corresponds to $M$.

DEFINITION 5.4.14. Let $M$ be closed and oriented, $F: M \rightarrow M$, and $X$ a vector field. The oriented Lefschetz number and Euler characteristic are defined by

$$
L(F)=I\left(\left(i d_{M}, F\right), \Delta\right)
$$

and

$$
\chi(M)=I\left(X, M_{0}\right) .
$$

We can now reprove proposition 5.3.6
Proposition 5.4.15. For a closed and oriented manifold

$$
L\left(i d_{M}\right)=\chi(M) .
$$

Proof. The proof is the same after we note that the identification of $T M$ with $N(\Delta)$ respects the orientation choices we have made. Given a positively oriented basis $e_{1}, \ldots, e_{m}$ we assume that $\Delta$ is oriented by $\left(e_{1}, e_{1}\right), \ldots,\left(e_{m}, e_{m}\right)$ and claim that $N(\Delta)$ is oriented by $\left(e_{1},-e_{1}\right), \ldots,\left(e_{m},-e_{m}\right)$. We use column operations to verify this, noting that adding multiples of vectors to other vectors can't change orientations. Starting with

$$
\left(e_{1},-e_{1}\right), \ldots,\left(e_{m},-e_{m}\right),\left(e_{1}, e_{1}\right), \ldots,\left(e_{m}, e_{m}\right)
$$

we can add the last $m$ vectors to the first $m$ and obtain

$$
\left(e_{1}, 0\right), \ldots,\left(e_{m}, 0\right),\left(e_{1}, e_{1}\right), \ldots,\left(e_{1}, e_{m}\right)
$$

and then subtract the first $m$ vectors from the last $m$ vectors to get our standard basis

$$
\left(e_{1}, 0\right), \ldots,\left(e_{m}, 0\right),\left(0, e_{1}\right), \ldots,\left(0, e_{m}\right)
$$

Corollary 5.4.16. For an odd dimensional manifold $L\left(i d_{M}\right)=\chi(M)=0$.
In order to do calculations we need a way of checking orientations at intersection points.

Lemma 5.4.17. Let $M$ be closed and oriented and $F: M \rightarrow M$. The map $\left(i d_{M}, F\right)$ : $M \rightarrow M \times M$ is transverse to $\Delta$ at $(p, p)$ if and only if $\left.D F\right|_{p}: T_{p} M \rightarrow T_{p} M$ only has $0_{p}$ as a fixed point, i.e., +1 is not an eigenvalue for $\left.D F\right|_{p}$. Moreover, in this case the intersection number is given by the sign of $\operatorname{det}\left(i d_{T_{p} M}-\left.D F\right|_{p}\right)$.

Proof. Fix an oriented basis $e_{1}, \ldots, e_{m}$ for $T_{p} M$ and for convenience denote $\left.D F\right|_{p}=A$. The tangent space to the graph of $F$ is spanned by

$$
\left(e_{1}, A\left(e_{1}\right)\right), \ldots,\left(e_{m}, A\left(e_{m}\right)\right)
$$

so transversality comes down to checking if

$$
\left(e_{1}, A\left(e_{1}\right)\right), \ldots,\left(e_{m}, A\left(e_{m}\right)\right),\left(e_{1}, e_{1}\right), \ldots,\left(e_{m}, e_{m}\right)
$$

is a basis and the intersection number is determined by whether this is a positively oriented basis. We subtract the first $m$ vectors from the last $m$ to obtain

$$
\left(e_{1}, A\left(e_{1}\right)\right), \ldots,\left(e_{m}, A\left(e_{m}\right)\right),\left(0, e_{1}-A\left(e_{1}\right)\right), \ldots,\left(0, e_{m}-A\left(e_{m}\right)\right)
$$

This can only be a basis if the last $m$ vectors are linearly independent, i.e., $\operatorname{det}\left(i d_{T_{p} M}-\left.D F\right|_{p}\right) \neq$ 0 . Moreover, when this happens then suitable linear combinations of the last $m$ vectors can be used to obtain the basis

$$
\left(e_{1}, 0\right), \ldots,\left(e_{m}, 0\right),\left(0, e_{1}-A\left(e_{1}\right)\right), \ldots,\left(0, e_{m}-A\left(e_{m}\right)\right)
$$

which is positively oriented only if

$$
e_{1}-A\left(e_{1}\right), \ldots, e_{m}-A\left(e_{m}\right)
$$

is positively oriented.
REMARK 5.4.18. This lemma also shows that we don't have to know or use the orientation of $T_{p} M$ to calculate the intersection number as the sign of $\operatorname{det}\left(i d_{T_{p} M}-\left.D F\right|_{p}\right)$ does not depend on a choice of basis. This makes it particularly easy to calculate Lefschetz numbers.

Let us calculate the Lefschetz numbers for linear maps on projective spaces. The first general observation is that a map $A \in \operatorname{Aut}(V)$ has a fixed point $p \in \mathbb{P}(V)$ iff $p$ is an invariant one dimensional subspace for $A$. In other words fixed points for $A$ on $\mathbb{P}(V)$ correspond to eigenvectors, but without information about eigenvalues.

We start with the complex case as it is a bit simpler. The claim is that any $A \in \operatorname{Aut}(V)$ with distinct eigenvalues is a Lefschetz map on $\mathbb{P}(V)$ with $L(A)=\operatorname{dim} V$. Since such maps are diagonalizable we can restrict attention to $V=\mathbb{C}^{n+1}$ and the diagonal matrix

$$
A=\left[\begin{array}{lll}
\lambda_{0} & & 0 \\
& \ddots & \\
0 & & \lambda_{n}
\end{array}\right]
$$

By symmetry we need only study the fixed point $p=[1: 0: \cdots: 0]$. Note that the eigenvalues are assumed to be distinct and none of then vanish. To check the action of $A$ on a neighborhood of $p$ we use the coordinates $\left[1: z^{1}: \cdots: z^{n}\right]$ and observe that

$$
\begin{aligned}
A\left[1: z^{1}: \cdots: z^{n}\right] & =\left[\lambda_{0} 1: \lambda_{1} z^{1}: \cdots: \lambda_{n} z^{n}\right] \\
& =\left[1: \frac{\lambda_{1}}{\lambda_{0}} z^{1}: \cdots: \frac{\lambda_{n}}{\lambda_{0}} z^{n}\right]
\end{aligned}
$$

This is already (complex) linear in these coordinates so the differential at $p$ must be represented by the complex $n \times n$ matrix

$$
\left.D A\right|_{p}=\left[\begin{array}{ccc}
\frac{\lambda_{1}}{\lambda_{0}} & & 0 \\
& \ddots & \\
0 & & \frac{\lambda_{n}}{\lambda_{0}}
\end{array}\right]
$$

As the eigenvalues are all distinct 1 is not an eigenvalue of this matrix, showing that $A$ really is a Lefschetz map. Next we need to check the differential of $\operatorname{det}\left(I-\left.D A\right|_{p}\right)$. Since $G l_{n}(\mathbb{C})$ is connected it must lie in $G l_{2 n}^{+}(\mathbb{R})$ as a real matrix, i.e., complex matrices always have positive determinant when viewed as real matrices. Since $\left.D A\right|_{p}$ is complex it must follow that $\operatorname{det}\left(I-\left.D A\right|_{p}\right)>0$. So all local Lefschetz numbers are 1 . This shows that $L(A)=n+1$. Since $G l_{n+1}(\mathbb{C})$ is connected any linear map is homotopic to a linear Lefschetz map and must therefore also have Lefschetz number $n+1$.

In particular, we have shown that all invertible complex linear maps must have eigenvectors. Note that this fact is obvious for maps that are not invertible. This could be one of the most convoluted ways of proving the Fundamental Theorem of Algebra. We used
the fact that $G l_{n}(\mathbb{C})$ is connected. This in turn follows from the polar decomposition of matrices, which in turn follows from the Spectral Theorem. Finally we observe that the Spectral Theorem can be proven without invoking the Fundamental Theorem of Algebra.

The alternate observation that the above Lefschetz maps are dense in $G l_{n}(\mathbb{C})$ is also quite useful in many situations.

The real projective spaces can be analyzed in a similar way but we need to consider the parity of the dimension as well as the sign of the determinant of the linear map.

For $A \in G L_{2 n+2}^{+}(\mathbb{R})$ we might not have any eigenvectors whatsoever as $A$ could be $n+1$ rotations. Since $G L_{2 n+2}^{+}(\mathbb{R})$ is connected this means that $L(A)=0$ on $\mathbb{R} \mathbb{P}^{2 n+1}$ if $A \in G L_{2 n+2}^{+}(\mathbb{R})$. On the other hand any $A \in G L_{2 n+2}^{-}(\mathbb{R})$ must have at least two eigenvalues of opposite sign. Since $G L_{2 n+2}^{-}(\mathbb{R})$ is connected we just need to check what happens for a specific

$$
\begin{aligned}
A & =\left[\begin{array}{cccccc}
1 & 0 & & & & \\
0 & -1 & & & & \\
& & 0 & -1 & & \\
& & 1 & 0 & & \\
& & & & \ddots & \\
& & & & & 0
\end{array}\right)-1 \\
& =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & R
\end{array}\right]
\end{aligned}
$$

We have two fixed points

$$
\begin{aligned}
p & =[1: 0: \cdots: 0] \\
q & =[0: 1: \cdots: 0] .
\end{aligned}
$$

For $p$ we can quickly guess that

$$
D A_{p}=\left[\begin{array}{cc}
-1 & 0 \\
0 & R
\end{array}\right]
$$

This matrix doesn't have 1 as an eigenvalue and

$$
\begin{aligned}
\operatorname{det}\left(I-\left[\begin{array}{cc}
-1 & 0 \\
0 & R
\end{array}\right]\right) & =\operatorname{det}\left[\begin{array}{ccc}
2 & 0 \\
0 & I-R
\end{array}\right] \\
& =\operatorname{det}\left[\begin{array}{ccccc}
2 & & & \\
& 1 & 1 & & \\
& -1 & 1 & & \\
& & & \ddots & \\
& & & & 1 \\
& & & 1 \\
& =2^{n+1} .
\end{array}\right.
\end{aligned}
$$

So we see that the determinant is positive. For $q$ we use the coordinates $\left[z^{0}: 1: z^{2}: \cdots: z^{n}\right]$ and easily see that the differential is

$$
\left[\begin{array}{cc}
-1 & 0 \\
0 & -R
\end{array}\right]
$$

which also doesn't have 1 as an eigenvalue and again gives us positive determinant for $I-D A_{q}$. This shows that $L(A)=2$ if $A \in G L_{2 n+2}^{-}(\mathbb{R})$.

In case $A \in G l_{2 n+1}(\mathbb{R})$ it is only possible to compute the Lefschetz number mod 2 as $\mathbb{R P}^{2 n}$ isn't orientable. We can select

$$
A^{ \pm}=\left[\begin{array}{cc} 
\pm 1 & 0 \\
0 & R
\end{array}\right] \in G L_{2 n+1}^{ \pm}(\mathbb{R})
$$

with $R$ as above. In either case we have only one fixed point and it is a Lefschetz fixed point since $D A_{p}^{ \pm}= \pm R$. Thus $L\left(A^{ \pm}\right)=1$ and all $A \in G(2 n+1, \mathbb{R})$ have $L(A)=1$.

This last example can also be used to calculate the Euler characteristic of even dimensional spheres. In fact the matrix

$$
A=\left[\begin{array}{ll}
1 & 0 \\
0 & R
\end{array}\right] \in S O(2 n+1)
$$

is orthogonal and preserves the sphere with two fixed points $( \pm 1,0, \ldots, 0)$ and is homotopic to the identity map. The intersection numbers are both calculated as the sign of $\operatorname{det}\left(i d_{T_{p} S^{2 n}}-R\right)>0$. So the Lefschetz number and the Euler characteristic are both 2.
5.4.4. Isotopies and Poincaré-Hopf-Lefschetz. We start with a useful localization procedure showing that any finite collection of points in a connected manifold lie in an open set diffeomorphic to $\mathbb{R}^{n}$.

DEFINITION 5.4.19. An isotopy is a homotopy of diffeomorphisms $H:[0,1] \times M \rightarrow$ $M$, i.e., for each $t$, the map $x \mapsto H(t, x)$ is a diffeomorphism. It is said to be compactly supported if there is a compact set $C \subset M$, such that $H(t, x)=x$ for all $t$ and $x \in M-C$. Note that we can alter any such homotopy, using a function $\lambda:[0,1] \rightarrow[0,1]$, to a new homotopy $H(\lambda(t), x)$. If $\lambda=0$ for $t<\varepsilon$ and $\lambda=1$ for $t>1-\varepsilon$, then the new homotopy becomes stationary at the ends. This allows us to smoothly concatenate homotopies provided $H_{1}(1, x)=H_{2}(0, x)$.

PROPOSITION 5.4.20. If $p, q \in \mathbb{R}^{n}$, then there exists a compactly supported isotopy such that $H(0, x)=x$ for all $x \in \mathbb{R}^{n}$ and $H(1, p)=q$.

Proof. Simply select a suitable compactly supported function $\phi: \mathbb{R}^{n} \rightarrow[0,1]$ with $\phi(p)=1$ and define

$$
H_{t}(x)=H(t, x)=x+t \phi(x)(q-p) .
$$

This map is proper since it is the identity outside a compact set, it is also nonsingular provided $|d \phi|<\frac{1}{|q-p|}$. Thus corollary 1.4 .37 shows that it is a diffeomorphism.

Lemma 5.4.21. Let $M$ be connected and with $\operatorname{dim} M \geq 2$. If $p_{1}, \ldots, p_{k} \in M$ are distinct and $q_{1}, \ldots, q_{k} \in M$ are distinct, then there exists a compactly supported isotopy such that $H(0, x)=x$ for all $x \in M$ and $H\left(1, p_{i}\right)=q_{i}$.

Proof. The proof is by induction on $k$.
For $k=1$ we create a relation by saying that $p, q$ are related provided the statement of the lemma holds. This is clearly an equivalence relation. The previous proposition shows that the equivalence classes are open. The fact that $M$ is connected then finishes the proof.

Now assume the statement holds for $k-1$ points. Since $\operatorname{dim} M \geq 2$ we know that $M-\left\{p_{k}, q_{k}\right\}$ and $M-\left\{p_{1}, \ldots, p_{k-1}, q_{1}, \ldots, q_{k-1}\right\}$ are connected. Therefore, there exist compactly supported isotopies $H$ on $M-\left\{p_{k}, q_{k}\right\}$ and $G$ on $M-\left\{p_{1}, \ldots, p_{k-1}, q_{1}, \ldots, q_{k-1}\right\}$ that are the identity when $t=0$ and with $H\left(1, p_{i}\right)=q_{i}, i=1, \ldots, k-1$ and $G\left(1, p_{k}\right)=q_{k}$. As
they are compactly supported they extend to all of $M$. Now $H$ fixes $p_{i}, q_{i}, i=1, \ldots, k-1$ and $G$ fixes $p_{k}, q_{k}$. We can then compose $H(t, G(t, x))$ to obtain the desired isotopy.

Corollary 5.4.22. Any finite collection of points in a connected manifold lies in an open set diffeomorphic to $\mathbb{R}^{n}$.

Proof. The above lemma settles this when $n=\operatorname{dim} M \geq 2$. When $\operatorname{dim} M=1$, it follows from our classification of one-manifolds.

Consider a vector field $X$ on $\mathbb{R}^{n}$ where $p$ is an isolated zero. Trivializing the tangent bundle $T \mathbb{R}^{n}=\mathbb{R}^{n} \times \mathbb{R}^{n}$ we can think of $X: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, where $p$ is an isolated zero. We define the index of $X$ :

$$
\operatorname{ind}_{p} X=W\left(\left.X\right|_{\partial B(p, \varepsilon)}, 0\right)
$$

Similarly, if a map $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ has an isolated fixed point $p$, then consider $X(x)=x-$ $F(x)$ and define the Lefschetz number:

$$
L_{p}(F)=\operatorname{ind}_{p} X
$$

These definitions make sense for all small $\varepsilon$ and by remark 5.4 .12 will give the same answer for all $\varepsilon$. In fact, instead of $B(p, \varepsilon)$ we could have used any closed neighborhood, $M$, around $p$ with smooth boundary and with the property that $p$ is the only zero or fixed point in $M$. Both definitions also match the intersection numbers as discussed in the previous subsection when everything is transverse.

We can now define the index of an isolated zero of a vector field and the Lefschetz number of an isolated fixed point of a function on an oriented manifold. Simply select a positively oriented chart around the point and then use the definition from Euclidean space. It is easy to prove that any two positively oriented charts give the same number. We just need to check that the definition in Euclidean space is independent of diffeomorphisms that fix, say, the origin. Such maps have an expansion $G(x)=\left.D G\right|_{0} x+o(x)$ and are thus isotopic to $\left.D G\right|_{0}$ on a small neighborhood of the origin. As $\left.D G\right|_{0}$ is an orientation preserving linear map it is in turn isotopic to the identity.

THEOREM 5.4.23 (Poincaré-Hopf). If $X$ is a vector field with finitely many zeros on an oriented compact oriented manifold, then

$$
\chi(M)=\sum_{p, X(p)=0} \operatorname{ind}_{p} X .
$$

THEOREM 5.4.24 (Lefschetz). If $F: M \rightarrow M$ is a map with finitely many fixed points on a compact oriented manifold, then

$$
L(F)=\sum_{p, F(p)=p} L_{p}(F)
$$

Proof. The two proofs are virtually identical after restricting to an open set $U \subset M$ diffeomorphic to $\mathbb{R}^{n}$ that contains all the zeros or fixed points. We focus on the second as it is more general. As such, we consider a map $F: M \rightarrow M$ that has a finite number of fixed points $p_{1}, \ldots, p_{k}$ in the interior of a closed ball $B=\bar{B}(0, R) \subset \mathbb{R}^{n} \simeq U \subset M$. To calculate the relevant winding numbers we consider the auxiliary vector field $X(x)=x$ $F(x)$ on $B$ whose zeros are precisely the fixed points of $F$. From remark 5.4.12 we have for sufficiently small $\varepsilon>0$ that

$$
\sum_{i} L_{p_{i}}(F)=\sum_{i} W\left(\left.X\right|_{\partial B\left(p_{i}, \varepsilon\right)}, 0\right)=W\left(\left.X\right|_{\partial B}, 0\right)
$$

We can now select a new function $G: M \rightarrow M$ that is homotopic to $F$, agrees with $F$ on $M-\operatorname{int} B$, and such that the graph of $G$ is transverse to the diagonal. This implies that 0 is a regular value for $Y(x)=x-G(x)$. We can then invoke theorem 5.4.11 to conclude

$$
\sum_{i} L_{p_{i}}(F)=W\left(\left.X\right|_{\partial B}, 0\right)=W\left(\left.Y\right|_{\partial B}, 0\right)=I(Y,\{0\})=L(G)=L(F) .
$$

It is tempting to use the above constructions to define Lefschetz numbers for maps on noncompact manifolds. But even on $\mathbb{R}^{n}$ this runs in to some trouble. Clearly all maps are homotopic. However, there are maps without fixed points such as translations and maps with nontrivial Lefschetz numbers such as rotations in the plane. The same issue occurs for vector fields, as there exist vector fields that vanish only at the origin but with any integer as index. Similar issues occur for vector fields on compact manifolds with boundary such a closed ball in $\mathbb{R}^{n}$.

Finally we give an outline of how the Euler characteristic ties in with the traditional combinatorial definition. This works in all dimensions but is a little easier to define for surfaces.

DEFINITION 5.4.25. A polygonal subdivision of a surface $M$ is a decomposition $M=$ $\cup P_{\alpha}$ such that each $P_{\alpha}$ is diffeomorphic to a polygon in the plane and such that $P_{\alpha} \cap P_{\beta}$ is a vertex or union of edges. The fact that $M$ is a manifold without boundary means that each edge is the edge of exactly two polygons.

With respect to such a decomposition it is easy to visualize a vector field that is tangent to the edges, has a sink at each vertex, a saddle at exactly one interior point of each edge, and a source at exactly one point in the interior of each polygon. As sinks and sources have index 1 , while saddles have index -1 we end up with the formula

$$
\chi(M)=V-E+F
$$

where $V$ is the number of vertices, $E$ the number of edges, and $F$ the number of polygons, e.g., faces.

We shall in section 8.2 show a more general formula for the Euler characteristic and Lefschetz number which only depends on the cohomology of the space. This formula makes sense on a much broader class of compact spaces, but it is less obvious why a map with nonzero Lefschetz number must have a fixed point. This topological Lefschetz number is invariant under homotopies. In particular, translations have Lefschetz number 1 so compactness is a crucial assumption in order to guarantee fixed points.
5.4.5. Hopf's Degree Theorem. The Hopf degree theorem states that maps from a closed, connected, oriented $n$-manifold to the $n$-sphere are homotopic if and only if they have the same degree. The same statement holds for nonorientable manifolds if we use the $\bmod 2$ degree. Since the result is also important when $n=1$ and has a much more direct proof we start with that case.

THEOREM 5.4.26. A map $F: S^{1} \rightarrow S^{1}$ is homotopic to $z \mapsto z^{\operatorname{deg} F}$.
Proof. We can use the covering map $\pi: \mathbb{R} \rightarrow S^{1}$ given by the function $\theta \mapsto e^{2 \pi i \theta}$ of period 1 to lift $F$ to a map $\bar{F}: \mathbb{R} \rightarrow \mathbb{R}$ such that $F\left(e^{2 \pi i \theta}\right)=e^{2 \pi i \bar{F}(\theta)}$. Clearly $\bar{F}(\theta+1)-$ $\bar{F}(\theta) \in \mathbb{Z}$ so it follows that it is a constant, say, $k$. We will show that $F$ and $z \mapsto z^{k}$ are homotopic. In $\mathbb{R}$ we have an obvious linear homotopy

$$
\bar{H}(t, \theta)=(1-t) \bar{F}(\theta)+t k \theta
$$

Since

$$
\bar{H}(t, \theta+1)-\bar{H}(t, \theta)=k
$$

it induces a homotopy

$$
H\left(t, e^{2 \pi i \theta}\right)=e^{2 \pi i \bar{H}(t, \theta)}
$$

between $F$ and $z \mapsto z^{k}$. As the latter map has degree $n$ the theorem follows.
Before moving on to the general case we start with two easy extension results.
Proposition 5.4.27. Let $B \subset \mathbb{R}^{n}$ be an open ball and $N$ a manifold. If $F: \mathbb{R}^{n}-B \rightarrow N$ has the property that $\left.F\right|_{\partial B}: \partial B \rightarrow N$ is homotopic to a constant, then there is an extension $\bar{F}: \mathbb{R}^{n} \rightarrow N$ that agrees with $F$ on $\mathbb{R}^{n}-B$.

Proof. Let $H(t, x):[0,1] \times \partial B \rightarrow N$ be a smooth homotopy with $H(1, x)=F(x)$ and $H(0, x)=p$ for some $p \in N$. We can further assume that for $t<\varepsilon$ we have $H(t, x)=p$ and for $t>1-\varepsilon$ we have $H(t, x)=F(x)$. Parametrizing $B$ by $[0,1] \times \partial B \rightarrow B$ then shows that $H$ induces a smooth map on $B$ that is $F$ near the boundary and thus smoothly extends $F$.

Lemma 5.4.28. Let $B$ be a manifold with smooth boundary $\partial B=M$. Any map $F$ : $M \rightarrow \mathbb{R}^{n}$ extends to a smooth map $G: B \rightarrow \mathbb{R}^{n}$ where $\partial G=M$.

Proof. We can assume that there is a proper embedding $B \subset \mathbb{R}^{k}$ and instead show that we can extend $F$ to be defined on all of $\mathbb{R}^{k}$. The desired $G$ is then gotten by restricting to $B$.

Select a retract $\pi: U \rightarrow M$ on a tubular neighborhood $U \supset M$ and a bump function $\lambda$ : $\mathbb{R}^{k} \rightarrow[0,1]$ which is 1 on $M$ ( $M$ is a closed subset as it is properly embedded) and 0 outside a neighborhood $V \supset M$ with $\bar{V} \subset U$. The extension is given by $G(x)=\lambda(x) F(\pi(x))$. This is certainly an extension to $U$ and as it vanishes outside $V$ it is well-defined on all of $\mathbb{R}^{k}$.

To prove the Hopf degree statement we start by considering maps of degree 0 .
THEOREM 5.4.29. Let $M^{n}$ be a closed, connected, oriented n-manifold. If $F: M^{n} \rightarrow S^{n}$ has degree 0 , then $F$ is homotopic to a constant map.

This has an immediate consequence
COROLLARY 5.4.30. Let $M^{n}$ be a closed, connected, oriented $n$-manifold. If $F: M^{n} \rightarrow$ $\mathbb{R}^{n+1}-\{0\}$ has $W(F, 0)=0$, then $F$ is homotopic to a constant map in $\mathbb{R}^{n+1}-\{0\}$.

Proof. Theorem 5.4.29 shows that $\frac{F}{|F|}: M^{n} \rightarrow S^{n}$ is homotopic to a constant.
PROOF OF THEOREM5.4.29. The proof is by induction on $n$ and recall that we did the full theorem when $n=1$ above. For the purpose of the induction step note that the above corollary holds in dimension $n-1$ provided the theorem holds in dimension $n-1$.

Consider a map $F: M^{n} \rightarrow S^{n}$ of degree 0 . If the map is not surjective, then it is clearly homotopic to a constant map. Otherwise select two regular values $p, q \in S^{n}$ and an open set $U \simeq \mathbb{R}^{n}$ that contains $F^{-1}(p)$ and is disjoint from $F^{-1}(q)$, i.e., $\left.F\right|_{U}: U \rightarrow S^{n}-\{q\}$. We have diffeomorphisms $A: S^{n}-\{q\} \rightarrow \mathbb{R}^{n}$ that map $p$ to the origin and $B: \mathbb{R}^{n} \rightarrow U$ with the property that $F^{-1}(p) \subset G(B(0,1))$. The composition $G=A \circ F \circ B: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ has 0 as a regular value so by assumption and theorem 5.4.11 we have

$$
0=\operatorname{deg} F=I(G,\{0\})=W\left(\left.G\right|_{S^{n-1}=\partial B(0,1)}, 0\right)
$$

By induction it follows that $\left.G\right|_{S^{n-1}}: S^{n-1} \rightarrow \mathbb{R}^{n}-\{0\}$ is homotopically trivial. Proposition 5.4.27 then gives us an extension of $\left.G\right|_{\mathbb{R}^{n}-B(0,1)}$ to a map $\bar{G}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}-\{0\}$. This gives us a map $A^{-1} \circ \bar{G} \circ B^{-1}: U \rightarrow S^{n}-\{p, q\}$ that agrees with $F$ outside a compact set in $U$ and thus induces a map $\bar{F}: S^{n} \rightarrow S^{n}-\{p\}$.

Here $\bar{G}, G: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ are clearly homotopic via a linear homotopy that is independent of $t$ on $\mathbb{R}^{n}-B(0,1)$. Thus there is a similar homotopy of $A^{-1} \circ \bar{G} \circ B^{-1}$ and $\left.F\right|_{U}$ that maps into $S^{n}-\{q\}$ and is independent of $t$ outside a compact set. This shows that $F$ and $\bar{F}$ are homotopic. As $\bar{F}$ is not surjective it is homotopic to a constant.

This theorem implies an important extension that is a partial converse to theorem 5.4.1
Corollary 5.4.31. Let $N^{n+1}$ be a compact, connected, oriented manifold with boundary. A map $F: \partial N \rightarrow S^{n}$ has an extension to $G: N \rightarrow S^{n}$ with $\partial G=F$, provided $\operatorname{deg} F=0$.

Proof. By lemma 5.4.28 we can find an extension $\bar{F}: N \rightarrow \mathbb{R}^{n+1}$ with $\partial \bar{F}=F$. We can further assume that 0 is a regular value for $\bar{F}$ and that $F^{-1}(0) \subset B$ where $B$ is an open ball with smooth boundary $\partial B$. The map $\left.\bar{F}\right|_{\partial B}: \partial B \rightarrow \mathbb{R}^{n+1}-\{0\}$ has winding number 0 by theorem 5.4.11 and is thus homotopic to a constant in $\mathbb{R}^{n+1}-\{0\}$. By proposition 5.4.27 we can then extend $\bar{F}: N-B \rightarrow \mathbb{R}^{n+1}-\{0\}$ to a smooth map $G: N \rightarrow \mathbb{R}^{n+1}-\{0\}$. This map agrees with $F$ on $\partial N$ and can thus be normalized to create the desired extension.

The full version of Hopf's theorem now follows.
THEOREM 5.4.32. Let $M^{n}$ be a connected, closed, and oriented manifold. Two maps $F_{0}, F_{1}: M \rightarrow S^{n}$ are homotopic if they have the same degree.

Proof. Let $N=[0,1] \times M$ with it natural orientation so that the boundaries have opposite orientations. Thus $F_{0}, F_{1}$ yield a map $\partial N \rightarrow S^{n}$ of degree 0 . We can then apply the above corollary.

Our final result follows along similar lines:
THEOREM 5.4.33. If $M$ is a compact, connected, and oriented manifold, then $\chi(M)=$ 0 if and only if $M$ admits a nowhere vanishing vector field.

Proof. Clearly a nonzero vector field leads to vanishing Euler characteristic. Conversely select a vector field $X$ that is transverse to the zero section and an open ball $B$ with smooth boundary such that these zeros are contained in $B$. On a neighborhood of $\bar{B}$ we can trivialize the tangent bundle and write $X(x)=(x, F(x))$. Now 0 is a regular value for $F: \bar{B} \rightarrow \mathbb{R}^{n}$ and

$$
0=\chi(X)=I\left(F^{-1}(0),\{0\}\right)
$$

Thus $\left.F\right|_{\partial B}$ has a smooth extension to a map $\bar{F}: \bar{B} \rightarrow \mathbb{R}^{n}-\{0\}$ that we can assume is smoothly joined to $X$ outside $B$. This gives us a nonvanishing vector field.

### 5.5. Exercises

(1) Let $S \subset \mathbb{R}^{n}-\{0\}$ be a submanifold. Show that almost all $k$-dimensional subspaces are transverse to $S$. Hint: consider the map

$$
\left(\alpha^{1}, \ldots, \alpha^{k}, v_{1}, \ldots, v_{k}\right) \mapsto \sum \alpha^{i} v_{i}
$$

where $v_{1}, \ldots, v_{k}$ are linearly independent.
(2) Given maps between compact, connected, oriented n-manifolds:

$$
L \xrightarrow{F} M \xrightarrow{G} N
$$

show that

$$
\operatorname{deg}(G \circ F)=\operatorname{deg} G \operatorname{deg} F
$$

(3) Let $M^{m}, N^{n} \subset \mathbb{R}^{n+m+1}$ be two closed, oriented, disjoint submanifolds and define the linking number

$$
l(M, N)=\operatorname{deg}\left(F: M \times N \rightarrow S^{n+m}\right), F(x, y)=\frac{x-y}{|x-y|}
$$

(a) Show that $l(M, N)=(-1)^{(m+1)(n+1)} l(N, M)$.
(b) Show that $l(M, N)=0$ if $M=\partial B$, where $B$ is compact, oriented, and disjoint from $N$.
(4) Starting with $S^{1}$ show that there is a map of degree $k$ on $S^{n}$ for every integer $k$.
(5) What is the degree of a rational map $\frac{p}{q}$ on $\mathbb{C P}^{1}$, where $p, q \in \mathbb{C}[X]$ have no roots in common?
(6) Let $M$ be a closed, connected, and oriented $n$-manifold. Show that there is a map $M \rightarrow S^{n}$ of degree $k$ for every integer $k$.
(7) If $M \rightarrow N$ is a $k$-fold covering of closed manifolds, then $\chi(M)=k \chi(N)$.
(8) If $M, N$ are manifolds, then $\chi(M \times N)=\chi(M) \chi(N)$.
(9) Let $M$ be connected and $p_{1}, \ldots, p_{k}$ and $q_{1}, \ldots, q_{k}$ two collections of distinct points as in lemma 5.4.21. Show that if $v_{i} \in T_{p_{i}} M-\{0\}$ and $w_{i} \in T_{q_{i}} M-\{0\}$, then there is a compactly supported isotopy $H$ from $i d_{M}$ such that $H_{1}(x)=H(1, x)$ satisfies:

$$
\left.D H_{1}\right|_{p_{i}}\left(v_{i}\right)=w_{i}
$$

(10) Calculate the intersection number of $\mathbb{C P}^{k}, \mathbb{C P}^{n-k} \subset \mathbb{C P}^{n}$.
(11) Let $X$ be a vector field on $\mathbb{C}$ given by a complex polynomial that has no repeated roots. What is the index at each zero?
(12) Calculate the indices at $0 \in \mathbb{C}$ of the vector fields given by $X(z)=z^{m}$ and $X(z)=$ $\bar{z}^{m}, m=1,2,3 \ldots$
(13) Calculate the Lefschetz numbers at $0 \in \mathbb{C}$ of the maps $F(z)=z-z^{m}$ and $F(z)=$ $z-\bar{z}^{m}, m=1,2,3 \ldots$
(14) Show that there are antipodal points on Earth with the same the temperature and barometric pressure.
(15) Let $U_{i} \subset \mathbb{R}^{n}, i=1, \ldots, n$ be open, bounded, and connected. Show that there exists a hyperplane $H$ that bisects the $n$ open sets, i.e., if $\mathbb{R}^{n}=A \cup B$, where $A \cap B=H$, then

$$
\operatorname{vol}\left(U_{i} \cap A\right)=\operatorname{vol}\left(U_{i} \cap B\right), i=1, \ldots, n
$$

Hint: You can use that Borsuk-Ulam holds for continuous functions.
(16) Assume $L^{n-1} \subset M^{n}$ is properly embedded, both manifolds are connected, and $M$ is simply connected. Show that $M-L$ has exactly two components.
(17) Consider the maps on $\mathbb{C P}^{2}$ :

$$
\begin{aligned}
& F_{k}\left(\left[z_{0}: z_{1}: z_{2}\right]\right)=\left[z_{0}^{k}: z_{1}^{k}: z_{2}^{k}\right], \\
& \bar{F}_{k}\left(\left[z_{0}: z_{1}: z_{2}\right]\right)=\left[z_{0}^{k}: z_{1}^{k}: z_{2}^{k}\right],
\end{aligned}
$$

where $k=1,2,3, \ldots$
(a) Show that if $U_{k}$ is the group of the $k$ th roots of unity then $U_{k}^{3} / \Delta$ acts transitively on the preimages of these maps, here $\Delta=\left\{(\zeta, \zeta, \zeta) \mid \zeta \in U_{k}\right\}$.
(b) Show that the degree is $k^{2}$.
(c) What happens with the analogous question on $\mathbb{R} \mathbb{P}^{2}$ ?
(d) Show that $F_{k}$ is transverse to

$$
\mathbb{C P}^{1}=\left\{\left[w_{0}: w_{1}: w_{2}\right] \in \mathbb{C P}^{2} \mid w_{0}+w_{1}+w_{2}=0\right\}
$$

and let $M$ be the preimage so that we obtain a map $G_{k}: M \rightarrow \mathbb{C P}^{1}$.
(e) Show that $U_{k}^{3} / \Delta$ acts transitively on the preimages of $G_{k}$.
(f) Show that except for three points where there are $k$ preimages all other points have $k^{2}$ preimages.
(g) Use this to show that $\chi(M)=k(3-k)$ (hint: the image is the union of two triangles whose vertices are the special three points with $k$ preimages).
(h) Is $F_{k}$ transverse to

$$
\mathbb{C P}^{1}=\left\{\left[w_{0}: w_{1}: w_{2}\right] \in \mathbb{C P}^{2} \mid w_{0}=0\right\} ?
$$

## CHAPTER 6

## Basic Tensor Analysis

### 6.1. Tensors

Define tensors, basic tensoriality conditions, tensor products, tensors with symmetry conditions.

### 6.2. The Lie Derivative and Its Uses

Let $X$ be a vector field and $\Phi^{t}=\Phi_{X}^{t}$ the corresponding locally defined flow on a smooth manifold $M$. Thus $\Phi^{t}(p)$ is defined for small $t$ and the curve $t \mapsto \Phi^{t}(p)$ is the integral curve for $X$ that goes through $p$ at $t=0$. The Lie derivative of a tensor in the direction of $X$ is defined as the first order term in a suitable Taylor expansion of the tensor when it is moved by the flow of $X$.
6.2.1. Definitions and Properties. Let us start with a function $f: M \rightarrow \mathbb{R}$. Then

$$
f\left(\Phi^{t}(p)\right)=f(p)+t\left(L_{X} f\right)(p)+o(t)
$$

where the Lie derivative $L_{X} f$ is just the directional derivative $D_{X} f=d f(X)$. We can also write this as

$$
\begin{aligned}
f \circ \Phi^{t} & =f+t L_{X} f+o(t) \\
L_{X} f & =D_{X} f=d f(X)
\end{aligned}
$$

When we have a vector field $Y$ things get a little more complicated. We wish to consider $\left.Y\right|_{\Phi^{t}}$, but this can't be directly compared to $Y$ as the vectors live in different tangent spaces. Thus we look at the curve $t \rightarrow D \Phi^{-t}\left(\left.Y\right|_{\Phi^{t}(p)}\right)$ that lies in $T_{p} M$. We can expand for $t$ near 0 to get

$$
D \Phi^{-t}\left(\left.Y\right|_{\Phi^{t}(p)}\right)=\left.Y\right|_{p}+\left.t\left(L_{X} Y\right)\right|_{p}+o(t)
$$

for some vector $\left.\left(L_{X} Y\right)\right|_{p} \in T_{p} M$. If we compare this with proposition 2.2.6 then we see that $L_{X} Y$ measures how far $Y$ is from being $\Phi^{t}$ related to itself for small $t$. This is made more precise in the next result.

Proposition 6.2.1. Consider two vector fields $X, Y$ on $M$. The following are equivalent:
(1) $\Phi_{X}^{t} \circ \Phi_{Y}^{s}=\Phi_{Y}^{S} \circ \Phi_{X}^{t}$,
(2) $D \Phi_{X}^{t}(Y)=Y \circ \Phi_{X}^{t}$, i.e., $Y$ is $\Phi_{X}^{t}$-related to itself,
(3) $L_{X} Y=0$ on $M$.

Proof. The fact that (1) and (2) are equivalent follows from proposition 2.2.6. The fact that (2) implies (3) follows from

$$
L_{X} Y=\lim _{t \rightarrow 0} \frac{\left.Y\right|_{\Phi_{X}^{t}}-D \Phi_{X}^{t}(Y)}{t}
$$

Conversely, consider the curve $c(t)=D \Phi_{X}^{-t}\left(\left.Y\right|_{\Phi_{X}^{t}(p)}\right) \in T_{p} M$. Its velocity at $t_{0}$ is calculated by considering the difference:

$$
\begin{aligned}
D \Phi_{X}^{-t}\left(\left.Y\right|_{\Phi_{X}^{t}(p)}\right)-D \Phi_{X}^{-t_{0}}\left(\left.Y\right|_{\Phi_{X}^{t_{0}}(p)}\right) & =D \Phi_{X}^{-t_{0}}\left(D \Phi^{-\left(t-t_{0}\right)}\left(\left.Y\right|_{\Phi_{X}^{t-t_{0}}\left(\Phi_{X}^{t_{0}}(p)\right)}\right)\right)-D \Phi_{X}^{-t_{0}}\left(\left.Y\right|_{\Phi_{X}^{t_{0}}(p)}\right) \\
& =D \Phi_{X}^{-t_{0}}\left(D \Phi^{-\left(t-t_{0}\right)}\left(\left.Y\right|_{\Phi_{X}^{t-t_{0}}\left(\Phi_{X}^{t_{0}}(p)\right)}\right)-\left.Y\right|_{\Phi_{X}^{t_{0}}(p)}\right) \\
& =D \Phi_{X}^{-t_{0}}\left(\left.\left(t-t_{0}\right) L_{X} Y\right|_{\Phi_{X}^{t_{0}}(p)}+o\left(t-t_{0}\right)\right) \\
& =o\left(t-t_{0}\right) .
\end{aligned}
$$

Showing that the curve is constant and consequently that (2) holds provided the Lie bracket vanishes.

This Lie derivative of a vector field is in fact the Lie bracket.
Proposition 6.2.2. For vector fields $X, Y$ on $M$ we have

$$
L_{X} Y=[X, Y]
$$

Proof. We see that the Lie derivative satisfies

$$
D \Phi^{-t}\left(\left.Y\right|_{\Phi^{t}}\right)=Y+t L_{X} Y+o(t)
$$

or equivalently

$$
\left.Y\right|_{\Phi^{t}}=D \Phi^{t}(Y)+t D \Phi^{t}\left(L_{X} Y\right)+o(t) .
$$

It is therefore natural to consider the directional derivative of a function $f$ in the direction of $\left.Y\right|_{\Phi^{t}}-D \Phi^{t}(Y)$.

$$
\begin{aligned}
D_{\left(\left.Y\right|_{\Phi^{t}}-D \Phi^{t}(Y)\right)} f= & D_{\left.Y\right|_{\Phi^{t}}} f-D_{D_{\Phi^{t}(Y)}} f \\
= & \left(D_{Y} f\right) \circ \Phi^{t}-D_{Y}\left(f \circ \Phi^{t}\right) \\
= & D_{Y} f+t D_{X} D_{Y} f+o(t) \\
& -D_{Y}\left(f+t D_{X} f+o(t)\right) \\
= & t\left(D_{X} D_{Y} f-D_{Y} D_{X} f\right)+o(t) \\
= & t D_{[X, Y]} f+o(t)
\end{aligned}
$$

This shows that

$$
\begin{aligned}
L_{X} Y & =\lim _{t \rightarrow 0} \frac{\left.Y\right|_{\Phi^{t}}-D \Phi^{t}(Y)}{t} \\
& =[X, Y] .
\end{aligned}
$$

We are now ready to define the Lie derivative of a $(0, p)$-tensor $T$ and also give an algebraic formula for this derivative. We define

$$
\left(\Phi^{t}\right)^{*} T=T+t\left(L_{X} T\right)+o(t)
$$

or more precisely

$$
\begin{aligned}
\left(\left(\Phi^{t}\right)^{*} T\right)\left(Y_{1}, \ldots, Y_{p}\right) & =T\left(D \Phi^{t}\left(Y_{1}\right), \ldots, D \Phi^{t}\left(Y_{p}\right)\right) \\
& =T\left(Y_{1}, \ldots, Y_{p}\right)+t\left(L_{X} T\right)\left(Y_{1}, \ldots, Y_{p}\right)+o(t) .
\end{aligned}
$$

Proposition 6.2.3. If $X$ is a vector field and $T a(0, p)$-tensor on $M$, then

$$
\left(L_{X} T\right)\left(Y_{1}, \ldots, Y_{p}\right)=D_{X}\left(T\left(Y_{1}, \ldots, Y_{p}\right)\right)-\sum_{i=1}^{p} T\left(Y_{1}, \ldots, L_{X} Y_{i}, \ldots, Y_{p}\right)
$$

Proof. We restrict attention to the case where $p=1$. The general case is similar but requires more notation. Using that

$$
\left.Y\right|_{\Phi^{t}}=D \Phi^{t}(Y)+t D \Phi^{t}\left(L_{X} Y\right)+o(t)
$$

we get

$$
\begin{aligned}
\left(\left(\Phi^{t}\right)^{*} T\right)(Y) & =T\left(D \Phi^{t}(Y)\right) \\
& =T\left(\left.Y\right|_{\Phi^{t}}-t D \Phi^{t}\left(L_{X} Y\right)\right)+o(t) \\
& =T(Y) \circ \Phi^{t}-t T\left(D \Phi^{t}\left(L_{X} Y\right)\right)+o(t) \\
& =T(Y)+t D_{X}(T(Y))-t T\left(D \Phi^{t}\left(L_{X} Y\right)\right)+o(t)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(L_{X} T\right)(Y) & =\lim _{t \rightarrow 0} \frac{\left(\left(\Phi^{t}\right)^{*} T\right)(Y)-T(Y)}{t} \\
& =\lim _{t \rightarrow 0}\left(D_{X}(T(Y))-T\left(D \Phi^{t}\left(L_{X} Y\right)\right)\right) \\
& =D_{X}(T(Y))-T\left(L_{X} Y\right)
\end{aligned}
$$

Finally we have that Lie derivatives satisfy all possible product rules. From the above propositions this is already obvious when multiplying functions with vector fields or $(0, p)$ tensors. However, it is less clear when multiplying tensors.

Proposition 6.2.4. Let $T_{1}$ and $T_{2}$ be $\left(0, p_{i}\right)$-tensors, then

$$
L_{X}\left(T_{1} \cdot T_{2}\right)=\left(L_{X} T_{1}\right) \cdot T_{2}+T_{1} \cdot\left(L_{X} T_{2}\right)
$$

Proof. Recall that for 1-forms and more general $(0, p)$-tensors we define the product as

$$
T_{1} \cdot T_{2}\left(X_{1}, \ldots, X_{p_{1}}, Y_{1}, \ldots, Y_{p_{2}}\right)=T_{1}\left(X_{1}, \ldots, X_{p_{1}}\right) \cdot T_{2}\left(Y_{1}, \ldots, Y_{p_{2}}\right) .
$$

The proposition is then a simple consequence of the previous proposition and the product rule for derivatives of functions.

Proposition 6.2.5. Let $T$ be $a(0, p)$-tensor and $f: M \rightarrow \mathbb{R}$ a function, then

$$
L_{f X} T\left(Y_{1}, \ldots, Y_{p}\right)=f L_{X} T\left(Y_{1}, \ldots, Y_{p}\right)+d f\left(Y_{i}\right) \sum_{i=1}^{p} T\left(Y_{1}, \ldots, X, \ldots, Y_{p}\right)
$$

Proof. We have that

$$
\begin{aligned}
L_{f X} T\left(Y_{1}, \ldots, Y_{p}\right)= & D_{f X}\left(T\left(Y_{1}, \ldots, Y_{p}\right)\right)-\sum_{i=1}^{p} T\left(Y_{1}, \ldots, L_{f X} Y_{i}, \ldots, Y_{p}\right) \\
= & f D_{X}\left(T\left(Y_{1}, \ldots, Y_{p}\right)\right)-\sum_{i=1}^{p} T\left(Y_{1}, \ldots,\left[f X, Y_{i}\right], \ldots, Y_{p}\right) \\
= & f D_{X}\left(T\left(Y_{1}, \ldots, Y_{p}\right)\right)-f \sum_{i=1}^{p} T\left(Y_{1}, \ldots,\left[X, Y_{i}\right], \ldots, Y_{p}\right) \\
& +d f\left(Y_{i}\right) \sum_{i=1}^{p} T\left(Y_{1}, \ldots, X, \ldots, Y_{p}\right)
\end{aligned}
$$

The case where $\left.X\right|_{p}=0$ is of special interest when computing Lie derivatives. We note that $\Phi^{t}(p)=p$ for all $t$. Thus $D \Phi^{t}: T_{p} M \rightarrow T_{p} M$ and

$$
\begin{aligned}
\left.L_{X} Y\right|_{p} & =\lim _{t \rightarrow 0} \frac{D \Phi^{-t}\left(\left.Y\right|_{p}\right)-\left.Y\right|_{p}}{t} \\
& =\left.\frac{d}{d t}\left(D \Phi^{-t}\right)\right|_{t=0}\left(\left.Y\right|_{p}\right)
\end{aligned}
$$

This shows that $L_{X}=\left.\frac{d}{d t}\left(D \Phi^{-t}\right)\right|_{t=0}$ when $\left.X\right|_{p}=0$. From this we see that if $\theta$ is a 1-form, then $L_{X} \theta=-\theta \circ L_{X}$ at points $p$ where $\left.X\right|_{p}=0$.

Before moving on to some applications of Lie derivatives we introduce the concept of interior product, it is simply evaluation of a vector field in the first argument of a tensor:

$$
i_{X} T\left(X_{1}, \ldots, X_{k}\right)=T\left(X, X_{1}, \ldots, X_{k}\right)
$$

We list 4 general properties of Lie derivatives.

$$
\begin{aligned}
L_{[X, Y]} & =L_{X} L_{Y}-L_{Y} L_{X}, \\
L_{X}(f T) & =L_{X}(f) T+f L_{X} T, \\
L_{X}[Y, Z] & =\left[L_{X} Y, Z\right]+\left[Y, L_{X} Z\right], \\
L_{X}\left(i_{Y} T\right) & =i_{L_{X} Y} T+i_{Y}\left(L_{X} T\right) .
\end{aligned}
$$

6.2.2. Lie Groups. Lie derivatives also come in handy when working with Lie groups. For a Lie group $G$ we have the inner automorphism $\mathrm{Ad}_{h}: x \rightarrow h x h^{-1}$ and its differential at $x=e$ denoted by the same letters

$$
\operatorname{Ad}_{h}: \mathfrak{g} \rightarrow \mathfrak{g}
$$

LEmmA 6.2.6. The differential of $h \rightarrow \operatorname{Ad}_{h}$ is given by $U \rightarrow \operatorname{ad}_{U}(X)=[U, X]$
Proof. If we write $\operatorname{Ad}_{h}(x)=R_{h^{-1}} L_{h}(x)$, then its differential at $x=e$ is given by $\operatorname{Ad}_{h}=D R_{h^{-1}} D L_{h}$. Now let $\Phi^{t}$ be the flow for $U$. Then $\Phi^{t}(g)=g \Phi^{t}(e)=L_{g}\left(\Phi^{t}(e)\right)$ as both curves go through $g$ at $t=0$ and have $U$ as tangent everywhere since $U$ is a leftinvariant vector field. This also shows that $D \Phi^{t}=D R_{\Phi^{t}(e)}$. Thus

$$
\begin{aligned}
\left.\operatorname{ad}_{U}(X)\right|_{e} & =\left.\frac{d}{d t} D R_{\Phi^{-t}(e)} D L_{\Phi^{t}(e)}\left(\left.X\right|_{e}\right)\right|_{t=0} \\
& =\left.\frac{d}{d t} D R_{\Phi^{-t}(e)}\left(\left.X\right|_{\Phi^{t}(e)}\right)\right|_{t=0} \\
& =\left.\frac{d}{d t} D \Phi^{-t}\left(\left.X\right|_{\Phi^{t}(e)}\right)\right|_{t=0} \\
& =L_{U} X=[U, X]
\end{aligned}
$$

This is used in the next Lemma.
LEMMA 6.2.7. Let $G=G l(V)$ be the Lie group of invertible matrices on $V$. The Lie bracket structure on the Lie algebra $\mathfrak{g l}(V)$ of left invariant vector fields on $G l(V)$ is given by commutation of linear maps. i.e., if $X, Y \in T_{I} G l(V)$, then

$$
\left.[X, Y]\right|_{I}=X Y-Y X
$$

Proof. Since $x \mapsto h x h^{-1}$ is a linear map on the space hom $(V, V)$ we see that $\operatorname{Ad}_{h}(X)=$ $h X h^{-1}$. The flow of $U$ is given by $\Phi^{t}(g)=g(I+t U+o(t))$ so we have

$$
\begin{aligned}
{[U, X] } & =\left.\frac{d}{d t}\left(\Phi^{t}(I) X \Phi^{-t}(I)\right)\right|_{t=0} \\
& =\left.\frac{d}{d t}((I+t U+o(t)) X(I-t U+o(t)))\right|_{t=0} \\
& =\left.\frac{d}{d t}(X+t U X-t X U+o(t))\right|_{t=0} \\
& =U X-X U
\end{aligned}
$$

6.2.3. The Hessian. Lie derivatives are also useful for defining Hessians of functions. We start with a Riemannian manifold $\left(M^{m}, g\right)$. The Riemannian structure immediately identifies vector fields with 1 -forms. If $X$ is a vector field, then the corresponding 1 -form is denoted $\omega_{X}$ and is defined by

$$
\omega_{X}(v)=g(X, v)
$$

In local coordinates this looks like

$$
\begin{aligned}
X & =a^{i} \partial_{i} \\
\omega_{X} & =g_{i j} a^{i} d x^{j}
\end{aligned}
$$

This also tells us that the inverse operation in local coordinates looks like

$$
\begin{aligned}
\phi & =a_{j} d x^{j} \\
& =\delta_{j}^{k} a_{k} d x^{j} \\
& =g_{j i} g^{i k} a_{k} d x^{j} \\
& =g_{i j}\left(g^{i k} a_{k}\right) d x^{j}
\end{aligned}
$$

so the corresponding vector field is $X=g^{i k} a_{k} \partial_{i}$. If we introduce an inner product on 1forms that makes this correspondence an isometry

$$
g\left(\omega_{X}, \omega_{Y}\right)=g(X, Y)
$$

Then we see that

$$
\begin{aligned}
g\left(d x^{i}, d x^{j}\right) & =g\left(g^{i k} \partial_{k}, g^{j l} \partial_{l}\right) \\
& =g^{i k} g^{j l} g_{k l} \\
& =\delta_{l}^{i} g^{j l} \\
& =g^{j i}=g^{i j}
\end{aligned}
$$

Thus the inverse matrix to $g_{i j}$, the inner product of coordinate vector fields, is simply the inner product of the coordinate 1 -forms.

With all this behind us we define the gradient $\operatorname{grad} f$ of a function $f$ as the vector field corresponding to $d f$, i.e.,

$$
\begin{aligned}
d f(v) & =g(\operatorname{grad} f, v) \\
\omega_{\operatorname{grad} f} & =d f \\
\operatorname{grad} f & =g^{i j} \partial_{i} f \partial_{j}
\end{aligned}
$$

This correspondence is a bit easier to calculate in orthonormal frames $E_{1}, \ldots, E_{m}$, i.e., $g\left(E_{i}, E_{j}\right)=\delta_{i j}$, such a frame can always be constructed from a general frame using the Gram-Schmidt procedure. We also have a dual frame $\phi^{1}, \ldots, \phi^{m}$ of 1-forms, i.e., $\phi^{i}\left(E_{j}\right)=$ $\delta_{j}^{i}$. First we observe that

$$
\phi^{i}(X)=g\left(X, E_{i}\right)
$$

thus

$$
\begin{aligned}
X & =a^{i} E_{i}=\phi^{i}(X) E_{i}=g\left(X, E_{i}\right) E_{i} \\
\omega_{X} & =\delta_{i j} a^{i} \phi^{j}=a^{i} \phi^{i}=g\left(X, E_{i}\right) \phi^{i}
\end{aligned}
$$

In other words the coefficients don't change. The gradient of a function looks like

$$
\begin{aligned}
d f & =a_{i} \phi^{i}=\left(D_{E_{i}} f\right) \phi^{i} \\
\operatorname{grad} f & =g\left(\operatorname{grad} f, E_{i}\right) E_{i}=\left(D_{E_{i}} f\right) E_{i} .
\end{aligned}
$$

In Euclidean space we know that the usual Cartesian coordinates $\partial_{i}$ also form an orthonormal frame and hence the differentials $d x^{i}$ yield the dual frame of 1-forms. This makes it particularly simple to calculate in $\mathbb{R}^{n}$. One other manifold with the property is the torus $T^{n}$. In this case we don't have global coordinates, but the coordinates vector fields and differentials are defined globally. This is precisely what we are used to in vector calculus, where the vector field $X=P \partial_{x}+Q \partial_{y}+R \partial_{z}$ corresponds to the 1-form $\omega_{X}=P d x+Q d y+R d z$ and the gradient is given by $\partial_{x} f \partial_{x}+\partial_{y} f \partial_{y}+\partial_{z} f \partial_{z}$.

Having defined the gradient of a function the next goal is to define the Hessian of $F$. This is a bilinear form, like the metric, $\operatorname{Hess} f(X, Y)$ that measures the second order change of $f$. It is defined as the Lie derivative of the metric in the direction of the gradient. Thus it seems to measure how the metric changes as we move along the flow of the gradient

$$
\operatorname{Hess} f(X, Y)=\frac{1}{2}\left(L_{\operatorname{grad} f} g\right)(X, Y)
$$

We will calculate this in local coordinates to check that it makes some sort of sense:

$$
\begin{aligned}
\operatorname{Hess} f\left(\partial_{i}, \partial_{j}\right)= & \frac{1}{2}\left(L_{\operatorname{grad} f} g\right)\left(\partial_{i}, \partial_{j}\right) \\
= & \frac{1}{2} L_{\operatorname{grad} f} g_{i j}-\frac{1}{2} g\left(L_{\operatorname{grad} f} \partial_{i}, \partial_{j}\right)-\frac{1}{2} g\left(\partial_{i}, L_{\operatorname{grad} f} \partial_{j}\right) \\
= & \frac{1}{2} L_{\operatorname{grad} f} g_{i j}-\frac{1}{2} g\left(\left[\operatorname{grad} f, \partial_{i}\right], \partial_{j}\right)-\frac{1}{2} g\left(\partial_{i},\left[\operatorname{grad} f, \partial_{j}\right]\right) \\
= & \frac{1}{2} L_{g^{k l} \partial_{l} f \partial_{k}} g_{i j}-\frac{1}{2} g\left(\left[g^{k l} \partial_{l} f \partial_{k}, \partial_{i}\right], \partial_{j}\right)-\frac{1}{2} g\left(\partial_{i},\left[g^{k l} \partial_{l} f \partial_{k}, \partial_{j}\right]\right) \\
= & \frac{1}{2} g^{k l} \partial_{l} f \partial_{k}\left(g_{i j}\right)+\frac{1}{2} g\left(\partial_{i}\left(g^{k l} \partial_{l} f\right) \partial_{k}, \partial_{j}\right)+\frac{1}{2} g\left(\partial_{i}, \partial_{j}\left(g^{k l} \partial_{l} f\right) \partial_{k}\right) \\
= & \frac{1}{2} g^{k l} \partial_{l} f \partial_{k}\left(g_{i j}\right)+\frac{1}{2} \partial_{i}\left(g^{k l} \partial_{l} f\right) g_{k j}+\frac{1}{2} \partial_{j}\left(g^{k l} \partial_{l} f\right) g_{i k} \\
= & \frac{1}{2} g^{k l} \partial_{k}\left(g_{i j}\right) \partial_{l} f+\frac{1}{2} \partial_{i}\left(g^{k l}\right) g_{k j} \partial_{l} f+\frac{1}{2} \partial_{j}\left(g^{k l}\right) g_{i k} \partial_{l} f \\
& +\frac{1}{2} g^{k l} \partial_{i}\left(\partial_{l} f\right) g_{k j}+\frac{1}{2} g^{k l} \partial_{j}\left(\partial_{l} f\right) g_{i k} \\
= & \frac{1}{2} g^{k l} \partial_{k}\left(g_{i j}\right) \partial_{l} f-\frac{1}{2} g^{k l} \partial_{i}\left(g_{k j}\right) \partial_{l} f-\frac{1}{2} g^{k l} \partial_{j}\left(g_{i k}\right) \partial_{l} f \\
& +\frac{1}{2} \delta_{j}^{l} \partial_{i} \partial_{l} f+\frac{1}{2} \delta_{i}^{l}\left(\partial_{j} \partial_{l} f\right) \\
= & \frac{1}{2} g^{k l}\left(\partial_{k} g_{i j}-\partial_{i} g_{k j}-\partial_{j} g_{i k}\right) \partial_{l} f+\partial_{i} \partial_{j} f .
\end{aligned}
$$

So if the metric coefficients are constant, as in Euclidean space with Cartesian coordinates, or we are at a critical point, this gives us the old fashioned Hessian.

It is worth pointing out that these more general definitions and formulas are useful even in Euclidean space. The minute we switch to some more general coordinates, such as polar, cylindrical, spherical etc, the metric coefficients are no longer all constant. Thus the above formulas are our only way of calculating the gradient and Hessian in such general coordinates.

### 6.3. Operations on Forms

6.3.1. General Properties. Given $p 1$-forms $\omega_{i} \in \Omega^{1}(M)$ on a manifold $M$ we define

$$
\left(\omega_{1} \wedge \cdots \wedge \omega_{p}\right)\left(v_{1}, \ldots, v_{p}\right)=\operatorname{det}\left(\left[\omega_{i}\left(v_{j}\right)\right]\right)
$$

where $\left[\omega_{i}\left(v_{j}\right)\right]$ is the matrix with entries $\omega_{i}\left(v_{j}\right)$. We can then extend the wedge product to all forms using linearity and associativity. This gives the wedge product operation

$$
\begin{aligned}
\Omega^{p}(M) \times \Omega^{q}(M) & \rightarrow \Omega^{p+q}(M) \\
(\omega, \psi) & \rightarrow \omega \wedge \psi
\end{aligned}
$$

This operation is bilinear and antisymmetric in the sense that:

$$
\omega \wedge \psi=(-1)^{p q} \psi \wedge \omega
$$

The wedge product of a function and a form is simply standard multiplication.
The exterior derivative of a form is defined by

$$
\left.\begin{array}{rl}
d \omega\left(X_{0}, \ldots, X_{k}\right)= & \sum_{i=0}^{k}(-1)^{i} L_{X_{i}}\left(\omega\left(X_{0}, \ldots, \widehat{X}_{i}, \ldots, X_{k}\right)\right) \\
& -\sum_{i<j}(-1)^{i} \omega\left(X_{0}, \ldots, \widehat{X}_{i}, \ldots, L_{X_{i}} X_{j}, \ldots, X_{k}\right) \\
= & \sum_{i=0}^{k}(-1)^{i} L_{X_{i}}\left(\omega\left(X_{0}, \ldots, \widehat{X}_{i}, \ldots, X_{k}\right)\right) \\
& +\sum_{i<j}(-1)^{i+j} \omega\left(L_{X_{i}} X_{j}, X_{0}, \ldots, \widehat{X}_{i}, \ldots, \widehat{X}_{j}, \ldots, X_{k}\right) \\
= & \frac{1}{2} \sum_{i=0}^{k}(-1)^{i}\binom{\left(L_{X_{i}} \omega\right)\left(X_{0}, \ldots, \widehat{X}_{i}, \ldots, X_{k}\right)}{+L_{X_{i}}\left(\omega\left(X_{0}, \ldots, \widehat{X_{i}}, \ldots, X_{k}\right)\right.}
\end{array}\right) .
$$

Lie derivatives, interior products, wedge products and exterior derivatives on forms are related as follows:

$$
\begin{aligned}
d(\omega \wedge \psi) & =(d \omega) \wedge \psi+(-1)^{p} \omega \wedge(d \psi) \\
i_{X}(\omega \wedge \psi) & =\left(i_{X} \omega\right) \wedge \psi+(-1)^{p} \omega \wedge\left(i_{X} \psi\right) \\
L_{X}(\omega \wedge \psi) & =\left(L_{X} \omega\right) \wedge \psi+\omega \wedge\left(L_{X} \psi\right)
\end{aligned}
$$

and the composition properties

$$
\begin{aligned}
d \circ d & =0, \\
i_{X} \circ i_{X} & =0, \\
L_{X} & =d \circ i_{X}+i_{X} \circ d, \\
L_{X} \circ d & =d \circ L_{X}, \\
i_{X} \circ L_{X} & =L_{X} \circ i_{X} .
\end{aligned}
$$

The third property $L_{X}=d \circ i_{X}+i_{X} \circ d$ is also known a $H$. Cartan's formula (son of the geometer E. Cartan). It is behind the definition of exterior derivative we gave above. If we know how $d$ is defined on $p$-forms, then we can define $d$ on ( $p+1$ )-forms by

$$
i_{X_{0}} \circ d=L_{X_{0}}-d \circ i_{X_{0}} .
$$

6.3.2. The Volume Form. We are now ready to explain how forms are used to unify some standard concepts from differential vector calculus. We shall work on a Riemannian manifold $(M, g)$ and use orthonormal frames $E_{1}, \ldots, E_{m}$ as well as the dual frame $\phi^{1}, \ldots, \phi^{m}$ of 1 -forms.

The local volume form is defined as:

$$
\mathrm{vol}=\operatorname{vol}_{g}=\phi^{1} \wedge \cdots \wedge \phi^{m}
$$

We see that if $\psi^{1}, \ldots, \psi^{m}$ is another collection of 1-forms coming from an orthonormal frame $F_{1}, \ldots, F_{m}$, then

$$
\begin{aligned}
\psi^{1} \wedge \cdots \wedge \psi^{m}\left(E_{1}, \ldots, E_{m}\right) & =\operatorname{det}\left(\psi^{i}\left(E_{j}\right)\right) \\
& =\operatorname{det}\left(g\left(F_{i}, E_{j}\right)\right) \\
& = \pm 1
\end{aligned}
$$

The sign depends on whether or not the two frames define the same orientation. In case $M$ is oriented and we only use positively oriented frames we will get a globally defined volume form. Next we calculate the local volume form in local coordinates assuming that the frame and the coordinates are both positively oriented:

$$
\begin{aligned}
\operatorname{vol}\left(\partial_{1}, \ldots \partial_{m}\right) & =\operatorname{det}\left(\phi^{i}\left(\partial_{j}\right)\right) \\
& =\operatorname{det}\left(g\left(E_{i}, \partial_{j}\right)\right)
\end{aligned}
$$

As $E_{i}$ hasn't been eliminated we have to work a little harder. To this end we note that

$$
\begin{aligned}
\operatorname{det}\left(g\left(\partial_{i}, \partial_{j}\right)\right) & =\operatorname{det}\left(g\left(g\left(\partial_{i}, E_{k}\right) E_{k}, g\left(\partial_{j}, E_{l}\right) E_{l}\right)\right) \\
& =\operatorname{det}\left(g\left(\partial_{i}, E_{k}\right) g\left(\partial_{j}, E_{l}\right) \delta_{k l}\right) \\
& =\operatorname{det}\left(g\left(\partial_{i}, E_{k}\right) g\left(\partial_{j}, E_{k}\right)\right) \\
& =\operatorname{det}\left(g\left(\partial_{i}, E_{k}\right)\right) \operatorname{det}\left(g\left(\partial_{j}, E_{k}\right)\right) \\
& =\left(\operatorname{det}\left(g\left(E_{i}, \partial_{j}\right)\right)\right)^{2} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\operatorname{vol}\left(\partial_{1}, \ldots \partial_{m}\right) & =\sqrt{\operatorname{det} g_{i j}} \\
\operatorname{vol} & =\sqrt{\operatorname{det} g_{i j}} d x^{1} \wedge \cdots \wedge d x^{m}
\end{aligned}
$$

6.3.3. Divergence. The divergence of a vector field is defined as the change in the volume form as we flow along the vector field. Note the similarity with the Hessian.

$$
L_{X} \mathrm{vol}=\operatorname{div}(X) \operatorname{vol}
$$

In coordinates using that $X=a^{i} \partial_{i}$ we get

$$
\begin{aligned}
L_{X} \text { vol }= & L_{X}\left(\sqrt{\operatorname{det} g_{k l}} d x^{1} \wedge \cdots \wedge d x^{m}\right) \\
= & L_{X}\left(\sqrt{\operatorname{det} g_{k l}}\right) d x^{1} \wedge \cdots \wedge d x^{m} \\
& +\sqrt{\operatorname{det} g_{k l}} \sum_{i} d x^{1} \wedge \cdots \wedge L_{X}\left(d x^{i}\right) \wedge \cdots \wedge d x^{m} \\
= & a^{i} \partial_{i}\left(\sqrt{\operatorname{det} g_{k l}}\right) d x^{1} \wedge \cdots \wedge d x^{m} \\
& +\sqrt{\operatorname{det} g_{k l}} \sum_{i} d x^{1} \wedge \cdots \wedge d\left(L_{X} x^{i}\right) \wedge \cdots \wedge d x^{m} \\
= & a^{i} \partial_{i}\left(\sqrt{\operatorname{det} g_{k l}}\right) d x^{1} \wedge \cdots \wedge d x^{m} \\
& +\sqrt{\operatorname{det} g_{k l} \sum_{i} d x^{1} \wedge \cdots \wedge d\left(a^{i}\right) \wedge \cdots \wedge d x^{m}} \\
= & a^{i} \partial_{i}\left(\sqrt{\operatorname{det} g_{k l}}\right) d x^{1} \wedge \cdots \wedge d x^{m} \\
& +\sqrt{\operatorname{det} g_{k l} \sum_{i} d x^{1} \wedge \cdots \wedge\left(\partial_{j} a^{i} d x^{j}\right) \wedge \cdots \wedge d x^{m}} \\
& a^{i} \partial_{i}\left(\sqrt{\operatorname{det} g_{k l}}\right) d x^{1} \wedge \cdots \wedge d x^{m} \\
& +\sqrt{\operatorname{det} g_{k l} \sum_{i} d x^{1} \wedge \cdots \wedge\left(\partial_{i} a^{i} d x^{i}\right) \wedge \cdots \wedge d x^{m}} \\
= & \left(a^{i} \partial_{i}\left(\sqrt{\operatorname{det} g_{k l}}\right)+\sqrt{\operatorname{det} g_{k l}} \partial_{i} a^{i}\right) d x^{1} \wedge \cdots \wedge d x^{m} \\
= & \frac{\partial_{i}\left(a^{i} \sqrt{\operatorname{det} g_{k l}}\right)}{\sqrt{\operatorname{det} g_{k l}}} \sqrt{\operatorname{det} g_{k l}} d x^{1} \wedge \cdots \wedge d x^{m} \\
= & \frac{\partial_{i}\left(a^{i} \sqrt{\operatorname{det} g_{k l}}\right)}{\sqrt{\operatorname{det} g_{k l}}} \operatorname{vol}
\end{aligned}
$$

We see again that in case the metric coefficients are constant we get the familiar divergence from vector calculus.
H. Cartan's formula for the Lie derivative of forms gives us a different way of finding the divergence

$$
\begin{aligned}
\operatorname{div}(X) \mathrm{vol} & =L_{X} \mathrm{vol} \\
& =d i_{X} \mathrm{vol}+i_{X} d \mathrm{vol} \\
& =d i_{X} \mathrm{vol}
\end{aligned}
$$

in particular $\operatorname{div}(X)$ vol is always exact.
This formula suggests that we should study the correspondence that takes a vector field $X$ to the $(n-1)$-form $i_{X}$ vol. Using the orthonormal frame this correspondence is

$$
\begin{aligned}
i_{X} \mathrm{vol} & =i_{g\left(X, E_{j}\right) E_{j}}\left(\phi^{1} \wedge \cdots \wedge \phi^{m}\right) \\
& =g\left(X, E_{j}\right) i_{E_{j}}\left(\phi^{1} \wedge \cdots \wedge \phi^{m}\right) \\
& =\sum(-1)^{j+1} g\left(X, E_{j}\right) \phi^{1} \wedge \cdots \wedge \widehat{\phi^{j}} \wedge \cdots \wedge \phi^{m}
\end{aligned}
$$

while in coordinates

$$
\begin{aligned}
i_{X}(d \mathrm{vol}) & =i_{a j \partial_{j}}\left(\sqrt{\operatorname{det} g_{k l}} d x^{1} \wedge \cdots \wedge d x^{m}\right) \\
& =\sqrt{\operatorname{det} g_{k l}} \sum a^{j} i_{\partial_{j}}\left(d x^{1} \wedge \cdots \wedge d x^{m}\right) \\
& =\sqrt{\operatorname{det} g_{k l}} \sum(-1)^{j+1} a^{j} d x^{1} \wedge \cdots \wedge \widehat{d x^{j}} \wedge \cdots \wedge d x^{m}
\end{aligned}
$$

If we compute $d i_{X}$ vol using this formula we quickly get back our coordinate formula for $\operatorname{div}(X)$.

In vector calculus this gives us the correspondence

$$
\begin{aligned}
i_{\left(P \partial_{x}+Q \partial_{y}+R \partial_{z}\right)} d x \wedge d y \wedge d z= & P i_{\partial_{x}} d x \wedge d y \wedge d z \\
& +Q i_{\partial_{y}} d x \wedge d y \wedge d z \\
& +R i_{\partial_{z}} d x \wedge d y \wedge d z \\
= & P d y \wedge d z-Q d x \wedge d z+R d x \wedge d y \\
= & P d y \wedge d z+Q d z \wedge d x+R d x \wedge d y
\end{aligned}
$$

If we compose the grad and div operations we get the Laplacian:

$$
\operatorname{div}(\operatorname{grad} f)=\Delta f
$$

For this to make sense we should check that it is the "trace" of the Hessian. This is most easily done using an orthonormal frame $E_{i}$. On one hand the trace of the Hessian is:

$$
\begin{aligned}
\sum_{i} \operatorname{Hess} f\left(E_{i}, E_{i}\right) & =\sum_{i} \frac{1}{2}\left(L_{\operatorname{grad} f} g\right)\left(E_{i}, E_{i}\right) \\
& =\sum_{i} \frac{1}{2} L_{\operatorname{grad} f}\left(g\left(E_{i}, E_{i}\right)\right)-\frac{1}{2} g\left(L_{\operatorname{grad} f} E_{i}, E_{i}\right)-\frac{1}{2} g\left(E_{i}, L_{\operatorname{grad} f} E_{i}\right) \\
& =-\sum_{i} g\left(L_{\operatorname{grad} f} E_{i}, E_{i}\right)
\end{aligned}
$$

While the divergence is calculated as

$$
\begin{aligned}
\operatorname{div}(\operatorname{grad} f) & =\operatorname{div}(\operatorname{grad} f) d \operatorname{vol}\left(E_{1}, \ldots, E_{m}\right) \\
& =\left(L_{\operatorname{grad} f} \phi^{1} \wedge \cdots \wedge \phi^{m}\right)\left(E_{1}, \ldots, E_{m}\right) \\
& =\sum\left(\phi^{1} \wedge \cdots \wedge L_{\operatorname{grad} f} \phi^{i} \wedge \cdots \wedge \phi^{m}\right)\left(E_{1}, \ldots, E_{m}\right) \\
& =\sum\left(L_{\operatorname{grad} f} \phi^{i}\right)\left(E_{i}\right) \\
& =\sum L_{\operatorname{grad} f}\left(\phi^{i}\left(E_{i}\right)\right)-\phi^{i}\left(L_{\operatorname{grad} f} E_{i}\right) \\
& =-\sum \phi^{i}\left(L_{\operatorname{grad} f} E_{i}\right) \\
& =-\sum g\left(L_{\operatorname{grad} f} E_{i}, E_{i}\right)
\end{aligned}
$$

6.3.4. Curl. While the gradient and divergence operations work on any Riemannian manifold, the curl operator is specific to oriented 3 dimensional manifolds. It uses the above two correspondences between vector fields and 1-forms as well as 2-forms:

$$
d\left(\omega_{X}\right)=i_{\mathrm{curl} X} \mathrm{vol}
$$

If $X=P \partial_{x}+Q \partial_{y}+R \partial_{z}$ and we are on $\mathbb{R}^{3}$ we can easily see that

$$
\operatorname{curl} X=\left(\partial_{y} R-\partial_{z} Q\right) \partial_{x}+\left(\partial_{z} P-\partial_{x} R\right) \partial_{y}+\left(\partial_{x} Q-\partial_{y} P\right) \partial_{z}
$$

Taken together these three operators are defined as follows:

$$
\begin{aligned}
\omega_{\operatorname{grad} f} & =d f \\
i_{\operatorname{curl} X} \operatorname{vol} & =d \omega_{X} \\
\operatorname{div}(X) \text { vol } & =d i_{X} \text { vol. }
\end{aligned}
$$

Using that $d \circ d=0$ on all forms we obtain the classical vector analysis formulas

$$
\begin{aligned}
\operatorname{curl}(\operatorname{grad} f) & =0, \\
\operatorname{div}(\operatorname{curl} X) & =0,
\end{aligned}
$$

from

$$
\begin{aligned}
i_{\operatorname{curl}(\operatorname{grad} f)} \operatorname{vol} & =d \omega_{\operatorname{grad} f}=d d f \\
\operatorname{div}(\operatorname{curl} X) \operatorname{vol} & =d i_{\operatorname{curl} X} \mathrm{vol}=d d \omega_{X}
\end{aligned}
$$

### 6.4. Orientability

Recall that two ordered bases of a finite dimensional vector space are said to represent the same orientation if the transition matrix from one to the other is of positive determinant. This evidently defines an equivalence relation with exactly two equivalence classes. A choice of such an equivalence class is called an orientation for the vector space.

Given a smooth manifold each tangent space has two choices for an orientation. Thus we obtain a two fold covering map $O_{M} \rightarrow M$, where the preimage of each $p \in M$ consists of the two orientations for $T_{p} M$. A connected manifold is said to be orientable if the orientation covering is disconnected. For a disconnected manifold, we simply require that each connected component be connected. A choice of sheet in the covering will correspond to a choice of an orientation for each tangent space.

To see that $O_{M}$ really is a covering just note that if we have a chart $\left(x^{1}, x^{2}, \ldots, x^{n}\right): U \subset$ $M \rightarrow \mathbb{R}^{n}$, where $U$ is connected, then we have two choices of orientations over $U$, namely, the class determined by the framing $\left(\partial_{1}, \partial_{2}, \ldots, \partial_{n}\right)$ and by the framing $\left(-\partial_{1}, \partial_{2}, \ldots, \partial_{n}\right)$. Thus $U$ is covered by two sets each diffeomorphic to $U$ and parametrized by these two different choices of orientation. Observe that this tells us that $\mathbb{R}^{n}$ is orientable and has a canonical orientation given by the standard Cartesian coordinate frame $\left(\partial_{1}, \partial_{2}, \ldots, \partial_{n}\right)$.

Note that since simply connected manifolds only have trivial covering spaces they must all be orientable. Thus $S^{n}, n>1$ is always orientable.

An other important observation is that the orientation covering $O_{M}$ is an orientable manifold since it is locally the same as $M$ and an orientation at each tangent space has been picked for us.

THEOREM 6.4.1. The following conditions for a connected $n$-manifold $M$ are equivalent.

1. $M$ is orientable.
2. Orientation is preserved moving along loops.
3. $M$ admits an atlas where the Jacobians of all the transitions functions are positive.
4. $M$ admits a nowhere vanishing n-form.

Proof. $1 \Leftrightarrow 2$ : The unique path lifting property for the covering $O_{M} \rightarrow M$ tells us that orientation is preserved along loops if and only if $O_{M}$ is disconnected.
$1 \Rightarrow 3$ : Pick an orientation. Take any atlas $\left(U_{\alpha}, F_{\alpha}\right)$ of $M$ where $U_{\alpha}$ is connected. As in our description of $O_{M}$ from above we see that either each $F_{\alpha}$ corresponds to the chosen orientation, otherwise change the sign of the first component of $F_{\alpha}$. In this way we get
an atlas where each chart corresponds to the chosen orientation. Then it is easily checked that the transition functions $F_{\alpha} \circ F_{\beta}^{-1}$ have positive Jacobian as they preserve the canonical orientation of $\mathbb{R}^{n}$.
$3 \Rightarrow 4$ : Choose a locally finite partition of unity $\left(\lambda_{\alpha}\right)$ subordinate to an atlas $\left(U_{\alpha}, F_{\alpha}\right)$ where the transition functions have positive Jacobians. On each $U_{\alpha}$ we have the nowhere vanishing form $\omega_{\alpha}=d x_{\alpha}^{1} \wedge \ldots \wedge d x_{\alpha}^{n}$. Now note that if we are in an overlap $U_{\alpha} \cap U_{\beta}$ then

$$
\begin{aligned}
d x_{\alpha}^{1} \wedge \ldots \wedge d x_{\alpha}^{n}\left(\frac{\partial}{\partial x_{\beta}^{1}}, \ldots, \frac{\partial}{\partial x_{\beta}^{n}}\right) & =\operatorname{det}\left(d x_{\alpha}^{i}\left(\frac{\partial}{\partial x_{\beta}^{j}}\right)\right) \\
& =\operatorname{det}\left(D\left(F_{\alpha} \circ F_{\beta}^{-1}\right)\right) \\
& >0 .
\end{aligned}
$$

Thus the globally defined form $\omega=\sum \lambda_{\alpha} \omega_{\alpha}$ is always nonnegative when evaluated on $\left(\frac{\partial}{\partial x_{\beta}^{1}}, \ldots, \frac{\partial}{\partial x_{\beta}^{n}}\right)$. What is more, at least one term must be positive according to the definition of partition of unity.
$4 \Rightarrow 1$ : Pick a nowhere vanishing $n$-form $\omega$. Define the two sets $O_{ \pm}$according to whether $\omega$ is positive or negative when evaluated on a basis. This yields two disjoint open sets in $O_{M}$ which cover all of $M$.

With this result behind us we can try to determine which manifolds are orientable and which are not. Conditions 3 and 4 are often good ways of establishing orientability. To establish non-orientability is a little more tricky. However, if we suspect a manifold to be non-orientable then 1 tells us that there must be a non-trivial 2 -fold covering map $\pi: \hat{M} \rightarrow$ $M$, where $\hat{M}$ is oriented and the two given orientations at points over $p \in M$ are mapped to different orientations in $M$ via $D \pi$. A different way of recording this information is to note that for a two fold covering $\pi: \hat{M} \rightarrow M$ there is only one nontrivial deck transformation $A: \hat{M} \rightarrow \hat{M}$ with the properties: $A(x) \neq x, A \circ A=i d_{M}$, and $\pi \circ A=\pi$. With this is mind we can show

Proposition 6.4.2. Let $\pi: \hat{M} \rightarrow M$ be a non-trivial 2-fold covering and $\hat{M}$ an oriented manifold. Then $M$ is orientable if and only if A preserves the orientation on $\hat{M}$.

Proof. First suppose $A$ preserves the orientation of $\hat{M}$. Then given a choice of orientation $e_{1}, \ldots, e_{n} \in T_{x} \hat{M}$ we can declare $D \pi\left(e_{1}\right), \ldots, D \pi\left(e_{n}\right) \in T_{\pi(x)} M$ to be an orientation at $\pi(x)$. This is consistent as $D A\left(e_{1}\right), \ldots, D A\left(e_{n}\right) \in T_{I(x)} \hat{M}$ is mapped to $D \pi\left(e_{1}\right), \ldots, D \pi\left(e_{n}\right)$ as well (using $\pi \circ A=\pi$ ) and also represents the given orientation on $\hat{M}$ since $A$ was assumed to preserve this orientation.

Suppose conversely that $M$ is orientable and choose an orientation for $M$. Since we assume that both $\hat{M}$ and $M$ are connected the projection $\pi: \hat{M} \rightarrow M$, being nonsingular everywhere, must always preserve or reverse the orientation. We can without loss of generality assume that the orientation is preserved. Then we just use $\pi \circ A=\pi$ as in the first part of the proof to see that $A$ must preserve the orientation on $\hat{M}$.

We can now use these results to check some concrete manifolds for orientability.
We already know that $S^{n}, n>1$ are orientable, but what about $S^{1}$ ? One way of checking that this space is orientable is to note that the tangent bundle is trivial and thus a uniform choice of orientation is possible. This clearly generalizes to Lie groups and other parallelizable manifolds. Another method is to find a nowhere vanishing form. This can be
done on all spheres $S^{n}$ by considering the $n$-form

$$
\omega=\sum_{i=1}^{n+1}(-1)^{i+1} x^{i} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n+1}
$$

on $\mathbb{R}^{n+1}$. This form is a generalization of the 1-form $x d y-y d x$, which is $\pm$ the angular form in the plane. Note that if $X=x^{i} \partial_{i}$ denotes the radial vector field, then we have (see also the section below on the classical integral theorems)

$$
i_{X}\left(d x^{1} \wedge \cdots \wedge d x^{n+1}\right)=\omega
$$

From this it is clear that if $v_{2}, \ldots, v_{n}$ form a basis for a tangent space to the sphere, then

$$
\begin{aligned}
\omega\left(v_{2}, \ldots, v_{n}\right) & =d x^{1} \wedge \cdots \wedge d x^{n+1}\left(X, v_{2}, \ldots, v_{n+1}\right) \\
& \neq 0
\end{aligned}
$$

Thus we have found a nonvanishing $n$-form on all spheres regardless of whether or not they are parallelizable or simply connected. As another exercise people might want to use one of the several coordinate atlases known for the spheres to show that they are orientable.

Recall that $\mathbb{R} \mathbb{P}^{n}$ has $S^{n}$ as a natural double covering with the antipodal map as a natural deck transformation. Now this deck transformation preserves the radial field $X=x^{i} \partial_{i}$ and thus its restriction to $S^{n}$ preserves or reverses orientation according to what it does on $\mathbb{R}^{n+1}$. On the ambient Euclidean space the map is linear and therefore preserves the orientation iff its determinant is positive. This happens iff $n+1$ is even. Thus we see that $\mathbb{R}^{P^{n}}$ is orientable iff $n$ is odd.

Using the double covering lemma show that the Klein bottle and the Möbius band are non-orientable.

### 6.5. Integration of Forms

We shall assume that $M$ is an oriented $n$-manifold. Thus, $M$ comes with a covering of charts $\varphi_{\alpha}=\left(x_{\alpha}^{1}, \ldots, x_{\alpha}^{n}\right): U_{\alpha} \longleftrightarrow B(0,1) \subset \mathbb{R}^{n}$ such that the transition functions $\varphi_{\alpha} \circ$ $\varphi_{\beta}^{-1}$ preserve the usual orientation on Euclidean space, i.e., $\operatorname{det}\left(D\left(\varphi_{\alpha} \circ \varphi_{\beta}^{-1}\right)\right)>0$. In addition, we shall also assume that a partition of unity with respect to this covering is given. In other words, we have smooth functions $\phi_{\alpha}: M \rightarrow[0,1]$ such that $\phi_{\alpha}=0$ on $M-U_{\alpha}$ and $\sum_{\alpha} \phi_{\alpha}=1$. For the last condition to make sense, it is obviously necessary that the covering be also locally finite.

Given an $n$-form $\omega$ on $M$ we wish to define the integral:

$$
\int_{M} \omega .
$$

When $M$ is not compact, it might be necessary to assume that the form has compact support, i.e., it vanishes outside some compact subset of $M$.

In each chart we can write

$$
\omega=f_{\alpha} d x_{\alpha}^{1} \wedge \cdots \wedge d x_{\alpha}^{n}
$$

Using the partition of unity, we then obtain

$$
\begin{aligned}
\omega & =\sum_{\alpha} \phi_{\alpha} \omega \\
& =\sum_{\alpha} \phi_{\alpha} f_{\alpha} d x_{\alpha}^{1} \wedge \cdots \wedge d x_{\alpha}^{n}
\end{aligned}
$$

where each of the forms $\phi_{\alpha} f_{\alpha} d x_{\alpha}^{1} \wedge \cdots \wedge d x_{\alpha}^{n}$ has compact support in $U_{\alpha}$. Since $U_{\alpha}$ is identified with $\bar{U}_{\alpha} \subset \mathbb{R}^{n}$, we simply declare that

$$
\int_{U_{\alpha}} \phi_{\alpha} f_{\alpha} d x_{\alpha}^{1} \wedge \cdots \wedge d x_{\alpha}^{n}=\int_{\bar{U}_{\alpha}} \phi_{\alpha} f_{\alpha} d x^{1} \cdots d x^{n}
$$

Here the right-hand side is simply the integral of the function $\phi_{\alpha} f_{\alpha}$ viewed as a function on $\bar{U}_{\alpha}$. Then we define

$$
\int_{M} \omega=\sum_{\alpha} \int_{U_{\alpha}} \phi_{\alpha} f_{\alpha} d x_{\alpha}^{1} \wedge \cdots \wedge d x_{\alpha}^{n}
$$

whenever this sum converges. Using the standard change of variables formula for integration on Euclidean space, we see that this definition is indeed independent of the choice of coordinates.

With these definitions behind us, we can now state and prove Stokes' theorem for manifolds with boundary.

THEOREM 6.5.1. For any $\omega \in \Omega^{n-1}(M)$ with compact support we have

$$
\int_{M} d \omega=\int_{\partial M} \omega
$$

Proof. If we use the trick

$$
d \omega=\sum_{\alpha} d\left(\phi_{\alpha} \omega\right),
$$

then we see that it suffices to prove the theorem in the case $M=L^{n}$ and $\omega$ has compact support. In that case we can write

$$
\omega=\sum_{i=1}^{n} f_{i} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n}
$$

The differential of $\omega$ is now easily computed:

$$
\begin{aligned}
d \omega & =\sum_{i=1}^{n}\left(d f_{i}\right) \wedge d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n} \\
& =\sum_{i=1}^{n}\left(\frac{\partial f_{i}}{\partial x^{i}}\right) d x^{i} \wedge d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n} \\
& =\sum_{i=1}^{n}(-1)^{i-1} \frac{\partial f_{i}}{\partial x^{i}} d x^{1} \wedge \cdots \wedge d x^{i} \wedge \cdots \wedge d x^{n} .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{L^{n}} d \omega & =\int_{L^{n}} \sum_{i=1}^{n}(-1)^{i-1} \frac{\partial f_{i}}{\partial x^{i}} d x^{1} \wedge \cdots \wedge d x^{n} \\
& =\sum_{i=1}^{n}(-1)^{i-1} \int_{L^{n}} \frac{\partial f_{i}}{\partial x^{i}} d x^{1} \cdots d x^{n} \\
& =\sum_{i=1}^{n}(-1)^{i-1} \int\left(\int\left(\frac{\partial f_{i}}{\partial x^{i}}\right) d x^{i}\right) d x^{1} \cdots \widehat{d x^{i}} \cdots d x^{n}
\end{aligned}
$$

The fundamental theorem of calculus tells us that

$$
\begin{aligned}
\int_{-\infty}^{\infty}\left(\frac{\partial f_{i}}{\partial x^{i}}\right) d x^{i} & =0, \text { for } i>1 \\
\int_{-\infty}^{0}\left(\frac{\partial f_{1}}{\partial x^{1}}\right) d x^{1} & =f_{1}\left(0, x^{2}, \ldots, x^{n}\right)
\end{aligned}
$$

Thus

$$
\int_{L^{n}} d \omega=\int_{\partial L^{n}} f_{1}\left(0, x^{2}, \ldots, x^{n}\right) d x^{2} \wedge \cdots \wedge d x^{n}
$$

Since $d x^{1}=0$ on $\partial L^{n}$ it follows that

$$
\left.\omega\right|_{\partial L^{n}}=f_{1} d x^{2} \wedge \cdots \wedge d x^{n}
$$

This proves the theorem.
We get a very nice corollary out of Stokes' theorem.
THEOREM. If $M$ is a compact manifold with nonempty boundary, then there is no retract $r: M \rightarrow \partial M$.

Proof. Note that if $\partial M$ is not connected such a retract clearly can't exists so we need only worry about having connected boundary.

If $M$ is oriented and $\omega$ is a volume form on $\partial M$, then we have

$$
\begin{aligned}
0 & <\int_{\partial M} \omega \\
& =\int_{\partial M} r^{*} \omega \\
& =\int_{M} d\left(r^{*} \omega\right) \\
& =\int_{M} r^{*} d \omega \\
& =0
\end{aligned}
$$

If $M$ is not orientable, then we lift the situation to the orientation cover and obtain a contradiction there.

We shall briefly discuss how the classical integral theorems of Green, Gauss, and Stokes follow from the general version of Stokes' theorem presented above.

Green's theorem in the plane is quite simple.
THEOREM 6.5.2. (Green's Theorem) Let $\Omega \subset \mathbb{R}^{2}$ be a domain with smooth boundary $\partial \Omega$. If $X=P \partial_{x}+Q \partial_{y}$ is a vector field defined on a region containing $\Omega$ then

$$
\int_{\Omega}\left(\partial_{x} Q-\partial_{y} P\right) d x d y=\int_{\partial \Omega} P d x+Q d y .
$$

Proof. Note that the integral on the right-hand side is a line integral, which can also be interpreted as the integral of the 1-form $\omega=P d x^{1}+Q d x^{2}$ on the 1-manifold $\partial \Omega$. With this in mind we just need to observe that $d \omega=\left(\partial_{1} Q-\partial_{2} P\right) d x^{1} \wedge d x^{2}$ in order to establish the theorem.

Gauss' Theorem is quite a bit more complicated, but we did some of the ground work when we defined the divergence above. The context is a connected, compact, oriented Riemannian manifold $M$ with boundary, but the example to keep in mind is a domain $M \subset \mathbb{R}^{n}$ with smooth boundary

THEOREM 6.5.3. (The divergence theorem or Gauss' theorem) Let $X$ be a vector field defined on $M$ and $N$ the outward pointing unit normal field to $\partial M$, then

$$
\int_{M}(\operatorname{div} X) \operatorname{vol}_{g}=\int_{\partial M} g(X, N) \operatorname{vol}_{\left.g\right|_{\partial M}}
$$

Proof. We know that

$$
\operatorname{div} X \operatorname{vol}_{g}=d i_{X} \operatorname{vol}_{g}
$$

So by Stokes' theorem it suffices to show that

$$
\left.i_{X} \operatorname{vol}_{g}\right|_{\partial M}=g(X, N) \operatorname{vol}_{\mathrm{g}_{\partial \mathrm{M}}}
$$

The orientation on $T_{p} \partial M$ is so that $v_{2}, \ldots, v_{n}$ is a positively oriented basis for $T_{p} \partial M$ iff $N, v_{2}, \ldots, v_{n}$ is a positively oriented basis for $T_{p} M$. Therefore, the natural volume form for $\partial M$ denoted $\operatorname{vol}_{\left.g\right|_{\partial M}}$ is given by $i_{N} \operatorname{vol}_{g}$. If $v_{2}, \ldots, v_{n} \in T_{p} \partial M$ is a basis, then

$$
\begin{aligned}
\left.i_{X} \operatorname{vol}_{g}\right|_{\partial M}\left(v_{2}, \ldots, v_{n}\right) & =\operatorname{vol}_{g}\left(X, v_{2}, \ldots, v_{n}\right) \\
& =\operatorname{vol}_{g}\left(g(X, N) N, v_{2}, \ldots, v_{n}\right) \\
& =g(X, N) \operatorname{vol}_{g}\left(N, v_{2}, \ldots, v_{n}\right) \\
& =g(X, N) i_{N} \operatorname{vol}_{g} \\
& =\left.g(X, N) \operatorname{vol}_{g}\right|_{\partial M}
\end{aligned}
$$

where we used that $X-g(X, N) X$, the component of $X$ tangent $T_{p} \partial M$, is a linear combination of $v_{2}, \ldots, v_{n}$ and therefore doesn't contribute to the form.

Stokes' Theorem is specific to 3 dimensions. Classically it holds for an oriented surface $S \subset \mathbb{R}^{3}$ with smooth boundary but can be formulated for oriented surfaces in oriented Riemannian 3-manifolds.

THEOREM 6.5.4. (Stokes' theorem) Let $S \subset M^{3}$ be an oriented surface with boundary $\partial S$. If $X$ is a vector field defined on a region containing $S$ and $N$ is the unit normal field to $S$, then

$$
\int_{S} g(\operatorname{curl} X, N) \operatorname{vol}_{\left.g\right|_{S}}=\int_{\partial S} \omega_{X}
$$

Proof. Recall that $\omega_{X}$ is the 1-form defined by

$$
\omega_{X}(v)=g(X, v)
$$

This form is related to curl $X$ by

$$
d \omega_{X}=i_{\mathrm{curl} X} \operatorname{vol}_{g}
$$

So Stokes' Theorem tells us that

$$
\int_{\partial S} \omega_{X}=\int_{S} i_{\mathrm{curl} X \operatorname{vol}_{g}}
$$

The integral on the right-hand side can now be understood in a manner completely analogous to our discussion of $\left.i_{X} \mathrm{vol}_{g}\right|_{\partial M}$ in the Divergence Theorem. We note that $N$ is chosen perpendicular to $T_{p} S$ in such a way that $N, v_{2}, v_{3} \in T_{p} M$ is positively oriented iff $v_{2}, v_{3} \in T_{p} S$ is positively oriented. Thus we have again that

$$
\operatorname{vol}_{\left.g\right|_{S}}=i_{N} \operatorname{vol}_{g}
$$

and consequently

$$
i_{\mathrm{curl} X} \operatorname{vol}_{g}=g(\operatorname{curl} X, N) \operatorname{vol}_{\left.g\right|_{S}}
$$

### 6.6. Frobenius

In this section we prove the theorem of Frobenius for vector fields and relate it to equivalent versions for forms and differential equations that involve Lie derivatives. Use the section from Spivak vol 1 ed 3 .

## CHAPTER 7

## Basic Cohomology Theory

### 7.1. De Rham Cohomology

Throughout we let $M$ be an $n$-manifold. Using that $d \circ d=0$, we see that the exact forms

$$
B^{p}(M)=d\left(\Omega^{p-1}(M)\right)
$$

are a subset of the closed forms

$$
Z^{p}(M)=\left\{\omega \in \Omega^{p}(M) \mid d \omega=0\right\}
$$

The de Rham cohomology is defined as the quotient space:

$$
H^{p}(M)=\frac{Z^{p}(M)}{B^{p}(M)}
$$

Given a closed form $\psi$, we let $[\psi]$ denote the corresponding cohomology class.
The first simple property comes from the fact that any function with zero differential must be locally constant. On a connected manifold we therefore have

$$
H^{0}(M)=\mathbb{R}
$$

Given a smooth map $F: M \rightarrow N$ the pull-back operation on forms induces a map in cohomology:

$$
\begin{aligned}
H^{p}(N) & \rightarrow H^{p}(M), \\
F^{*}([\psi]) & =\left[F^{*} \psi\right] .
\end{aligned}
$$

This definition is independent of the choice of $\psi$, since $F^{*}$ commutes with $d$.
The two key results that are needed for a deeper understanding of de Rham cohomology are the Mayer-Vietoris sequence and homotopy invariance of the pull-back map.

Lemma 7.1.1. (The Mayer-Vietoris Sequence) If $M=A \cup B$ for open sets $A, B \subset M$, then there is a long exact sequence

$$
\cdots \rightarrow H^{p}(M) \rightarrow H^{p}(A) \oplus H^{p}(B) \rightarrow H^{p}(A \cap B) \rightarrow H^{p+1}(M) \rightarrow \cdots
$$

Proof. The proof is given in outline, as it is exactly the same as the corresponding proof in algebraic topology. We start by defining a short exact sequence

$$
0 \rightarrow \Omega^{p}(M) \rightarrow \Omega^{p}(A) \oplus \Omega^{p}(B) \rightarrow \Omega^{p}(A \cap B) \rightarrow 0
$$

The map $\Omega^{p}(M) \rightarrow \Omega^{p}(A) \oplus \Omega^{p}(B)$ is simply restriction $\omega \mapsto\left(\left.\omega\right|_{A},\left.\omega\right|_{B}\right)$. The second is given by $(\omega, \psi) \mapsto\left(\left.\omega\right|_{A \cap B}-\left.\psi\right|_{A \cap B}\right)$. With these definitions it is clear that $\Omega^{p}(M) \rightarrow$ $\Omega^{p}(A) \oplus \Omega^{p}(B)$ is injective and that the sequence is exact at $\Omega^{p}(A) \oplus \Omega^{p}(B)$. It is a bit less obvious why $\Omega^{p}(A) \oplus \Omega^{p}(B) \rightarrow \Omega^{p}(A \cap B)$ is surjective. To see this select a partition of unity $\lambda_{A}, \lambda_{B}$ with respect to the covering $A, B$. Given $\omega \in \Omega^{p}(A \cap B)$ we see that $\lambda_{A} \omega$ defines a form on $B$, while $\lambda_{B} \omega$ defines a form on $A$. Then $\left(\lambda_{B} \omega,-\lambda_{A} \omega\right) \mapsto \omega$.

These maps induce maps in cohomology

$$
H^{p}(M) \rightarrow H^{p}(A) \oplus H^{p}(B) \rightarrow H^{p}(A \cap B)
$$

such that this sequence is exact. The connecting homomorphisms

$$
\delta: H^{p}(A \cap B) \rightarrow H^{p+1}(M)
$$

are constructed using the diagram

$$
\left.\begin{array}{cccccccc}
0 & \rightarrow & \Omega^{p+1}(M) & \rightarrow & \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) & \rightarrow & \Omega^{p+1}(A \cap B) & \rightarrow
\end{array}\right) 0 .
$$

If we take a form $\omega \in \Omega^{p}(A \cap B)$, then $\left(\lambda_{B} \omega,-\lambda_{A} \omega\right) \in \Omega^{p}(A) \oplus \Omega^{p}(B)$ is mapped onto $\omega$. If $d \omega=0$, then

$$
\begin{aligned}
d\left(\lambda_{B} \omega,-\lambda_{A} \omega\right) & =\left(d \lambda_{B} \wedge \omega,-d \lambda_{A} \wedge \omega\right) \\
& \in \Omega^{p+1}(A) \oplus \Omega^{p+1}(B)
\end{aligned}
$$

vanishes when mapped to $\Omega^{p+1}(A \cap B)$. So we obtain a well-defined form

$$
\begin{aligned}
\delta \omega & = \begin{cases}d \lambda_{B} \wedge \omega & \text { on } A \\
-d \lambda_{A} \wedge \omega & \text { on } B\end{cases} \\
& \in \Omega^{p+1}(M) .
\end{aligned}
$$

It is easy to see that this defines a map in cohomology that makes the Mayer-Vietoris sequence exact.

The construction here is fairly concrete, but it is a very general homological construction.

The first part of the Mayer-Vietoris sequence

$$
0 \rightarrow H^{0}(M) \rightarrow H^{0}(A) \oplus H^{0}(B) \rightarrow H^{0}(A \cap B) \rightarrow H^{1}(M)
$$

is particularly simple since we know what the zero dimensional cohomology is. In case $A \cap B$ is connected it must be a short exact sequence

$$
0 \rightarrow H^{0}(M) \rightarrow H^{0}(A) \oplus H^{0}(B) \rightarrow H^{0}(A \cap B) \rightarrow 0
$$

so the Mayer-Vietoris sequence for higher dimensional cohomology starts with

$$
0 \rightarrow H^{1}(M) \rightarrow H^{1}(A) \oplus H^{1}(B) \rightarrow \cdots
$$

To study what happens when we have homotopic maps between manifolds we have to figure out how forms on the product $[0,1] \times M$ relate to forms on $M$.

On the product $[0,1] \times M$ we have the vector field $\partial_{t}$ tangent to the first factor and the corresponding one-form $d t$. In local coordinates forms on $[0,1] \times M$ can be written

$$
\omega=a_{I} d x^{I}+b_{J} d t \wedge d x^{J}
$$

if we use summation convention and multi index notation

$$
\begin{aligned}
a_{I} & =a_{i \cdots i_{k}} \\
d x^{I} & =d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}}
\end{aligned}
$$

For each form the $d t$ factor can be integrated out as follows

$$
\mathscr{I}(\omega)=\int_{0}^{1} \omega=\int_{0}^{1} b_{J} d t \wedge d x^{J}=\left(\int_{0}^{1} b_{J} d t\right) d x^{J}
$$

Thus giving a map

$$
\Omega^{k+1}([0,1] \times M) \rightarrow \Omega^{k}(M)
$$

To see that this is well-defined note that it can be expressed as

$$
\mathscr{I}(\omega)=\int_{0}^{1} d t \wedge i_{\partial_{t}} \omega
$$

since

$$
i_{\partial_{t}}(\omega)=b_{J} d x^{J} .
$$

Lemma 7.1.2. Let $j_{t}: M \rightarrow[0,1] \times M$ be the map $j_{t}(x)=(t, x)$, then

$$
\mathscr{I}(d \omega)+d \mathscr{I}(\omega)=j_{1}^{*}(\omega)-j_{0}^{*}(\omega)
$$

Proof. The key is to prove that

$$
\mathscr{I}(d \omega)+d \mathscr{I}(\omega)=\int_{0}^{1} d t \wedge L_{\partial_{t}} \omega
$$

Given this it follows that the right hand side is

$$
\begin{aligned}
\int_{0}^{1} d t \wedge L_{\partial_{t}} \omega & =\int_{0}^{1} d t \wedge L_{\partial_{t}}\left(a_{I} d x^{I}+b_{J} d t \wedge d x^{J}\right) \\
& =\int_{0}^{1} d t \wedge\left(\partial_{t} a_{I} d x^{I}+\partial_{t} b_{J} d t \wedge d x^{J}\right) \\
& =\int_{0}^{1} d t \wedge\left(\partial_{t} a_{I}\right) d x^{I} \\
& =\left(\int_{0}^{1} d t \partial_{t} a_{I}\right) d x^{I} \\
& =\left(a_{I}(1, x)-a_{I}(0, x)\right) d x^{I} \\
& =j_{1}^{*}(\omega)-j_{0}^{*}(\omega)
\end{aligned}
$$

The first formula follows by noting that

$$
\begin{aligned}
\mathscr{I}(d \omega)+d \mathscr{I}(\omega) & =\int_{0}^{1} d t \wedge i_{\partial_{t}} d \omega+d\left(\int_{0}^{1} d t \wedge i_{\partial_{t}} \omega\right) \\
& =\int_{0}^{1} d t \wedge i_{\partial_{t}} d \omega+\int_{0}^{1} d t \wedge d i_{\partial_{t}} \omega \\
& =\int_{0}^{1} d t \wedge\left(i_{\partial_{t}} d \omega+d i_{\partial_{t}} \omega\right) \\
& =\int_{0}^{1} d t \wedge\left(L_{\partial_{t}} \omega\right)
\end{aligned}
$$

The one tricky move here is the identity

$$
d\left(\int_{0}^{1} d t \wedge i_{\partial_{t}} \omega\right)=\int_{0}^{1} d t \wedge d i_{\partial_{t}} \omega
$$

On the left hand side it is clear what $d$ does, but on the right hand side we are computing $d$ of a form on the product. However, as we are wedging with $d t$ this does not become an issue. Specifically, if $d$ is exterior differentiation on $[0,1] \times M$ and $d_{x}$ exterior differentiation
on $M$, then

$$
\begin{aligned}
d_{x}\left(\int_{0}^{1} d t \wedge i_{\partial_{t}} \omega\right) & =d_{x}\left(\int_{0}^{1} b_{J} d t\right) \wedge d x^{J} \\
& =\sum_{i} \frac{\partial \int_{0}^{1} b_{J} d t}{\partial x^{i}} \wedge d x^{i} \wedge d x^{J} \\
& =\sum_{i} \int_{0}^{1} \frac{\partial b_{J}}{\partial x^{i}} d t \wedge d x^{i} \wedge d x^{J} \\
& =\left(\int_{0}^{1} d t \wedge\left(\sum_{i} \frac{\partial b_{J}}{\partial x^{i}} d x^{i}\right)\right) \wedge d x^{J} \\
& =\left(\int_{0}^{1} d t \wedge\left(d_{x} b_{J}\right)\right) \wedge d x^{J} \\
& =\left(\int_{0}^{1} d t \wedge\left(d b_{J}-\partial_{t} b_{J} d t\right)\right) \wedge d x^{J} \\
& =\left(\int_{0}^{1} d t \wedge d b_{J}\right) \wedge d x^{J} \\
& =\int_{0}^{1} d t \wedge d i_{\partial_{t}} \omega
\end{aligned}
$$

We can now establish homotopy invariance.
PROPOSITION 7.1.3. If $F_{0}, F_{1}: M \rightarrow N$ are smoothly homotopic, then they induce the same maps on de Rham cohomology.

Proof. The formula

$$
\mathscr{I}(d \omega)+d \mathscr{I}(\omega)=j_{1}^{*}(\omega)-j_{0}^{*}(\omega)
$$

shows that $j_{1}^{*}(\omega)-j_{0}^{*}(\omega)$ is exact provided $d \omega=0$. In particular, $j_{0}$ and $j_{1}$ induce the same maps in cohomology:

$$
j_{0}^{*}=j_{1}^{*}: H^{*}([0,1] \times M) \rightarrow H^{*}(M)
$$

Assuming we have a homotopy $H:[0,1] \times M \rightarrow N$, such that $F_{0}=H \circ j_{0}$ and $F_{1}=$ $H \circ j_{1}$ it follows that

$$
F_{1}^{*}(\omega)-F_{0}^{*}(\omega)=\left(H \circ j_{1}\right)^{*}(\omega)-\left(H \circ j_{0}\right)^{*}(\omega)=0
$$

COROLLARY 7.1.4. If two manifolds, possibly of different dimension, are homotopy equivalent, then they have the same de Rham cohomology.

Proof. This follows from having maps $F: M \rightarrow N$ and $G: N \rightarrow M$ such that $F \circ G$ and $G \circ F$ are homotopic to the identity maps.

Lemma 7.1.5. (The Poincaré Lemma) The cohomology of a contractible manifold M is

$$
\begin{aligned}
H^{0}(M) & =\mathbb{R} \\
H^{p}(M) & =\{0\} \text { for } p>0
\end{aligned}
$$

In particular, convex sets in $\mathbb{R}^{n}$ have trivial de Rham cohomology.

Proof. Being contractible is the same as being homotopy equivalent to a point.
While we can't definitely relate the cohomology of a covering space to its base there is a simple relationship.

LEmmA 7.1.6. Let $F: M \rightarrow N$ be a finite covering map, then

$$
F^{*}: H^{p}(N) \rightarrow H^{p}(M)
$$

is an injection.
Proof. The trick lies in finding a so called transgression map $\tau: \Omega^{p}(M) \rightarrow \Omega^{p}(N)$ that commutes with exterior differentiation, $d \circ \tau=\tau \circ d$ and such that $\tau \circ F^{*}=i d_{\Omega^{p}(N)}$. This will induce a map $\tau^{*}: H^{*}(M) \rightarrow H^{*}(N)$ such that $\tau^{*} \circ F^{*}=i d_{H^{*}(N)}$, which shows in particular that $F^{*}$ is an injection.

While it'd be natural to try to average forms on $M$ to make them descend to $N$, this won't work unless we have a finite group that acts transitively on the fibers. Instead we do the averaging in $N$. If $\omega \in \Omega^{p}(M)$ and $y \in N$ is covered by the points $x_{i} \in M, i=1, \ldots, k$, then we can push each of the linear forms $\left.\omega\right|_{x_{i}}$ on $T_{x_{i}} M$ via $\left.D F\right|_{x_{i}}$ to a linear $p$-form on $T_{y} N$ and then define

$$
\left.\tau(\omega)\right|_{y}=\left.\frac{1}{k} \sum\left(\left(\left.D F\right|_{x_{i}}\right)^{-1}\right)^{*} \omega\right|_{x_{i}}
$$

This yields a smooth form as each point in $N$ is evenly covered by $k$ diffeomorphic sets. The composition property is immediate and the commutation with $d$ follows from the fact that $d$ commutes with pull backs of maps, in this case the locally defined inverse of $F$.

### 7.2. Examples of Cohomology Groups

We calculate the cohomology of spheres and projective spaces in two ways. First the traditional way using Mayer-Vietoris and then by a completely different approach using the large group of symmetries on these spaces.
7.2.1. Spheres. For $S^{n}$ we use that

$$
\begin{aligned}
S^{n} & =\left(S^{n}-\{p\}\right) \cup\left(S^{n}-\{-p\}\right), \\
S^{n}-\{ \pm p\} & \simeq \mathbb{R}^{n}, \\
\left(S^{n}-\{p\}\right) \cap\left(S^{n}-\{-p\}\right) & \simeq \mathbb{R}^{n}-\{0\} .
\end{aligned}
$$

Since $\mathbb{R}^{n}-\{0\}$ deformation retracts onto $S^{n-1}$ this allows us to compute the cohomology of $S^{n}$ by induction using the Mayer-Vietoris sequence. We start with $S^{1}$, which is a bit different as the intersection has two components. The Mayer-Vietoris sequence starting with $p=0$ looks like

$$
0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow H^{1}\left(S^{1}\right) \rightarrow 0
$$

Showing that $H^{1}\left(S^{1}\right) \simeq \mathbb{R}$. For $n \geq 2$ the intersection is connected so the connecting homomorphism

$$
H^{p-1}\left(S^{n-1}\right) \rightarrow H^{p}\left(S^{n}\right)
$$

must be an isomorphism for $p \geq 1$. Thus

$$
H^{p}\left(S^{n}\right)= \begin{cases}0, & p \neq 0, n \\ \mathbb{R}, & p=0, n\end{cases}
$$

7.2.2. Projective Spaces. For $\mathbb{P}^{n}$ we use the decomposition

$$
\mathbb{P}^{n}=\left(\mathbb{P}^{n}-\mathbb{P}^{n-1}\right) \cup\left(\mathbb{P}^{n}-p\right),
$$

where

$$
\begin{aligned}
p & =[1: 0: \cdots: 0] \\
\mathbb{P}^{n-1} & =\mathbb{P}\left(p^{\perp}\right)=\left\{\left[0: z^{1}: \cdots: z^{n}\right] \mid\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{F}^{n}-\{0\}\right\},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\mathbb{P}^{n}-p & =\left\{\left[z: z^{1}: \cdots: z^{n}\right] \mid\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{F}^{n}-\{0\} \text { and } z \in \mathbb{F}\right\} \simeq \mathbb{P}^{n-1}, \\
\mathbb{P}^{n}-\mathbb{P}^{n-1} & =\left\{\left[1: z^{1}: \cdots: z^{n}\right] \mid\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{F}^{n}\right\} \simeq \mathbb{F}^{n}, \\
\left(\mathbb{P}^{n}-\mathbb{P}^{n-1}\right) \cap\left(\mathbb{P}^{n}-p\right) & =\left\{\left[1: z^{1}: \cdots: z^{n}\right] \mid\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{F}^{n}-\{0\}\right\} \simeq \mathbb{F}^{n}-\{0\} .
\end{aligned}
$$

We have already identified $\mathbb{P}^{1}$ so we don't need to worry about having a disconnected intersection when $\mathbb{F}=\mathbb{R}$ and $n=1$. Using that $\mathbb{F}^{n}-\{0\}$ deformation retracts to the unit sphere $S$ of dimension $\operatorname{dim}_{\mathbb{R}} \mathbb{F}^{n}-1$ we see that the Mayer-Vietoris sequence reduces to

$$
\begin{aligned}
0 & \rightarrow H^{1}\left(\mathbb{P}^{n}\right) \rightarrow H^{1}\left(\mathbb{P}^{n-1}\right) \rightarrow H^{1}(S) \rightarrow \cdots \\
\cdots & \rightarrow H^{p-1}(S) \rightarrow H^{p}\left(\mathbb{P}^{n}\right) \rightarrow H^{p}\left(\mathbb{P}^{n-1}\right) \rightarrow H^{p}(S) \rightarrow \cdots
\end{aligned}
$$

for $p \geq 2$. To get more information we need to specify the scalars and in the real case even distinguish between even and odd $n$. First assume that $\mathbb{F}=\mathbb{C}$. Then $S=S^{2 n-1}$ and $\mathbb{C P}^{1} \simeq S^{2}$. A simple induction then shows that

$$
H^{p}\left(\mathbb{C P}^{n}\right)=\left\{\begin{array}{lc}
0, \quad p=1,3, \ldots, 2 n-1 \\
\mathbb{R}, & p=0,2,4, \ldots, 2 n
\end{array}\right.
$$

When $\mathbb{F}=\mathbb{R}$, we have $S=S^{n-1}$ and $\mathbb{R} \mathbb{P}^{1} \simeq S^{1}$. This shows that $H^{p}\left(\mathbb{R} \mathbb{P}^{n}\right)=0$ when $p=1, \ldots, n-2$. The remaining cases have to be extracted from the last part of the sequence

$$
0 \rightarrow H^{n-1}\left(\mathbb{R}^{p}\right) \rightarrow H^{n-1}\left(\mathbb{R} \mathbb{P}^{n-1}\right) \rightarrow H^{n-1}\left(S^{n-1}\right) \rightarrow H^{n}\left(\mathbb{R P}^{n}\right) \rightarrow 0
$$

where we know that

$$
H^{n-1}\left(S^{n-1}\right)=\mathbb{R}
$$

This shows that $H^{n}\left(\mathbb{R P}^{n}\right)$ is either 0 or $\mathbb{R}$. Next we observe that the natural map

$$
H^{k}\left(\mathbb{R} \mathbb{P}^{n}\right) \rightarrow H^{k}\left(S^{n}\right)
$$

is an injection by lemma 7.1.6. This means that we obtain the simpler exact sequence

$$
0 \rightarrow H^{n-1}\left(\mathbb{R}^{n-1}\right) \rightarrow H^{n-1}\left(S^{n-1}\right) \rightarrow H^{n}\left(\mathbb{R} \mathbb{P}^{n}\right) \rightarrow 0
$$

From this we conclude that $H^{n}\left(\mathbb{R} \mathbb{P}^{n}\right)=0$ iff $H^{n-1}\left(\mathbb{R} \mathbb{P}^{n-1}\right)=\mathbb{R}$. Given that $H^{1}\left(\mathbb{R} \mathbb{P}^{1}\right)=\mathbb{R}$ we then obtain the cohomology groups:

$$
\begin{gathered}
H^{p}\left(\mathbb{R}^{2 n}\right)= \begin{cases}0, & p \geq 1, \\
\mathbb{R}, & p=0,\end{cases} \\
H^{p}\left(\mathbb{R} \mathbb{P}^{2 n+1}\right)=\left\{\begin{array}{lc}
0, & 2 n \geq p \geq 1, \\
\mathbb{R}, & p=0,2 n+1 .
\end{array}\right.
\end{gathered}
$$

7.2.3. Invariant Cohomology. There is a very powerful general principle that allows us to calculate the cohomology of all of the above spaces and more using only homotopy invariance.

The general set-up is a manifold $M$ with an action by a group $G$ of diffeomorphisms. The action of each group element $g \in G$ will be denoted by

$$
\begin{aligned}
A_{g}: M & \rightarrow M \\
x & \mapsto A_{g}(x)=g x .
\end{aligned}
$$

The $G$-invariant $p$-forms are defined by

$$
\Omega_{G}^{p}(M)=\left\{\omega \in \Omega^{p}(M) \mid A_{g}^{*} \omega=\omega \text { for all } g \in G\right\} .
$$

As $A_{g}^{*} \circ d=d \circ A_{g}^{*}$ we obtain a complex

$$
\Omega_{G}^{0}(M) \xrightarrow{d} \Omega_{G}^{1}(M) \xrightarrow{d} \Omega_{G}^{2}(M) \xrightarrow{d} \cdots
$$

and a corresponding $G$-invariant cohomology

$$
H_{G}^{p}(M)=\frac{\operatorname{ker}\left(\Omega_{G}^{p}(M) \xrightarrow{d} \Omega_{G}^{p+1}(M)\right)}{\operatorname{im}\left(\Omega_{G}^{p-1}(M) \xrightarrow{d} \Omega_{G}^{p}(M)\right)} .
$$

The inclusion

$$
\Omega_{G}^{p}(M) \rightarrow \Omega^{p}(M)
$$

induces a natural map

$$
H_{G}^{p}(M) \rightarrow H^{p}(M)
$$

which need not be an isomorphism or even an injection.
Example 7.2.1. On $\mathbb{R}$ consider the action that translates by integers $\mathbb{Z} \subset \mathbb{R}$. The invariant 1 -forms are simply the forms $f(x) d x$ where $f$ is a function with period 1 . For such a form to be exact with respect to invariant forms requires that $f d x=d h$ for some function $h$ with period 1 . This however implies that

$$
\int_{0}^{1} f d x=h(1)-h(0)=0 .
$$

So if $f \equiv 1$, then $[d x] \in H_{\mathbb{Z}}^{1}(\mathbb{R})$ creates a nontrivial cohomology class which becomes trivial in $H^{1}(\mathbb{R})$. In this case we have in fact that

$$
H_{\mathbb{Z}}^{*}(\mathbb{R}) \simeq H^{*}(\mathbb{R} / \mathbb{Z})=H^{*}\left(S^{1}\right)
$$

THEOREM 7.2.2. If $G$ is a compact Lie group, in particular a finite group, then $H_{G}^{p}(M) \rightarrow H^{p}(M)$ is an injection. Moreover, if in addition $G \subset G^{*}$, where $G^{*}$ is a connected Lie group that also acts on $M$, then $H_{G}^{p}(M) \rightarrow H^{p}(M)$ is an isomorphism.

Proof. Select a left invariant volume form $\operatorname{vol}_{G}$ on $G$. By compactness we can assume for simplicity that $\int_{G} \mathrm{vol}_{G}=1$. On a finite group integration is merely averaging over the elements in the group.

Integration of vector valued functions on $G$ allows us to create a left inverse to the inclusion $\Omega_{G}^{p}(M) \rightarrow \Omega^{p}(M)$. For $\omega \in \Omega^{p}(M)$ fix a point $x \in M$ and average $\left.\left(A_{g}^{*} \omega\right)\right|_{x}$ over $g \in G$ :

$$
\left.\bar{\omega}\right|_{x}=\left.\int_{G}\left(A_{g}^{*} \omega\right)\right|_{x} \operatorname{vol}_{G}
$$

When $\omega \in \Omega_{G}^{p}(M)$ it follows that

$$
\left.\bar{\omega}\right|_{x}=\left.\int_{G}\left(A_{g}^{*} \omega\right)\right|_{x} \operatorname{vol}_{G}=\left.\int_{G} \omega\right|_{x} \operatorname{vol}_{G}=\left.\omega\right|_{x} \int_{G} \operatorname{vol}_{G}=\left.\omega\right|_{x} .
$$

Thus averaging really is a left inverse. To check that the averaged form is invariant we have to use that the volume form is left invariant:

$$
\begin{aligned}
A_{h}^{*} \bar{\omega} & =\int_{G} A_{h}^{*} A_{g}^{*} \omega \operatorname{vol}_{G} \\
& =\int_{G} A_{h g}^{*} \omega l_{h}^{*} \operatorname{vol}_{G} \\
& =\int_{G} A_{l_{h g}}^{*} \omega l_{h}^{*} \operatorname{vol}_{G} \\
& =\int_{G}\left(A_{g}^{*} \omega\right) \operatorname{vol}_{G}
\end{aligned}
$$

Finally note that averaging also commutes with the exterior derivative on forms

$$
d \bar{\omega}=\int_{G} d A_{g}^{*} \omega \operatorname{vol}_{G}=\int_{G} A_{g}^{*} d \omega \operatorname{vol}_{G}=\overline{d \omega} .
$$

Thus averaging also induces a left inverse in cohomology. In particular, the induced map $H_{G}^{p}(M) \rightarrow H^{p}(M)$ is an injection.

When the elements $g \in G \subset G^{*}$ are part of a larger connected group, a path from $g$ to $e$ creates a homotopy from $A_{g}$ to $A_{e}=i d_{M}$. Thus the cohomology classes satisfy $\left[A_{g}^{*} \omega\right]=[\omega]$ for all $g \in G$. This shows that $[\bar{\omega}]=[\omega]$ and in particular that $H_{G}^{p}(M) \rightarrow H^{p}(M)$ is an isomorphism.

EXAMPLE 7.2.3. On $S^{2 n}$ the antipodal map $A$ is orientation reversing. Since $A^{2}=i d_{S^{2 n}}$ we can average over the group $\left\{i d_{S^{2 n}}, A\right\}$. For any volume form $\omega \in \Omega^{2 n}\left(S^{2 n}\right)$ we have $\int_{S^{2 n}} A^{*} \omega=-\int_{S^{2 n}} \omega$. Thus averaging volume forms simply results in a form that integrates to zero. As we shall see in section 7.5.1 this implies that the cohomology class of an averaged volume form vanishes.

In order to calculate the cohomology of some basic examples it is convenient to reduce the task. We will consider manifolds $M$ with a transitive action of a compact connected Lie group $G$, i.e., for each $x, y \in M$ there exists $g \in G$ such that $A_{g} x=g x=y$. The isotropy at a fixed point $x \in M$ is the closed subgroup $H=\left\{g \in G \mid A_{g} x=g x=x\right\}$. Since each $A_{h}$ fixes $x$, when $h \in H$, the differential $\left.D A_{h}\right|_{x}$ acts on $T_{x} M$. Thus $H$ induces a linear action on $T_{x} M$. Since the action is transitive any $G$-invariant form $\omega$ is completely determined by its value at $x$. The linear form $\left.\omega\right|_{x}$ on $T_{x} M$ is in addition invariant under the action of $H$ on $T_{x} M$ :

$$
\left.\omega\right|_{x}\left(v_{1}, \ldots, v_{p}\right)=\left.\omega\right|_{x}\left(\left.D A_{h}\right|_{x}\left(v_{1}\right), \ldots,\left.D A_{h}\right|_{x}\left(v_{p}\right)\right)
$$

for all $v_{1}, \ldots, v_{p} \in T_{x} M$ and $h \in H$. We claim that the converse is also true, i.e., any linear form $\omega_{x}$ on $T_{x} M$ that is invariant under the action of $H$ extends to a $G$-invariant form on $M$. Note that if $y=g x=g^{\prime} x$, then $g^{\prime}=g h$, where $h=g^{-1} g^{\prime} \in H$. We define

$$
\left.\omega\right|_{y}\left(\left.D A_{g}\right|_{x}\left(v_{1}\right), \ldots,\left.D A_{g}\right|_{x}\left(v_{p}\right)\right)=\omega_{x}\left(v_{1}, \ldots, v_{p}\right)
$$

or more succinctly $\left.\left.\omega\right|_{y} \circ D A_{g}\right|_{x}=\omega_{x}$ or equivalently $\left.\omega\right|_{y}=\left.\omega_{x} \circ D A_{g^{-1}}\right|_{y}$. This is welldefined as

$$
\begin{aligned}
\left.\left.\omega\right|_{y} \circ D A_{g}\right|_{x} & =\omega_{x} \\
& =\left.\omega_{x} \circ D A_{h}\right|_{x} \\
& =\left.\left.\left.\omega\right|_{y} \circ D A_{g}\right|_{x} \circ D A_{h}\right|_{x} \\
& =\left.\left.\omega\right|_{y} \circ D A_{g h}\right|_{x} .
\end{aligned}
$$

It is also easy to see that it is a $G$-invariant form.
Example 7.2.4. We can use the action of $S O(n+1)$ on $S^{n}$ to calculate the cohomology of spheres. The action is by orthogonal transformations of unit vectors in $\mathbb{R}^{n+1}$ that transform the standard basis $e_{0}, \ldots, e_{n}$ to the other positively oriented orthonormal bases of $\mathbb{R}^{n+1}$. In particular, the action is transitive on $S^{n}$. We fix $x=e_{0}$ as the first basis vector in the ambient Euclidean space. The elements of $S O(n+1)$ that fix $x$ can be identified with $S O(n)$. So we consider the action of $S O(n)$ on $T_{x} S^{n}=x^{\perp}=\operatorname{span}\left\{e_{1}, \ldots, e_{n}\right\}$.

This reduces the problem to checking which constant coefficient $p$-forms

$$
\omega=\sum_{i_{1}<\cdots<i_{p}} \omega_{i_{1} \cdots i_{p}} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{p}}, \omega_{i_{1} \cdots i_{p}} \in \mathbb{R}
$$

on $\mathbb{R}^{n}$ are invariant under $S O(n)$. Clearly constant functions and the standard volume form have this property. So we have to consider the case where $0<p<n$. Evaluating on $e_{i_{1}}, \ldots, e_{i_{p}}$ we can select $j \neq i_{1}, \ldots, i_{p}$ and an element $g \in S O(n)$ such that $g\left(e_{i_{1}}\right)=-e_{i_{1}}$, $g\left(e_{j}\right)=-e_{j}, g\left(e_{i}\right)=e_{i}$, for $i \neq i_{1}, j$. This shows that

$$
\omega_{i_{1} \cdots i_{p}}=\omega\left(e_{i_{1}}, \ldots, e_{i_{p}}\right)=\omega\left(-e_{i_{1}}, \ldots, e_{i_{p}}\right)=-\omega_{i_{1} \cdots i_{p}} .
$$

Thus all linear $p$-forms that are $S O(n)$ invariant vanish.
We conclude that $\Omega_{S O(n+1)}^{0}\left(S^{n}\right)=\mathbb{R}, \Omega_{S O(n+1)}^{p}\left(S^{n}\right)=0$, for $0<p<n$, and $\Omega_{S O(n+1)}^{n}\left(S^{n}\right)=$ $\mathbb{R}$ with a generator

$$
\sum_{i}(-1)^{i} d x^{0} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n}
$$

This generator restricts to the standard volume form $d x^{1} \wedge \cdots \wedge d x^{n}$ on $T_{x} S^{n}$. The invariant 0 -forms are clearly all closed. This shows that the invariant $n$-forms are not exact when $n=1$. For $n>1$ the $n$-forms can't be exact as there are no nontrivial invariant $(n-1)$ forms. This calculates the cohomology of $S^{n}$ and agrees with the previous calculations:

$$
H^{*}\left(S^{n}\right) \simeq H_{S O(n+1)}^{*}\left(S^{n}\right)=\Omega_{S O(n+1)}^{*}\left(S^{n}\right)
$$

EXAMPLE 7.2.5. The previous example can be used to calculate the cohomology of $\mathbb{R}^{\mathbb{P}^{n}}$ also using the action of $S O(n+1)$. We will think of points $x \in \mathbb{R}^{p n}$ as antipodal pairs $\{ \pm y\} \in S^{n}$. In this way $T_{x} \mathbb{R}^{p}$ also becomes equivalence classes in $T_{ \pm y} S^{n}=\{ \pm y\}^{\perp}$ where $(y, v) \in S^{n} \times\{y\}^{\perp} \subset S^{n} \times \mathbb{R}^{n+1}$ is identified with $(-y,-v) \in S^{n} \times\{-y\}^{\perp}$. The action of $g \in S O(n+1)$ on $S^{n}$ becomes an action on $T S^{n}$ :

$$
g \cdot(y, v)=\left(A_{g} y, D A_{g} v\right)=(g y, g v) .
$$

As $-g \cdot(y, v)=g \cdot(-y,-v)$ this also tells us how $g$ acts on $T \mathbb{R} \mathbb{P}^{n}$ :

$$
g \cdot( \pm y, \pm v)=( \pm g y, \pm g v)
$$

As for $S^{n}$ it follows that $S O(n)$ fixes the point $x=\left\{ \pm e_{0}\right\}$, however, the full isotropy consists of $S(O(1) \times O(n))$ which has two components. The other component consists of the
orthogonal transformations that send $e_{0}$ to $-e_{0}$ and act on span $\left\{e_{1}, \ldots, e_{n}\right\}$ with determinant -1 .

We thus immediately obtain: $\Omega_{S O(n+1)}^{0}\left(\mathbb{R}^{n}\right)=\mathbb{R}, \Omega_{S O(n+1)}^{p}\left(\mathbb{R} \mathbb{P}^{n}\right)=0$, for $0<p<n$. For $\Omega_{S O(n+1)}^{n}\left(\mathbb{R} \mathbb{P}^{n}\right)$ the answer depends on the parity of $n$. In case $n=2 k+1$ is odd, the antipodal map $-I \in S O(2 k+2)$ acts trivially on $\mathbb{R} \mathbb{P}^{2 k+1}$. The isotropy of $x$ thus consists of elements $g \in S O(2 k+1)$ and $-I g \in S(O(1) \times O(2 k+1))$ in the other component. As the actions of $g$ and $-I g$ are the same on $\mathbb{R P}^{2 k+1}$ we only need to consider the linear action of $S O(n)$ on $\mathbb{R}^{n}$ in order to understand the linear action of the isotropy on $T_{x} \mathbb{R} \mathbb{P}^{2 k+1}$. As for the sphere we conclude that $\Omega_{S O(2 k+2)}^{2 k+1}\left(\mathbb{R P}^{2 k+1}\right)=\mathbb{R}$. When $n=2 k$ is even the antipodal map is not an element of $S O(2 k+1)$ and thus the action on $\mathbb{R} \mathbb{P}^{2 k}$ is effective, i.e., no element fixes all points. This means that the isotropy $S(O(1) \times O(2 k))$ is also effective and in particular contains an element that is orientation reversing on $T_{x} \mathbb{R}^{2} \mathbb{P}^{2 k}$, e.g., the element that maps $e_{0}$ to $-e_{0}, e_{1}$ to $-e_{1}$, and fixes all other basis vectors (on $\mathbb{R} \mathbb{P}^{2 k+1}$ this map preserves the orientation!). Since the linear volume form on $\mathbb{R}^{n}$ is not preserved by orientation reversing orthogonal transformations we conclude that $\Omega_{S O(2 k+1)}^{2 k}\left(\mathbb{R}^{2 k}\right)=0$. All in all we have shown that

$$
H^{p}\left(\mathbb{R}^{P^{n}}\right) \simeq H_{S O(n+1)}^{p}\left(\mathbb{R}^{n}\right)=\Omega_{S O(n+1)}^{p}\left(\mathbb{R}^{p}\right)
$$

EXAMPLE 7.2.6. On complex projective space $\mathbb{P}^{n}$ we can use the transitive action of $U(n+1)$ that maps complex lines to complex lines in $\mathbb{C}^{n+1}$. We use the standard complex basis $c_{0}, \ldots, c_{n}$ which gives a real basis $e_{0}, f_{0}, \ldots, e_{n}, f_{n}$, where $c_{i}=e_{i}+\sqrt{-1} f_{i}$. The isotropy at $x=\operatorname{span}\left\{c_{0}\right\}=[1: 0: \cdots: 0]$ is $U(1) \times U(n)$, where $U(1)$ acts trivially as it is simply multiplication by complex scalars on span $\left\{c_{0}\right\}$ and $U(n)$ acts on the tangent space in the way $U(n)$ acts on $\mathbb{C}^{n}=\operatorname{span}\left\{c_{1}, \ldots c_{n}\right\}$.

We can then again simply check which constant coefficient forms on $\mathbb{C}^{n}$ are invariant under $U(n)$. Consider the unitary transformations $g_{i} \in U(n)$ such that $g_{i}\left(c_{i}\right)=-c_{i}$ and $g_{i}\left(c_{j}\right)=c_{j}$ for $j \neq i$. Using these transformations it follows that a $(p+q)$-form, $0<$ $p+q<2 n$, vanishes if it is evaluated on $e_{i_{1}}, \ldots, e_{i_{p}}, f_{j_{1}}, \ldots, f_{j_{q}}$ where one of $e_{i_{k}}$ (resp. $f_{j_{l}}$ ) in the collection does not have its partner $f_{i_{k}}$ (resp. $e_{j_{l}}$ ) in the collection. This means that we can restrict attention to the cases where $p=q$ and $i_{1}=j_{1}, \ldots, i_{p}=j_{p}$. However, in all of these cases the value $\omega\left(e_{i_{1}}, \ldots, e_{i_{p}}, f_{i_{1}}, \ldots, f_{i_{p}}\right)$ must be the same as permutations of the complex basis vectors $c_{1}, \ldots, c_{n}$ are also unitary transformations.

This shows that $\Omega_{U(n+1)}^{2 p+1}\left(\mathbb{P}^{n}\right)=0$ and $\Omega_{U(n+1)}^{2 p}\left(\mathbb{P}^{n}\right)=\mathbb{R}$ with a generator that when restricted to $T_{x} \mathbb{P}^{n}$ is given by

$$
\left(\sum_{i=1}^{n} d x^{i} \wedge d y^{i}\right)^{p}=\binom{n}{p} \sum_{i_{1}<\cdots<i_{p}} d x^{i_{1}} \wedge d y^{i_{1}} \wedge \cdots \wedge d x^{i_{p}} \wedge d y^{i_{p}}
$$

Since there are no invariant forms of odd degree all of the invariant forms of even degree are closed but not exact. This calculates the cohomology of complex projective space and in addition gives us generators $\left[\omega^{p}\right] \in H^{2 p}$ that can be calculated from any generator $[\omega] \in H^{2}:$

$$
H^{*}\left(\mathbb{P}^{n}\right) \simeq H_{U(n+1)}^{*}\left(\mathbb{P}^{n}\right)=\Omega_{U(n+1)}^{*}\left(\mathbb{P}^{n}\right)
$$

EXAMPLE 7.2.7. The final example will be the torus $T^{n}=S^{1} \times \cdots \times S^{1} \subset \mathbb{C}^{n}$ which acts on itself via multiplication of unit complex numbers in each factor. This action is also transitive but has trivial isotropy. The 1 -forms $d \theta^{i}, i=1, \ldots, n$ that are the standard volume forms on the factors are invariant under this action as are all of their wedge products. This
shows that

$$
\Omega_{T^{n}}^{p}\left(T^{n}\right)=\operatorname{span}\left\{d \theta^{i_{1}} \wedge \cdots \wedge d \theta^{i_{p}} \mid 1 \leq i_{1}<\cdots<i_{p} \leq n\right\}
$$

As these forms are all closed we have calculated the cohomology of $T^{n}$ to be

$$
H^{*}\left(T^{n}\right) \simeq H_{T^{n}}^{*}\left(T^{n}\right)=\Omega_{T^{n}}^{*}\left(T^{n}\right)
$$

Based on this last example it is tempting to think that the cohomology of a connected compact Lie group is equally simple. It is true that we can select a basis of left invariant 1forms and their wedge products to obtain similar spaces for $\Omega_{G}^{p}(G)$. However, these forms are not necessarily closed. For example, if we evaluate the differential of a left invariant 1-form on left invariant fields we obtain

$$
d \omega(X, Y)=D_{X} \omega(Y)-D_{Y} \omega(X)-\omega([X, Y])=-\omega([X, Y]) .
$$

Thus the differential is dictated by the Lie algebra, which in case of the torus was Abelian. Nevertheless with a good choice of basis for the Lie algebra it does become possible to calculate the cohomology.

### 7.3. Axiomatic Cohomology

In this section we specify the most basic properties of cohomology theories for manifolds.

In section 7.1 we introduced the functor

$$
M \mapsto H^{*}(M)=H^{0}(M) \oplus H^{1}(M) \oplus \cdots \oplus H^{n}(M)
$$

that maps an $n$-manifold to a graded vector space. The morphisms are the smooth maps between manifolds and the functor is contravariant as $F: M \rightarrow N$ induces a pull-back $F^{*}: H^{*}(N) \rightarrow H^{*}(M)$. Pull-back maps are natural in the sense that $(G \circ F)^{*}=F^{*} \circ G^{*}$. We established the basic, but not elementary, properties:

- (Point Axiom)

$$
H^{*}(\{p\})=H^{0}(\{p\})=\mathbb{R}
$$

- (Mayer-Vietoris) If $A, B \subset M$ are open subsets, then we obtain a connecting homomorphism $\delta: H^{*}(A \cap B) \rightarrow H^{*+1}(A \cup B)$ that yields a long exact sequence

$$
\cdots \rightarrow H^{*}(A \cup B) \rightarrow H^{*}(A) \oplus H^{*}(B) \rightarrow H^{*}(A \cap B) \rightarrow H^{*+1}(A \cup B) \rightarrow \cdots
$$

- (Homotopy Invariance) The projection $\pi: \mathbb{R} \times M \rightarrow M$ induces an isomorphism

$$
H^{*}(M) \rightarrow H^{*}(\mathbb{R} \times M)
$$

The inclusions $j_{t}: M \rightarrow \mathbb{R} \times M$ given by $j_{t}(x)=(t, x)$ all induce inverses to $\pi$ in cohomology.
The last statement clearly follows from homotopy invariance of de Rham cohomology (proposition7.1.3). It is also a slightly better formulation of homotopy invariance as it also works when $M$ has boundary.

The above properties hold for all cohomology theories on topological spaces and essentially characterizes them. On manifolds, or even just for all (orientable) $n$-manifolds, we can simplify these axioms to better align with theorem 1.3.11. This will also guide us in how to establish several isomorphism results.

First we narrow down the category. The objects can be all manifolds, all $n$-manifolds, or all oriented $n$-manifolds. The morphisms are the inclusion maps $A \subset M$ of open sets in manifolds. A cohomology functor on manifolds and inclusions

$$
M \mapsto \mathscr{H}^{*}(M)=\mathscr{H}^{0}(M) \oplus \mathscr{H}^{1}(M) \oplus \cdots \oplus \mathscr{H}^{n}(M)
$$

is now only natural under inclusions:

$$
j_{A \subset B}^{*} \circ j_{B \subset M}^{*}=\left(j_{B \subset M} \circ j_{A \subset B}\right)^{*}=\left(j_{A \subset M}\right)^{*} .
$$

We impose the additional requirement that the cohomologies of diffeomorphic manifolds are isomorphic. The modified axioms for (oriented) $n$-manifolds now become:
(1) (Poincaré Lemma)

$$
\mathscr{H}^{*}\left(\mathbb{R}^{n}\right)=\mathscr{H}^{0}\left(\mathbb{R}^{n}\right)=\mathbb{R}
$$

(2) (Mayer-Vietoris) If $A, B \subset M$ are open subsets, then we obtain a connecting homomorphism $\delta: \mathscr{H}^{*}(A \cap B) \rightarrow \mathscr{H}^{*+1}(A \cup B)$ that yields a long exact sequence

$$
\cdots \rightarrow \mathscr{H}^{*}(A \cup B) \rightarrow \mathscr{H}^{*}(A) \oplus \mathscr{H}^{*}(B) \rightarrow \mathscr{H}^{*}(A \cap B) \rightarrow \mathscr{H}^{*+1}(A \cup B) \rightarrow \cdots .
$$

(3) (Countable Disjointness) If $A_{\alpha} \subset M$ form a countable collection of pairwise disjoint open subsets, then

$$
\mathscr{H}^{*}\left(\bigcup_{\alpha} A_{\alpha}\right)=\times_{\alpha} \mathscr{H}^{*}\left(A_{\alpha}\right)
$$

We've already established all but the last property for de Rham cohomology. For last last property just note that any form on the union naturally restricts to forms on each of the open sets. Moreover, the form is exact iff it is exact on each set $A_{\alpha}$.

The last axiom is really a countable version of Mayer-Vietoris for disjoint sets and is necessary for when we wish to prove results for general noncompact manifolds. Note that there is no homotopy invariance axiom, but the three axioms together actually imply homotopy invariance. In fact theorem 1.3.11 immediately implies that these properties uniquely determines cohomology on $n$-manifolds. This is also known as the de Rham isomorphism theorem.

THEOREM 7.3.1. Consider a cohomology theory $\mathscr{H}^{*}$ on (oriented) n-manifolds that satisfies (1),(2), (3) and a natural map $\mathscr{H}^{*}(M) \rightarrow H^{*}(M)$ that respects inclusions:


The map $\mathscr{H}^{*}(M) \rightarrow H^{*}(M)$ is an isomorphism for all (oriented) $n$-manifolds provided it is an isomorphism when $M=\mathbb{R}^{n}$.

Proof. Given that $\mathrm{P}\left(\mathbb{R}^{n}\right)$ is true we still need to establish the other two conditions in theorem 1.3.11. Assume that $A, B \subset M$ are open and that $P(A), P(B)$, and $P(A \cap B)$ are true. Using the Mayer-Vietoris property we obtain

$$
\begin{array}{ccccccccc}
\mathscr{H}^{*-1}(A) \oplus \mathscr{H}^{*-1}(B) & \rightarrow & \mathscr{H}^{*-1}(A \cap B) & \rightarrow & \mathscr{H}^{*}(A \cup B) & \rightarrow & \mathscr{H}^{*}(A) \oplus \mathscr{H}^{*}(B) & \rightarrow & \mathscr{H}^{*}(A \cap B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^{*-1}(A) \oplus H^{*-1}(B) & \rightarrow & H^{*-1}(A \cap B) & \rightarrow & H^{*}(A \cup B) & \rightarrow & H^{*}(A) \oplus H^{*}(B) & \rightarrow & H^{*}(A \cap B)
\end{array}
$$

Each square in this diagram is commutative and all vertical arrows, except for the middle one, are assumed to be isomorphisms. It follows by a simple diagram chase that the middle arrow is also an isomorphism. More precisely, the five lemma asserts that if we have a commutative diagram:

$$
\begin{array}{ccccccccc}
A_{1} & \rightarrow & A_{2} & \rightarrow & A_{3} & \rightarrow & A_{4} & \rightarrow & A_{5} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_{1} & \rightarrow & B_{2} & \rightarrow & B_{3} & \rightarrow & B_{4} & \rightarrow & B_{5}
\end{array}
$$

where the two horizontal rows are exact and $A_{i} \rightarrow B_{i}$ are isomorphisms for $i=1,2,4,5$, then $A_{3} \rightarrow B_{3}$ is an isomorphism.

Finally assume that $A_{\alpha} \subset M$ are pairwise disjoint open sets and $P\left(A_{\alpha}\right)$ are true. It follows that

$$
\mathscr{H}^{*}\left(\bigcup A_{\alpha}\right)=\times \mathscr{H}^{*}\left(A_{\alpha}\right) \simeq \times H^{*}\left(A_{\alpha}\right)=H^{*}\left(\bigcup A_{\alpha}\right)
$$

and $P\left(\bigcup A_{\alpha}\right)$ is true.
REMARK 7.3.2. Note that condition that $\mathscr{H}^{*}\left(\mathbb{R}^{n}\right) \rightarrow H^{*}\left(\mathbb{R}^{n}\right)$ is an isomorphism is almost trivial as the two vector spaces are isomorphic to $\mathbb{R}$. Thus the natural map merely has to be nontrivial for it to become an isomorphism.

The squares in the five-lemma were all assumed to be commutative. This depends on how the horizontal maps in the Mayer-Vietoris sequence are defined. It can happen that a sign makes the squares anti-commute, but this does not affect the validity of the statement.

### 7.4. Generalized Cohomology Theories

We introduce several cohomology theories that can assist in calculating the cohomology of spaces. Only compactly supported cohomology is needed for subsequent sections.
7.4.1. Compactly Supported Cohomology. Compactly supported cohomology is not a cohomology theory in the sense of theorem 7.3.1. In the next section we will see how it can be dualized to better fit in with cohomology. Here we establish the basic properties.

DEFINITION 7.4.1. Compactly supported cohomology is defined as follows: Let $\Omega_{c}^{p}(M)$ denote the compactly supported $p$-forms. With this we have the compactly supported exact and closed forms $B_{c}^{p}(M) \subset Z_{c}^{p}(M)$ (note that $d: \Omega_{c}^{p}(M) \rightarrow \Omega_{c}^{p+1}(M)$ ) and define

$$
H_{c}^{p}(M)=\frac{Z_{c}^{p}(M)}{B_{c}^{p}(M)}
$$

Needless to say, for closed manifolds the two cohomology theories are identical. For connected open manifolds, on the other hand, we have that the closed 0 -forms must be zero, as they also have to have compact support. Thus $H_{c}^{0}(M)=\{0\}$ if $M$ has no compact connected components.

Note that only proper maps $F: M \rightarrow N$ have the property that they map $F^{*}: \Omega_{c}^{p}(N) \rightarrow$ $\Omega_{c}^{p}(M)$. In particular, if $A \subset M$ is open, we do not have a restriction map. Instead, we observe that there is a natural inclusion $\Omega_{c}^{p}(A) \rightarrow \Omega_{c}^{p}(M)$, which induces

$$
H_{c}^{p}(A) \rightarrow H_{c}^{p}(M) .
$$

Thus compactly supported cohomology looks more like a homology theory.
We start by establishing a version of the Poincaré lemma for this new cohomology theory.

LEMMA 7.4.2. The compactly supported cohomology of Euclidean space is

$$
H_{c}^{p}\left(\mathbb{R}^{n}\right)= \begin{cases}\mathbb{R} & \text { when } p=n \\ 0 & \text { when } p \neq n\end{cases}
$$

Proof. We focus on the case where $p=n$, the other cases will be handled in a similar way.

First observe that for any oriented $n$-manifold, $M$, the map

$$
\begin{aligned}
\Omega_{c}^{n}(M) & \rightarrow \mathbb{R} \\
\omega & \mapsto \int_{M} \omega
\end{aligned}
$$

vanishes on closed forms by Stokes' theorem. Thus it induces a map

$$
\begin{aligned}
H_{c}^{n}(M) & \rightarrow \mathbb{R}, \\
{[\omega] } & \mapsto \int_{M} \omega .
\end{aligned}
$$

It is also onto, since any form with the property that it is positive when evaluated on a positively oriented frame is integrated to a positive number.

Case 1: $M=S^{n}$. We know that $H^{n}\left(S^{n}\right)=\mathbb{R}$, so $\int: H^{n}\left(S^{n}\right) \rightarrow \mathbb{R}$ must be an isomorphism.

Case 2: $M=\mathbb{R}^{n}$. We can think of $M=S^{n}-\{x\}$. Any compactly supported form $\omega$ on $M$ is thus also a form on $S^{n}$. Given that $\int_{M} \omega=0$, we further note that $\int_{S^{n}} \omega=0$. In particular, $\omega$ must be exact on $S^{n}$. Let $\psi \in \Omega^{n-1}\left(S^{n}\right)$ be chosen such that $d \psi=\omega$. Use again that $\omega$ is compactly supported to find an open disc $U$ around $x$ such that $\omega$ vanishes on $U$ and $U \cup M=S^{n}$. Then $\psi$ is clearly closed on $U$ and must by the Poincaré lemma be exact. Thus, we can find $\theta \in \Omega^{n-2}(U)$ with $d \theta=\psi$ on $U$. This form doesn't necessarily extend to $S^{n}$, but we can select a bump function $\lambda: S^{n} \rightarrow[0,1]$ that vanishes on $S^{n}-U$ and is 1 on some smaller neighborhood $V \subset U$ around $x$. Now observe that $\psi-d(\lambda \theta)$ is actually defined on all of $S^{n}$. It vanishes on $V$ and clearly

$$
d(\psi-d(\lambda \theta))=d \psi=\omega
$$

The case for $p$-forms proceeds in a similar way using that $H^{p}\left(S^{n}\right)=0$ for $1<p<n$. When $p=1$, we obtain $\omega=d \psi$, where $\psi \in \Omega^{0}\left(S^{n}\right)$. Thus $\psi$ is constant in a neighborhood of $x$ and we can use $\psi-\psi(x)$ as a function with compact support in $S^{n}-\{x\}$ whose differential is $\omega$.

Finally $H_{c}^{0}(M)=0$ for all connected non-compact manifolds.

This result together with the fact that compactly supported cohomology respects inclusions of compact sets indicates that for an $n$-manifold we should consider

$$
\mathscr{H}^{p}(M)=\operatorname{Hom}\left(H_{c}^{n-p}(M), \mathbb{R}\right)
$$

This in fact defines a cohomology functor. Clearly, $\mathscr{H}^{*}$ is the same for diffeomorphic manifolds and is contravariant under inclusions:

$$
\mathscr{H}^{p}(M)=\operatorname{Hom}\left(H_{c}^{n-p}(M), \mathbb{R}\right) \rightarrow \operatorname{Hom}\left(H_{c}^{n-p}(A), \mathbb{R}\right)=\mathscr{H}^{p}(A)
$$

The above lemma tells us that the Poincaré lemma holds:

$$
\mathscr{H}^{*}\left(\mathbb{R}^{n}\right)=\mathscr{H}^{0}\left(\mathbb{R}^{n}\right)=\operatorname{Hom}\left(H_{c}^{n}\left(\mathbb{R}^{n}\right), \mathbb{R}\right)=\mathbb{R}
$$

Since $\Omega_{c}^{*}\left(\bigcup_{\alpha} A_{\alpha}\right)=\oplus \alpha \Omega_{c}^{*}\left(A_{\alpha}\right)$ for a pairwise disjoint union of open sets $A_{\alpha} \subset M$, we also obtain

$$
\begin{aligned}
\mathscr{H}^{*}\left(\bigcup_{\alpha} A_{\alpha}\right) & =\operatorname{Hom}\left(H_{c}^{n-*}\left(\left(\bigcup_{\alpha} A_{\alpha}\right)\right), \mathbb{R}\right) \\
& =\operatorname{Hom}\left(\bigoplus_{\alpha} H_{c}^{n-*}\left(\left(A_{\alpha}\right)\right), \mathbb{R}\right) \\
& =\underset{\alpha}{\times \operatorname{Hom}\left(H_{c}^{n-*}\left(\left(A_{\alpha}\right)\right), \mathbb{R}\right)} \\
& =\underset{\alpha}{\times \mathscr{H}^{*}\left(A_{\alpha}\right)}
\end{aligned}
$$

Finally we need a Mayer-Vietoris sequence for open sets $A, B \subset M$ with $M=A \cup B$. This starts with the observation that we have exact sequences:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega_{c}^{*+1}(A \cap B) & \rightarrow & \Omega_{c}^{*+1}(A) \oplus \Omega_{c}^{*+1}(B) & \rightarrow & \Omega_{c}^{*+1}(M) & \rightarrow & 0 \\
& & \uparrow d & & \uparrow & & & & \\
0 & \rightarrow & \Omega_{c}^{*}(A \cap B) & \rightarrow & \Omega_{c}^{*}(A) \oplus \Omega_{c}^{*}(B) & \rightarrow & \Omega_{c}^{*}(M) & \rightarrow & \\
0 & \rightarrow & & &
\end{array}
$$

where the horizontal arrows are defined by:

$$
\begin{aligned}
\Omega_{c}^{*}(A \cap B) & \rightarrow \Omega_{c}^{*}(A) \oplus \Omega_{c}^{*}(B), \\
{[\omega] } & \mapsto([\omega],[\omega]),
\end{aligned}
$$

and

$$
\begin{array}{rll}
\Omega_{c}^{*}(A) \oplus \Omega_{c}^{*}(B) & \rightarrow & \Omega_{c}^{*}(M) \\
\left(\left[\omega_{A}\right],\left[\omega_{B}\right]\right) & \mapsto & {\left[\omega_{A}-\omega_{B}\right] .}
\end{array}
$$

This certainly leads to a long exact Mayer-Vietoris sequence:

$$
\cdots \rightarrow H_{c}^{*}(A \cap B) \rightarrow H_{c}^{*}(A) \oplus H_{c}^{*}(B) \rightarrow H_{c}^{*}(M) \rightarrow H_{c}^{*+1}(A \cap B) \rightarrow \cdots
$$

However, we can also dualize to obtain a short exact sequence that algebraically looks similar (even with the sign choices) to the sequence used for Mayer-Vietoris:
$0 \rightarrow \operatorname{Hom}\left(\Omega_{c}^{n-p}(M), \mathbb{R}\right) \rightarrow \operatorname{Hom}\left(\Omega_{c}^{n-p}(A), \mathbb{R}\right) \oplus \operatorname{Hom}\left(\Omega_{c}^{n-p}(B), \mathbb{R}\right) \rightarrow \operatorname{Hom}\left(\Omega_{c}^{n-p}(A \cap B), \mathbb{R}\right) \rightarrow 0$ and differentials that map

$$
d: \operatorname{Hom}\left(\Omega_{c}^{n-p}(M), \mathbb{R}\right) \rightarrow \operatorname{Hom}\left(\Omega_{c}^{n-p-1}(M), \mathbb{R}\right)=\operatorname{Hom}\left(\Omega_{c}^{n-(p+1)}(M), \mathbb{R}\right)
$$

This gives us a connecting homomorphism $\delta: \mathscr{H}^{*}(A \cap B) \rightarrow \mathscr{H}^{*+1}(M)$ and a long exact sequence

$$
\cdots \rightarrow \mathscr{H}^{*}(M) \rightarrow \mathscr{H}^{*}(A) \oplus \mathscr{H}^{*}(B) \rightarrow \mathscr{H}^{*}(A \cap B) \rightarrow \mathscr{H}^{*+1}(M) \rightarrow \cdots
$$

Finally we can also prove lemma 7.1.6 for compactly supported cohomology.
Lemma 7.4.3. If $F: M \rightarrow N$ is a finite covering map, then

$$
F^{*}: H_{c}^{p}(N) \rightarrow H_{c}^{p}(M)
$$

is an injection.
Proof. The proof uses the same transgression map after we note that it maps $\tau$ : $\Omega_{c}^{p}(M) \rightarrow \Omega_{c}^{p}(N)$ since $F$ takes compact sets to compact sets.
7.4.2. Relative Cohomology. Compactly supported cohomology can be used very effectively to define relative cohomology and also simplifies the calculation of some of the cohomology groups we have seen.

We start with the simplest and most important situation where $S \subset M$ is a closed submanifold of a closed manifold.

PROPOSITION 7.4.4. If $S \subset M$ is a closed submanifold of a closed manifold, then
(1) The restriction map $i^{*}: \Omega^{p}(M) \rightarrow \Omega^{p}(S)$ is surjective.
(2) If $\theta \in \Omega^{p-1}(S)$ is closed, then there exists $\psi \in \Omega^{p-1}(M)$ such that $\theta=i^{*} \psi$ and $d \psi \in \Omega_{c}^{p}(M-S)$.
(3) If $\omega \in \Omega^{p}(M)$ with $d \omega \in \Omega_{c}^{p+1}(M-S)$ and $i^{*} \omega \in \Omega^{p}(S)$ is exact, then there exists $\theta \in \Omega^{p-1}(M)$ such that $\omega-d \theta \in \Omega_{c}^{p}(M-S)$.

Proof. Select a neighborhood $S \subset U \subset M$ that deformation retracts $\pi: U \rightarrow S$. Then $i^{*}: H^{p}(U) \rightarrow H^{p}(S)$ is an isomorphism. We also need a function $\lambda: M \rightarrow[0,1]$ that is compactly supported in $U$ and is 1 on a neighborhood of $S$.

1. Given $\omega \in \Omega^{p}(S)$ let $\bar{\omega}=\lambda \pi^{*}(\omega)$.
2. This also shows that $d\left(\lambda \pi^{*} \theta\right)=d \lambda \wedge \pi^{*} \theta+\lambda d \pi^{*} \theta$ has compact support in $M-S$.
3. Conversely assume that $\omega \in \Omega^{p}(M)$ has $d \omega \in \Omega_{c}^{p+1}(M-S)$. By possibly shrinking $U$ we can assume that it is disjoint from the support of $d \omega$. Thus, $\left.d \omega\right|_{U}=0$ since $i: S \rightarrow U$ is an isomorphism in cohomology and we assume that $i^{*} \omega$ is exact, it follows that $\left.\omega\right|_{U}=$ $d \psi$ for some $\psi \in \Omega^{p-1}(U)$. Define $\theta=\lambda \psi$ and then note that

$$
\begin{aligned}
\omega-d \theta & =\omega-\lambda d \psi-d \lambda \wedge \psi \\
& =\omega-\left.\lambda \omega\right|_{U}-d \lambda \wedge \theta \\
& \in \Omega_{c}^{p}(M-S)
\end{aligned}
$$

THEOREM 7.4.5. Assume $S \subset M$ is a closed submanifold of a closed manifold, then

$$
\rightarrow H_{c}^{p}(M-S) \rightarrow H^{p}(M) \rightarrow H^{p}(S) \rightarrow H_{c}^{p+1}(M-S) \rightarrow
$$

is a long exact sequence of cohomology groups.
Proof. Part (1) of the above proposition shows that we have a short exact sequence

$$
\begin{aligned}
0 & \rightarrow \Omega^{p}(M, S) \rightarrow \Omega^{p}(M) \rightarrow \Omega^{p}(S) \rightarrow 0, \\
\Omega^{p}(M, S) & =\operatorname{ker}\left(i^{*}: \Omega^{p}(M) \rightarrow \Omega^{p}(S)\right) .
\end{aligned}
$$

We claim that (2) and (3) show that the natural inclusion

$$
\Omega_{c}^{p}(M-S) \rightarrow \Omega^{p}(M, S)
$$

induces an isomorphism $H_{c}^{p}(M-S) \rightarrow H^{p}(M, S)$.
To show that it is injective consider $\omega \in \Omega_{c}^{p}(M-S)$, such that $\omega=d \theta$, where $\theta \in$ $\Omega^{p-1}(M, S)$. We can apply (3) to $\theta$ to find $\psi \in \Omega^{p-2}(M)$ such that $\theta-d \psi \in \Omega_{c}^{p-1}(M-S)$. This shows that $\omega=d(\theta-d \psi)$ for a form $\theta-d \psi \in \Omega_{c}^{p-1}(M-S)$.

To show that it is surjective consider $\omega \in \Omega^{p}(M, S)$ with $d \omega=0$. By (3) we can find $\theta \in \Omega^{p-1}(M)$ such that $\omega-d \theta \in \Omega_{c}^{p}(M-S)$, but we don't know that $\theta \in \Omega^{p-1}(M, S)$. To fix that problem use (2) to find $\psi \in \Omega^{p-1}(M)$ such that $i^{*} \theta=i^{*} \psi$ and $d \psi \in \Omega_{c}^{p}(M-S)$. Then $\omega-d(\theta-\psi)=(\omega-d \theta)-d \psi \in \Omega_{c}^{p}(M-S)$ and $\theta-\psi \in \Omega^{p-1}(M, S)$.

Good examples are $S^{n-1} \subset S^{n}$ with $S^{n}-S^{n-1}$ being two copies of $\mathbb{R}^{n}$ and $\mathbb{P}^{n-1} \subset \mathbb{P}^{n}$ where $\mathbb{P}^{n}-\mathbb{P}^{n-1} \simeq \mathbb{F}^{n}$. This gives us a slightly different inductive method for computing the cohomology of these spaces. Conversely, given the cohomology groups of those spaces, it computes the compactly supported cohomology of $\mathbb{R}^{n}$.

It can also be used on manifolds with boundary:

$$
\rightarrow H_{c}^{p}(\operatorname{int} M) \rightarrow H^{p}(M) \rightarrow H^{p}(\partial M) \rightarrow H_{c}^{p+1}(\operatorname{int} M) \rightarrow
$$

where we can specialize to $M=D^{n} \subset \mathbb{R}^{n}$, the closed unit ball. The Poincaré lemma computes the cohomology of $D^{n}$ so we get that

$$
H_{c}^{p+1}\left(B^{n}\right) \simeq H^{p}\left(S^{n-1}\right)
$$

For general connected compact manifolds with boundary we also obtain some interesting information.

THEOREM 7.4.6. If $M$ is a connected compact n-manifold with boundary, then

$$
H^{n}(M)=0 .
$$

Proof. If $M$ is oriented, then we know that $\partial M$ is also oriented and that

$$
\begin{aligned}
H^{n}(M, \partial M) & =H_{c}^{n}(\operatorname{int} M) \simeq \mathbb{R} \\
H^{n}(\partial M) & =\{0\} \\
H^{n-1}(\partial M) & \simeq \mathbb{R}^{k},
\end{aligned}
$$

where $k$ is the number of components of $\partial M$. The connecting homomorphism $H^{n-1}(\partial M) \rightarrow$ $H_{c}^{n}(\operatorname{int} M)$ can be analyzed from the diagram

$$
\left.\begin{array}{cccccccc}
0 & \rightarrow & \Omega^{n}(M, \partial M) & \rightarrow & \Omega^{n}(M) & \rightarrow & \Omega^{n}(\partial M) & \rightarrow
\end{array}\right) 0
$$

Evidently any $\omega \in \Omega^{n-1}(\partial M)$ is the restriction of some $\bar{\omega} \in \Omega^{n-1}(M)$, where we can further assume that $d \bar{\omega} \in \Omega_{c}^{n}(\operatorname{int} M)$. Stokes' theorem then tells us that

$$
\int_{M} d \bar{\omega}=\int_{\partial M} \bar{\omega}=\int_{\partial M} \omega .
$$

This shows that the map $H^{n-1}(\partial M) \rightarrow H_{c}^{n}(\operatorname{int} M)$ is nontrivial and hence surjective, which in turn implies that $H^{n}(M)=\{0\}$.

If $M$ is not orientable then we can use lemma 7.1.6 on the orientation covering.
It is possible to extend the above long exact sequence to the case where $M$ is noncompact by using compactly supported cohomology on $M$. This gives us the long exact sequence

$$
\rightarrow H_{c}^{p}(M-S) \rightarrow H_{c}^{p}(M) \rightarrow H^{p}(S) \rightarrow H_{c}^{p+1}(M-S) \rightarrow
$$

It is even possible to also have $S$ be non-compact if we assume that the embedding is proper and then also use compactly supported cohomology on $S$

$$
\rightarrow H_{c}^{p}(M-S) \rightarrow H_{c}^{p}(M) \rightarrow H_{c}^{p}(S) \rightarrow H_{c}^{p+1}(M-S) \rightarrow
$$

We can generalize even further to a situation where $S$ is simply a compact subset of $M$. In that case we define the deRham-Cech cohomology groups $\check{H}^{p}(S)$ using

$$
\check{\Omega}^{p}(S)=\frac{\left\{\omega \in \Omega^{p}(M)\right\}}{\omega_{1} \sim \omega_{2} \text { iff } \omega_{1}=\omega_{2} \text { on a ngbd of } S},
$$

i.e., the elements of $\check{\Omega}^{p}(S)$ are germs of forms on $M$ at $S$. We now obtain a short exact sequence

$$
0 \rightarrow \Omega_{c}^{p}(M-S) \rightarrow \Omega_{c}^{p}(M) \rightarrow \check{\Omega}^{p}(S) \rightarrow 0 .
$$

This in turn gives us a long exact sequence

$$
\rightarrow H_{c}^{p}(M-S) \rightarrow H_{c}^{p}(M) \rightarrow \check{H}^{p}(S) \rightarrow H_{c}^{p+1}(M-S) \rightarrow
$$

Finally we can define a more general relative cohomology group. We take a differentiable map $F: S \rightarrow M$ between manifolds. It could, e.g., be an embedding of $S \subset M$, but $S$ need not be closed. Define

$$
\Omega^{p}(F)=\Omega^{p}(M) \oplus \Omega^{p-1}(S)
$$

and the differential

$$
\begin{aligned}
d: \Omega^{p}(F) & \rightarrow \Omega^{p+1}(F) \\
d(\omega, \psi) & =\left(d \omega, F^{*} \omega-d \psi\right)
\end{aligned}
$$

Note that $d^{2}=0$ so we get a complex and cohomology groups $H^{p}(F)$. These "forms" fit into a sort exact sequence

$$
0 \rightarrow \Omega^{p-1}(S) \rightarrow \Omega^{p}(F) \rightarrow \Omega^{p}(M) \rightarrow 0
$$

where the maps are just the natural inclusion and projection. When we include the differential we arrive at a large diagram where the left square is anti-commutative and the right one commutative

$$
\begin{array}{cccccccc}
0 & \rightarrow & \Omega^{p}(S) & \rightarrow & \Omega^{p+1}(M) \oplus \Omega^{p}(S) & \rightarrow & \Omega^{p+1}(M) & \rightarrow \\
& & \uparrow d & & \uparrow\left(d, F^{*}-d\right) & & 0 \\
0 & \rightarrow & \Omega^{p-1}(S) & \rightarrow & \Omega^{p}(M) \oplus \Omega^{p-1}(S) & \rightarrow & \Omega^{p}(M) & \rightarrow
\end{array}
$$

This still leads us to a long exact sequence

$$
\rightarrow H^{p-1}(S) \rightarrow H^{p}(F) \rightarrow H^{p}(M) \rightarrow H^{p}(S) \rightarrow
$$

The connecting homomorphism $H^{p}(M) \rightarrow H^{p}(S)$ is in fact the pull-back map $F^{*}$ as can be seen by a simple diagram chase.

In case $i: S \subset M$ is an embedding we also use the notation $H^{p}(M, S)=H^{p}(i)$. In this case it'd seem that the connecting homomorphism is more naturally defined to be $H^{p-1}(S) \rightarrow H^{p}(M, S)$, but we don't have a short exact sequence

$$
0 \rightarrow \Omega^{p}(M) \oplus \Omega^{p-1}(S) \rightarrow \Omega^{p}(M) \rightarrow \Omega^{p}(S) \rightarrow 0
$$

hence the tricky shift in the groups.
We can easily relate the new relative cohomology to the one defined above. This shows that the relative cohomology, while trickier to define, is ultimately more general and useful.

Proposition 7.4.7. If $i: S \subset M$ is a closed submanifold of a closed manifold, then the natural map

$$
\begin{aligned}
\Omega_{c}^{p}(M-S) & \rightarrow \Omega^{p}(M) \oplus \Omega^{p-1}(S) \\
\omega & \rightarrow(\omega, 0)
\end{aligned}
$$

defines an isomorphism

$$
H_{c}^{p}(M-S) \simeq H^{p}(i)
$$

Proof. Simply observe that we have two long exact sequences

$$
\begin{aligned}
\rightarrow H^{p}(i) & \rightarrow H^{p}(M) \rightarrow H^{p}(S) \rightarrow H^{p+1}(i) \rightarrow \\
\rightarrow H_{c}^{p}(M-S) & \rightarrow H^{p}(M) \rightarrow H^{p}(S) \rightarrow H_{c}^{p+1}(M-S) \rightarrow
\end{aligned}
$$

where two out of three terms are equal.
Now that we have a fairly general relative cohomology theory we can establish the well-known excision property.

THEOREM 7.4.8. If $M=U \cup V$, where $U$ and $V$ are open, then the restriction map

$$
H^{p}(M, U) \rightarrow H^{p}(V, U \cap V)
$$

is an isomorphism.
Proof. First select a partition of unity $\lambda_{U}, \lambda_{V}$ relative to $U, V$.
We start with injectivity. Take a class $[(\omega, \psi)] \in H^{p}(M, U)$, i.e.,

$$
\begin{aligned}
d \omega & =0 \\
\left.\omega\right|_{U} & =d \psi
\end{aligned}
$$

If the restriction to $(V, U \cap V)$ is exact, then we can find $(\bar{\omega}, \bar{\psi}) \in \Omega^{p-1}(V) \oplus \Omega^{p-2}(U \cap V)$ such that

$$
\begin{aligned}
\left.\omega\right|_{V} & =d \bar{\omega} \\
\left.\psi\right|_{U \cap V} & =\left.\bar{\omega}\right|_{U \cap V}-d \bar{\psi}
\end{aligned}
$$

Using that $\bar{\psi}=\lambda_{U} \bar{\psi}+\lambda_{V} \bar{\psi}$ we obtain

$$
\begin{aligned}
\left.\left(\psi+d\left(\lambda_{V} \bar{\psi}\right)\right)\right|_{U \cap V} & =\left.\left(\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right)\right)\right|_{U \cap V} \\
\psi+d\left(\lambda_{V} \bar{\psi}\right) & \in \Omega^{p-1}(U) \\
\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right) & \in \Omega^{p-1}(V)
\end{aligned}
$$

Thus we have a form $\tilde{\omega} \in \Omega^{p-1}(M)$ defined by $\psi+d\left(\lambda_{V} \bar{\psi}\right)$ on $U$ and $\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right)$ on $V$. Clearly $d \tilde{\omega}=\omega$ and $\psi=\left.\tilde{\omega}\right|_{U}-d\left(\lambda_{V} \bar{\psi}\right)$ so we have shown that $(\omega, \psi)$ is exact.

For surjectivity select $(\bar{\omega}, \bar{\psi}) \in \Omega^{p}(V) \oplus \Omega^{p-1}(U \cap V)$ that is closed:

$$
\begin{aligned}
d \bar{\omega} & =0 \\
\left.\bar{\omega}\right|_{U \cap V} & =d \bar{\psi} .
\end{aligned}
$$

Using

$$
\begin{aligned}
\left.\bar{\omega}\right|_{U \cap V}-d\left(\lambda_{U} \bar{\psi}\right) & =d\left(\lambda_{V} \bar{\psi}\right), \\
\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right) & \in \Omega^{p}(V), \\
d\left(\lambda_{V} \bar{\psi}\right) & \in \Omega^{p}(U)
\end{aligned}
$$

we can define $\omega$ as $\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right)$ on $V$ and $d\left(\lambda_{V} \bar{\psi}\right)$ on $U$. Clearly $\omega$ is closed and $\left.\omega\right|_{U}=$ $d\left(\lambda_{V} \bar{\psi}\right)$. Thus we define $\psi=\lambda_{V} \bar{\psi}$ in order to get a closed form $(\omega, \psi) \in \Omega^{p}(M) \oplus$ $\Omega^{p-1}(U)$. Restricting this form to $\Omega^{p}(V) \oplus \Omega^{p-1}(U \cap V)$ yields $\left(\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right), \lambda_{V} \bar{\psi}\right)$ which is not $(\bar{\omega}, \bar{\psi})$. However, the difference is exact:

$$
\begin{aligned}
(\bar{\omega}, \bar{\psi})-\left(\bar{\omega}-d\left(\lambda_{U} \bar{\psi}\right), \lambda_{V} \bar{\psi}\right) & =\left(d\left(\lambda_{U} \bar{\psi}\right), \lambda_{U} \bar{\psi}\right) \\
& =d\left(\lambda_{U} \bar{\psi}, 0\right) .
\end{aligned}
$$

Thus $[(\omega, \psi)] \in H^{p}(M, U)$ is mapped to $[(\bar{\omega}, \bar{\psi})] \in H^{p}(V, U \cap V)$.

### 7.5. Poincaré Duality and its Consequences

We explain several interesting results that follow from Poincaré Duality and begin to connect the numerical invariants defined geometrically using transversality to algebraic concepts in cohomology. We start by explaining Poincaré duality, then give some examples of its consequences and finish by showing how it can be used to calculate the degree of a map using integration.
7.5.1. Poincaré Duality. The last piece of information we need to understand is how the wedge product acts on cohomology. It is easy to see that we have a map

$$
\begin{aligned}
H^{p}(M) \times H^{q}(M) & \rightarrow H^{p+q}(M), \\
([\omega],[\psi]) & \mapsto[\omega \wedge \psi] .
\end{aligned}
$$

This is well-defined as

$$
(\omega+d \theta) \wedge(\psi+d \phi)=\omega \wedge \psi+d(\theta \wedge(\psi+d \phi)) \pm d(\omega \wedge \phi)
$$

Thus the wedge product induces a ring structure on $H^{*}(M)$ that in a suitable sense will be shown to be dual to the intersection theory developed using transversality.To that end we are particularly interested in understanding what happens in case $p+q=n$ as that will create a natural map from the cohomology functor $\mathscr{H}^{*}(M)$ to de Rham cohomology.

Note that this ring structure also gives us a well-defined map:

$$
H^{p}(M) \times H_{c}^{q}(M) \rightarrow H_{c}^{p+q}(M) .
$$

When $M$ is oriented and $p+q=n$ we can in addition integrate to obtain a pairing:

$$
H^{p}(M) \times H_{c}^{q}(M) \rightarrow H_{c}^{n}(M) \xrightarrow{\int} \mathbb{R}
$$

THEOREM 7.5.1 (Poincaré Duality). Let $M$ be an oriented n-manifold. The pairing

$$
\begin{aligned}
H^{p}(M) \times H_{c}^{n-p}(M) & \rightarrow \mathbb{R}, \\
([\omega],[\psi]) & \mapsto \int_{M} \omega \wedge \psi
\end{aligned}
$$

is well-defined and non-degenerate. In particular, the two cohomology groups $H^{p}(M)$ and $H_{c}^{n-p}(M)$ are dual to each other and consequently have the same dimension when they are finite dimensional.

Proof. The bilinear form defines a linear map on all oriented $n$-manifolds:

$$
H^{p}(M) \rightarrow \operatorname{Hom}\left(H_{c}^{n-p}(M), \mathbb{R}\right)=\mathscr{H}^{p}(M)
$$

We claim that this map is an isomorphism for all orientable, but not necessarily connected, manifolds. This will follow from theorem 7.3 .1 provided we can show that it is an isomorphism when $M=\mathbb{R}^{n}$. This case follows from the proof of lemma 7.4.2

There is also a map

$$
H_{c}^{n-p}(M) \rightarrow \operatorname{Hom}\left(H^{p}(M), \mathbb{R}\right)
$$

which is an isomorphism when $H_{c}^{n-p}(M)$ is finite dimensional, but not necessarily otherwise. In fact the countable disjointness property generally fails in this case.

### 7.5.2. Consequences of Poincaré Duality.

COROLLARY 7.5.2. If $M^{n}$ is contractible, then

$$
H_{c}^{p}(M)= \begin{cases}\mathbb{R} & \text { when } p=n \\ 0 & \text { when } p \neq n\end{cases}
$$

THEOREM 7.5.3. On a closed oriented n-manifold $M$ the cohomology groups $H^{p}(M)$ and $H^{n-p}(M)$ are isomorphic.

Proof. This requires that we know that $H^{p}(M)$ is finite dimensional for all $p$.
First note that if $O \subset \mathbb{R}^{k}$ is a finite union of open boxes, then the de Rham cohomology groups are finite dimensional. The proof of this uses Mayer-Vietoris and induction on the number of boxes. Specifically if $M=A \cup B$, where $H^{*}(A), H^{*}(B)$, and $H^{*}(A \cap B)$ are finite dimensional, then also $H^{*}(M)$ is finite dimensional. To see this consider the part of the long exact sequence:

$$
H^{*-1}(A \cap B) \rightarrow H^{*}(M) \rightarrow H^{*}(A) \oplus H^{*}(B)
$$

Here the image of $H^{*-1}(A \cap B) \rightarrow H^{*}(M)$ is finite dimensional and as the sequence is exact any complement to the image is mapped injectively into $H^{*}(A) \oplus H^{*}(B)$ and is thus also finite dimensional. Next if $B_{1}, \ldots, B_{k} \subset \mathbb{R}^{k}$ are boxes, then the intersection

$$
B_{k} \cap\left(B_{1} \cup \cdots \cup B_{k-1}\right)=\left(B_{k} \cap B_{1}\right) \cup \cdots \cup\left(B_{k} \cap B_{k-1}\right)
$$

consists of at most $k-1$ boxes. This allows us to complete the induction step.
This will give the result for $M \subset \mathbb{R}^{k}$ as we can find a tubular neighborhood $M \subset U \subset \mathbb{R}^{k}$ and a retract $r: U \rightarrow M$, i.e., $\left.r\right|_{M}=i d_{M}$. Now cover $M$ by open boxes that lie in $U$ and use compactness of $M$ to find $M \subset O \subset U$ with $O$ being a union of finitely many open boxes. Since $\left.r\right|_{M}=i d_{M}$ the retract $r^{*}: H^{p}(M) \rightarrow H^{p}(O)$ is an injection so it follows that $H^{p}(M)$ is finite dimensional.

Note that $\mathbb{R} \mathbb{P}^{2}$ does not satisfy this duality between $H^{0}$ and $H^{2}$. In fact we always have
THEOREM 7.5.4. If $M$ is a connected n-manifold that is not orientable, then

$$
H_{c}^{n}(M)=0 .
$$

Proof. We use the two-fold orientation cover $F: \hat{M} \rightarrow M$ and the involution $A$ : $\hat{M} \rightarrow \hat{M}$ such that $F=F \circ A$. The fact that $M$ is not orientable means that $A$ is orientation reversing. This implies that pull-back by $A$ changes integrals by a sign:

$$
\int_{\hat{M}} \eta=-\int_{\hat{M}} A^{*} \eta, \eta \in \Omega_{c}^{n}(\hat{M})
$$

To prove the theorem select $\omega \in \Omega_{c}^{n}(M)$ and consider the pull-back $F^{*} \omega \in \Omega_{c}^{n}(\hat{M})$. Since $F=F \circ A$ this form is invariant under pull-back by $A$

$$
\int_{\hat{M}} F^{*} \omega=\int_{\hat{M}} A^{*} \circ F^{*} \omega
$$

On the other hand, as $A$ reverses orientation we must also have

$$
\int_{\hat{M}} F^{*} \omega=-\int_{\hat{M}} A^{*} \circ F^{*} \omega
$$

Thus

$$
\int_{\hat{M}} F^{*} \omega=0
$$

This shows that the pull-back is exact

$$
F^{*} \omega=d \psi, \psi \in \Omega_{c}^{n-1}(\hat{M})
$$

On the other hand, from lemma 7.4.3 we know that $F^{*}: H_{c}^{n}(M) \rightarrow H_{c}^{n}(\hat{M})$ is an injection. This shows the claim.

Corollary 7.5.5. If $M$ is an open connected n-manifold, then

$$
H^{n}(M)=0 .
$$

Proof. By lemma 7.4.3 it suffices to prove this for orientable manifolds. In this case it follows from Poincare duality that

$$
H^{n}(M) \simeq \operatorname{Hom}\left(H_{c}^{0}(M), \mathbb{R}\right) \simeq 0
$$

There are many more interesting results for compactly supported cohomology. In case of oriented manifolds Poincaré duality is a natural way of proving them, but without that result one can often proven them using theorem 1.3.11. A good example is the compactly supported version of homotopy invariance

$$
H_{c}^{*}(M) \simeq H_{c}^{*+1}(\mathbb{R} \times M)
$$

7.5.3. Degrees of Maps. Given the simple nature of the top cohomology class of a manifold, we see that maps between manifolds of the same dimension can act only by multiplication on the top cohomology class. We shall see that this multiplicative factor is in fact an integer, called the degree of the map.

To be precise, suppose we have two connected oriented $n$-manifolds $M$ and $N$ and also a proper map $F: M \rightarrow N$. Then we get a diagram

$$
\begin{array}{ccc}
H_{c}^{n}(N) & \xrightarrow{F^{*}} & H_{c}^{n}(M) \\
\downarrow \int & & \downarrow \int \\
\mathbb{R} & \xrightarrow{d} & \mathbb{R} .
\end{array}
$$

Since the vertical arrows are isomorphisms, the induced map $F^{*}$ yields a unique map $d$ : $\mathbb{R} \rightarrow \mathbb{R}$. This map must be multiplication by some number, which we call the degree of $F$, denoted by $\operatorname{deg} F$. Clearly, the degree is defined by the property

$$
\int_{M} F^{*} \omega=\operatorname{deg} F \cdot \int_{N} \omega
$$

From the functorial properties of the induced maps on cohomology we see that

$$
\operatorname{deg}(F \circ G)=\operatorname{deg}(F) \operatorname{deg}(G) .
$$

PROPOSITION 7.5.6. If $F: M \rightarrow N$ is a diffeomorphism between oriented $n$-manifolds, then $\operatorname{deg} F= \pm 1$, depending on whether $F$ preserves or reverses orientation.

Proof. Note that our definition of integration of forms is independent of coordinate changes. It relies only on a choice of orientation. If this choice is changed then the integral changes by a sign. This clearly establishes the lemma.

THEOREM 7.5.7. If $F: M \rightarrow N$ is a proper map between oriented n-manifolds, then $\operatorname{deg} F$ is an integer and agrees with the oriented degree.

Proof. The proof will also give a recipe for computing the degree. By Sard's theorem there is a regular value $y \in N$. Lemma 1.4 .28 there exists a neighborhood $V$ around $y$ such that $F^{-1}(V)=\bigcup_{k=1}^{n} U_{k}$, where $U_{k}$ are mutually disjoint and $F: U_{k} \rightarrow V$ is a diffeomorphism. Now select $\omega \in \Omega_{c}^{n}(V)$ with $\int \omega=1$ and note that:

$$
F^{*} \omega=\left.\sum_{i=1}^{k} F^{*} \omega\right|_{U_{i}},
$$

where each $\left.F^{*} \omega\right|_{U_{i}}$ has compact support in $U_{i}$. The above lemma now tells us that

$$
\left.\int_{U_{i}} F^{*} \omega\right|_{U_{i}}= \pm 1
$$

Hence,

$$
\begin{aligned}
\operatorname{deg} F & =\operatorname{deg} F \cdot \int_{N} \omega \\
& =\operatorname{deg} F \cdot \int_{U} \omega \\
& =\int_{M} F^{*} \omega \\
& =\left.\sum_{i=1}^{k} \int_{U_{i}} F^{*} \omega\right|_{U_{i}}
\end{aligned}
$$

is an integer. Here $\left.\int_{U_{i}} F^{*} \omega\right|_{U_{i}}= \pm 1$ depending simply on whether $F$ preserves or reverses the orientations at $x_{i}$. Thus, the cohomologically defined degree also counts the number of preimages for regular values with sign just as the oriented degree from section5.4.2

On an oriented Riemannian manifold $(M, g)$ we always have a canonical volume form denoted by $d \mathrm{vol}_{g}$. Using this form, we see that the degree of a map between closed Riemannian manifolds $F:(M, g) \rightarrow(N, h)$ can be computed as

$$
\operatorname{deg} F=\frac{\int_{M} F^{*}\left(d \operatorname{vol}_{h}\right)}{\operatorname{vol}(N)}
$$

In case $F$ is locally a Riemannian isometry, we must have that:

$$
F^{*}\left(d \mathrm{vol}_{h}\right)= \pm d \mathrm{vol}_{g}
$$

Hence,

$$
\operatorname{deg} F= \pm \frac{\operatorname{vol} M}{\operatorname{vol} N}
$$

This gives the well-known formula for the relationship between the volumes of Riemannian manifolds that are related by a finite covering map.

On $\mathbb{R}^{n}-\{0\}$ there is an interesting closed $(n-1)$-form

$$
\omega=r^{-n} \sum_{i=1}^{n}(-1)^{i+1} x^{i} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n}
$$

where $r^{2}=\sum_{i=1}^{n}\left(x^{i}\right)^{2}$. If we restrict this to a sphere of radius $\varepsilon$ around the origin, then

$$
\begin{aligned}
\int_{S^{n-1}(\varepsilon)} \omega & =\varepsilon^{-n} \int_{S^{n-1}(\varepsilon)} \sum_{i=1}^{n}(-1)^{i+1} x^{i} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n} \\
& =\varepsilon^{-n} \int_{\bar{B}(0, \varepsilon)} d\left(\sum_{i=1}^{n}(-1)^{i+1} x^{i} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{n}\right) \\
& =\varepsilon^{-n} \int_{\bar{B}(0, \varepsilon)} n d x^{1} \wedge \cdots \wedge d x^{n} \\
& =n \varepsilon^{-n \operatorname{vol} \bar{B}(0, \varepsilon)} \\
& =n \operatorname{vol} \bar{B}(0,1) \\
& =\operatorname{vol}_{n-1} S^{n-1}(1)
\end{aligned}
$$

More generally if $F: M^{n-1} \rightarrow \mathbb{R}^{n}-\{0\}$ is a smooth map, then it is clearly homotopic to the map $F_{1}: M^{n-1} \rightarrow S^{n-1}(1)$ defined by $F_{1}=F /|F|$ so we obtain an integral formula for the winding number

$$
\begin{aligned}
W(F, 0) & =\operatorname{deg} F_{1} \\
& =\frac{1}{\operatorname{vol}_{n-1} S^{n-1}(1)} \int_{M} F_{1}^{*} \omega \\
& =\frac{1}{\operatorname{vol}_{n-1} S^{n-1}(1)} \int_{M} F^{*} \omega
\end{aligned}
$$

### 7.6. The Künneth-Leray-Hirsch Theorem

In this section we shall compute the cohomology of a fibration under certain simplifying assumptions. We start with the trivial fiber bundles $E=F \times B$. The standard projection for any fiber bundle is denoted $\pi: E \rightarrow B$ and when the bundle is trivial we also have a projection $\bar{\pi}: E \rightarrow F$ on to the fiber.

THEOREM 7.6.1 (Künneth). If $H^{*}(F)$ is finite dimensional, then there is an isomorphism:

$$
\bigoplus_{p+q=r} H^{p}(F) \otimes H^{q}(B) \rightarrow H^{r}(E)
$$

where the map $H^{p}(F) \otimes H^{q}(B) \rightarrow H^{p+q}(E)$ is defined by $\psi \otimes \omega \mapsto \bar{\pi}^{*}(\psi) \wedge \pi^{*}(\omega)$.
Proof. We fix $F$ and use theorem 1.3.11 with the statement $P(B)$ being that the theorem is true.

When $B=\mathbb{R}^{n}$, we have $H^{*}(B)=H^{0}(B)=\mathbb{R}$. Thus

$$
\bigoplus_{p+q=r} H^{p}(F) \otimes H^{q}(B)=H^{r}(F)
$$

and the statement follows from homotopy invariance of cohomology.
For condition (2) in theorem 1.3.11 assume that the result holds for open sets $A_{1}, A_{2}, A_{1} \cap$ $A_{2} \subset B$, then we can use the same strategy as in the proof of theorem 7.3.1 to verify the statement for $A_{1} \cup A_{2}$.

Finally for condition (3), assume the statement holds for pairwise disjoint open sets: $A_{\alpha} \subset B$. We have to show it also holds for the union. This depends crucially on $H^{p}(F)$ being finite dimensional as tensor products do not, in general, respect infinite products (see example below). Specifically, we use that

$$
\operatorname{Hom}\left(\operatorname{Hom}\left(H^{p}(F), \mathbb{R}\right), V\right) \simeq \operatorname{Hom}\left(\operatorname{Hom}\left(H^{p}(F), \mathbb{R}\right), \mathbb{R}\right) \otimes V \simeq H^{p}(F) \otimes V
$$

In particular, if $V=\times_{\alpha} V_{\alpha}$, then

$$
\begin{aligned}
H^{p}(F) \otimes\left(\times_{\alpha} V_{\alpha}\right) & =\operatorname{Hom}\left(\operatorname{Hom}\left(H^{p}(F), \mathbb{R}\right), \times_{\alpha} V_{\alpha}\right) \\
& =\times_{\alpha} \operatorname{Hom}\left(\operatorname{Hom}\left(H^{p}(F), \mathbb{R}\right), V_{\alpha}\right) \\
& =\times_{\alpha}\left(H^{p}(F) \otimes V_{\alpha}\right)
\end{aligned}
$$

This leads us to the desired isomorphism:

$$
\begin{aligned}
\bigoplus_{p+q=r} H^{p}(F) \otimes H^{q}\left(\bigcup A_{\alpha}\right) & =\bigoplus_{p+q=r} H^{p}(F) \otimes\left(\times_{\alpha} H^{q}\left(A_{\alpha}\right)\right) \\
& =\bigoplus_{p+q=r} \times_{\alpha}\left(H^{p}(F) \otimes H^{q}\left(A_{\alpha}\right)\right) \\
& =\times_{\alpha} \bigoplus_{p+q=r}\left(H^{p}(F) \otimes H^{q}\left(A_{\alpha}\right)\right) \\
& =\times_{\alpha} H^{k}\left(\pi^{-1}\left(A_{\alpha}\right)\right) \\
& =H^{k}\left(\pi^{-1}\left(\bigcup_{\alpha} A_{\alpha}\right)\right)
\end{aligned}
$$

EXAMPLE 7.6.2. In case both factors have infinite dimensional cohomology the result does not necessarily hold. Consider two 0 -dimensional manifolds $A, B$, i.e., they are finite or countable sets. Here $H^{0}(A \times B)$ is isomorphic the the space of functions $A \times B \rightarrow \mathbb{R}$, while $H^{0}(A) \otimes H^{0}(B)$ consists of finite sums of elements of the form $f_{A} \otimes f_{B}$, where $f_{C}$ : $C \rightarrow \mathbb{R}$. Thus the map $H^{0}(A) \otimes H^{0}(B) \rightarrow H^{0}(A \times B)$ is only an isomorphism when $A$ or $B$ is finite. To address the construction in the above proof note that

$$
H^{0}(A) \otimes H^{0}(B)=H^{0}(A) \otimes \times_{b \in B} H^{0}(b)
$$

while

$$
\times_{b \in B} H^{0}(A) \otimes H^{0}(b)=\times_{b \in B} H^{0}(A) \otimes \mathbb{R}=\times_{b \in B} H^{0}(A)=\times_{a \in A, b \in B} \mathbb{R}=H^{0}(A \times B)
$$

Künneth's theorem also has a direct counter part for compactly supported cohomology:

$$
\bigoplus_{p+q=r} H_{c}^{p}(F) \otimes H_{c}^{q}(B)=H_{c}^{r}(F \times B)
$$

as long as $H_{c}^{*}(F)$ is finite dimensional. The proof is similar with the caveat that homotopy invariance is replaced by

$$
H_{c}^{*+n}\left(F \times \mathbb{R}^{n}\right) \simeq H_{c}^{*}(F)
$$

We now assume that $\pi: E \rightarrow B$ is a submersion-fibration where the fibers are diffeomorphic to a manifold $F$. The key condition that is needed is that the restriction to any fiber $\pi^{-1}(p) \cong F$ is a surjection in cohomology

$$
H^{*}(E) \rightarrow N^{*}\left(\pi^{-1}(p)\right) \rightarrow 0, \text { for all } p \in B
$$

In the case of a product this obviously holds since the projection $\bar{\pi}: F \times B \rightarrow F$ is a right inverse to the inclusions $F \rightarrow F \times\{s\} \subset F \times B$. The restriction assumption does not hold in general, e.g., the fibration $S^{3} \rightarrow S^{2}$ is a good counter example.

It seems a daunting task to check the condition for all fibers in a general situation. Assuming we know it is true for a specific fiber $F=\pi^{-1}(p)$ we can select a neighborhood $A$ around $p$ such that $\pi^{-1}(A)=F \times A$. As long as $A$ is contractible we see that $\pi^{-1}(A)$ and $F$ are homotopy equivalent and so the restriction to any of the fibers over $A$ will also give a
surjection in cohomology. When $B$ is connected a covering of such contractible sets shows that $H^{*}(E) \rightarrow N^{*}\left(\pi^{-1}(x)\right)$ is a surjection for all $x \in B$. In fact, this construction gives us a bit more. We assume that for a specific fiber $F$ there is a subspace $\mathscr{V}^{*} \in H^{*}(E)$ that is isomorphic to $H^{*}(F)$. The construction then shows that $\mathscr{V}^{*}$ is isomorphic to $H^{*}\left(\pi^{-1}(x)\right)$ for all $x \in B$ as long as $B$ is connected.

THEOREM 7.6.3 (Leray-Hirsch). Assume we have $\mathscr{V}^{*} \subset H^{*}(E)$ that is isomorphic to $H^{*}\left(\pi^{-1}(x)\right)$ via restriction for all $x \in B$. If $H^{*}(F)$ is finite dimensional, then there is an isomorphism:

$$
\bigoplus_{p+q=r} \mathscr{V}^{p} \otimes H^{q}(B) \rightarrow H^{r}(E)
$$

where the map $\mathscr{V}^{p} \otimes H^{q}(B) \rightarrow H^{p+q}(E)$ is defined by $\psi \otimes \omega \mapsto \psi \wedge \pi^{*}(\omega)$.
REMARK 7.6.4. Observe that for any map $E \rightarrow B$ the space $H^{*}(E)$ is naturally a $H^{*}(B)$ module:

$$
H^{*}(B) \times H^{*}(E) \rightarrow H^{*}(E)
$$

via pull-back $H^{*}(B) \rightarrow H^{*}(E)$ and wedge product in $H^{*}(E)$. The statement of the theorem can then be rephrased as offering a condition for when $H^{*}(E)$ is a free $H^{*}(B)$-module.

Proof. Note that for each open $A \subset B$ there is a natural restriction

$$
\left.\mathscr{V}^{*} \subset H^{*}(E) \rightarrow \mathscr{V}^{*}\right|_{A} \subset H^{*}\left(\pi^{-1}(A)\right)
$$

This shows that the assumption of the theorem holds for all of the bundles $\pi^{-1}(A) \rightarrow A$, where $A \subset B$ is open.

With these constructions in mind we can employ the strategy from corollary 1.3.12 To that end, restrict attention to open subsets $A \subset M$ with the statement $P(A)$ being that for all $r$ the map

$$
\left.\bigoplus_{p+q=r} \mathscr{V}^{p}\right|_{A} \otimes H^{q}(A) \rightarrow H^{r}\left(\pi^{-1}(A)\right)
$$

is an isomorphism.
To check condition (1) note that the statement holds for any $A \subset B$ that is diffeomorphic to $\mathbb{R}^{\operatorname{dim} B}$ and where the bundle is trivial $\pi^{-1}(A) \cong F \times A$. In particular, the statement also holds for any box in $A$.

Condition (2) in corollary 1.3.12 is established as in theorem 7.6.1 and the proof of theorem 7.3.1.

Finally for condition (3) we simply replace $H^{p}(F)$ with $\mathscr{V}^{p}$ and proceed as in the proof of theorem 7.6.1.

In the general case of a fiber bundle the obvious generalization to a compactly supported result runs into some logistical problems. The best version uses forms on $E$ that are compactly supported on fibers $\Omega_{c v}^{*}(E)$, thus $\Omega_{c}^{*}(E) \subset \Omega_{c v}^{*}(E) \subset \Omega^{*}(E)$. This leads to a cohomology theory $H_{c v}^{*}(E)$ that has the natural property that for $A \subset B$ there is a restriction $\operatorname{map} H_{c v}^{*}(E) \rightarrow H_{c v}^{*}\left(\pi^{-1}(A)\right)$. The proof from above can then be used again to show.

THEOREM 7.6.5. Assume we have $\mathscr{V}_{c v}^{*} \subset H_{c v}^{*}(E)$ that is isomorphic to $H_{c}^{*}\left(\pi^{-1}(x)\right)$ via restriction for all $x \in B$. If $H_{c}^{*}(F)$ is finite dimensional, then there is an isomorphism:

$$
\bigoplus_{p+q=r} \mathscr{V}_{c v}^{p} \otimes H^{q}(B) \rightarrow H_{c v}^{r}(E)
$$

where the map $\mathscr{V}_{c v}^{p} \otimes H^{q}(B) \rightarrow H_{c v}^{p+q}(E)$ is defined by $\psi \otimes \omega \mapsto \psi \wedge \pi^{*}(\omega)$.

The important special case is when $B$ is compact where the formulation becomes more natural.

Corollary 7.6.6. Assume that $B$ is compact and $\mathscr{V}_{c}^{*} \subset H_{c}^{*}(E)$ is isomorphic to $H_{c}^{*}\left(\pi^{-1}(x)\right)$ via restriction for all $x \in B$. If $H_{c}^{*}(F)$ is finite dimensional, then there is an isomorphism:

$$
\bigoplus_{p+q=r} \mathscr{V}_{c}^{p} \otimes H^{q}(B) \rightarrow H_{c}^{r}(E)
$$

where the map $\mathscr{V}_{c}^{p} \otimes H^{q}(B) \rightarrow H_{c}^{p+q}(E)$ is defined by $\psi \otimes \omega \mapsto \psi \wedge \pi^{*}(\omega)$.
This corollary follows directly from Poincare duality when both $E$ and $B$ are oriented. The corresponding $\mathscr{V}^{*}$ is defined via the Poincare duality isomorphism $H^{*} \rightarrow$ $\operatorname{Hom}\left(H_{c}^{n-*}, \mathbb{R}\right)$, i.e., $\mathscr{V}^{p} \simeq \operatorname{Hom}\left(\mathscr{V}_{c}^{n-p}, \mathbb{R}\right)$, where $n=\operatorname{dim} E$.

### 7.7. Exercises

(1) Show that the one-form $f \cdot(-y d x+x d y-d z)$ cannot be closed for any $f: \mathbb{R}^{3} \rightarrow$ $(0, \infty)$.
(2) Calculate the cohomology of the torus using Mayer-Vietoris and induction on dimension.
(3) Let $\omega \in \Omega^{1}(M)$.
(a) Define $\int_{c} \omega$ for a piecewise smooth curve $c:[a, b] \rightarrow M$.
(b) If $d \omega=0$, then $\int_{c_{0}} \omega=\int_{c_{1}} \omega$, where $c_{0,1}:[a, b] \rightarrow M$ agree at the end points and are homotopic via a homotopy that fixes the end points.
(c) Show that $\omega$ is exact provided $\int_{c} \omega$ only depends on $c(a)$ and $c(b)$.
(d) Show that a simply connected manifold has $H^{1}(M)=0$.
(4) Let $G / H$ be a homogeneous space where $G$ is compact and simply connected and $H$ is connected. Show that $G / H$ is simply connected, e.g., $S U(n) / S O(n)$ is simply connected. Hint: Lift a loop based at the equivalence class $H$ to a path in $G$ that begins and ends in $H$.
(5) Let $G / H$ be an $n$-dimensional homogeneous space where $G$ is compact and connected. Show that:

$$
\operatorname{dim} H^{p}(G / H) \leq\binom{ n}{p}
$$

(6) Let $G / H$ be an $n$-dimensional homogeneous space where $G$ is compact and connected. Show that if the linear action of $H$ on $T_{H} G / H$ contains an orientation reversing element, then $\Omega_{G}^{n}(G / H)=0$ and $G / H$ is not orientable.
(7) Let $M$ be a closed $n$-manifold. Calculate $H^{*}(M-\{p\})$ in terms of $H^{*}(M)$.
(8) Show that if $F: M \rightarrow N$ is homotopic to a constant map then $F^{*}(\omega)$ is exact for any closed form $\omega$ on $N$.
(9) Show that if $F: M \rightarrow N$ admits a section $s: N \rightarrow M$, i.e., $F \circ s=i d_{N}$, then $F^{*}: H^{*}(N) \rightarrow H^{*}(M)$ is an injection.
(10) Let $G$ be a finite group that acts on $M$ with trivial isotropy, i.e., if $g x=x$ for any $x \in M$ and $g \in G$, then $g=e$. Show that $M \rightarrow M / G$ defines a covering map and that $H^{*}(M / G)=H_{G}^{*}(M)$.
(11) Show that there is a natural isomorphism

$$
H_{c}^{*}(M) \simeq H_{c}^{*+1}(\mathbb{R} \times M)
$$

(12) Show (in two ways) that the cohomology of $S^{p} \times S^{q}$ is generated by a form $\omega_{1} \in \Omega^{p}\left(S^{p} \times S^{q}\right)$, a form $\omega_{2} \in \Omega^{q}\left(S^{p} \times S^{q}\right)$, and $\omega_{1} \wedge \omega_{2}$. Hint: Use Künneth's theorem or that the action by $S O(p+1) \times S O(q+1)$ is transitive.
(13) Let $M=T^{n}=S^{1} \times \cdots \times S^{1}$ and let $\theta$ be a generator for $H^{1}\left(S^{1}\right)$. Define $\theta_{i}=$ $\pi_{i}^{*}(\theta)$, where $\pi_{i}: T^{n} \rightarrow S^{1}$ is the projection onto the $i^{\text {th }}$ factor. Use Künneth's theorem to show that $H^{p}\left(T^{n}\right)$ has a basis of the form $\theta_{i_{1}} \wedge \cdots \wedge \theta_{i_{p}}, i_{1}<\cdots<i_{p}$. Conclude that $\operatorname{dim} H^{p}\left(T^{n}\right)=\binom{n}{p}$.
(14) Show that if $\omega \in \Omega^{2}\left(\mathbb{C P}^{n}\right)$ generates $H^{2}\left(\mathbb{C P}{ }^{n}\right)$, then $\omega^{k}$ generates $H^{2 k}\left(\mathbb{C P}^{n}\right)$.
(15) Show that any map $S^{p+q} \rightarrow S^{p} \times S^{q}$ has degree 0 .
(16) Let $p, q \in \mathbb{N}$. Show that any map $S^{2 p} \times S^{2 q} \rightarrow \mathbb{C} \mathbb{P}^{p+q}$ has degree 0 unless $p=$ $q=1$.
(17) A symplectic form $\omega \in \Omega^{2}\left(M^{2 n}\right)$ is a closed form that is nondegenerate, i.e., for every $v$ the linear function $w \mapsto \omega(w, v)$ is not trivial.
(a) Show $\omega \in \Omega^{2}(M)$ is nondegenerate if and only if $\operatorname{dim} M$ is even and $\omega^{n}$ is a volume form where $2 n=\operatorname{dim} M$. Hint: This is linear algebra. Find a normal form on a vector space for any skew-symmetric bilinear form.
(b) Show that when $M$ is closed, then a symplectic form generates a nontrivial element in cohomology.
(18) Let $M^{4 n+2}$ be closed and oriented. Show that $\operatorname{dim} H^{2 n+1}(M)$ is even.
(19) Let $S \subset \mathbb{R}^{n}$ be a closed or properly embedded oriented submanifold of codimension 1 .
(a) Use the long exact sequence for the pair $\left(\mathbb{R}^{n}, S\right)$ to show that the number of components of $\mathbb{R}^{n}-S$ can be calculated with the formula:

$$
\operatorname{dim} H^{0}\left(\mathbb{R}^{n}-S\right)=1+\operatorname{dim} H_{c}^{n-1}(S)
$$

(b) Generalize (a) to the case where $\mathbb{R}^{n}$ is replaced by a connected oriented manifold $M^{n}$ with $H^{1}(M)=0$.
(c) Give examples where (b) fails if one or both manifolds are not orientable.
(20) For a smooth function $f: M^{n} \rightarrow \mathbb{R}$ define

$$
\begin{aligned}
d_{f}: \Omega^{p}(M) & \rightarrow \Omega^{p+1}(M) \\
d_{f}(\omega) & =d \omega+d f \wedge \omega
\end{aligned}
$$

and

$$
\begin{aligned}
m_{f}: \Omega^{p}(M) & \rightarrow \Omega^{p}(M) \\
m_{f}(\omega) & =e^{f} \omega
\end{aligned}
$$

(a) Show that $d_{f}=m_{-f} \circ d \circ m_{f}$ and $d_{f} \circ d_{f}=0$.
(b) Show that the cohomology groups defined by $d_{f}$ are isomorphic to de Rham cohomology.
(21) For a 1-form $\theta \in \Omega^{1}(M)$ define

$$
\begin{aligned}
d_{\theta}: \Omega^{p}(M) & \rightarrow \Omega^{p+1}(M) \\
d_{\theta}(\omega) & =d \omega+\theta \wedge \omega
\end{aligned}
$$

(a) Show that if $d \theta=0$, then $d_{\theta} \circ d_{\theta}=0$.
(b) Show that if $\theta$ is closed but not exact, then the cohomology defined by $d_{\theta}$ is not necessarily isomorphic to de Rham cohomology. Hint: Show that the $d_{\theta}$-cohomology of $S^{1}$ is trivial.

## CHAPTER 8

## Characteristic Classes

### 8.1. Intersection Theory and the Poincaré Dual

Let $S^{k} \subset N^{n}$ be a closed oriented submanifold of an oriented manifold with finite dimensional de Rham cohomology. The codimension is denoted by $m=n-k$. By integrating $k$-forms on $N$ over $S$ we obtain a linear functional $H^{k}(N) \rightarrow \mathbb{R}$. By theorem 7.5.1 we have $\operatorname{Hom}\left(H^{k}(N), \mathbb{R}\right) \simeq H_{c}^{m}(N)$. The Poincaré dual to this functional is the cohomology class $\left[\eta_{S}^{N}\right] \in H_{c}^{m}(N)$ such that

$$
\int_{S} \omega=\int_{N} \eta_{S}^{N} \wedge \omega
$$

for all $\omega \in H^{k}(N)$. Any representative $\eta_{S}^{N} \in\left[\eta_{S}^{N}\right]$ is called a Poincaré dual to $S \subset N$. The obvious defect of this definition is that several natural submanifolds might not have nontrivial duals for the simple reason that $H_{c}^{m}(N)$ vanishes, e.g., $N=S^{n}$.

To find a nontrivial dual we observe that $\int_{S} \omega$ only depends on the values of $\omega$ in a neighborhood of $S$. Thus we can consider duals supported in any neighborhood $U$ of $S$ in $N$, i.e., $\left[\eta_{S}^{U}\right] \in H_{c}^{m}(U)$. We normally select a tubular neighborhood so that there is a deformation retraction $\pi: U \rightarrow S$, where the fibers $\pi^{-1}(p)$ are diffeomorphic to $\mathbb{R}^{m}$ for all $p \in S$. In particular,

$$
\pi^{*}: H^{k}(S) \rightarrow H^{k}(U)
$$

is an isomorphism and $\left[\eta_{S}^{U}\right] \in H_{c}^{m}(U)$ is characterized as the dual to integration of $k$-forms on $S$, i.e., for all $\omega \in \Omega^{k}(S)$ we have

$$
\int_{S} \omega=\int_{U} \eta_{S}^{N} \wedge \pi^{*}(\omega)
$$

Example 8.1.1. When $S=p$ is a point integration over $S$ is simply evaluation of functions at $p$. The Poincaré dual is represented by any compactly supported $n$-form that integrates to 1 .

EXAMPLE 8.1.2. When $S=S^{1} \subset S^{1} \times(-1,1) \subset S^{2}$ we first note that $\left[\eta_{S^{1}}^{S^{2}}\right]=0$ while $\left[\eta_{S^{1}}^{S^{1} \times(-1,1)}\right] \in H_{c}^{1}\left(S^{1} \times(-1,1)\right)$ where homotopy invariance implies that $H_{c}^{1}\left(S^{1} \times(-1,1)\right) \simeq$ $H_{c}^{0}\left(S^{1}\right)$. The Poincaré dual can be represented by $\pi_{2}^{*} \eta$, where $\pi_{2}: S^{1} \times(-1,1) \rightarrow(-1,1)$ is the projection and $\eta \in \Omega_{c}^{1}((-1,1))$ any form with integral 1 .

EXAMPLE 8.1.3. Consider the embedded submanifold $S_{p, q} \subset T^{2}=S^{1} \times S^{1}$ defined by $F\left(e^{i \theta}\right)=\left(e^{i p \theta}, e^{i q \theta}\right)$, where $p, q \in \mathbb{Z}$ only have $\pm 1$ as common divisors. Let $d t=\frac{d \theta}{2 \pi} \in$ $\Omega^{1}\left(S^{1}\right)$ be the volume form with integral 1 and $\pi_{1,2}: T^{2} \rightarrow S^{1}$ the projections onto the two factors. We obtain two forms $\eta_{1,2}=\pi_{1,2}^{*}(d t)$ that generate $H^{1}\left(T^{2}\right)$ and yield a volume form $\eta_{1} \wedge \eta_{2}$ that integrates to 1 . To find a representative $\eta_{S_{p, q}}^{T^{2}}=\alpha \eta_{1}+\beta \eta_{2}$ we simply
need to check that

$$
\begin{aligned}
\int_{T^{2}}\left(\alpha \eta_{1}+\beta \eta_{2}\right) \wedge \eta_{1} & =\int_{S_{p, q}} \eta_{1} \\
\int_{T^{2}}\left(\alpha \eta_{1}+\beta \eta_{2}\right) \wedge \eta_{2} & =\int_{S_{p, q}} \eta_{2}
\end{aligned}
$$

Here the left hand sides are $-\beta$ and $\alpha$ respectively, while the left hand sides are $p$ and $q$ respectively. Thus

$$
\eta_{S_{p, q}}^{T^{2}}=-p \eta_{1}+q \eta_{2}
$$

The dual gives us an interesting isomorphism called the Thom isomorphism. A more general and abstract version was presented in corollary 7.6.6

LEMMA 8.1.4 (Thom Isomorphism). Recall that $k+m=n$. If $\pi: U \rightarrow S$ is a tubular neighborhood, then the map

$$
\begin{aligned}
H_{c}^{*}(S) & \rightarrow H_{c}^{*+m}(U), \\
{[\omega] } & \mapsto\left[\eta_{S}^{U} \wedge \pi^{*}(\omega)\right]
\end{aligned}
$$

is an isomorphism.
Proof. Using Poincaré duality twice we see that

$$
\begin{aligned}
H_{c}^{*+m}(U) & \simeq \operatorname{Hom}\left(H^{n-*-m}(U), \mathbb{R}\right) \\
& \simeq \operatorname{Hom}\left(H^{k-*}(S), \mathbb{R}\right) \\
& \simeq H_{c}^{*}(S)
\end{aligned}
$$

Thus it suffices to show that the map

$$
\begin{aligned}
H_{c}^{*}(S) & \rightarrow H_{c}^{*+m}(U) \\
{[\omega] } & \mapsto\left[\eta_{S}^{N} \wedge \pi^{*}(\omega)\right]
\end{aligned}
$$

is injective. When $p=k$ this follows from the construction of the dual. For $p<k$ select a nontrivial $\left[\omega \in H^{p}(S)\right]$ and using Poincaré duality $\tau \in H^{k-p}(S)$, such that $[\omega \wedge \tau] \in H^{k}(S)$ is nontrivial. This shows that $\left[\eta_{S}^{N} \wedge \pi^{*}(\omega) \wedge \pi^{*}(\tau)\right]$ is nontrivial. This in turn implies that $\left[\eta_{S}^{N} \wedge \pi^{*}(\omega)\right]$ is nontrivial.

The next goal is to find a characterization of $\eta_{S}^{U}$, this characterization is valid as long as $\pi: U \rightarrow S$ is merely a retract with connected preimages, i.e., $\pi \circ i=i d_{S}$, where $i: S \rightarrow N$ is the inclusion. However, we will only use it for tubular neighborhoods. The characterization makes it possible to construct the dual in many situations and also shows why the Thom isomorphism follows from corollary 7.6.6

THEOREM 8.1.5. The dual is characterized as a closed form with compact support that integrates to 1 along fibers $\pi^{-1}(p)$ for all $p \in S$. In particular, when $U$ is a tubular neighborhood the dual generates the cohomology of the fibers $H_{c}^{*}\left(\pi^{-1}(p)\right)=H_{c}^{*}\left(\mathbb{R}^{m}\right)$.

Proof. The characterization requires a choice of orientation for the fibers. It is chosen so that $T_{p} \pi^{-1}(p) \oplus T_{p} S$ and $T_{p} N$ have the same orientation (this is consistent with [Guillemin-Pollack], but not with several other texts.) For $\omega \in \Omega^{k}(S)$ we note that $\pi^{*} \omega$ is constant on $\pi^{-1}(p), p \in S$. Therefore, if $\eta$ is a closed compactly supported form that integrates to 1 along all fibers, then

$$
\int_{U} \eta \wedge \pi^{*} \omega=\int_{S} \int_{\pi^{-1}(p)} \eta \wedge \pi^{*} \omega=\int_{S} \omega
$$

as desired.
Conversely we define

$$
\begin{aligned}
f & : \quad S \rightarrow \mathbb{R} \\
f(p) & =\int_{\pi^{-1}(p)} \eta_{S}^{U}
\end{aligned}
$$

and note that

$$
\int_{S} \omega=\int_{U} \eta_{S}^{U} \wedge \pi^{*} \omega=\int_{S} f \omega
$$

for all $\omega$. Since the support of $\omega$ can be chosen to be in any open subset of $S$, this shows that $f=1$ on $S$.

Unless explicitly stated, we assume that duals of submanifolds are calculated inside tubular neighborhoods. Given the structure of the dual on the fibers we shall generally use the notation $\eta_{S}$ with the implicit assumption that it is defined in some tubular neighborhood of $S$. Note that tubular neighborhoods are constructed to be naturally diffeomorphic to a tube around the zero section of a normal bundle of $S \subset N$ (theorem 3.2.6). With that in mind it is natural to focus attention on oriented vector bundles $E \rightarrow S$ with oriented base $S$.

Corollary 8.1.6. If $F: S^{\prime} \rightarrow S$ is a map between closed oriented manifolds and $\pi: E \rightarrow S$ is an oriented m-dimensional vector bundle, then

$$
F^{*}\left(\eta_{S}^{E}\right)=\eta_{S^{\prime}}^{F^{*}(E)}
$$

Proof. The pullback vector bundle is given by

$$
F^{*}(E)=\left\{(x, v) \in S^{\prime} \times E \mid F(x)=\pi(v)\right\}
$$

and thus has the same the same fibers as $E$. This also naturally orients $F^{*}(E)$. When restricting $F^{*}\left(\eta_{S}^{E}\right)$ to a fiber $F^{*}(E)_{x} \simeq E_{F(x)}$ we see that

$$
\int_{F^{*}(E)_{x}} F^{*}\left(\eta_{S}^{E}\right)=\int_{E_{F(x)}} \eta_{S}^{E}=1
$$

Theorem 8.1.5 then implies the claim.
Corollary 8.1.7. If $F: M \rightarrow N$ is proper and transverse to $S$, then for suitable tubular neighborhoods we have

$$
\left[F^{*}\left(\eta_{S}\right)\right]=\left[\eta_{F^{-1}(S)}\right]
$$

Proof. We can assume that both $M$ and $N$ are embedded in Euclidean space so that the tangent spaces come with inner product structures. The key is simply to observe that if $v(S \subset N)$ is the normal bundle, then the pullback bundle $F^{*} v$ is isomorphic to the normal bundle $v^{\prime}=v\left(F^{-1}(S) \subset M\right)$. Since $F$ is transverse to $S$ it follows that each fiber $v_{x}^{\prime}$ is mapped to a subspace $D F\left(v_{x}^{\prime}\right)$ that is a complement to $T_{F(x)} S \subset T_{F(x)} N$. We can then orthogonally project it onto $v_{F(x)}$ to obtain an isomorphism

$$
\left.\operatorname{proj}_{v} \circ D F\right|_{v^{\prime}}: v^{\prime} \rightarrow F^{*}(v) .
$$

This isomorphism is orientation preserving as the orientation on $v_{x}^{\prime}$ is chosen to agree with the orientation for $D F\left(v_{x}^{\prime}\right)$ as in section 5.4.1.

A special interesting case of naturality occurs for submanifolds.

COROLLARY 8.1.8. If $S_{1}^{k_{1}}, S_{2}^{k_{2}} \subset N$ are compact, transverse and oriented, then with suitable orientations on $S_{1} \cap S_{2}$ the dual is given by

$$
\left[\eta_{S_{1}} \wedge \eta_{S_{2}}\right]=\left[\eta_{S_{1} \cap S_{2}}\right]
$$

Proof. We have the inclusions $S_{1} \cap S_{2} \subset S_{1} \subset U$ and $\eta_{S_{1} \cap S_{s}}=i^{*}\left(\eta_{S_{2}}\right)$ since the inclusion $i: S_{1} \rightarrow N$ is transverse to $S_{2}$. Thus for $\omega \in \Omega^{n-k_{1}-k_{2}}(N)$ we see that

$$
\int_{N} \eta_{S_{1}} \wedge \eta_{S_{2}} \wedge \omega=\int_{S_{1}} i^{*}\left(\eta_{S_{2}} \wedge \omega\right)=\int_{S_{1}} \eta_{S_{1} \cap S_{s}}^{S_{1}} \wedge i^{*}(\omega)=\int_{S_{1} \cap S_{s}} \omega
$$

showing that $\eta_{S_{1}} \wedge \eta_{S_{2}}$ represents the dual to $S_{1} \cap S_{s} \subset N$.
We can also apply the naturality of the dual to obtain a new formula for intersection numbers.

Corollary 8.1.9. If $\operatorname{dim} M+\operatorname{dim} S=\operatorname{dim} N$, and $F: M \rightarrow N$ is proper, then

$$
I(F, S)=\int_{M} F^{*}\left(\eta_{S}\right)=\int_{M} F^{*}\left(\eta_{S}^{N}\right)
$$

Proof. We can assume that $F$ is transverse to $S$ as in corollary 8.1.7. Here $F^{-1}(S)$ is a finite collection of points and its normal bundle $v^{\prime}$ is simply the tangent spaces at these points. Similarly, pullback bundle $F^{*}(v)$ consists of finitely many vector spaces that can be identified with the tangent spaces to $M$ via $\operatorname{proj}_{v} \circ D F$. The orientation of the fibers $v_{x}^{\prime} \simeq F^{*}(v)_{x} \simeq v_{F(x)}$ are chosen so that the isomorphisms are orientation preserving. This might not agree with the orientation of $T_{x} M$ (as in section 5.4.1) thus assigning sign ${ }_{x}= \pm 1$ as an orientation for each $x \in F^{-1}(S)$. The sum of these signs is precisely the intersection number. Next identify $v^{\prime}$ with a tubular neighborhood $V \supset F^{-1}(S)$, i.e., a finite collection of pairwise disjoint discs $V_{x}$, and use proposition 1.4 .20 to select a tubular neighborhood $U \supset S$ corresponding to $v$ such that $F^{-1}(U) \subset V$. The orientation choice of $x$ gives us the crucial difference between integrating $F^{*}\left(\eta_{S}^{v}\right)$ and $F^{*}\left(\eta_{S}^{U}\right)$ :

$$
\int_{V_{x}^{\prime}} F^{*}\left(\eta_{S}^{v}\right)=\operatorname{sign}_{x} \int_{V_{x}} F^{*}\left(\eta_{S}^{U}\right)
$$

and thus

$$
I(F, S)=\int_{V} F^{*}\left(\eta_{S}^{U}\right)
$$

Finally,

$$
I(F, S)=\int_{V} F^{*}\left(\eta_{S}^{U}\right)=\int_{M} F^{*}\left(\eta_{S}^{M}\right)
$$

since $\eta_{S} \in \Omega_{c}^{m}(U) \subset \Omega_{c}^{m}(N)$ can be used as a representative for $\left[\eta_{S}^{N}\right]$.
Note the the integral vanishes when $F$ doesn't intersect $S$ or when $\left[\eta_{S}^{N}\right]=0$. The advantage of this formula is that the right-hand side can be calculated even when $F$ isn't transverse to $S$. As both sides are invariant under proper homotopies of $F$ this gives us a more general way of calculating intersection numbers.

### 8.2. The Hopf-Lefschetz Formulas

We are going to relate the Euler characteristic and Lefschetz numbers to the cohomology of the space.

THEOREM 8.2.1. (Hopf-Poincaré) If $M$ is a closed oriented $n$-manifold, then

$$
\chi(M)=I(\Delta, \Delta)=\sum(-1)^{p} \operatorname{dim} H^{p}(M)
$$

Proof. If we consider the map

$$
\begin{aligned}
&(i d, i d): \quad M \rightarrow \Delta \\
&(i d, i d)(x)= \\
&(x, x)
\end{aligned}
$$

then the Euler characteristic can be computed as the intersection number

$$
\begin{aligned}
\chi(M) & =I(\Delta, \Delta) \\
& =I((i d, i d), \Delta) \\
& =\int_{M}(i d, i d)^{*}\left(\eta_{\Delta}^{M \times M}\right) .
\end{aligned}
$$

Thus we need a formula for the Poincaré dual $\eta_{\Delta}=\eta_{\Delta}^{M \times M}$. To find this formula we use Künneth's formula for the cohomology of the product. To this end select a basis $\omega_{i}$ for the cohomology theory $H^{*}(M)$ as well as a dual basis $\tau_{i}$, i.e.,

$$
\int_{M} \omega_{i} \wedge \tau_{j}=\delta_{i j}
$$

where the integral is assumed to be zero if the form $\omega_{i} \wedge \tau_{j}$ doesn't have degree $n$.
By Künneth's theorem $\pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{j}\right)$ is a basis for $H^{*}(M \times M)$. The dual basis is up to a sign given by $\pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right)$ as we can see by calculating

$$
\begin{aligned}
& \int_{M \times M} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{j}\right) \wedge \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) \\
= & (-1)^{\operatorname{deg} \tau_{j} \operatorname{deg} \tau_{k}} \int_{M \times M} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\tau_{j}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) \\
= & (-1)^{\operatorname{deg} \tau_{j}\left(\operatorname{deg} \tau_{k}+\operatorname{deg} \omega_{l}\right)} \int_{M \times M} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) \wedge \pi_{2}^{*}\left(\tau_{j}\right) \\
= & (-1)^{\operatorname{deg} \tau_{j}\left(\operatorname{deg} \tau_{k}+\operatorname{deg} \omega_{l}\right)}\left(\int_{M} \omega_{i} \wedge \tau_{k}\right)\left(\int_{M} \omega_{l} \wedge \tau_{j}\right) \\
= & (-1)^{\operatorname{deg} \tau_{j}\left(\operatorname{deg} \tau_{k}+\operatorname{deg} \omega_{l}\right)} \delta_{i k} \delta_{l j}
\end{aligned}
$$

Clearly this vanishes unless $i=k$ and $l=j$.
This can be used to compute $\eta_{\Delta}$ for $\Delta \subset M \times M$. We assume that

$$
\eta_{\Delta}=\sum c_{i j} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{j}\right)
$$

On one hand

$$
\begin{aligned}
& \int_{M \times M} \eta_{\Delta} \wedge \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) \\
= & \sum c_{i j} \int_{M \times M} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{j}\right) \wedge \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) \\
= & \sum c_{i j}(-1)^{\operatorname{deg} \tau_{j}\left(\operatorname{deg} \tau_{k}+\operatorname{deg} \omega_{l}\right)} \delta_{k i} \delta_{j l} \\
= & c_{k l}(-1)^{\operatorname{deg} \tau_{l}\left(\operatorname{deg} \tau_{k}+\operatorname{deg} \omega_{l}\right)}
\end{aligned}
$$

While on the other hand the fact that $(i d, i d): M \rightarrow \Delta$ is a map of degree 1 tells us that

$$
\begin{aligned}
\int_{M \times M} \eta_{\Delta} \wedge \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) & =\int_{\Delta} \pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right) \\
& =\int_{M}(i d, i d)^{*}\left(\pi_{1}^{*}\left(\tau_{k}\right) \wedge \pi_{2}^{*}\left(\omega_{l}\right)\right) \\
& =\int_{M} \tau_{k} \wedge \omega_{l} \\
& =(-1)^{\operatorname{deg}\left(\tau_{k}\right) \operatorname{deg}\left(\omega_{l}\right)} \delta_{k l}
\end{aligned}
$$

Thus

$$
c_{k l}(-1)^{\operatorname{deg} \tau_{l}\left(\operatorname{deg} \omega_{l}+\operatorname{deg} \tau_{k}\right)}=(-1)^{\operatorname{deg} \tau_{k} \operatorname{deg} \omega_{l}} \delta_{k l}
$$

or in other words $c_{k l}=0$ unless $k=l$ and in that case

$$
\begin{aligned}
c_{k k} & =(-1)^{\operatorname{deg} \tau_{k}\left(2 \operatorname{deg} \omega_{k}+\operatorname{deg} \tau_{k}\right)} \\
& =(-1)^{\operatorname{deg} \tau_{k} \operatorname{deg} \tau_{k}} \\
& =(-1)^{\operatorname{deg} \tau_{k}}
\end{aligned}
$$

This yields the formula

$$
\eta_{\Delta}=\sum(-1)^{\operatorname{deg} \tau_{i}} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{i}\right)
$$

The Euler characteristic can now be computed as follows

$$
\begin{aligned}
\chi(M) & =\int_{M}(i d, i d)^{*}\left(\eta_{\Delta}^{M \times M}\right) \\
& =\int_{M}(i d, i d)^{*}\left(\sum(-1)^{\operatorname{deg} \tau_{i}} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{i}\right)\right) \\
& =\sum(-1)^{\operatorname{deg} \tau_{i}} \int_{M} \omega_{i} \wedge \tau_{i} \\
& =\sum(-1)^{\operatorname{deg} \tau_{i}} \\
& =\sum(-1)^{p} \operatorname{dim} H^{p}(M)
\end{aligned}
$$

A generalization of this leads us to a similar formula for the Lefschetz number of a $\operatorname{map} F: M \rightarrow M$.

THEOREM 8.2.2. (Lefschetz) If $F: M \rightarrow M$, then

$$
L(F)=I(\operatorname{graph}(F), \Delta)=\sum(-1)^{p} \operatorname{tr}\left(F^{*}: H^{p}(M) \rightarrow H^{p}(M)\right) .
$$

Proof. This time we use the map $(i d, F): M \rightarrow \operatorname{graph}(F)$ sending $x$ to $(x, F(x))$ to compute the Lefschetz number

$$
\begin{aligned}
I(\operatorname{graph}(F), \Delta) & =\int_{M}(i d, F)^{*} \eta_{\Delta} \\
& =\int_{M}(i d, F)^{*}\left(\sum(-1)^{\operatorname{deg} \tau_{i}} \pi_{1}^{*}\left(\omega_{i}\right) \wedge \pi_{2}^{*}\left(\tau_{i}\right)\right) \\
& =\sum(-1)^{\operatorname{deg} \tau_{i}} \int_{M} \omega_{i} \wedge F^{*} \tau_{i} \\
& =\sum(-1)^{\operatorname{deg} \tau_{i}} \int_{M} \omega_{i} \wedge F_{i j} \tau_{j} \\
& =\sum(-1)^{\operatorname{deg} \tau_{i}} F_{i j} \delta_{i j} \\
& =\sum(-1)^{\operatorname{deg} \tau_{i}} F_{i i} \\
& =\sum(-1)^{p} \operatorname{tr}\left(F^{*}: H^{p}(M) \rightarrow H^{p}(M)\right) .
\end{aligned}
$$

The definition $I(\operatorname{graph}(F), \Delta)$ for the Lefschetz number is not consistent with Guillemin-Pollack. But if we use their definition, then the formula we just established would have a sign $(-1)^{\operatorname{dim} M}$ on it. This is a very common confusion in the general literature.

### 8.3. Examples of Lefschetz Numbers

It is in fact true that $\operatorname{tr}\left(F^{*}: H^{p}(M) \rightarrow H^{p}(M)\right)$ is always an integer, but to see this requires that we know more algebraic topology. In the cases we study here this can be established directly. Two cases where we do know this to be true are when $p=0$ or $p=\operatorname{dim} M$ and $M$ is compact, connected and oriented, in those cases

$$
\begin{aligned}
\operatorname{tr}\left(F^{*}: H^{0}(M) \rightarrow H^{0}(M)\right) & =1, \\
\operatorname{tr}\left(F^{*}: H^{n}(M) \rightarrow H^{n}(M)\right) & =\operatorname{deg} F .
\end{aligned}
$$

8.3.1. Spheres and Real Projective Spaces. The simplicity of the cohomology of spheres and odd dimensional projective spaces now immediately give us the Lefschetz number in terms of the degree.

When $F: S^{n} \rightarrow S^{n}$ we have $L(F)=1+(-1)^{n} \operatorname{deg} F$. This confirms that any map without fixed points must be homotopic to the antipodal map and therefore have degree $(-1)^{n+1}$.

When $F: \mathbb{R} \mathbb{P}^{2 n+1} \rightarrow \mathbb{R} \mathbb{P}^{2 n+1}$ we have $L(F)=1-\operatorname{deg}(F)$. This also conforms with our feeling for what happens with orthogonal transformations. Namely, if $F \in G l_{2 n+2}^{+}(\mathbb{R})$, then it is possible to not have a fixed point as $F: \mathbb{R}^{2 n+2} \rightarrow \mathbb{R}^{2 n+2}$ might not have an eigenvector. On the other hand, if $F \in G l_{2 n+2}^{-}(\mathbb{R})$, then there should be at least two fixed points.

The even dimensional version $F: \mathbb{R} \mathbb{P}^{2 n} \rightarrow \mathbb{R} \mathbb{P}^{2 n}$ is a bit trickier as the manifold isn't orientable and thus our above approach doesn't work. However, as the only nontrivial cohomology group is when $p=0$ we would expect the mod 2 Lefschetz number to be 1 for all $F$. When $F \in G l_{2 n+1}(\mathbb{R})$, this is indeed true as such maps have an odd number of real eigenvalues. For general $F$ we can lift to a map $\tilde{F}: S^{2 n} \rightarrow S^{2 n}$ satisfying the symmetry condition

$$
\tilde{F}(-x)= \pm \tilde{F}(x) .
$$

The sign $\pm$ must be consistent on the entire sphere. If it is + then we have that $\tilde{F} \circ A=\tilde{F}$, where $A$ is the antipodal map. This shows that $\operatorname{deg} \tilde{F} \cdot(-1)^{2 n+1}=\operatorname{deg} \tilde{F}$, and hence that
$\operatorname{deg} \tilde{F}=0$. In particular, $\tilde{F}$ and also $F$ must have a fixed point. If the sign is - and we assume that $\tilde{F}$ doesn't have a fixed point, then the homotopy to the antipodal map

$$
H(x, t)=\frac{(1-t) \tilde{F}(x)-t x}{|(1-t) \tilde{F}(x)-t x|}
$$

must also be odd

$$
\begin{aligned}
H(-x, t) & =\frac{(1-t) \tilde{F}(-x)-t(-x)}{|(1-t) \tilde{F}(-x)-t(-x)|} \\
& =-\frac{(1-t) \tilde{F}(x)-t(x)}{|(1-t) \tilde{F}(x)-t(x)|} \\
& =-H(x, t)
\end{aligned}
$$

This implies that $F$ is homotopic to the identity on $\mathbb{R}^{2 n}$ and thus $L(F)=L(i d)=1$.
8.3.2. Tori. Next let us consider $M=T^{n}$. The torus is a product of $n$ circles. If we let $d \theta$ be a generator for $H^{1}\left(S^{1}\right)$ and $d \theta_{i}=\pi_{i}^{*}(d \theta)$, where $\pi_{i}: T^{n} \rightarrow S^{1}$ is the projection onto the $i^{\text {th }}$ factor, then example 7.2.7 or Künneth's formula (theorem7.6.1) tells us that $H^{p}\left(T^{n}\right)$ has a basis of the form $d \theta_{i_{1}} \wedge \cdots \wedge d \theta_{i_{p}}, i_{1}<\cdots<i_{p}$. Thus $F^{*}$ is entirely determined by knowing what $F^{*}$ does to $d \theta_{i}$. We write $F^{*}\left(d \theta_{i}\right)=\alpha_{i j} d \theta_{j}$. The action of $F^{*}$ on the basis $d \theta_{i_{1}} \wedge \cdots \wedge d \theta_{i_{p}}, i_{1}<\cdots<i_{p}$ is

$$
\begin{aligned}
F^{*}\left(d \theta_{i_{1}} \wedge \cdots \wedge d \theta_{i_{p}}\right) & =F^{*}\left(d \theta_{i_{1}}\right) \wedge \cdots \wedge F^{*}\left(d \theta_{i_{p}}\right) \\
& =\alpha_{i_{1} j_{1}} d \theta_{j_{1}} \wedge \cdots \wedge \alpha_{i_{p} j_{p}} d \theta_{j_{p}} \\
& =\left(\alpha_{i_{1} j_{1}} \cdots \alpha_{i_{p} j_{p}}\right) d \theta_{j_{1}} \wedge \cdots \wedge d \theta_{j_{p}}
\end{aligned}
$$

this is zero unless $j_{1}, \ldots, j_{p}$ are distinct. Even then, these indices have to be reordered thus introducing a sign. Note also that there are $p$ ! ordered $j_{1}, \ldots, j_{p}$ that when reordered to be increasing are the same. To find the trace we are looking for the "diagonal" entries, i.e., those $j_{1}, \ldots, j_{p}$ that when reordered become $i_{1}, \ldots, i_{p}$. If $S\left(i_{1}, \ldots, i_{p}\right)$ denotes the set of permutations of $i_{1}, \ldots, i_{p}$ then we have shown that

$$
\left.\operatorname{tr} F^{*}\right|_{H^{p}\left(T^{n}\right)}=\sum_{i_{1}<\cdots<i_{p}} \sum_{\sigma \in S\left(i_{1}, \ldots, i_{p}\right)} \operatorname{sign}(\sigma) \alpha_{i_{1} \sigma\left(i_{1}\right)} \cdots \alpha_{i_{p} \sigma\left(i_{p}\right)} .
$$

This leads us to the formula

$$
L(F)=\sum_{p=0}^{n}(-1)^{p} \sum_{i_{1}<\cdots<i_{p}} \sum_{\sigma \in S\left(i_{1}, \ldots, i_{p}\right)} \operatorname{sign}(\sigma) \alpha_{i_{1} \sigma\left(i_{1}\right)} \cdots \alpha_{i_{p} \sigma\left(i_{p}\right)}
$$

We claim that this can be simplified considerably by making the observation

$$
\begin{aligned}
\operatorname{det}\left(\delta_{i j}-\alpha_{i j}\right) & =\sum_{\sigma \in S(1, \ldots, n)} \operatorname{sign}(\sigma)\left(\delta_{1 \sigma(1)}-\alpha_{1 \sigma(1)}\right) \cdots\left(\delta_{n \sigma(n)}-\alpha_{n \sigma(n)}\right) \\
& =\sum_{\sigma \in S(1, \ldots, n)} \operatorname{sign}(\sigma)(-1)^{p} \alpha_{i_{1} \sigma\left(i_{1}\right)} \cdots \alpha_{i_{p} \sigma\left(i_{p}\right)} \delta_{i_{p+1} \sigma\left(i_{p+1}\right)} \cdots \delta_{i_{n} \sigma\left(i_{n}\right)}
\end{aligned}
$$

where in the last sum $\left\{i_{1}, \ldots, i_{p}, i_{p+1}, \ldots, i_{n}\right\}=\{1, \ldots, n\}$. Since the terms vanish unless the permutation fixes $i_{p+1}, \ldots, i_{n}$ we have shown that

$$
L(F)=\operatorname{det}\left(\delta_{i j}-\alpha_{i j}\right)
$$

Finally we claim that the $n \times n$ matrix $\left[\alpha_{i j}\right]$ has integer entries. To see this first lift $F$ to $\tilde{F}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and think of $T^{n}=\mathbb{R}^{n} / \mathbb{Z}^{n}$ where $\mathbb{Z}^{n}$ is the usual integer lattice. Let $e_{i}$ be the canonical basis for $\mathbb{R}^{n}$ and observe that $e_{i} \in \mathbb{Z}^{n}$. The fact that $\tilde{F}$ is a lift of a map in $T^{n}$
means that $\tilde{F}\left(x+e_{i}\right)-\tilde{F}(x) \in \mathbb{Z}^{n}$ for all $x$ and $i=1, \ldots, n$. Since $\tilde{F}$ is continuous we see that

$$
\tilde{F}\left(x+e_{i}\right)-\tilde{F}(x)=\tilde{F}\left(e_{i}\right)-\tilde{F}(0)=A e_{i} \in \mathbb{Z}^{n}
$$

For some $A=\left[a_{i j}\right] \in \operatorname{Mat}_{n \times n}(\mathbb{Z})$. We can then construct a linear homotopy

$$
H(x, t)=(1-t) \tilde{F}(x)+t(A x)
$$

Since

$$
\begin{aligned}
H\left(x+e_{i}, t\right) & =(1-t) \tilde{F}\left(x+e_{i}\right)+t A\left(x+e_{i}\right) \\
& =(1-t)\left(\tilde{F}(x)+A e_{i}\right)+t\left(A x+A e_{i}\right) \\
& =(1-t)(\tilde{F}(x))+t(A x)+A e_{i} \\
& =H(x, t)+A e_{i}
\end{aligned}
$$

we see that this defines a homotopy on $T^{n}$ as well. Thus showing that $F$ is homotopic to the linear map $A$ on $T^{n}$. This means that $F^{*}=A^{*}$. Since $A^{*}\left(d \theta_{i}\right)=a_{j i} d \theta_{j}$, we have shown that $\left[\alpha_{i j}\right]$ is an integer valued matrix.
8.3.3. Complex Projective Space. The cohomology groups of $\mathbb{P}^{n}=\mathbb{C} \mathbb{P}^{n}$ vanish in odd dimensions and are one dimensional in even dimensions. The trace formula for the Lefschetz number therefore can't be too complicated. It turns out to be even simpler and completely determined by the action of the map on $H^{2}\left(\mathbb{P}^{n}\right)$, analogously with what happened on tori. To establish this we need to show that any generator $[\omega] \in H^{2}\left(\mathbb{P}^{n}\right)$ has the property that $\left[\omega^{k}\right] \in H^{2 k}\left(\mathbb{P}^{n}\right)$ is a generator (see also example 7.2 .6 for a different proof). We can use induction on $n$ to show this. Fix $\mathbb{P}^{n-1} \subset \mathbb{P}^{n}$ and recall from section 7.2 that $H^{2 k}\left(\mathbb{P}^{n}\right) \rightarrow H^{2 k}\left(\mathbb{P}^{n-1}\right)$ is an isomorphism for $k \leq n-1$. We can now use the induction hypothesis to claim that $\left[\left.\omega^{k}\right|_{\mathbb{P}^{n-1}}\right] \in H^{2 k}\left(\mathbb{P}^{n-1}\right)$ are nontrivial for $k \leq n-1$. This in turn shows that $\left[\omega^{k}\right] \in H^{2 k}\left(\mathbb{P}^{n}\right)$ are nontrivial for $k \leq n-1$. Finally, since the duality pairing

$$
\begin{aligned}
H^{2}\left(\mathbb{P}^{n}\right) \times H^{2(n-1)}\left(\mathbb{P}^{n}\right) & \rightarrow H^{2 n}\left(\mathbb{P}^{n}\right), \\
\left(\left[\omega_{1}\right],\left[\omega_{2}\right]\right) & \mapsto
\end{aligned}\left[\omega_{1} \wedge \omega_{2}\right] .
$$

is nondegenerate it follows that $\left[\omega^{n}\right]=\left[\omega \wedge \omega^{n-1}\right] \in H^{2 n}\left(\mathbb{P}^{n}\right)$ is a generator.
Now let $F: \mathbb{P}^{n} \rightarrow \mathbb{P}^{n}$ and define $\lambda$ by $F^{*}(\omega)=\lambda \omega$. Then $F^{*}\left(\omega^{k}\right)=\lambda^{k} \omega^{k}$ and

$$
L(F)=1+\lambda+\cdots+\lambda^{n} .
$$

If $\lambda=1$ this gives us $L(F)=n+1$, which was the answer we got for maps from $G l_{n+1}(\mathbb{C})$. In particular, the Euler characteristic $\chi\left(\mathbb{P}^{n}\right)=n+1$. When $\lambda \neq 1$, the formula simplifies to

$$
L(F)=\frac{1-\lambda^{n+1}}{1-\lambda}
$$

Since $\lambda$ is real we note that this can't vanish unless $\lambda=-1$ and $n+1$ is even. Thus all maps on $\mathbb{P}^{2 n}$ have fixed points, just as on $\mathbb{R}^{2 n}$. On the other hand $\mathbb{P}^{2 n+1}$ does admit a map without fixed points, it just can't come from a complex linear map. Instead we just select a real linear map without fixed points that still yields a map on $\mathbb{P}^{2 n+1}$

$$
I\left(\left[z^{0}: z^{1}: \cdots\right]\right)=\left[-\bar{z}^{1}: \bar{z}^{0}: \cdots\right] .
$$

If $I$ fixes a point then

$$
\begin{aligned}
-\lambda \bar{z}^{1} & =z^{0} \\
\lambda \bar{z}^{0} & =z^{1}
\end{aligned}
$$

which implies

$$
-|\lambda|^{2} z^{i}=z^{i}
$$

for all $i$. Since this is impossible the map does not have any fixed points.
Finally we should justify why $\lambda$ is an integer. Let $F_{1}=\left.F\right|_{\mathbb{P}^{1}}: \mathbb{P}^{1} \rightarrow \mathbb{P}^{n}$ and observe that

$$
\lambda[\omega]=\left[F^{*}(\omega)\right]=\left[F_{1}^{*}(\omega)\right] .
$$

We now claim that $F_{1}$ is homotopic to a map $\mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$. To see this note that $F_{1}\left(\mathbb{P}^{1}\right) \subset \mathbb{P}^{n}$ is compact and has measure 0 by Sard's theorem. Thus we can find $p \notin \operatorname{im}\left(F_{1}\right) \cup \mathbb{P}^{1}$. This allows us to deformation retract $\mathbb{P}^{n}-p$ to a $\mathbb{P}^{n-1} \supset \mathbb{P}^{1}$. This $\mathbb{P}^{n-1}$ might not be perpendicular to $p$ in the usual metric, but one can always select a metric where $p$ and $\mathbb{P}^{1}$ are perpendicular and then use the $\mathbb{P}^{n-1}$ that is perpendicular to $p$. Thus $F_{1}: \mathbb{P}^{1} \rightarrow \mathbb{P}^{n}$ is homotopic to a map $F_{2}: \mathbb{P}^{1} \rightarrow \mathbb{P}^{n-1}$. We can repeat this argument until we obtain a map $F_{n}: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$ homotopic to the original $F_{1}$. This implies that

$$
\lambda[\omega]=\left[F^{*}(\omega)\right]=\left[F_{n}^{*}(\omega)\right]
$$

and consequently $\lambda=\operatorname{deg}\left(F_{n}\right)$.
The next two examples show two different approaches to finding a specific form $\omega$. The first example is an abstract construction that yields a unique form, the second offers a concrete calculation of the form in coordinates.

EXAMPLE 8.3.1. The form $\omega$ that generates $H^{2}\left(\mathbb{P}^{2}\right)$ can be constructed to have the property that $\int_{\mathbb{P}^{1}} \omega=1$ for all $\mathbb{P}^{1} \subset \mathbb{P}^{n}$. Recall that we showed in example 7.2.6 that the space of $U(n+1)$ invariant 2-forms is 1-dimensional. So it is clear that we can find $\omega \in \Omega_{U(n+1)}^{2}\left(\mathbb{P}^{n}\right)$ such that $\int_{\mathbb{P}^{1}} \omega=1$ for a specific $\mathbb{P}^{1} \subset \mathbb{P}^{n}$. However, $U(n+1)$ also acts transitively on the space of $\mathbb{P}^{1} \mathrm{~s}$ in $\mathbb{P}^{n}$. Specifically, a $\mathbb{P}^{1}$ corresponds to a complex subspace of dimension 2 in $\mathbb{C}^{n+1}$ and for any two such subspaces there is a unitary transformation that take one into the other. This shows that our chosen 2 -form also integrates to 1 on all other $\mathbb{P}^{1} \subset \mathbb{P}^{n}$.

EXAMPLE 8.3.2. With a bit of complex analysis notation we obtain a more concrete construction.

Using the submersion $\mathbb{C}^{n+1}-\{0\} \rightarrow \mathbb{P}^{n}$ that sends $\left(z^{0}, \ldots, z^{n}\right)$ to $\left[z^{0}: \cdots: z^{n}\right]$ we should be able to construct $\omega$ on $\mathbb{C}^{n+1}-\{0\}$. A bit of auxiliary notation is needed to define the desired 2-form $\omega$ on $\mathbb{C}^{n+1}-\{0\}$ :

$$
\begin{aligned}
d z^{i} & =d x^{i}+\sqrt{-1} d y^{i} \\
d \bar{z}^{i} & =d x^{i}-\sqrt{-1} d y^{i} \\
\frac{\partial f}{\partial z^{i}} & =\frac{1}{2}\left(\frac{\partial f}{\partial x^{i}}-\sqrt{-1} \frac{\partial f}{\partial y^{i}}\right) \\
\frac{\partial f}{\partial \bar{z}^{i}} & =\frac{1}{2}\left(\frac{\partial f}{\partial x^{i}}+\sqrt{-1} \frac{\partial f}{\partial y^{i}}\right) \\
\partial f & =\frac{\partial f}{\partial z^{i}} d z^{i} \\
\bar{\partial} f & =\frac{\partial f}{\partial \bar{z}^{i}} d \bar{z}^{i}
\end{aligned}
$$

The factor $\frac{1}{2}$ and strange signs ensure that the complex differentials work as one would think

$$
\begin{aligned}
d z^{j}\left(\frac{\partial}{\partial z^{i}}\right) & =\frac{\partial z_{j}}{\partial z^{i}}=\delta_{i}^{j}=\frac{\partial \bar{z}_{j}}{\partial \bar{z}^{i}}=d \bar{z}^{j}\left(\frac{\partial}{\partial \bar{z}^{i}}\right) \\
d z^{j}\left(\frac{\partial}{\partial \bar{z}^{i}}\right) & =0=d \bar{z}^{j}\left(\frac{\partial}{\partial z^{i}}\right)
\end{aligned}
$$

More generally we can define $\partial \omega$ and $\bar{\partial} \omega$ for complex valued forms by simply computing $\partial$ and $\bar{\partial}$ of the coefficient functions just as the local coordinate definition of $d$, specifically

$$
\begin{aligned}
\partial\left(f d z^{i_{1}} \wedge \cdots \wedge d z^{i_{p}} \wedge d \bar{z}^{j_{1}} \wedge \cdots \wedge d \bar{z}^{j_{q}}\right) & =\partial f \wedge d z^{i_{1}} \wedge \cdots \wedge d z^{i_{p}} \wedge d \bar{z}^{j_{1}} \wedge \cdots \wedge d \bar{z}^{j_{q}} \\
\bar{\partial}\left(f d z^{i_{1}} \wedge \cdots \wedge d z^{i_{p}} \wedge d \bar{z}^{j_{1}} \wedge \cdots \wedge d \bar{z}^{j_{q}}\right) & =\bar{\partial} f \wedge d z^{i_{1}} \wedge \cdots \wedge d z^{i_{p}} \wedge d \bar{z}^{j_{1}} \wedge \cdots \wedge d \bar{z}^{j_{q}} .
\end{aligned}
$$

With this definition we see that

$$
\begin{aligned}
d & =\partial+\bar{\partial} \\
\partial^{2} & =\bar{\partial}^{2}=\partial \bar{\partial}+\bar{\partial} \partial=0
\end{aligned}
$$

and the Cauchy-Riemann equations for holomorphic functions can be stated as

$$
\bar{\partial} f=0
$$

Working on $\mathbb{C}^{n+1}-\{0\}$ define

$$
\begin{aligned}
\Phi(z) & =\log |z|^{2} \\
& =\log \left(z^{0} \bar{z}^{0}+\cdots+z^{n} \bar{z}^{n}\right)
\end{aligned}
$$

and

$$
\omega=\frac{\sqrt{-1}}{2 \pi} \partial \bar{\partial} \Phi
$$

As $|z|^{2}$ is invariant under $U(n+1)$ the form $\omega$ will also be invariant. If we multiply $z \in$ $\mathbb{C}^{n+1}-\{0\}$ by a nonzero scalar $\lambda$ then

$$
\begin{aligned}
\Phi(\lambda z) & =\log \left(|\lambda z|^{2}\right)=\log |\lambda|^{2}+\log |z|^{2} \\
& =\log |\lambda|^{2}+\Phi(z)
\end{aligned}
$$

so when taking derivatives the constant $\log |\lambda|^{2}$ disappears. This shows that the form $\omega$ becomes invariant under multiplication by complex scalars and so defines a form on $\mathbb{P}^{n}$. That said, it is not possible to define $\Phi$ on all of $\mathbb{P}^{n}$. We give a local coordinate representation below. It is called the potential, or Kähler potential, of $\omega$. Note that the form is exact on $\mathbb{C}^{n+1}-\{0\}$ since

$$
\partial \bar{\partial}=(\partial+\bar{\partial}) \bar{\partial}=d \bar{\partial}
$$

To show that $\omega$ is a nontrivial element of $H^{2}\left(\mathbb{P}^{n}\right)$ it suffices to show that $\int_{\mathbb{P}} \omega \neq 0$. By deleting a point from $\mathbb{P}^{1}$ we can coordinatize it by $\mathbb{C}$. Specifically we consider

$$
\mathbb{P}^{1}=\left[z^{0}: z^{1}: 0: \cdots: 0\right]
$$

and coordinatize $\mathbb{P}^{1}-\{[0: 1: 0: \cdots: 0]\}$ by $z \mapsto[1: z: 0: \cdots: 0]$. Then

$$
\begin{aligned}
\omega & =\frac{\sqrt{-1}}{2 \pi} \partial \bar{\partial} \log (1+z \bar{z}) \\
& =\frac{\sqrt{-1}}{2 \pi}\left(\partial\left(\frac{z d \bar{z}}{1+|z|^{2}}\right)\right) \\
& =\frac{\sqrt{-1}}{2 \pi}\left(\frac{\partial(z d \bar{z})}{1+|z|^{2}}-\left(\partial\left(1+|z|^{2}\right)\right) \wedge \frac{z d \bar{z}}{\left(1+|z|^{2}\right)^{2}}\right) \\
& =\frac{\sqrt{-1}}{2 \pi}\left(\frac{d z \wedge d \bar{z}}{1+|z|^{2}}-(\bar{z} d z) \wedge \frac{z d \bar{z}}{\left(1+|z|^{2}\right)^{2}}\right) \\
& =\frac{\sqrt{-1}}{2 \pi}\left(\frac{d z \wedge d \bar{z}}{1+|z|^{2}}-\frac{|z|^{2} d z \wedge d \bar{z}}{\left(1+|z|^{2}\right)^{2}}\right) \\
& =\frac{\sqrt{-1}}{2 \pi} \frac{d z \wedge d \bar{z}}{\left(1+|z|^{2}\right)^{2}} \\
& =\frac{\sqrt{-1}}{2 \pi} \frac{d(x+\sqrt{-1} y) \wedge d(x-\sqrt{-1} y)}{\left(1+x^{2}+y^{2}\right)^{2}} \\
& =\frac{\sqrt{-1}}{2 \pi} \frac{2 \sqrt{-1} d y \wedge d x}{\left(1+x^{2}+y^{2}\right)^{2}} \\
& =\frac{1}{\pi} \frac{d x \wedge d y}{\left(1+x^{2}+y^{2}\right)^{2}} \\
& =\frac{1}{\pi} \frac{r d r \wedge d \theta}{\left(1+r^{2}\right)^{2}}
\end{aligned}
$$

If we delete the $\pi$ in the formula this is the volume form for the sphere of radius $\frac{1}{2}$ in stereographic coordinates, or the volume form for that sphere in Riemann's conformally flat model. Specifically,

$$
\begin{aligned}
\int_{\mathbb{P}^{1}} \omega & =\int_{\mathbb{P}^{1}-\{[0: 1: 0 ; \cdots: 0]\}} \omega \\
& =\int_{\mathbb{C}} \frac{1}{2 \pi \sqrt{-1}} \frac{d \bar{z} \wedge d z}{\left(1+|z|^{2}\right)^{2}} \\
& =\int_{\mathbb{R}^{2}} \frac{1}{\pi} \frac{d x \wedge d y}{\left(1+x^{2}+y^{2}\right)^{2}} \\
& =\frac{1}{\pi} \int_{0}^{\infty} \int_{0}^{2 \pi} \frac{r d r \wedge d \theta}{\left(1+r^{2}\right)^{2}} \\
& =\int_{0}^{\infty} \frac{2 r d r}{\left(1+r^{2}\right)^{2}} \\
& =1
\end{aligned}
$$

This tells us that the concretely defined form is the unique form described abstractly in the previous example.

We can more generally calculate $\omega$ in the coordinates $z=\left(z^{1}, \ldots, z^{n}\right) \in \mathbb{C}^{n}$ corresponding to points $\left[1: z^{1}: \cdots: z^{n}\right] \in \mathbb{P}^{n}$.

$$
\begin{aligned}
\omega & =\frac{\sqrt{-1}}{2 \pi} \partial \bar{\partial} \log \left(1+z^{1} \bar{z}^{1}+\cdots+z^{n} \bar{z}^{n}\right) \\
& =\frac{\sqrt{-1}}{2 \pi} \partial \bar{\partial} \log \left(1+|z|^{2}\right) \\
& =\frac{\sqrt{-1}}{2 \pi}\left(\partial\left(\frac{\bar{\partial}|z|^{2}}{1+|z|^{2}}\right)\right) \\
& =\frac{\sqrt{-1}}{2 \pi}\left(\frac{\partial \bar{\partial}|z|^{2}}{1+|z|^{2}}-\frac{\partial|z|^{2} \wedge \bar{\partial}|z|^{2}}{\left(1+|z|^{2}\right)^{2}}\right) \\
& =\frac{\sqrt{-1}}{2 \pi\left(1+|z|^{2}\right)^{2}}\left(\left(1+|z|^{2}\right) \partial \bar{\partial}|z|^{2}-\partial|z|^{2} \wedge \bar{\partial}|z|^{2}\right)
\end{aligned}
$$

and in coordinates

$$
\begin{aligned}
\omega & =\frac{\sqrt{-1}}{2 \pi} \partial \bar{\partial} \log \left(1+|z|^{2}\right) \\
& =\frac{\sqrt{-1}}{2 \pi} \frac{\partial^{2} \log \left(1+|z|^{2}\right)}{\partial z^{i} \partial \bar{z}^{j}} d z^{i} \wedge d \bar{z}^{j} \\
& =\frac{\sqrt{-1}}{2 \pi} F_{i \bar{j}} d z^{i} \wedge d \bar{z}^{j}
\end{aligned}
$$

Here the matrix $\left[F_{i \bar{j}}\right]$ is Hermitian and in fact positive definite. The entries are given by

$$
F_{i \bar{j}}=\frac{\left(1+|z|^{2}\right) \delta_{i j}-z^{j} \bar{z}^{i}}{\left(1+|z|^{2}\right)^{2}}
$$

Here the matrix $\left[z^{j} z^{i}\right]=z \cdot z^{*}$, where $z^{*}$ is the adjoint of the column matrix $z$. In particular, the kernel of $z \cdot z^{*}$ consists of all the vectors orthogonal to $z$ and $z$ is an eigenvector with eigenvalue $|z|^{2}$. This also gives the eigenspace decomposition for $\left[F_{i j}\right]$. Specifically, $n-1$ eigenvectors with eigenvalue $\frac{1}{1+|z|^{2}}$ and one eigenvector with eigenvalue $\frac{1}{\left(1+|z|^{2}\right)^{2}}$. Thus $\operatorname{det}\left[F_{i \bar{j}}\right]=\left(1+|z|^{2}\right)^{-n-1}$.

We can now calculate

$$
\begin{aligned}
\omega^{n} & =\left(\frac{\sqrt{-1}}{2 \pi}\right)^{n}\left(F_{i \bar{j}} d z^{i} \wedge d \bar{z}^{j}\right)^{n} \\
& =\left(\frac{\sqrt{-1}}{2 \pi}\right)^{n}\left(F_{i_{1} \bar{j}_{1}} \cdots F_{i_{n} \bar{j}_{n}} d z^{i_{1}} \wedge d \bar{z}^{j_{1}} \wedge \cdots \wedge d z^{i_{n}} \wedge d \bar{z}^{j_{n}}\right)
\end{aligned}
$$

Now note that this vanishes unless all of the indices $i_{1}, \ldots, i_{n}$, as well as $j_{1}, \ldots, j_{n}$, are distinct. After rearraging we obtain

$$
\begin{aligned}
\omega^{n} & =\left(\frac{\sqrt{-1}}{2 \pi}\right)^{n} \operatorname{sign}\left(i_{1}, \ldots, i_{n}\right) \operatorname{sign}\left(j_{1}, \ldots, j_{n}\right) F_{i_{1} \bar{j}_{1}} \cdots F_{i_{n} \bar{j}_{n}} d z^{1} \wedge d \bar{z}^{1} \wedge \cdots \wedge d z^{n} \wedge d \bar{z}^{n} \\
& =\left(\frac{\sqrt{-1}}{2 \pi}\right)^{n} n!\operatorname{det}\left[F_{i \bar{j}}\right] d z^{1} \wedge d \bar{z}^{1} \wedge \cdots \wedge d z^{n} \wedge d \bar{z}^{n} \\
& =\frac{n!}{\pi^{n}\left(1+|z|^{2}\right)^{n+1}} d x^{1} \wedge d y^{1} \wedge \cdots \wedge d x^{n} \wedge d y^{n}
\end{aligned}
$$

and

$$
\int_{\mathbb{P}^{n}} \omega^{n}=\int_{\mathbb{P}^{n}-\mathbb{P}^{n-1}} \omega^{n}=\int_{\mathbb{R}^{2 n}} \pi n!\left(\frac{1}{\pi\left(1+|z|^{2}\right)}\right)^{n+1} d x^{1} \wedge d y^{1} \wedge \cdots \wedge d x^{n} \wedge d y^{n}>0
$$

This shows that $\omega^{n}$ is a volume form and that $\omega^{k} \in H^{2 k}\left(\mathbb{P}^{n}\right)$ is a generator for all $k=$ $0, \ldots, n$.

### 8.4. The Euler Class

We are interested in studying duals and in particular Euler classes in the special case where we have a vector bundle $\pi: E \rightarrow M$ and $M$ is thought of a submanifold of $E$ by embedding it into $E$ via the zero section. The total space $E$ is assumed oriented in such a way that a positive orientation for the fibers together with a positive orientation of $M$ gives us the orientation for $E$. The dimensions are set up so that the fibers of $E \rightarrow M$ have dimension $m$.

The dual $\eta_{M}^{E} \in H_{c}^{m}(E)$ is in this case usually called the Thom class of the bundle $E \rightarrow M$. The embedding $M \subset E$ is proper so by restriction to $M$ this dual defines a class $[e(E)]=i^{*}\left(\eta_{M}^{E}\right) \in H^{m}(M)$ called the Euler class (note that we only defined duals to closed submanifolds so $H_{c}(M)=H(M)$.) Since all sections $s: M \rightarrow E$ are homotopy equivalent we see that $e(E)=s^{*} \eta_{M}$. This immediately proves a very interesting theorem.

THEOREM 8.4.1. If a bundle $\pi: E \rightarrow M$ has a nowhere vanishing section then $e(E)=$ 0.

Proof. Let $s: M \rightarrow E$ be a section and consider $C \cdot s$ for a large constant $C$. Then the image of $C \cdot s$ must be disjoint from the compact support of $\eta_{M}$ and hence $s^{*}\left(\eta_{M}\right)=0$.

This Euler class is also natural
Proposition 8.4.2. Let $F: N \rightarrow M$ be a map that is covered by a vector bundle map $\bar{F}: E^{\prime} \rightarrow E$, i.e., $\bar{F}$ is a linear orientation preserving isomorphism on fibers. Then

$$
e\left(E^{\prime}\right)=F^{*}(e(E))
$$

An example is the pull-back vector bundle is defined by

$$
F^{*}(E)=\{(p, v) \in N \times E \mid \pi(v)=F(q)\} .
$$

Reversing orientation of fibers changes the sign of $\eta_{M}^{E}$ and hence also of $e(E)$. Using $F=i d$ and $\bar{F}(v)=-v$ yields an orientation reversing bundle map when $k$ is odd, showing that $e(E)=0$. Thus we usually only consider Euler classes for even dimensional bundles.

The Euler class can also be used to detect intersection numbers. In case $M$ and the fibers have the same dimension, we can define the intersection number $I(s, M)$ of a section $s: M \rightarrow E$ with the zero section or simply $M$. The formula is

$$
\begin{aligned}
I(s, M) & =\int_{M} s^{*}(e(E)) \\
& =\int_{M} e(E)
\end{aligned}
$$

since all sections are homotopy equivalent to the zero section.
In the special case of the tangent bundle to an oriented manifold $M$ we already know that the intersection number of a vector field $X$ with the zero section is the Euler characteristic. Thus

$$
\chi(M)=I(X, M)=\int_{M} e(T M)
$$

This result was first proven by Hopf and can be used to compute $\chi$ using a triangulation. This is explained in Guillemin-Pollack] and [Spivak].

THEOREM 8.4.3. The Euler class is characterized by

$$
\eta_{M}^{E} \wedge \pi^{*}\left(e_{M}^{E}\right)=\eta_{M}^{E} \wedge \eta_{M}^{E} \in H_{c}^{2 m}(E)
$$

In particular $e_{M}^{E}=0$ if $m$ is odd.
Proof. Since $\pi^{*}\left(e_{M}^{E}\right)$ and $\eta_{M}^{E}$ represent the same class in $H^{m}(E)$ we have that

$$
\pi^{*}\left(e_{M}^{E}\right)-\eta_{M}^{E}=d \omega
$$

Then

$$
\begin{aligned}
\eta_{M}^{E} \wedge \pi^{*}\left(e_{M}^{E}\right)-\eta_{M}^{E} \wedge \eta_{M}^{E} & =\eta_{M}^{E} \wedge(d \omega) \\
& =d\left(\eta_{M}^{E} \wedge \omega\right)
\end{aligned}
$$

Since $\eta_{M}^{E} \wedge \omega$ is compactly supported this shows that $\eta_{M}^{E} \wedge \pi^{*}\left(e_{M}^{E}\right)=\eta_{M}^{E} \wedge \eta_{M}^{E}$.
Moreover, as the map

$$
\begin{aligned}
H^{m}(M) & \rightarrow H_{c}^{2 m}(E) \\
e & \mapsto \eta_{S}^{E} \wedge \pi^{*}(e)
\end{aligned}
$$

is injective, it follows that that the relation $\eta_{S}^{E} \wedge \pi^{*}(e)=\eta_{S}^{E} \wedge \eta_{S}^{E}$ implies that $e=e_{M}^{E}$. In particular, $e_{S}=0$ when $\eta_{S}^{E} \wedge \eta_{S}^{E}=0$. This applies to the case when $m$ is odd as

$$
\eta_{S}^{E} \wedge \eta_{S}^{E}=-\eta_{S}^{E} \wedge \eta_{S}^{E}
$$

The Euler class has other natural properties when we do constructions with vector bundles.

THEOREM 8.4.4. Given two vector bundles $E \rightarrow M$ and $E^{\prime} \rightarrow M$, the Whitney sum has Euler class

$$
e\left(E \oplus E^{\prime}\right)=e(E) \wedge e\left(E^{\prime}\right)
$$

Proof. As we have a better characterization of duals we start with a more general calculation.

Let $\pi: E \rightarrow M$ and $\pi^{\prime}: E^{\prime} \rightarrow M^{\prime}$ be bundles and consider the product bundle $\pi \times \pi^{\prime}$ : $E \times E^{\prime} \rightarrow M \times M^{\prime}$. With this we have the projections $\pi_{1}: E \times E^{\prime} \rightarrow E$ and $\pi_{2}: E \times E^{\prime} \rightarrow E^{\prime}$.

Restricting to the zero sections gives the projections $\pi_{1}: M \times M^{\prime} \rightarrow M$ and $\pi_{2}: M \times M^{\prime} \rightarrow$ $M^{\prime}$. We claim that

$$
\eta_{M \times M^{\prime}}=(-1)^{n \cdot m^{\prime}} \pi_{1}^{*}\left(\eta_{M}\right) \wedge \pi_{2}^{*}\left(\eta_{M^{\prime}}\right) \in H_{c}^{m+m^{\prime}}\left(E \times E^{\prime}\right)
$$

Note that since the projections are not proper it is not clear that $\pi_{1}^{*}\left(\eta_{M}\right) \wedge \pi_{2}^{*}\left(\eta_{M^{\prime}}\right)$ has compact support. However, the support must be compact when projected to $E$ and $E^{\prime}$ and thus be compact in $E \times E^{\prime}$. To see the equality we select volume forms $\omega \in H^{n}(M)$ and $\omega^{\prime} \in H^{n^{\prime}}\left(M^{\prime}\right)$ that integrate to 1 . Then $\pi_{1}^{*}(\omega) \wedge \pi_{2}^{*}\left(\omega^{\prime}\right)$ is a volume form on $M \times M^{\prime}$ that integrates to 1 . Thus it suffices to compute

$$
\begin{aligned}
& \int_{E \times E^{\prime}} \pi_{1}^{*}\left(\eta_{M}\right) \wedge \pi_{2}^{*}\left(\eta_{M^{\prime}}\right) \wedge\left(\pi \times \pi^{\prime}\right)^{*}\left(\pi_{1}^{*}(\omega) \wedge \pi_{2}^{*}\left(\omega^{\prime}\right)\right) \\
= & \int_{E \times E^{\prime}} \pi_{1}^{*}\left(\eta_{M}\right) \wedge \pi_{2}^{*}\left(\eta_{M^{\prime}}\right) \wedge \pi_{1}^{*}\left(\pi^{*}(\omega)\right) \wedge \pi_{2}^{*}\left(\left(\pi^{\prime}\right)^{*}\left(\omega^{\prime}\right)\right) \\
= & (-1)^{n \cdot m^{\prime}} \int_{E \times E^{\prime}} \pi_{1}^{*}\left(\eta_{M}\right) \wedge \pi_{1}^{*}\left(\pi^{*}(\omega)\right) \wedge \pi_{2}^{*}\left(\eta_{M^{\prime}}\right) \wedge \pi_{2}^{*}\left(\left(\pi^{\prime}\right)^{*}\left(\omega^{\prime}\right)\right) \\
= & (-1)^{n \cdot m^{\prime}}\left(\int_{E} \eta_{M} \wedge \pi^{*}(\omega)\right)\left(\int_{E^{\prime}} \eta_{M^{\prime}} \wedge\left(\pi^{\prime}\right)^{*}\left(\omega^{\prime}\right)\right) \\
= & (-1)^{n \cdot m^{\prime}} .
\end{aligned}
$$

When we consider Euler classes this gives us

$$
e\left(E \times E^{\prime}\right)=\pi_{1}^{*}(e(E)) \wedge \pi_{2}^{*}\left(e\left(M^{\prime}\right)\right) \in H_{c}^{m+m^{\prime}}\left(M \times M^{\prime}\right)
$$

The sign is now irrelevant since $e\left(M^{\prime}\right)=0$ if $m^{\prime}$ is odd.
The Whitney sum $E \oplus E^{\prime} \rightarrow M$ of two bundles over the same space is gotten by taking direct sums of the vector space fibers over points in $M$. This means that $E \oplus E^{\prime}=$ $(i d, i d)^{*}\left(E \times E^{\prime}\right)$ where $(i d, i d): M \rightarrow M \times M$ since

$$
(i d, i d)^{*}\left(E \times E^{\prime}\right)=\left\{\left(p, v, v^{\prime}\right) \in M \times E \times E^{\prime}: \pi(v)=p=\pi^{\prime}\left(v^{\prime}\right)\right\}=E \oplus E^{\prime}
$$

Thus we get the formula

$$
e\left(E \oplus E^{\prime}\right)=e(E) \wedge e\left(E^{\prime}\right) .
$$

This implies
COROLLARY 8.4.5. If a bundle $\pi: E \rightarrow M$ admits an orientable odd dimensional sub-bundle $F \subset E$, then $e(E)=0$.

Proof. We have that $E=F \oplus E / F$ or if $E$ carries an inner product structure $E=$ $F \oplus F^{\perp}$. Now orient $F$ and then $E / F$ so that $F \oplus E / F$ and $E$ have compatible orientations. Then $e(E)=e(F) \wedge e(E / F)=0$.

Note that if there is a nowhere vanishing section, then there is a 1 dimensional orientable subbundle. So this recaptures our earlier vanishing theorem. Conversely any orientable 1 dimensional bundle is trivial and thus yields a nowhere vanishing section.

A meaningful theory of invariants for vector bundles using forms should try to avoid odd dimensional bundles altogether. The simplest way of doing this is to consider vector bundles where the vector spaces are complex and then insist on using only complex and Hermitian constructions. This will be investigated further below.

The trivial bundles $\mathbb{R}^{m} \oplus M$ all have $e\left(\mathbb{R}^{m} \oplus M\right)=0$. This is because these bundles are all pull-backs of the bundle $\mathbb{R}^{m} \oplus\{0\}$, where $\{0\}$ is the 1 point space.

To compute $e\left(\tau\left(\mathbb{P}^{n}\right)\right)$ recall that $\tau\left(\mathbb{P}^{n}\right)$ is the conjugate of $\mathbb{P}^{n+1}-\{p\} \rightarrow \mathbb{P}^{n}$ which has dual $\eta_{\mathbb{P}^{n}}=\omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e\left(\tau\left(\mathbb{P}^{n}\right)\right)=-\omega$.

Since $\chi\left(\mathbb{P}^{n}\right)=n+1$ we know that $e\left(T \mathbb{P}^{n}\right)=(n+1) \omega^{n}$.
We go on to describe how the dual and Euler class can be calculated locally. Assume that $M$ is covered by sets $U_{k}$ such that $\left.E\right|_{U_{k}}$ is trivial and that there is a partition of unit $\lambda_{k}$ relative to this covering.

First we analyze what the dual restricted to the fibers might look like. For that purpose we assume that the fiber is isometric to $\mathbb{R}^{m}$. We select a volume form $\psi \in \Omega^{m-1}\left(S^{m-1}\right)$ that integrates to 1 and a bump function $\rho:[0, \infty) \rightarrow[-1,0]$ that is -1 on a neighborhood of 0 and has compact support. Then extend $\psi$ to $\mathbb{R}^{m}-\{0\}$ and consider

$$
d(\rho \psi)=d \rho \wedge \psi
$$

Since $d \rho$ vanishes near the origin this is a globally defined form with total integral

$$
\begin{aligned}
\int_{\mathbb{R}^{m}} d \rho \wedge \psi & =\int_{0}^{\infty} d \rho \int_{S^{m-1}} \psi \\
& =(\rho(\infty)-\rho(0)) \\
& =1
\end{aligned}
$$

Each fiber of $E$ carries such a form. The bump function $\rho$ is defined on all of $E$ by $\rho(v)=$ $\rho(|v|)$, but the "angular" form $\psi$ is not globally defined. As we shall see, the Euler class is the obstruction for $\psi$ to be defined on $E$. Over each $U_{k}$ the bundle is trivial so we do get a closed form $\psi_{k} \in \Omega^{m-1}\left(S\left(\left.E\right|_{U_{k}}\right)\right)$ that restricts to the angular form on fibers. As these forms agree on the fibers the difference depends only on the footpoints:

$$
\psi_{k}-\psi_{l}=\pi^{*} \phi_{k l}
$$

where $\phi_{k l} \in \Omega^{m-1}\left(U_{k} \cap U_{l}\right)$ are closed. These forms satisfy the cocycle conditions

$$
\begin{aligned}
\phi_{k l} & =-\phi_{l k}, \\
\phi_{k i}+\phi_{i l} & =\phi_{k l} .
\end{aligned}
$$

Now define

$$
\varepsilon_{k}=\sum_{i} \lambda_{i} \phi_{k i} \in \Omega^{m-1}\left(U_{k}\right)
$$

and note that the cocycle conditions show that

$$
\begin{aligned}
\varepsilon_{k}-\varepsilon_{l} & =\sum_{i} \lambda_{i} \phi_{k i}-\sum_{i} \lambda_{i} \phi_{l i} \\
& =\sum_{i} \lambda_{i}\left(\phi_{k i}-\phi_{l i}\right) \\
& =\sum_{i} \lambda_{i} \phi_{k l} \\
& =\phi_{k l} .
\end{aligned}
$$

Thus we have a globally defined form $e=d \varepsilon_{k}$ on $M$ since $d\left(\varepsilon_{k}-\varepsilon_{l}\right)=d \phi_{k l}=0$. This will turn out to be the Euler form

$$
e=d\left(\sum_{i} \lambda_{i} \phi_{k i}\right)=\sum_{i} d \lambda_{i} \wedge \phi_{k i}
$$

Next we observe that

$$
\pi^{*} \varepsilon_{k}-\pi^{*} \varepsilon_{l}=\psi_{k}-\psi_{l}
$$

So

$$
\psi=\psi_{k}-\pi^{*} \varepsilon_{k}
$$

defines a form on $E$. This is our global angular form. We now claim that

$$
\begin{aligned}
\eta & =d(\rho \psi) \\
& =d \rho \wedge \psi+\rho d \psi \\
& =d \rho \wedge \psi-\rho \pi^{*} d \varepsilon_{k} \\
& =d \rho \wedge \psi-\rho \pi^{*} e
\end{aligned}
$$

is the dual. First we note that it is defined on all of $E$, is closed, and has compact support. It yields $e$ when restricted to the zero section as $\rho(0)=-1$. Finally when restricted to a fiber we can localize the expression

$$
\eta=d \rho \wedge \psi_{k}-d \rho \wedge \pi^{*} \varepsilon_{k}-\rho \pi^{*} e
$$

But both $\pi^{*} \varepsilon_{k}$ and $\pi^{*} e$ vanish on fibers so $\eta$, when restricted to a fiber, is simply the form we constructed above whose integral was 1 . This shows that $\eta$ is the dual to $M$ in $E$ and that $e$ is the Euler class.

We are now going to specialize to complex line bundles with a Hermitian structure on each fiber. Since an oriented Euclidean plane has a canonical complex structure this is the same as studying oriented 2-plane bundles. The complex structure just helps in setting up the formulas.

The angular form is usually denoted $d \theta$ as it is the differential of the locally defined angle. To make sense of this we select a unit length section $s_{k}: U_{k} \rightarrow S\left(\left.E\right|_{U_{k}}\right)$. For $v \in$ $S\left(\left.E\right|_{U_{k}}\right)$ the angle can be defined by

$$
v=h_{k}(v) s_{k}=e^{\sqrt{-1}} \theta_{k} s_{k} .
$$

This shows that the angular form is given by

$$
\begin{aligned}
d \theta_{k} & =-\sqrt{-1} \frac{d h_{k}}{h_{k}} \\
& =-\sqrt{-1} d \log h_{k} .
\end{aligned}
$$

Since we want the unit circles to have unit length we normalize this and define

$$
\psi_{k}=-\frac{\sqrt{-1}}{2 \pi} d \log h_{k}
$$

On $U_{k} \cap U_{l}$ we have that

$$
h_{l} s_{l}=v=h_{k} s_{k}
$$

So

$$
\left(h_{l}\right)^{-1} h_{k} s_{k}=s_{l} .
$$

But $\left(h_{l}\right)^{-1} h_{k}$ now only depends on the base point in $U_{k} \cap U_{l}$ and not on where $v$ might be in the unit circle. Thus

$$
\pi^{*} g_{k l}=g_{k l} \circ \pi=h_{k}\left(h_{l}\right)^{-1}
$$

where $g_{k l}: U_{k} \cap U_{l} \rightarrow S^{1}$ satisfy the cocycle conditions

$$
\begin{aligned}
\left(g_{k l}\right)^{-1} & =g_{l k} \\
g_{k i} g_{i l} & =g_{k l} .
\end{aligned}
$$

Taking logarithmic differentials then gives us

$$
\begin{aligned}
-\frac{\sqrt{-1}}{2 \pi} \pi^{*} \frac{d g_{k l}}{g_{k l}} & =-\frac{\sqrt{-1}}{2 \pi} \pi^{*} d \log \left(g_{k l}\right) \\
& =\left(-\frac{\sqrt{-1}}{2 \pi} d \log \left(h_{k}\right)\right)-\left(-\frac{\sqrt{-1}}{2 \pi} d \log \left(h_{l}\right)\right) \\
& =\left(-\frac{\sqrt{-1}}{2 \pi} \frac{d h_{k}}{h_{k}}\right)-\left(-\frac{\sqrt{-1}}{2 \pi} \frac{d h_{l}}{h_{l}}\right) .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\varepsilon_{k} & =-\frac{\sqrt{-1}}{2 \pi} \sum_{i} \lambda_{i} d \log \left(g_{k i}\right) \\
\psi & =\left(-\frac{\sqrt{-1}}{2 \pi} \frac{d h_{k}}{h_{k}}\right)-\pi^{*} \varepsilon_{k} \\
e & =d \varepsilon_{k} \\
& =d\left(\frac{\sqrt{-1}}{2 \pi} \sum_{i} \lambda_{i} d \log \left(g_{k i}\right)\right) \\
& =\frac{\sqrt{-1}}{2 \pi} \sum_{i} d \lambda_{i} \wedge d \log \left(g_{k i}\right)
\end{aligned}
$$

This can be used to prove an important result.
Lemma 8.4.6. Let $E \rightarrow M$ and $E^{\prime} \rightarrow M$ be complex line bundles, then

$$
\begin{aligned}
e\left(\operatorname{hom}\left(E, E^{\prime}\right)\right) & =-e(E)+e\left(E^{\prime}\right) \\
e\left(E \otimes E^{\prime}\right) & =e(E)+e\left(E^{\prime}\right)
\end{aligned}
$$

Proof. Note that the sign ensures that the Euler class vanishes when $E=E^{\prime}$.
Select a covering $U_{k}$ such that $E$ and $E^{\prime}$ have unit length sections $s_{k}$ respectively $t_{k}$ on $U_{k}$. If we define $L_{k} \in \operatorname{hom}\left(E, E^{\prime}\right)$ such that $L_{k}\left(s_{k}\right)=t_{k}$, then $h_{k}$ is a unit length section of hom $\left(E, E^{\prime}\right)$ over $U_{k}$. The transitions functions are

$$
\begin{aligned}
g_{k l} s_{k} & =s_{l} \\
\bar{g}_{k l} t_{k} & =t_{l}
\end{aligned}
$$

For $\operatorname{hom}\left(E, E^{\prime}\right)$ we see that

$$
\begin{aligned}
L_{l}\left(s_{k}\right) & =h_{k}\left(g_{l k} s_{l}\right) \\
& =g_{l k} L_{l}\left(s_{l}\right) \\
& =g_{l k} t_{l} \\
& =g_{l k} \bar{g}_{k l} t_{k} \\
& =\left(g_{k l}\right)^{-1} \bar{g}_{k l} t_{k}
\end{aligned}
$$

Thus

$$
L_{l}=\left(g_{k l}\right)^{-1} \bar{g}_{k l} L_{k} .
$$

This shows that

$$
\begin{aligned}
e\left(\operatorname{hom}\left(E, E^{\prime}\right)\right) & =-\frac{\sqrt{-1}}{2 \pi} \sum_{i} d \lambda_{i} \wedge d \log \left(\left(g_{k i}\right)^{-1} \bar{g}_{k i}\right) \\
& =\frac{\sqrt{-1}}{2 \pi} \sum_{i} d \lambda_{i} \wedge d \log \left(g_{k i}\right)-\frac{\sqrt{-1}}{2 \pi} \sum_{i} d \lambda_{i} \wedge d \log \left(\bar{g}_{k i}\right) \\
& =-e(E)+e\left(E^{\prime}\right) .
\end{aligned}
$$

The proof is similar for tensor products using

$$
\begin{aligned}
s_{l} \otimes t_{l} & =\left(g_{k l} s_{k}\right) \otimes\left(\bar{g}_{k l} t_{k}\right) \\
& =g_{k l} \bar{g}_{k l}\left(s_{k} \otimes t_{k}\right) .
\end{aligned}
$$

### 8.5. Characteristic Classes

All vector bundles will be complex and for convenience also have Hermitian structures. Dimensions etc will be complex so a little bit of adjustment is sometimes necessary when we check where classes live. Note that complex bundles are always oriented since $G l_{m}(\mathbb{C}) \subset G l_{2 m}^{+}(\mathbb{R})$.

We are looking for a characteristic class $c(E) \in H^{*}(M)$ that can be written as

$$
\begin{aligned}
c(E)= & c_{0}(E)+c_{1}(E)+c_{2}(E)+\cdots, \\
c_{0}(E)= & 1 \in H^{0}(M) \\
c_{1}(E) \in & H^{2}(M) \\
c_{2}(E) \in & H^{4}(M) \\
& \vdots \\
c_{m}(E) \in & H^{2 m}(M) \\
c_{l}(E)= & 0, l>m
\end{aligned}
$$

For a 1 dimensional or line bundle we simply define $c(E)=1+c_{1}(E)=1+e(E)$. There are two more general properties that these classes should satisfy. First they should be natural in the sense that

$$
c(E)=F^{*}\left(c\left(E^{\prime}\right)\right)
$$

where $F: M \rightarrow M^{\prime}$ is covered by a complex bundle map $E \rightarrow E^{\prime}$ that is an isomorphism on fibers. Second, they should satisfy the product formula

$$
\begin{aligned}
c\left(E \oplus E^{\prime}\right) & =c(E) \wedge c\left(E^{\prime}\right) \\
& =\sum_{p=0}^{m+m^{\prime}} \sum_{i=0}^{p} c_{i}(E) \wedge c_{p-i}\left(E^{\prime}\right)
\end{aligned}
$$

for Whitney sums.
There are two approaches to defining $c(E)$. In Milnor-Stasheff] an inductive method is used in conjunction with the Gysin sequence for the unit sphere bundle. As this approach doesn't seem to have any advantage over the one we shall give here we will not present it. The other method is more abstract, clean, and does not use the Hermitian structure. It is analogous to the construction of splitting fields in Galois theory and is due to Grothendieck.

First we need to understand the cohomology of $H^{*}(\mathbb{P}(E))$. Note that we have a natural fibration $\pi: \mathbb{P}(E) \rightarrow M$ and a canonical line bundle $\tau(\mathbb{P}(E))$. The Euler class of the line bundle is for simplicity denoted

$$
e=e(\tau(\mathbb{P}(E))) \in H^{2}(\mathbb{P}(E))
$$

The fibers of $\mathbb{P}(E) \rightarrow M$ are $\mathbb{P}^{m-1}$ and we note that the natural inclusion $i: \mathbb{P}^{m-1} \rightarrow \mathbb{P}(E)$ is also natural for the tautological bundles

$$
i^{*}(\tau(\mathbb{P}(E)))=\tau\left(\mathbb{P}^{m-1}\right)
$$

thus showing that

$$
i^{*}(e)=e\left(\tau\left(\mathbb{P}^{m-1}\right)\right)
$$

As $e\left(\tau\left(\mathbb{P}^{m-1}\right)\right)$ generates the cohomology of the fiber we have shown that the Leray-Hirch formula for the cohomology of the fibration $\mathbb{P}(E) \rightarrow M$ can be applied. Thus any element $\omega \in H^{*}(\mathbb{P}(E))$ has an expression of the form

$$
\omega=\sum_{i=1}^{m} \pi^{*}\left(\omega_{i}\right) \wedge e^{m-i}
$$

where $\omega_{i} \in H^{*}(M)$ are unique. In particular we can write:

$$
\begin{aligned}
0 & =(-e)^{m}+\pi^{*}\left(c_{1}(E)\right) \wedge(-e)^{m-1}+\cdots+\pi^{*}\left(c_{m-1}(E)\right) \wedge(-e)+\pi^{*}\left(c_{m}(E)\right) \\
& =\sum_{i=0}^{m} \pi^{*}\left(c_{i}(E)\right) \wedge(-e)^{m-i}
\end{aligned}
$$

This means that $H^{*}(\mathbb{P}(E))$ is an extension of $H^{*}(M)$ with a unique monic polynomial

$$
p_{E}(t)=t^{m}+c_{1}(E) t^{m-1}+\cdots+c_{m-1}(E) t+c_{m}(E)
$$

such that $p_{E}(-e)=0$. Moreover, the total Chern class is defined as

$$
p_{E}(1)=c(E)=1+c_{1}(E)+\cdots+c_{m}(E) .
$$

The reason for using $-e$ rather than $e$ is that $-e$ restricts to the form $\omega$ on the fibers of $\mathbb{P}(E)$.

THEOREM 8.5.1. Assume that we have vector bundles $E \rightarrow M$ and $E^{\prime} \rightarrow M^{\prime}$ both of rank $m$, and a smooth map $F: M \rightarrow M^{\prime}$ that is covered by a bundle map that is fiberwise an isomorphism. Then

$$
c(E)=F^{*}\left(c\left(E^{\prime}\right)\right)
$$

Proof. We start by selecting a Hermitian structure on $E^{\prime}$ and then transfer it to $E$ by the bundle map. In that way the bundle map preserves the unit sphere bundles. Better yet, we get a bundle map

$$
\pi^{*}(E) \rightarrow\left(\pi^{\prime}\right)^{*}\left(E^{\prime}\right)
$$

that also yields a bundle map

$$
\tau(\mathbb{P}(E)) \rightarrow \tau\left(\mathbb{P}\left(E^{\prime}\right)\right)
$$

Since the Euler classes for these bundles is natural we have

$$
F^{*}\left(e^{\prime}\right)=e
$$

and therefore

$$
\begin{aligned}
0 & =F^{*}\left(\sum_{i=0}^{m} c_{i}\left(E^{\prime}\right) \wedge\left(-e^{\prime}\right)^{m-i}\right) \\
& =\sum_{i=0}^{m} F^{*} c_{i}\left(E^{\prime}\right) \wedge(-e)^{m-i}
\end{aligned}
$$

Since $c_{i}(E)$ are uniquely defined by

$$
0=\sum_{i=0}^{m} c_{i}(E) \wedge(-e)^{m-i}
$$

we have shown that

$$
c_{i}(E)=F^{*} c_{i}\left(E^{\prime}\right)
$$

The trivial bundles $\mathbb{C}^{m} \oplus M$ all have $c\left(\mathbb{C}^{m} \oplus M\right)=1$. This is because these bundles are all pull-backs of the bundle $\mathbb{C}^{m} \oplus\{0\}$, where $\{0\}$ is the 1 point space.

To compute $e\left(\tau\left(\mathbb{P}^{n}\right)\right)$ recall that $\tau\left(\mathbb{P}^{n}\right)$ is the conjugate of $\mathbb{P}^{n+1}-\{p\} \rightarrow \mathbb{P}^{n}$ which has dual $\eta_{\mathbb{P}^{n}}=\omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e\left(\tau\left(\mathbb{P}^{n}\right)\right)=-\omega$.

The Whitney sum formula is established by proving the splitting principle.
THEOREM 8.5.2. If a bundle $\pi: E \rightarrow M$ splits $E=L_{1} \oplus \cdots \oplus L_{m}$ as a direct sum of line bundles, then

$$
c(E)=\prod_{i=1}^{m}\left(1+e\left(L_{i}\right)\right)
$$

Proof. We pull back all classes to $E$ without changing notation. We know that $c(E)=p_{E}(1)$ so it suffices to identify $p_{E}$ with the monic polynomial of degree $m$ defioned by $p(t)=\prod_{i=1}^{m}\left(t+e\left(L_{i}\right)\right)$. To prove this we need to show that

$$
p(-e)=\prod_{i=1}^{m}\left(-e+e\left(L_{i}\right)\right)=0
$$

Note that we can identify $-e+e\left(L_{i}\right)$ with the Euler class of hom $\left(\tau, L_{i}\right)$. With that in mind:

$$
\begin{aligned}
\prod_{i=1}^{m}\left(-e+e\left(L_{i}\right)\right) & =e\left(\bigoplus_{i=1}^{m} \operatorname{Hom}\left(\tau, L_{i}\right)\right) \\
& =e\left(\operatorname{Hom}\left(\tau, L_{1} \oplus \cdots \oplus L_{m}\right)\right) \\
& =e(\operatorname{Hom}(\tau, E)) \\
& =e\left(\operatorname{Hom}\left(\tau, \tau \oplus \tau^{\perp}\right)\right) \\
& =e(\operatorname{Hom}(\tau, \tau)) \wedge e\left(\operatorname{Hom}\left(\tau, \tau^{\perp}\right)\right) \\
& =0
\end{aligned}
$$

Where the last equality follows from the fact that $\operatorname{Hom}(\tau, \tau)$ has the identity map as a nowhere vanishing section.

The splitting principle can be used to compute $c\left(T \mathbb{P}^{n}\right)$. First note that $T \mathbb{P}^{n} \simeq \operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \tau\left(\mathbb{P}^{n}\right)^{\perp}\right)$. Thus

$$
\begin{aligned}
T \mathbb{P}^{n} \oplus \mathbb{C} & =\operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \tau\left(\mathbb{P}^{n}\right)^{\perp}\right) \oplus \mathbb{C} \\
& =\operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \tau\left(\mathbb{P}^{n}\right)^{\perp}\right) \oplus \operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \tau\left(\mathbb{P}^{n}\right)\right) \\
& =\operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \tau\left(\mathbb{P}^{n}\right)^{\perp} \oplus \tau\left(\mathbb{P}^{n}\right)\right) \\
& =\operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \mathbb{C}^{n+1}\right) \\
& =\operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \mathbb{C}\right) \oplus \cdots \oplus \operatorname{Hom}\left(\tau\left(\mathbb{P}^{n}\right), \mathbb{C}\right) .
\end{aligned}
$$

Thus

$$
\begin{aligned}
c\left(T \mathbb{P}^{n}\right) & =c\left(T \mathbb{P}^{n} \oplus \mathbb{C}\right) \\
& =(1+\omega)^{n+1}
\end{aligned}
$$

This shows that

$$
c_{i}\left(T \mathbb{P}^{n}\right)=\binom{n+1}{i} \omega^{i}
$$

which conforms with

$$
e\left(T \mathbb{P}^{n}\right)=c_{n}\left(T \mathbb{P}^{n}\right)=(n+1) \omega^{n}
$$

We can now finally establish the Whitney sum formula.
THEOREM 8.5.3. For two vector bundles $E \rightarrow M$ and $E^{\prime} \rightarrow M$ we have

$$
c\left(E \oplus E^{\prime}\right)=c(E) \wedge c\left(E^{\prime}\right) .
$$

Proof. First we repeatedly projectivize so as to create a map $\tilde{N} \rightarrow M$ with the property that it is an injection on cohomology and the pull-back of $E$ to $\tilde{N}$ splits as a direct sum of line bundles. Then repeat this procedure on the pull-back of $E^{\prime}$ to $\tilde{N}$ until we finally get a map $F: N \rightarrow M$ such that $F^{*}$ is an injection on cohomology and both of the bundles split

$$
\begin{aligned}
F^{*}(E) & =L_{1} \oplus \cdots \oplus L_{m}, \\
F^{*}\left(E^{\prime}\right) & =K_{1} \oplus \cdots \oplus K_{m^{\prime}}
\end{aligned}
$$

The splitting principle together with naturality then implies that

$$
\begin{aligned}
F^{*}\left(c\left(E \oplus E^{\prime}\right)\right) & =c\left(F^{*}\left(E \oplus E^{\prime}\right)\right) \\
& =c\left(L_{1}\right) \wedge \cdots \wedge c\left(L_{m}\right) \wedge c\left(K_{1}\right) \wedge \cdots \wedge c\left(K_{m^{\prime}}\right) \\
& =c\left(F^{*}(E)\right) \wedge c\left(F^{*}\left(E^{\prime}\right)\right) \\
& =F^{*} c(E) \wedge F^{*} c\left(E^{\prime}\right) \\
& =F^{*}\left(c(E) \wedge c\left(E^{\prime}\right)\right)
\end{aligned}
$$

Since $F^{*}$ is an injection this shows that

$$
c\left(E \oplus E^{\prime}\right)=c(E) \wedge c\left(E^{\prime}\right) .
$$

### 8.6. The Gysin Sequence

This sequence allows us to compute the cohomology of certain fibrations where the fibers are spheres. As we saw above, these fibrations are not necessarily among the ones where we can use the Hirch-Leray formula. This sequence uses the Euler class and will recapture the dual, or Thom class, from the Euler class.

We start with an oriented vector bundle $\pi: E \rightarrow M$. It is possible to put a smoothly varying inner product structure on the vector spaces of the fibration, using that such bundles are locally trivial and gluing inner products together with a partition of unity on $M$. The function $E \rightarrow \mathbb{R}$ that takes $v$ to $|v|^{2}$ is then smooth and the only critical value is 0 . As such we get a smooth manifold with boundary

$$
D(E)=\{v \in E:|v| \leq 1\}
$$

called the disc bundle with boundary

$$
S(E)=\partial D(E)=\{v \in E:|v|=1\}
$$

being the unit sphere bundle and interior

$$
\operatorname{int} D(E)=\{v \in E:|v|<1\}
$$

Two different inner product structures will yield different disc bundles, but it is easy to see that they are all diffeomorphic to each other. We also note that $\operatorname{int} D(E)$ is diffeomorphic to $E$, while $D(E)$ is homotopy equivalent to $E$. This gives us a diagram

$$
\begin{array}{ccccccccc}
\rightarrow & H_{c}^{p}(\operatorname{int} D(E)) & \rightarrow & H^{p}(D(E)) & \rightarrow & H^{p}(S(E)) & \rightarrow & H_{c}^{p+1}(\operatorname{int} D(E)) & \rightarrow \\
& \downarrow & & \uparrow & & & \uparrow & & \uparrow \\
& \rightarrow & H_{c}^{p}(E) & \rightarrow & H^{p}(E) & \rightarrow & H^{p}(S(E)) & \rightarrow & \rightarrow \\
& \rightarrow & H_{c}^{p+1}(E) & \rightarrow
\end{array}
$$

where the vertical arrows are simply pull-backs and all are isomorphims. The connecting homomorphism

$$
H^{p}(S(E)) \rightarrow H_{c}^{p+1}(\operatorname{int} D(E))
$$

then yields a map

$$
H^{p}(S(E)) \rightarrow H_{c}^{p+1}(E)
$$

that makes the bottom sequence a long exact sequence. Using the Thom isomorphism

$$
H^{p-m}(M) \rightarrow H_{c}^{p}(E)
$$

then gives us a new diagram

$$
\begin{array}{ccccccccc}
\rightarrow & H^{p-m}(M) & \xrightarrow{e \wedge} & H^{p}(M) & \rightarrow & H^{p}(S(E)) & \cdots & H^{p+1-m}(M) & \rightarrow \\
& \downarrow \eta_{M} \wedge \pi^{*}(\cdot) & & \downarrow & & \downarrow & & \downarrow & \\
\rightarrow & H_{c}^{p}(E) & \rightarrow & H^{p}(E) & \rightarrow & H^{p}(S(E)) & \rightarrow & H_{c}^{p+1}(E) & \rightarrow
\end{array}
$$

Most of the arrows are pull-backs and the vertical arrows are isomorphisms. The first square is commutative since $\pi^{*} i^{*}\left(\eta_{M}\right)=\pi^{*}(e)$ is represented by $\eta_{M}$ in $H^{m}(E)$. This is simply because the zero section $I: M \rightarrow E$ and projection $\pi: E \rightarrow M$ are homotopy equivalences. The second square is obviously commutative. Thus we get a map

$$
H^{p}(S(E))-\rightarrow H^{p+1-m}(M)
$$

making the top sequence exact. This is the Gysin sequence of the sphere bundle of an oriented vector bundle. The connecting homomorphism which lowers the degree by $m-1$ can be constructed explicitly and geometrically by integrating forms on $S(E)$ along the unit spheres, but we won't need this interpretation.

The Gysin sequence also tells us how the Euler class can be used to compute the cohomology of the sphere bundle from $M$.

To come full circle with the Leray-Hirch Theorem we now assume that $E \rightarrow M$ is a complex bundle of complex dimension $m$ and construct the projectivized bundle

$$
\mathbb{P}(E)=\left\{(p, L) \mid L \subset \pi^{-1}(p) \text { is a } 1 \text { dimensional subspace }\right\}
$$

This gives us projections

$$
S(E) \rightarrow \mathbb{P}(E) \rightarrow M
$$

There is also a tautological bundle

$$
\tau(\mathbb{P}(E))=\{(p, L, v) \mid v \in L\}
$$

The unit-sphere bundle for $\tau$ is naturally identified with $S(E)$ by

$$
\begin{aligned}
S(E) & \rightarrow S(\tau(\mathbb{P}(E))) \\
(p, v) & \rightarrow(p, \operatorname{span}\{v\}, v)
\end{aligned}
$$

This means that $S(E)$ is part of two Gysin sequences. One where $M$ is the base and one where $\mathbb{P}(E)$ is the base. These two sequences can be connected in a very interesting manner.

If we pull back $E$ to $\mathbb{P}(E)$ and let

$$
\tau^{\perp}=\left\{(p, L, w) \mid w \in L^{\perp}\right\}
$$

be the orthogonal complement then we have that

$$
\pi^{*}(e(E))=e\left(\pi^{*}(E)\right)=e(\tau(\mathbb{P}(E))) \wedge e\left(\tau^{\perp}\right) \in H^{*}(\mathbb{P}(E))
$$

Thus we obtain a commutative diagram


What is more we can now show in two ways that

$$
\operatorname{span}\left\{1, e, \ldots, e^{m-1}\right\} \otimes H^{*}(M) \rightarrow H^{*}(\mathbb{P}(E))
$$

is an isomorphism. First we can simply use the Leray-Hirch result by noting that the classes $1, e, \ldots, e^{m-1}$ when restricted to the fibers are the usual cohomology classes of the fiber $\mathbb{P}^{m}$. Or we can use diagram chases on the above diagram.

### 8.7. Further Study

There are several texts that expand on the material covered here. The book by Guillemin-Pollack] is the basic prerequisite for the material covered in the early chapters. The cohomology aspects we cover here correspond to a simplified version of [Bott-Tu]. Another text is the well constructed Madsen-Tornehave], which in addition explains how characteristic classes can be computed using curvature. The comprehensive text [Spivak vol. V] is also worth consulting for many aspects of the theory discussed here. For a more topological approach we recommend [Milnor-Stasheff]. Other useful texts are listed in the references.

### 8.8. Exercises

(1) Show that a map $F: S^{2} \times S^{2} \rightarrow \mathbb{C P}^{2}$ has even degree. Hint: Use suitable $\omega \in$ $\Omega^{2}\left(\mathbb{C P}^{2}\right)$ and $\omega^{\prime} \in \Omega^{2}\left(S^{2}\right)$ that can be used to generate volume forms on $\mathbb{C P}^{2}$ and $S^{2} \times S^{2}$.
(2) Show that there are classes in $H^{1}\left(T^{2}\right)$ which are not duals to closed 1-dimensional submanifolds of $T^{2}$.
(3) Show that if $E \rightarrow M$ is a vector bundle with $\operatorname{dim} E_{x}=m, x \in M$, then $H_{c}^{*}(M) \simeq$ $H_{c}^{*+m}(E)$, provided $M$ and $E$ are oriented manifolds. Show that the Möbius band is a counterexample in case $E$ is not orientable.

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