AN EXOTIC SPHERE WITH POSITIVE SECTIONAL CURVATURE

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In memory of Detlef Gromoll

During the 1950s, a famous theorem in geometry and some perplexing examples in topology were discovered that turned out to have unexpected connections. In geometry, the development was the Quarter Pinched Sphere Theorem. ([Berg1], [Kling], and [Rau])

Theorem *(Rauch-Berger-Klingenberg, 1952-1961) If a simply connected, complete manifold has sectional curvature between* $1/4$ *and* 1*, i.e.,*

$$1/4 < \sec \leq 1,$$

then the manifold is homeomorphic to a sphere.

The topological examples were [Miln]

Theorem *(Milnor, 1956) There are* 7*-manifolds that are homeomorphic to, but not diffeomorphic to, the* 7*-sphere.*

The latter result raised the question as to whether or not the conclusion in the former is optimal. After a long history of partial solutions, this problem has been finally solved.

Theorem *(Brendle-Schoen, 2007) Let* M *be a complete, Riemannian manifold and* $f : M \longrightarrow (0, \infty)$ *a* C^∞*–function so that at each point* x *of* M *the sectional curvature satisfies*

$$\frac{f(x)}{4} < \sec_x \leq f(x).$$

Then M *is diffeomorphic to a spherical space form.*

Prior to this major breakthrough, there were many partial results. Starting with Gromoll and Shikata ([Grom] and [Shik]) and more recently Suyama ([Suy]) it was shown that if one allows for a stronger pinching hypothesis $\delta \leq \sec \leq 1$ for some δ close to 1, then, in the simply connected case, the manifold is diffeomorphic to a sphere. In the opposite direction, Weiss showed that not all exotic spheres admit quarter pinched metrics [Weis].

Unfortunately, this body of technically difficult geometry and topology might have been about a vacuous subject. Until now there has not been a single example of an exotic sphere with positive sectional curvature.

To some extent this problem was alleviated in 1974 by Gromoll and Meyer [GromMey].

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Theorem *(Gromoll-Meyer, 1974) There is an exotic 7–sphere with nonnegative sectional curvature and positive sectional curvature at a point.*

A metric with this type of curvature is called *quasi-positively curved*, and positive curvature almost everywhere is referred to as *almost positive curvature*. In 1970 Aubin showed the following. (See [Aub] and also [Ehrl] for a similar result for scalar curvature.)

Theorem *(Aubin, 1970) Any complete metric with quasi-positive Ricci curvature can be perturbed to one with positive Ricci curvature.*

Coupled with the Gromoll-Meyer example, this raised the question of whether one could obtain a positively curved exotic sphere via a perturbation argument. Some partial justification for this came with Hamilton's Ricci flow and his observation that a metric with quasi-positive curvature operator can be perturbed to one with positive curvature operator (see [Ham]).

This did not change the situation for sectional curvature. For a long time, it was not clear whether the appropriate context for this problem was the Gromoll-Meyer sphere itself or more generally an arbitrary quasi-positively curved manifold. The mystery was due to an appalling lack of examples. For a 25–year period the Gromoll-Meyer sphere and the flag type example in [Esch1] were the only known examples with quasi-positive curvature that were not known to also admit positive curvature.

This changed around the year 2000 with the body of work [PetWilh], [Tapp1], [Wilh2], and [Wilk] that gave us many examples of almost positive curvature. In particular, [Wilk] gives examples with almost positive sectional curvature that do not admit positive sectional curvature, the most dramatic being a metric on $\mathbb{R}P^3 \times \mathbb{R}P^2$. We also learned in [Wilh2] that the Gromoll-Meyer sphere admits almost positive sectional curvature. (See [EschKer] for a more recent and much shorter proof.) Here we show that this space actually admits positive curvature.

Theorem *The Gromoll-Meyer exotic sphere admits positive sectional curvature.*

On the other hand, we know from the theorem of Brendle and Schoen that the Gromoll-Meyer sphere cannot carry pointwise, $\frac{1}{4}$–pinched, positive curvature. In addition, we know from [Weis] that it cannot carry

$$\sec \ \geq 1 \text{ and radius } > \frac{\pi}{2}$$

and from [GrovWilh] that it also can not admit

$$\sec \ \geq 1 \text{ and four points at pairwise distance} > \frac{\pi}{2}.$$

We still do not know whether any exotic sphere can admit

$$\sec \ \geq 1 \text{ and diameter } > \frac{\pi}{2}.$$

The Diameter Sphere Theorem says that such manifolds are topological spheres ([Berg3], [GrovShio]). We also do not know the diffeomorphism classification of "almost $\frac{1}{4}$–pinched", positively curved manifolds. According to [AbrMey] and [Berg4] such spaces are either diffeomorphic to CROSSes or topological spheres.

The class with $\sec \ \geq 1$ and diameter $> \frac{\pi}{2}$ includes the globally $\frac{1}{4}$–pinched, simply connected, class, apparently as a tiny subset. Indeed, globally $\frac{1}{4}$–pinched spheres

have uniform lower injectivity radius bounds, whereas manifolds with sec ≥ 1 and diameter $> \frac{\pi}{2}$ can be Gromov-Hausdorff close to intervals.

In contrast to the situation for sectional curvature, quite a bit is known about manifolds with positive scalar curvature, Ricci curvature, and curvature operator. Starting with the work of Hitchin, it became clear that not all exotic spheres can admit positive scalar curvature. In fact, the class of simply connected manifolds that admit positive scalar curvature is pretty well understood, thanks to work of Lichnerowicz, Hitchin, Schoen-Yau, Gromov-Lawson and most recently Stolz [Stol]. Since it is usually hard to understand metrics without any symmetries, it is also interesting to note that Lawson-Yau have shown that any manifold admitting a nontrivial S^3 action carries a metric of positive scalar curvature. In particular, exotic spheres that admit nontrivial S^3 actions carry metrics of positive scalar curvature. Poor and Wraith have also found a lot of exotic spheres that admit positive Ricci curvature ([Poor] and [Wrai]). By contrast Böhm-Wilking in [BohmWilk] showed that manifolds with positive curvature operator all admit metrics with constant curvature and hence no exotic spheres occur. This result is also a key ingredient in the differentiable sphere theorem by Brendle-Schoen mentioned above.

We construct our example as a deformation of a metric with nonnegative sectional curvature, so it is interesting to ponder the possible difference between the classes of manifolds with positive curvature and those with merely nonnegative curvature. For the three tensorial curvatures, much is known. For sectional curvature, the grim fact remains that there are no known differences between nonnegative and positive curvature for simply connected manifolds. Probably the most promising conjectured obstruction for passing from nonnegative to positive curvature is admitting a free torus action. Thus Lie groups of higher rank, starting with $S^3 \times S^3$, might be the simplest nonnegatively curved spaces that do not carry metrics with positive curvature. The Hopf conjecture about the Euler characteristic being positive for even dimensional positively curved manifolds is another possible obstruction to $S^3 \times S^3$ having positive sectional curvature. The other Hopf problem about whether or not $S^2 \times S^2$ admits positive sectional curvature is probably much more subtle.

Although our argument is very long, we will quickly establish that there is a good chance to have positive curvature on the Gromoll-Meyer sphere, Σ^7. Indeed, in the first section, we start with the metric from [Wilh2] and show that by scaling the fibers of the submersion $\Sigma^7 \longrightarrow S^4$, we get integrally positive curvature over the sections that have zero curvature in [Wilh2]. More precisely, the zero locus in [Wilh2] consists of a (large) family of totally geodesic 2–dimensional tori. We will show that after scaling the fibers of $\Sigma^7 \longrightarrow S^4$, the integral of the curvature over any of these tori becomes positive. The computation is fairly abstract, and the argument is made in these abstract terms, so no knowledge of the metric of [Wilh2] is required.

The difficulties of obtaining positive curvature after the perturbation of section 1 cannot be over stated. After scaling the fibers, the curvature is no longer nonnegative, and although the integral is positive, this positivity is to a higher order than the size of the perturbation. This higher order positivity is the best that we can hope for. Due to the presence of totally geodesic tori, there can be no perturbation of the metric that is positive to first order on sectional curvature [Stra]. The technical significance of this can be observed by assuming that one has a C^∞ family of

metrics $\{g_t\}_{t\in\mathbb{R}}$ with g_0 a metric of nonnegative curvature. If, in addition,

$$\frac{\partial}{\partial t}\sec_{g_t} P\Big|_{t=0} > 0$$

for all planes P so that $\sec_{g_0} P = 0$, then g_t has positive curvature for all sufficiently small $t > 0$. Since no such perturbation of the metric in [Wilh2] is possible, it will not be enough for us to consider the effect of our deformation on the set, Z, of zero planes of the metric in [Wilh2]. Instead we will have to check that the curvature becomes positive in an entire neighborhood of Z. This will involve understanding the change of the full curvature tensor.

According to recent work of Tapp, any zero plane in a Riemannian submersion of a biinvariant metric on a compact Lie group exponentiates to a flat. Thus any attempt at perturbing any of the known quasipositively curved examples to positive curvature would have to tackle this issue [Tapp2].

In contrast to the metric of [EschKer], the metric in [Wilh2] does not come from a left (or right) invariant metric on $Sp(2)$. So although the Gromoll–Meyer sphere is a quotient of the Lie group $Sp(2)$, we do not use Lie theory for any of our curvature computations or even for the definition of our metric. Our choice here is perhaps a matter of taste. The overriding idea is that although none of the metrics considered lift to left invariant ones on $Sp(2)$, there is still a lot of structure. Our goal is to exploit this structure to simplify the exposition as much as we can.

Our substitute for Lie theory is the pull-back construction of [Wilh1]. In fact, the current paper is a continuation of [PetWilh], [Wilh1], and [Wilh2]. The reader who wants a thorough understanding of our argument will ultimately want to read these earlier papers. We have, nevertheless, endeavored to make this paper as self-contained as possible by reviewing the basic definitions, notations, and results of [PetWilh], [Wilh1], and [Wilh2] in sections 2, 3, and 4. It should be possible to skip the earlier papers on a first read, recognizing that although most of the relevant results have been restated, the proofs and computations are not reviewed here. On the other hand, Riemannian submersions play a central role throughout the paper; so the reader will need a working knowledge of [On].

After establishing the existence of integrally positive curvature and reviewing the required background, we give a detailed and technical summary of the remainder of the argument in section 5. Unfortunately, aspects of the specific geometry of the Gromoll-Meyer sphere are scattered throughout the paper, starting with section 2; so it was not possible to write section 5 in a way that was completely independent of the review sections. Instead we offer the following less detailed summary with the hope that it will suffice for the moment.

Starting from the Gromoll-Meyer metric the deformations to get positive curvature are

- **(1):** The $(h_1 \oplus h_2)$–Cheeger deformation, described in section 3
- **(2):** The redistribution, described in section 6.
- **(3):** The $(U \oplus D)$–Cheeger deformation, described in section 3
- **(4):** The scaling of the fibers, described in section 1
- **(5):** The partial conformal change, described in section 10
- **(6):** The $\Delta(U, D)$ Cheeger deformation and a further h_1–deformation.

We let g_1, $g_{1,2}$, $g_{1,2,3}$, ect. be the metrics obtained after doing deformations (1), (1) and (2), or (1), (2), and (3) respectively.

It also makes sense to talk about metrics like $g_{1,3}$, i.e. the metric obtained from doing just deformations (1) and (3) without deformation (2).

All of the deformations occur on $Sp(2)$. So at each stage we verify invariance of the metric under the various group actions that we need. For the purpose of this discussion we let g_1, $g_{1,2}$, $g_{1,2,3}$, ect. stand for the indicated metric on both $Sp(2)$ and Σ^7.

$g_{1,3}$ is the metric of [Wilh2] that has almost positive curvature on Σ^7. $g_{1,2,3}$ is also almost positively curvature on Σ^7, and has precisely the same zero planes as $g_{1,3}$. Some specific positive curvatures of $g_{1,3}$ are redistributed in $g_{1,2,3}$. The reasons for this are technical, but as far as we can tell without deformation (2) our methods will not produce positive curvature. It does not seem likely that either $g_{1,2}$ or $g_{1,2,3}$ are nonnegatively curved on $Sp(2)$, but we have not verified this.

Deformation (4), scaling the fibers of $Sp(2) \longrightarrow S^4$, is the raison d'être of this paper. $g_{1,2,3,4}$ has some negative curvatures, but has the redeeming feature that the integral of the curvatures of the zero planes of $g_{1,3}$ is positive. In fact this integral is positive over any of the flat tori of $g_{1,3}$.

The role of deformation (5) is to even out the positive integral. The curvatures of the flat tori of $g_{1,3}$ are pointwise positive with respect to $g_{1,2,3,4,5}$.

To understand the role of deformation (6), recall that we have to check that we have positive curvature not only on the 0–planes of $g_{1,3}$, but in an entire neighborhood (of uniform size) of the zero planes of $g_{1,3}$. To do this suppose that our zero planes have the form

$$P = \text{span}\{\zeta, W\}.$$

We have to understand what happens when the plane is perturbed by moving its foot point, and also what happens when the plane moves within the fibers of the Grassmannian.

To deal with the foot points, we extend ζ and W to families of vectors $\mathcal{F}_\zeta$ and $\mathcal{F}_W$ on $Sp(2)$. These families can be multivalued and $\mathcal{F}_W$ contains some vectors that are not horizontal for the Gromoll-Meyer submersion. All pairs $\{\zeta, W\}$ that contain zero planes of $\left(\Sigma^7, g_{1,3}\right)$ are contained in these families, and the families are defined in a fixed neighborhood of the 0–locus of $g_{1,3}$. All of our arguments are valid for all pairs $\{z, V\}$ with $z \in \mathcal{F}_\zeta$ and $V \in \mathcal{F}_W$, provided z and V have the same foot point. In this manner, we can focus our attention on fiberwise deformations of the zero planes.

To do this we consider planes of the form

$$P = \text{span}\{\zeta + \sigma z, W + \tau V\}$$

where σ, τ are real numbers and z and V are tangent vectors. Ultimately we show that all values of all curvature polynomials

$$P(\sigma, \tau) = \text{curv}(\zeta + \sigma z, W + \tau V)$$

are positive.

Allowing σ, τ, z and V to range through all possible values describes an open dense subset in the Grassmannian fiber. The complement of this open dense set consists of planes that have either no z component or no W component. These curvatures can be computed as combinations of quartic, cubic, and quadratic terms in suitable polynomials $P(\sigma, \tau)$. In sections 12 and 13 we show that these combinations/curvatures do not decrease much under our deformations (in a proportional sense); so the entire Grassmannian is positively curved.

The role of the Cheeger deformations in (6) is that any fixed plane with a non-degenerate projection to the vertical space of $\Sigma^7 \longrightarrow S^4$ becomes positively curved, provided these deformations are carried out for a sufficiently long time. Although the zero planes $P = \operatorname{span}\{\zeta, W\}$ all have degenerate projections to the vertical space of $\Sigma^7 \longrightarrow S^4$, there are of course nearby planes whose projections are nondegenerate. Exploiting this idea we get

Proposition 0.1. *If all curvature polynomials whose corresponding planes have degenerate projection onto the vertical space of $\Sigma^7 \longrightarrow S^4$ are positive on $\left(\Sigma^7, g_{1,2,3,4,5}\right)$, then $\left(\Sigma^7 g_{1,2,3,4,5,6}\right)$ is positively curved, provided the Cheeger deformations in (6) are carried out for a sufficiently long time.*

Proof. The assumptions imply that a neighborhood N of the 0–locus of $g_{1,3}$ is positively curved with respect to $g_{1,2,3,4,5}$. The complement of this neighborhood is compact, so $g_{1,2,3,4,5,6}$ is positively curved on the whole complement, provided the Cheeger deformations in (6) are carried out for enough time. Since Cheeger deformations preserve positive curvature $g_{1,2,3,4,5,6}$ is also positively curved on N. So $g_{1,2,3,4,5,6}$ is positively curved. □

Thus the deformations in (6) allow us the computational convenience of assuming that the vector "z" is in the horizontal space of $\Sigma^7 \longrightarrow S^4$.

In the sequel, we will not use the notation $g_1, g_{1,2}, g_{1,2,3}$, ect. . Rather we will use more suggestive notation for these metrics, which we will specify in Section 5.

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1. Integrally Positive Curvature

Here we show that it is possible to perturb the metric from [Wilh2] to one that has more positive curvature but also has some negative curvatures. The sense in which the curvature has increased is specified in the theorem below. The idea is that if we integrate the curvatures of the planes that used to have zero curvature, then the answer is positive after the perturbation. The theorem is not specific to the Gromoll-Meyer sphere.

Theorem 1.1. *Let (M, g_0) be a Riemannian manifold with nonnegative sectional curvature and*

$$\pi : (M, g_0) \longrightarrow B$$

a Riemannian submersion. Further assume that G is an isometric group action on M that is by symmetries of π and that the intrinsic metrics on the principal orbits of G in B are homotheties of each other.

Let $T \subset M$ be a totally geodesic, flat torus spanned by geodesic fields X and W such that X is horizontal for π and $D\pi(W) = H_w$ is a Killing field for the G–action on B. We suppose further that X is invariant under G, $D\pi(X)$ is orthogonal to the orbits of G, and the normal distribution to the orbits of G on B is integrable. Let g_s be the metric obtained from g_0 by scaling the lengths of the fibers of π by

$$\sqrt{1 - s^2}.$$

Let c be an integral curve of $d\pi(X)$ from a zero of $|H_w|$ to a maximum of $|H_w|$ along c,whose interior passes through principle orbits. Then

$$\int_c \mathrm{curv}_{g_s}(X,W) = s^4 \int_c \left(D_X\left(|H_w|\right)\right)^2.$$

In particular, the curvature of span$\{X,W\}$ is integrally positive along c, provided H_w is not identically 0 along c.

Here and throughout the paper we set

$$\mathrm{curv}(X,W) \equiv R(X,W,W,X).$$

The formulas for the curvature tensor of metrics obtained by warping the fibers of a Riemannian submersion by a *function* on the base were computed by Detlef Gromoll and his Stony Brook students in various classes over the years. We were made aware of them via lecture notes by Carlos Duran [GromDur]. They will appear shortly in the textbook [GromWals]. In the case when the function is constant, these formulas are necessarily much simpler and can also be found in [Bes], where scaling the fibers by a constant is referred to as the "canonical variation". To ultimately get positive curvature on the Gromoll-Meyer sphere, we have to control the curvature tensor in an entire neighborhood in the Grassmannian, so we will need several of these formulas. In fact, since the particular "W" that we have in mind is neither horizontal nor vertical for π, we need multiple formulas just to find $\mathrm{curv}(X,W)$.

For vertical vectors $U,V\in\mathcal{V}$ and horizontal vectors $X,Y,Z\in\mathcal{H}$, for $\pi: M\to B$ we have

$$\begin{aligned}
\left(R^{g_s}(X,V)U\right)^{\mathcal{H}} &= \left(1-s^2\right)\left(R(X,V)U\right)^{\mathcal{H}} + \left(1-s^2\right)s^2 A_{A_XU}V\\
R^{g_s}(V,X)Y &= \left(1-s^2\right)R(V,X)Y + s^2\left(R(V,X)Y\right)^{\mathcal{V}} + s^2 A_X A_Y V\\
R^{g_s}(X,Y)Z &= \left(1-s^2\right)R(X,Y)Z + s^2\left(R(X,Y)Z\right)^{\mathcal{V}} + s^2 R^B(X,Y)Z \qquad (1.2)
\end{aligned}$$

The superscripts $^{\mathcal{H}}$ and $^{\mathcal{V}}$ denote the horizontal and vertical parts of the vectors, R and A are the curvature and A-tensors for the unperturbed metric g, R^{g_s} denotes the new curvature tensor of g_s, and R^B is the curvature tensor of the base.

To eventually understand the curvature in a neighborhood of the Gromoll-Meyer 0-locus, we will need formulas for

$$\begin{aligned}
&R^{g_s}(W,X)X \text{ and}\\
&\left(R^{g_s}(X,W)W\right)^{\mathcal{H}}
\end{aligned}$$

where X is as above and W is an arbitrary vector in TM.

Lemma 1.3. *Let*

$$\pi: (M,g_0)\longrightarrow B$$

be as above. Let X be a horizontal vector for π and let W be an arbitrary vector in TM. Then

$$\begin{aligned}
R^{g_s}(W,X)X &= \left(1-s^2\right)R(W,X)X + s^2\left(R(W,X)X\right)^{\mathcal{V}}\\
&\quad + s^2 R^B\left(W^{\mathcal{H}},X\right)X + s^2 A_X A_X W^{\mathcal{V}}\\
\left(R^{g_s}(X,W)W\right)^{\mathcal{H}} &= \left(1-s^2\right)\left(R(X,W)W\right)^{\mathcal{H}}\\
&\quad + \left(1-s^2\right)s^2 A_{A_XW^{\mathcal{V}}}W^{\mathcal{V}} + s^2 R^B\left(X,W^{\mathcal{H}}\right)W^{\mathcal{H}}
\end{aligned}$$

Remark 1.4. *Notice that the first curvature terms vanish in both formulas on the totally geodesic torus.*

Proof. We split $W = W^{\mathcal{V}} + W^{\mathcal{H}}$ and get

$$\begin{aligned} R^{g_s}(W,X)X &= R^{g_s}\left(W^{\mathcal{V}},X\right)X + R^{g_s}\left(W^{\mathcal{H}},X\right)X \\ &= \left(1-s^2\right)R(W^{\mathcal{V}},X)X + s^2\left(R(W^{\mathcal{V}},X)X\right)^{\mathcal{V}} + s^2 A_X A_X W^{\mathcal{V}} \\ &\quad + \left(1-s^2\right)R\left(W^{\mathcal{H}},X\right)X + s^2\left(R\left(W^{\mathcal{H}},X\right)X\right)^{\mathcal{V}} + s^2 R^B\left(W^{\mathcal{H}},X\right)X \\ &= \left(1-s^2\right)R(W,X)X + s^2\left(R(W,X)X\right)^{\mathcal{V}} + s^2 R^B\left(W^{\mathcal{H}},X\right)X + s^2 A_X A_X W^{\mathcal{V}} \end{aligned}$$

To find the other curvature we use

$$\begin{aligned} R^{g_s}(X,W)W &= R^{g_s}\left(X,W^{\mathcal{V}}\right)W^{\mathcal{V}} + R^{g_s}\left(X,W^{\mathcal{H}}\right)W^{\mathcal{V}} \\ &\quad + R^{g_s}\left(X,W^{\mathcal{V}}\right)W^{\mathcal{H}} + R^{g_s}\left(X,W^{\mathcal{H}}\right)W^{\mathcal{H}} \end{aligned}$$

Since $A_X A_{W^{\mathcal{H}}} W^{\mathcal{V}}$ and $A_{W^{\mathcal{H}}} A_X W^{\mathcal{V}}$ are vertical the above curvature formulas imply

$$\begin{aligned} \left(R^{g_s}\left(X,W^{\mathcal{H}}\right)W^{\mathcal{V}}\right)^{\mathcal{H}} &= \left(1-s^2\right)\left(R\left(X,W^{\mathcal{H}}\right)W^{\mathcal{V}}\right)^{\mathcal{H}} \\ \left(R^{g_s}\left(X,W^{\mathcal{V}}\right)W^{\mathcal{H}}\right)^{\mathcal{H}} &= \left(1-s^2\right)\left(R\left(X,W^{\mathcal{V}}\right)W^{\mathcal{H}}\right)^{\mathcal{H}}. \end{aligned}$$

In addition we have

$$\begin{aligned} \left(R^{g_s}\left(X,W^{\mathcal{V}}\right)W^{\mathcal{V}}\right)^{\mathcal{H}} &= \left(1-s^2\right)\left(R\left(X,W^{\mathcal{V}}\right)W^{\mathcal{V}}\right)^{\mathcal{H}} + \left(1-s^2\right)s^2 A_{A_X W^{\mathcal{V}}} W^{\mathcal{V}} \\ \left(R^{g_s}\left(X,W^{\mathcal{H}}\right)W^{\mathcal{H}}\right)^{\mathcal{H}} &= \left(1-s^2\right)\left(R\left(X,W^{\mathcal{H}}\right)W^{\mathcal{H}}\right)^{\mathcal{H}} + s^2 R^B\left(X,W^{\mathcal{H}}\right)W^{\mathcal{H}}. \end{aligned}$$

Therefore

$$\left(R^{g_s}(X,W)W\right)^{\mathcal{H}} = \left(1-s^2\right)\left(R(X,W)W\right)^{\mathcal{H}} + \left(1-s^2\right)s^2 A_{A_X W^{\mathcal{V}}} W^{\mathcal{V}} + s^2 R^B\left(X,W^{\mathcal{H}}\right)W^{\mathcal{H}}$$

as claimed. □

Now let X and W be as in the theorem. We set $H_w = D\pi\left(W^{\mathcal{H}}\right)$ and $V = W^{\mathcal{V}}$. To prove the theorem we need to find $\operatorname{curv}_B(X, H_w)$ and $A_X V$.

Lemma 1.5.

$$R^B(H_w,X)X = -\left(\frac{D_X D_X |H_w|}{|H_w|}\right)H_w$$

Proof. Since X is invariant under G, $[X, H_w] \equiv 0$. Since X is also a geodesic field

$$R^B(H_w,X)X = -\nabla_X \nabla_{H_w} X.$$

Similarly, since the normal distribution to the orbits of G on B is integrable we can extend any normal vector z to a G–invariant normal field Z, and get that all terms of the Koszul formula for

$$\langle \nabla_{H_w} X, Z \rangle$$

vanish. In particular, $\nabla_{H_w} X$ is tangent to the orbits of G.

If K is another Killing field we have that X commutes with K as well as H_w, and $[K, H_w]$ is perpendicular to X as it is again a Killing field. Combining this with our hypothesis that the intrinsic metrics on the principal orbits of G in B are

homotheties of each other, we see from Koszul's formula that $\nabla_{H_w} X$ is proportional to H_w and can be calculated by

$$\begin{aligned} \langle \nabla_{H_w} X, H_w \rangle &= \langle \nabla_X H_w, H_w \rangle \\ &= \frac{1}{2} D_X |H_w|^2 \\ &= |H_w| D_X |H_w|, \text{ so} \\ \nabla_{H_w} X &= \frac{D_X |H_w|}{|H_w|} H_w. \end{aligned}$$

Thus

$$\begin{aligned} R^B(H_w, X) X &= -\nabla_X \left(\frac{D_X |H_w|}{|H_w|} H_w \right) \\ &= -D_X \left(\frac{D_X |H_w|}{|H_w|} \right) H_w - \left(\frac{D_X |H_w|}{|H_w|} \nabla_X H_w \right) \\ &= -\left(\frac{|H_w| D_X D_X |H_w| - (D_X |H_w|)^2}{|H_w|^2} \right) H_w - \left(\frac{D_X |H_w|}{|H_w|} \right)^2 H_w \\ &= -\left(\frac{D_X D_X |H_w|}{|H_w|} \right) H_w. \end{aligned}$$

□

Lemma 1.6.

$$R^B(X, H_w) H_w = -|H_w| \nabla_X (\operatorname{grad} |H_w|).$$

Proof. Let Z be any vector field. Using that H_w is a Killing field we get

$$\begin{aligned} \langle \nabla_{H_w} H_w, Z \rangle &= -\langle \nabla_Z H_w, H_w \rangle \\ &= -\frac{1}{2} D_Z \langle H_w, H_w \rangle \\ &= -\frac{1}{2} D_Z |H_w|^2 \\ &= -|H_w| D_Z |H_w| \\ &= -\langle |H_w| \operatorname{grad} |H_w|, Z \rangle \end{aligned}$$

showing that

$$\nabla_{H_w} H_w = -|H_w| \operatorname{grad} |H_w|.$$

Thus

$$\begin{aligned} R^B(X, H_w) H_w &= \nabla_X \nabla_{H_w} H_w - \nabla_{H_w} \nabla_X H_w \\ &= -\nabla_X (|H_w| \operatorname{grad} |H_w|) - \nabla_{H_w} \left(\frac{D_X |H_w|}{|H_w|} H_w \right) \\ &= -(D_X |H_w|) \operatorname{grad} |H_w| - (|H_w| \nabla_X \operatorname{grad} |H_w|) - \frac{D_X |H_w|}{|H_w|} \nabla_{H_w} H_w \\ &= -(D_X |H_w|) \operatorname{grad} |H_w| - (|H_w| \nabla_X \operatorname{grad} |H_w|) + \frac{D_X |H_w|}{|H_w|} |H_w| \operatorname{grad} |H_w| \\ &= -(|H_w| \nabla_X \operatorname{grad} |H_w|) \end{aligned}$$

□

It follows that

$$\begin{aligned} \operatorname{curv}_B(X, H_w) &= \left\langle R^B(H_w, X)X, H_w \right\rangle \\ &= -\left(\frac{D_X D_X |H_w|}{|H_w|}\right) \langle H_w, H_w \rangle \\ &= -|H_w| \left(D_X D_X |H_w|\right). \end{aligned} \tag{1.7}$$

Next we focus on $|A_X V|^2$.

Lemma 1.8.

$$A_X V = -\frac{D_X |H_w|}{|H_w|} H_w.$$

Proof. Since X and W are commuting geodesic fields on a totally geodesic flat torus, $\nabla_X W = 0$.

So

$$\begin{aligned} A_X V &= (\nabla_X V)^{\mathcal{H}} \\ &= (\nabla_X W - \nabla_X H_w)^{\mathcal{H}} \\ &= -(\nabla_X H_w)^{\mathcal{H}} \\ &= -\nabla^B_{H_w} X \\ &= -\frac{D_X |H_w|}{|H_w|} H_w \end{aligned}$$

□

Combining this A–tensor formula with equation 1.7 and Lemma 1.3 yields

$$\begin{aligned} \operatorname{curv}_{g_s}(X, W) &= \left(1 - s^2\right) \operatorname{curv}(X, W) + s^2 \operatorname{curv}_B(X, H_w) - s^2 |A_X V|^2 + s^4 |A_X V|^2 \\ &= \left(1 - s^2\right) \operatorname{curv}(X, W) - s^2 \left(|H_w| \left(D_X D_X |H_w|\right)\right) - s^2 \left(D_X |H_w|\right)^2 + s^4 \left(D_X |H_w|\right)^2 \end{aligned}$$

Since $\operatorname{curv}(X, W) = 0$, this further simplifies to

$$\operatorname{curv}_{g_s}(X, W) = -s^2 \left(D_X \left(|H_w| D_X |H_w|\right)\right) + s^4 \left(D_X |H_w|\right)^2. \tag{1.9}$$

If c is an integral curve of X from a zero of H_w to a maximum of $|H_w|$ along c, then the first term integrates to 0 along c, yielding

$$\int_c \operatorname{curv}_{g_s}(X, W) = s^4 \int_c \left(D_X |H_w|\right)^2$$

as desired.

As we've mentioned, to get positive curvature on the Gromoll-Meyer sphere we will have to understand the full curvature tensor. Combining the calculations above we have

Lemma 1.10. *Let X and W be as in Theorem 1.1. Then*

$$\begin{aligned} R^{g_s}(W, X)X &= -s^2 \left(\frac{D_X D_X |H_w|}{|H_w|}\right) H_w - s^2 \frac{D_X |H_w|}{|H_w|} A_X H_w \\ \left(R^{g_s}(X, W)W\right)^{\mathcal{H}} &= -\left(1 - s^2\right) s^2 \frac{D_X |H_w|}{|H_w|} A_{H_w} W^{\mathcal{V}} - s^2 |H_w| \nabla_X \left(\operatorname{grad} |H_w|\right). \end{aligned}$$

Remark 1.11. *The two A–tensors $A_X H_w$ and $A_{H_w} W^{\mathcal{V}}$ involve derivatives of vectors that are not tangent or normal to the totally geodesic tori. They cannot be determined abstractly, and are in fact dependent on the particular geometry. We give estimates for them in the case of the Gromoll-Meyer sphere in Lemma 9.2 below.*

2. Review of the geometry of $Sp(2)$

The next three sections are a review of [PetWilh], [Wilh1], and [Wilh2].

We let $h : S^7 \longrightarrow S^4$ and $\tilde{h} : S^7 \longrightarrow S^4$ be the Hopf fibrations corresponding to the right A^h and left $A^{\tilde{h}}$ actions of S^3 on S^7.

Points on S^7 are denoted by pairs of quaternions written as column vectors. The quotient map for action on the right is

$$h : \begin{pmatrix} a \\ c \end{pmatrix} \mapsto (a\bar{c}, \frac{1}{2}(|a|^2 - |c|^2)),$$

and the quotient map for action on the left is

$$\tilde{h} : \begin{pmatrix} a \\ c \end{pmatrix} \mapsto (\bar{a}c, \frac{1}{2}(|a|^2 - |c|^2)).$$

The image is $S^4(\frac{1}{2}) \subset \mathbb{H} \oplus \mathbb{R}$ [Wilh1].

Proposition 2.1. *(The Pullback Identification) $Sp(2)$ is diffeomorphic to the total space of the pullback of the Hopf fibration $S^7 \xrightarrow{h} S^4$ via $S^7 \xrightarrow{-I \circ h} S^4$, where $S^4 \xrightarrow{-I} S^4$ is the antipodal map. In fact, the biinvariant metric on $Sp(2)$ is isometric (up to rescaling) to the subspace metric on the pullback*

$$(-I \circ h)^* \left(S^7\right) \subset S^7(1) \times S^7(1),$$

where $S^7(1)$ is the unit 7-sphere and $S^7(1) \times S^7(1)$ has the product metric.

In [GromMey] it was shown that Σ^7 is the quotient of the S^3-action on $Sp(2)$ given by

$$A_{2,-1}\left(q, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} qa\bar{q} & qb \\ qc\bar{q} & qd \end{pmatrix}.$$

We let $q_{2,-1} : Sp(2) \longrightarrow \Sigma^7$ denote the quotient map. It was observed by Gromoll and Meyer that Σ^7 is the S^3–bundle over S^4 of "type $(2,-1)$", using the classification convention of [Miln]. The submersion $p_{2,-1} : \Sigma^7 \longrightarrow S^4$ is induced by

$$\widetilde{h} \circ p_2|_{Sp(2)} : Sp(2) \longrightarrow S^4,$$

where $p_2 : S^7 \times S^7 \longrightarrow S^7$ is projection onto the second factor.

The Gromoll-Meyer metric on Σ^7 is induced by the biinvariant metric via $q_{2,-1}$. The metric studied in [Wilh2], $g_{\nu_1,\nu_2,l_1^u,l_1^d}$, is induced via $q_{2,-1}$ by the perturbation of the biinvariant metric that was studied in [PetWilh]. We will review the definition of this metric in the next section.

The isometry group of the metric discovered by Gromoll and Meyer is $O(2) \times SO(3)$. The $O(2)$-action is induced on Σ^7 by the action $A_{O(2)}$ on $Sp(2)$ defined as

$$\begin{array}{rcl} O(2) \times Sp(2) & \longrightarrow & Sp(2) \\ (A, U) & \mapsto & AU. \end{array}$$

The $SO(3)$-action is induced on Σ^7 by the S^3–action A^{h_2} on $Sp\,(2)$ defined as

$$\begin{array}{ccc} S^3 \times Sp\,(2) & \longrightarrow & Sp\,(2) \\ \left(q, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) & \longmapsto & \begin{pmatrix} a & b\bar{q} \\ c & d\bar{q} \end{pmatrix}. \end{array}$$

As in [Wilh1] we have

Proposition 2.2. *Every point in Σ^7 has a point in its orbit under $A_{SO(2)} \times A^{h_2}$ that can be represented in $Sp(2)$ by a point of the form*

$$\left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} p, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}\right)$$

with $t \in \left[0, \frac{\pi}{4}\right]$, $p, \alpha \in S^3 \subset \mathbb{H}$, and $\operatorname{Re}(\alpha) = 0$.

Since only A^{h_2} acts by isometries with respect to the metrics we study, the points in the previous proposition have to be multiplied by $SO\,(2)$ to get

Proposition 2.3. *Every point in Σ^7 has a representative point $(N_1 p, N_2)$ in its orbit under A^{h_2} that in $Sp(2)$ has the form*

$$\begin{aligned} (N_1 p, N_2) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} p, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} \cos\theta\cos t + \alpha \sin\theta \sin t \\ -\sin\theta\cos t + \alpha\cos\theta\sin t \end{pmatrix} p, \begin{pmatrix} \sin\theta\cos t + \alpha\cos\theta\sin t \\ \cos\theta\cos t - \alpha\sin\theta\sin t \end{pmatrix}\right) \end{aligned}$$

with $t \in \left[0, \frac{\pi}{4}\right]$, $\theta \in [0, \pi]$, $p, \alpha \in S^3$, and $\operatorname{Re}(\alpha) = 0$.

We have a similar representation in S^7.

Corollary 2.4. *Every point in S^7 has a point in its orbit under $A^{\tilde{h}} \times A^h$ of the form*

$$N = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} = \begin{pmatrix} \cos\theta\cos t + \alpha\sin\theta\sin t \\ -\sin\theta\cos t + \alpha\cos\theta\sin t \end{pmatrix}$$

with $t \in \left[0, \frac{\pi}{4}\right]$, $\theta \in [0, \pi]$, $\alpha \in S^3$, and $\operatorname{Re}(\alpha) = 0$.

The h–fiber of N consists of the points

$$\left\{Np : p \in S^3\right\}.$$

We need a basis for the tangent space of $Sp\,(2)$ that is well adapted to the Gromoll-Meyer sphere and its symmetry group. It turns out that a left invariant framing is ill suited for this purpose; rather we use a basis that comes from S^7 via the embedding $Sp\,(2) \subset S^7 \times S^7$. To get the correct basis we point out

Proposition 2.5. *$SO\,(2) \times A^h$ acts on S^7 by symmetries of $\tilde{h}$. The action induced on S^4 has $\mathbb{Z}_2$–kernel and induces an effective $SO\,(2) \times SO\,(3)$ action that respects the join decomposition $S^4 = S^1 * S^2$. The $SO\,(2)$–factor acts in the standard way on S^1 and as the identity on S^2. The $SO\,(3)$ action is standard on the S^2–factor and the identity on the S^1–factor. (See* [GluWarZil]*, cf also the proof of Proposition 1.2 in* [Wilh1]*.)*

Remark 2.6. *At a representative point*

$$\begin{aligned}(N_1p, N_2) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\left(\begin{pmatrix} \cos t \\ \alpha\sin t \end{pmatrix}p, \begin{pmatrix} \alpha\sin t \\ \cos t \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} \cos\theta\cos t + \alpha\sin\theta\sin t \\ -\sin\theta\cos t + \alpha\cos\theta\sin t \end{pmatrix}p, \begin{pmatrix} \sin\theta\cos t + \alpha\cos\theta\sin t \\ \cos\theta\cos t - \alpha\sin\theta\sin t \end{pmatrix}\right),\end{aligned}$$

the parameter θ, is the "S^1"–coordinate in $S^1 * S^2$, α is the S^2–coordinate, t is the distance to the singular S^1 in $S^1 * S^2$ and p parameterizes the fibers of $p_{2,-1} : \Sigma^7 \longrightarrow S^4$, giving us a partial coordinate system (t, θ, α, p) for Σ^7. We denote the singular S^1 in $S^1 * S^2$ by $S^1_{\mathbb{R}}$ and we denote the singular S^2 by S^2_{Im}. The points in $S^1_{\mathbb{R}}$ are represented in $Sp(2)$ by the points with $t = 0$, and S^2_{Im} corresponds to the set where $t = \frac{\pi}{4}$. Thus

$$\begin{aligned}S^1_{\mathbb{R}} &= \tilde{h} \circ p_2|_{Sp(2)} \left\{ \left(\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}p, \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\right) \in Sp(2) : \theta \in [0,\pi], p \in S^3 \right\} \text{ and} \\ S^2_{\mathrm{Im}} &= \tilde{h} \circ p_2|_{Sp(2)} \left\{ \left(\frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta + \alpha\sin\theta \\ -\sin\theta + \alpha\cos\theta \end{pmatrix}p, \frac{1}{\sqrt{2}}\begin{pmatrix} \sin\theta + \alpha\cos\theta \\ \cos\theta - \alpha\sin\theta \end{pmatrix}\right) \in Sp(2) : \right. \\ &\quad \left. \theta \in [0,\pi], \alpha, p \in S^3, \text{ and } \mathrm{Re}(\alpha) = 0 \right\}.\end{aligned}$$

Throughout the paper, γ_1 and γ_2 will be purely imaginary unit quaternions that satisfy $\gamma_1\gamma_2 = \alpha$. Using such a choice for γ_1 and γ_2 gets us a basis for the vertical space of h at $N \subset S^7$ by setting

$$\begin{aligned}\mathfrak{v} &= N\alpha p, \\ \vartheta_1 &= N\gamma_1 p, \\ \vartheta_2 &= N\gamma_2 p.\end{aligned}$$

The fibers of h and $\tilde{h}$ have a one-dimensional intersection when $t > 0$ and coincide when $t = 0$. $\mathfrak{v}$ is tangent to this intersection.

We get a basis for the horizontal space of h by selecting a suitable vector perpendicular to N. When $\theta = 0$ a natural choice is

$$\hat{N} = \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix}.$$

For general θ we just multiply by an element in $SO(2)$ and get

$$\begin{aligned}\hat{N} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \\ &= \begin{pmatrix} -\cos\theta\sin t + \alpha\sin\theta\cos t \\ \sin\theta\sin t + \alpha\cos\theta\cos t \end{pmatrix}.\end{aligned}$$

With this choice we define the basis for the horizontal space as

$$\begin{aligned}x &= \hat{N}p, \\ y &= \hat{N}\bar{\alpha}p \\ \eta_1 &= \hat{N}\gamma_1 p, \\ \eta_2 &= \hat{N}\gamma_2 p.\end{aligned}$$

These vectors are well-adapted to the Gromoll-Meyer sphere since x is normal to the $S^1 \times S^2$s in $S^1 * S^2 = S^4$, y is tangent to the S^1s in $S^1 \times S^2 \subset S^1 * S^2 = S^4$, and the ηs are tangent to the S^2s in $S^1 \times S^2 \subset S^1 * S^2 = S^4$.

We call x, y, and $\mathfrak{v}$, α–vectors, and we call η_1, η_2, ϑ_1, and ϑ_2, γ–vectors.

When $t = 0$, our formula for Np becomes

$$Np = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} p$$

which has no "α". So the vectors

$$\mathfrak{v}, \vartheta_1, \vartheta_2$$

become indistinguishable. This reflects the fact that the fibers of h and $\tilde{h}$ coincide when $t = 0$. Similarly our formulas for the vectors

$$x, \eta_1, \eta_2$$

become indistinguishable at $t = 0$. This reflects the fact that the set where $t = 0$ in S^4 is the "singular" $S^1 \subset S^1 * S^2 = S^4$, i.e. the place where the S^2s are "collapsed". On the other, hand at $t = 0$, y becomes

$$\begin{pmatrix} \alpha\sin\theta \\ \alpha\cos\theta \end{pmatrix} \bar{\alpha} p = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} p$$

and hence is well defined, reflecting the fact that y is tangent to the circles of the join decomposition.

Proposition 2.7. *On S^7 the "combined Hopf action" $A^{\tilde{h}} \times A^h$ leaves the splitting*

$$\operatorname{span}\{x, \eta_1, \eta_2\} \oplus \operatorname{span}\{y\} \oplus \operatorname{span}\{\mathfrak{v}, \vartheta_1, \vartheta_2\}$$

invariant and leaves the splitting

$$\operatorname{span}\{x\} \oplus \operatorname{span}\{y\} \oplus \operatorname{span}\{\eta_1, \eta_2\} \oplus \operatorname{span}\{\mathfrak{v}\} \oplus \operatorname{span}\{\vartheta_1, \vartheta_2\}$$

invariant when $t > 0$.

Proof. Since $A^{\tilde{h}}$ acts by symmetries of h, it at least preserves the horizontal and vertical splitting of h. But it also leaves its own horizontal and vertical spaces invariant. The $A^{\tilde{h}}$–invariance of $\operatorname{span}\{\mathfrak{v}\} \oplus \operatorname{span}\{\vartheta_1, \vartheta_2\}$ when $t > 0$ follows from the fact that $\operatorname{span}\{\mathfrak{v}\}$ is the intersection of the two vertical spaces and $\operatorname{span}\{\vartheta_1, \vartheta_2\}$ its orthogonal complement in the vertical space of h. The $A^{\tilde{h}}$–invariance of $\operatorname{span}\{x\} \oplus \operatorname{span}\{y\} \oplus \operatorname{span}\{\eta_1, \eta_2\}$ when $t > 0$ follows from the fact that at the level of S^4, $A^{\tilde{h}}$ preserves our join decomposition. Finally, $\operatorname{span}\{y\}$ is $A^{\tilde{h}}$–invariant when $t = 0$ since on S^4, the set where $t = 0$ is the fixed point set of $A^{\tilde{h}}$, and $\operatorname{span}\{y\}$ is the tangent space to this fixed point set.

A similar argument gives us the statement for A^h. □

As observed in [PetWilh], $TSp(2)$ has a splitting

$$TSp(2) = V_1 \oplus V_2 \oplus H,$$

where V_1 and V_2 are the vertical spaces for the Hopf fibrations that describe

$$Sp(2) \equiv (-I \circ h)^* \left(S^7\right) \subset S^7(1) \times S^7(1),$$

and H is the orthogonal complement of $V_1 \oplus V_2$ with respect to the biinvariant metric.

The vectors

$$\begin{aligned}(\mathfrak{v},0) &= (N_1\alpha p,0)\,,\\ (\vartheta_1,0) &= (N_1\gamma_1 p,0)\,,\\ (\vartheta_2,0) &= (N_1\gamma_2 p,0)\end{aligned}$$

form an orthogonal basis for V_1. Similarly,

$$\begin{aligned}(0,\mathfrak{v}) &= (0,N_2\alpha)\,,\\ (0,\vartheta_1) &= (0,N_2\gamma_1)\,,\\ (0,\vartheta_2) &= (0,N_2\gamma_2)\end{aligned}$$

form a orthogonal basis for V_2.

To get a basis for H at representative points we define

$$\begin{aligned}\hat{N}_1 &= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}-\sin t\\ \alpha\cos t\end{pmatrix},\\ \hat{N}_2 &= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\alpha\cos t\\ -\sin t\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}x^{2,0} &= \left(\hat{N}_1 p,\hat{N}_2\right),\\ y^{2,0} &= \left(\hat{N}_1\bar{\alpha}p,\hat{N}_2\alpha\right)\\ \left(\eta_1,\eta_1\right) &= \left(\hat{N}_1\gamma_1 p,\hat{N}_2\gamma_1\right)\\ \left(\eta_2,\eta_2\right) &= \left(\hat{N}_1\gamma_2 p,\hat{N}_2\gamma_2\right)\end{aligned}$$

We refer the reader to [Wilh1] for the computations that show that $x^{2,0}, y^{2,0}, \left(\eta_1,\eta_1\right)$, and $\left(\eta_2,\eta_2\right)$ are tangent to $Sp\left(2\right)$. A corollary of the previous proposition is

Corollary 2.8. *The Gromoll-Meyer action $A^{2,-1}\times A^{h_2}$ leaves*

$$\begin{aligned}&\operatorname{span}\left\{x^{2,0},\left(\eta_1,\eta_1\right),\left(\eta_2,\eta_2\right)\right\}\oplus\operatorname{span}\left\{y^{2,0}\right\}\\ &\oplus\operatorname{span}\left\{\left(\mathfrak{v},0\right),\left(\vartheta_1,0\right),\left(\vartheta_2,0\right)\right\}\oplus\operatorname{span}\left\{\left(0,\mathfrak{v}\right),\left(0,\vartheta_1\right),\left(0,\vartheta_2\right)\right\}\end{aligned}$$

invariant and leaves the splitting

$$\begin{aligned}&\operatorname{span}\left\{x^{2,0}\right\}\oplus\operatorname{span}\left\{y^{2,0}\right\}\oplus\operatorname{span}\left\{\left(\eta_1,\eta_1\right),\left(\eta_2,\eta_2\right)\right\}\\ &\oplus\operatorname{span}\left\{\left(\mathfrak{v},0\right)\right\}\oplus\operatorname{span}\left\{\left(\vartheta_1,0\right),\left(\vartheta_2,0\right)\right\}\oplus\operatorname{span}\left\{\left(0,\mathfrak{v}\right)\right\}\oplus\operatorname{span}\left\{\left(0,\vartheta_1\right),\left(0,\vartheta_2\right)\right\}\end{aligned}$$

invariant when $t>0$.

3. Cheeger Deformations

The metric studied in [Wilh2] is induced via $q_{2,-1}$ by the perturbation of the biinvariant metric that was studied in [PetWilh]. We start by reviewing its construction.

In [Cheeg] a general method for perturbing the metric g on a manifold M of nonnegative sectional curvature was proposed. Various special cases of this method were first studied in [Berg2], [BourDesSent], and [Wal].

If G is a compact group of isometries of (M,g), then we let G act on $G\times M$ by

$$q\cdot(p,m)=(pq^{-1},qm).$$

If b is a biinvariant metric on G, then for each $l > 0$ we get a product metric $l^2 b + g$ on $G \times M$. The quotient of this action then induces a new metric, g_l, of nonnegative sectional curvature on M. It was observed in [Cheeg] that we may expect the new metric to have fewer 0–curvatures and symmetries than the original metric, $g = g_\infty$. The quotient map of this action is denoted by

$$q_{G\times M} : G \times M \longrightarrow M. \tag{3.1}$$

In [PetWilh] we studied the effect of perturbing the biinvariant metric on $Sp(2)$ using Cheeger's method and the $S^3 \times S^3 \times S^3 \times S^3$ action induced by the commuting S^3-actions

$$\begin{aligned}
A^u\left(p_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} p_1 a & p_1 b \\ c & d \end{pmatrix}, \\
A^d\left(p_2, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} a & b \\ p_2 c & p_2 d \end{pmatrix}, \\
A^{h_1}\left(q_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} a\bar{q}_1 & b \\ c\bar{q}_1 & d \end{pmatrix}, \\
A^{h_2}\left(q_2, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} a & b\bar{q}_2 \\ c & d\bar{q}_2 \end{pmatrix}.
\end{aligned}$$

If $\xi \in TM$, then $\hat{\xi} \in T(G \times M)$ denotes the horizontal vector, with respect to $q_{G\times M}$, satisfying $dp_2\left(\hat{\xi}\right) = \xi$, where $p_2 : G \times M \longrightarrow M$ is the projection onto the second factor. Similarly if $P \subset TM$ is a tangent plane, then $\hat{P} \subset T(G \times M)$ is the horizontal plane satisfying $dp_2(\hat{P}) = P$. Cheeger's observation was that ([Cheeg], cf [PetWilh], Proposition 1.10)

Proposition 3.2. **(i):** *If the curvature of P is positive with respect to g_∞, then the curvature of*

$$dq_{G\times M}(\hat{P})$$

is positive with respect g_l.

(ii): *The curvature of $dq_{G\times M}(\hat{P})$ is positive with respect to g_l if the A-tensor of $q_{G\times M}$ is nonzero on $\hat{P}$.*

(iii): *If $G = S^3$, then the curvature of $dq_{G\times M}(\hat{P})$ is positive if the projection of P onto TO_G is nondegenerate.*

(iv): *If the curvature of $\hat{P}$ is 0 and $A^{q_{G\times M}}$ vanishes on $\hat{P}$, then the curvature of $dq_{G\times M}(\hat{P})$ is 0.*

Remark 3.3. *According to* [Tapp2]*, no new positive curvature can be created via (ii) if M is a Lie group with a biinvariant metric.*

Following [PetWilh] and [Wilh2], our computations will be based on deformations of the biinvariant metric on $Sp(2)$. The biinvariant metric induced by $Sp(2) \subset S^7(1) \times S^7(1)$ is called b. The biinvariant metric we use is scaled so that the vectors $x^{2,0}$ etc. have unit length. Thus we use $\frac{1}{2}b$, also called $b_{\frac{1}{\sqrt{2}}}$ in [PetWilh] and [Wilh2], which is induced by $Sp(2) \subset S^7(\frac{1}{\sqrt{2}}) \times S^7(\frac{1}{\sqrt{2}})$, where $S^7(\frac{1}{\sqrt{2}})$ is the sphere of radius $\frac{1}{\sqrt{2}}$.

The effect of the Cheeger perturbation $A^{h_1} \times A^{h_2}$ is to scale V_1 and V_2 and to preserve the splitting $V_1 \oplus V_2 \oplus H$ and $\frac{1}{2}b|_H$. The amount of the scaling is < 1 and converges to 1 as the scale on the S^3-factor in $\left(S^3 \times S^3\right) \times Sp(2)$ converges to ∞ and converges to 0 when the $S^3 \times S^3$ factor is scaled to a point. We will call

the resulting scales on V_1 and V_2, ν_1 and ν_2. To simplify the exposition, we set $\nu = \nu_1 = \nu_2$ and call the resulting metric g_ν.

It follows that g_ν is the restriction to $Sp(2)$ of the product metric $S^7_\nu \times S^7_\nu$ where S^7_ν denotes the Berger metric obtained from $S^7(\frac{1}{\sqrt{2}})$ by scaling the fibers of h by $\nu\sqrt{2}$.

The following results can be found in [PetWilh].

Proposition 3.4. *Let $g_{\nu,l}$ denote a metric obtained from the biinvariant metric on $Sp(2)$ via Cheeger's method using the $S^3\times S^3\times S^3\times S^3$-action, $A^u\times A^d\times A^{h_1}\times A^{h_2}$.*

Then $A^u\times A^d\times A^{h_1}\times A^{h_2}$ is by isometries with respect to $g_{\nu,l}$. In particular, $A_{2,-1}$ is by isometries with respect to $g_{\nu,l}$, and hence $g_{\nu,l}$ induces a metric of nonnegative curvature on the Gromoll-Meyer sphere, Σ^7.

Proposition 3.5. *Let $A_H : H \times M \longrightarrow M$ be an action that is by isometries with respect to both g_∞ and g_l. Let H_{A_H} denote the distribution of vectors that are perpendicular to the orbits of A_H.*

P is in H_{A_H} with respect to g_∞ if and only if $dq_{G\times M}(\hat{P})$ is in H_{A_H} with respect to g_l. In fact,

$$g_\infty(u,w) = g_l\left(u, dq_{G\times M}(\hat{w})\right)$$

for all $u, w \in TM$.

Notational Convention: Let

$$q_{G\times M} : G\times (M, g_\infty) \longrightarrow (M, g_l)$$

be a Cheeger submersion. Suppose that $\pi : M \longrightarrow B$ is a Riemannian submersion with respect to both g_∞ and g_l. It follows that z is horizontal for $\pi : M \longrightarrow B$ with respect to g_∞ if and only if $dq_{G\times M}(\hat{z})$ is horizontal for π with respect to g_l. To keep the notation simpler, we can think of this correspondence as a parameterization of the horizontal space, H_{π,g_l}, of π with respect to g_l by the horizontal space, $H_{\pi,\ g_\infty}$ of π with respect to g_∞. We can then denote vectors and planes in $H_{\pi,\ g_l}$ by the corresponding vectors and planes in $H_{\pi,\ g_\infty}$. We will do this for the $\left(A^u\oplus A^d\right)$–Cheeger deformation, but not for the $\left(A^{h_1}\oplus A^{h_2}\right)$–Cheeger deformation.

Note that if $t \in [0, \frac{\pi}{4})$ then the orthogonal projection $p_{V_h,V_{\tilde{h}}} : V_h \longrightarrow V_{\tilde{h}}$ with respect to the unit metric on S^7 is an isomorphism. In fact the matrix of $p_{V_h,V_{\tilde{h}}}$ with respect to the ordered bases $\mathfrak{v}$, ϑ_1, ϑ_2 and $\mathfrak{v}$, $\tilde{\vartheta}_1$, $\tilde{\vartheta}_2$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & 0 \\ 0 & 0 & \cos(2t) \end{pmatrix}.$$

The horizontal space of $q_{2,-1}$ with respect to g_ν is given by

Proposition 3.6. [Wilh2] *For $t \in [0, \frac{\pi}{4})$ the horizontal space of $q_{2,-1}$ with respect to g_ν at the representative point (N_1p, N_2) is spanned by*

$$\left\{ x^{2,0},\ y^{2,0},\ \left(\eta_1, \eta_1 + \tan(2t)\frac{\vartheta_1}{\nu^2}\right),\ \left(\eta_2, \eta_2 + \tan(2t)\frac{\vartheta_2}{\nu^2}\right), \right.$$
$$\left(-\frac{\mathfrak{v}}{\nu^2}, \frac{\mathfrak{v}}{\nu^2} - \frac{p^{-1}_{V_h,V_{\tilde{h}}}(\bar{p}\alpha p N_2)}{\nu^2}\right),\ \left(-\frac{\vartheta_1}{\nu^2}, \frac{\vartheta_1}{\nu^2} - \frac{p^{-1}_{V_h,V_{\tilde{h}}}(\bar{p}\gamma_1 p N_2)}{\nu^2}\right),$$
$$\left.\left(-\frac{\vartheta_2}{\nu^2}, \frac{\vartheta_2}{\nu^2} - \frac{p^{-1}_{V_h,V_{\tilde{h}}}(\bar{p}\gamma_2 p N_2)}{\nu^2}\right)\right\}$$

Notation: We will call the seven vectors in Proposition 3.6, $x^{2,0}$, $y^{2,0}$, $\eta_1^{2,0}$, $\eta_2^{2,0}$, $\mathfrak{v}^{2,-1}$, $\vartheta_1^{2,-1}$, and $\vartheta_2^{2,-1}$ respectively. We will call the span of the first four $H_{2,-1}$ and the span of the last three $V_{2,-1}$.

Although our partial framing of $TSp(2)$ is well adapted to study the Gromoll-Meyer sphere it is neither left nor right invariant. For example, the left invariant field that equals $x^{2,0}$ at

$$Q = \left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix}, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix} \right)$$

is

$$\begin{aligned} (L_Q)_* \left(\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \right) &= \begin{pmatrix} \cos t & \alpha \sin t \\ \alpha \sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sin t & \alpha \cos t \\ \alpha \cos t & -\sin t \end{pmatrix} \\ &= x^{2,0}. \end{aligned}$$

Since α varies, $x^{2,0}$ is not left invariant.

Note also that one should think of $\{\eta_1, \eta_2\}$ as defining a global distribution rather than as global vector fields. The fact that S^2 is not parallelizable corresponds to the fact that γ_1 is not canonically determined by α. Consequently, any statement that we make about a single unit $\gamma \in \operatorname{span}\{\gamma_1, \gamma_2\}$ is valid for any $\gamma \in \operatorname{span}\{\gamma_1, \gamma_2\}$. Similarly any statement about a single unit $\eta \in \operatorname{span}\{\eta_1, \eta_2\}$ is valid for any $\eta \in \operatorname{span}\{\eta_1, \eta_2\}$, and any statement about a single $\vartheta \in \operatorname{span}\{\vartheta_1, \vartheta_2\}$ is valid for any $\vartheta \in \operatorname{span}\{\vartheta_1, \vartheta_2\}$.

4. Zero Curvatures of Σ^7

In this final review section we discuss the zero curvatures of $\left(\Sigma^7, g_{\nu,l}\right)$. The description that we give is more geometric than that of [Wilh2]. We give a brief idea of why the zeros occur, but for a full justification we combine [Wilh2] and [Tapp2] with new computations of the zero curvatures when $t = 0$. These were not given in [Wilh2] because they were not needed. We give them here to fully justify our description and also because they give a flavor of some of the important issues of [Wilh2].

From Proposition 3.1, we see that a 0–plane for $g_{\nu,l}$ must have a degenerate projection onto the tangent spaces to the orbits of all four S^3–actions, A^u, A^d, A^{h_1}, and A^{h_2}.

There is a vector field tangent to $Sp(2)$ that is normal to the orbits of all four actions. We call this field ζ.When restricted to an S^7–factor, ζ is the field that is normal to the $S^3 \times S^3$s in the join decomposition $S^7 = S^3 * S^3$, that corresponds to writing a point in S^7 as

$$\begin{pmatrix} a \\ c \end{pmatrix} \text{ with } a, c \in \mathbb{H}.$$

ζ is of course in $\operatorname{span}\left\{x^{2,0}, y^{2,0}\right\}$, but the combination is quite complicated.

$$\zeta = \frac{(\sin 2t \cos 2\theta)\ x^{2,0} - (\sin 2\theta)\ y^{2,0}}{\sqrt{\sin^2 2t \cos^2 2\theta + \sin^2 2\theta}}.$$

So ζ does not have much to do with our join decomposition $S^4 = S^1_{\mathbb{R}} * S^2_{\mathrm{Im}}$. Rather it is the geodesic field that is the gradient of the distance from the point where

$(t,\theta)=(0,0)$. In our coordinate system for S^4, the antipodal point to $(t,\theta)=(0,0)$ is $(t,\theta)=\left(0,\frac{\pi}{2}\right)$. So ζ is the field that is tangent to the meridians between these two points. Thus ζ is multivalued at the two poles. This corresponds to the fact that our formula for ζ is $\frac{0}{0}$ at these poles.

Unfortunately ζ is everywhere normal to the Gromoll-Meyer action. Fortunately the vectors

$$ZV\equiv\left\{U\in TSp\left(2\right)|\mathrm{curv}_b\left(\zeta,U\right)=0\right\}$$

are typically not horizontal for the Gromoll-Meyer submersion $p_{2,-1}$. However, from [Tapp2] we know that every time a vector U is horizontal for $p_{2,-1}$, we get a zero plane in Σ^7, even with respect to $g_{\nu,l}$.

The projection to S^4 of the points in Σ^7 that have zero curvature planes containing ζ are

Theorem 4.1. *The points in S^4 over which there is a horizontal vector for $q_{2,-1}:Sp\left(2\right)\longrightarrow\Sigma^7$ that is in ZV are the meridians emmanating from $(t,\theta)=(0,0)$ that make an angle that is $\leq\frac{\pi}{6}$ with the meridians that go from $(t,\theta)=(0,0)$ through S^2_{Im}.*

The set is therefore 4–dimensional with a four dimensional complement. In [Wilh2] it is described as the sublevel set $L\left(t,\theta\right)\leq 1$, where $L:S^4\longrightarrow\mathbb{R}$ is

$$L(t,\theta)=\begin{cases}\frac{2\cos(2t)\sin(2\theta)}{\sqrt{\sin^2 2\theta+\sin^2 2t\cos^2 2\theta}} & \text{if } (t,\theta)\neq(0,0) \text{ or } \left(0,\frac{\pi}{2}\right)\\ 0 & \text{if } (t,\theta)=(0,0) \text{ or } \left(0,\frac{\pi}{2}\right)\end{cases}.$$

Combining this with the main theorem of [Tapp2] and Proposition 4.7 below gives us Theorem 4.1.

Of course there can also be zero planes that do not contain ζ. Since ζ (generically) spans the orthogonal complement of the orbit of the $S^3\times S^3\times S^3\times S^3$ action, such planes necessarily have a nondegenerate projection onto the tangent space to the entire orbit of $S^3\times S^3\times S^3\times S^3$, but a degenerate projection onto the orbit of each individual S^3–action. In addition, the plane must have zero curvature for the biinvariant metric and be horizontal for the Gromoll-Meyer submersion, it is not surprising that such planes are fairly rare.

Theorem 4.2. *The set of points $\mathcal{Z}$ in S^4 over which there is a 0–plane in Σ^7 is the union of the points described in Theorem 4.1 with the points where* $\cos 2\theta=0$.

To get a quick idea of how these other zeros occur, we point out that the horizontal vectors for $q_{2,-1}:Sp\left(2\right)\longrightarrow\Sigma^7$ that are also perpendicular to the orbits of $A^{h_1}\oplus A^{h_2}$ are

$$\mathrm{span}\left\{x^{2,0},y^{2,0}\right\}$$

when $t>0$ and

$$\mathrm{span}\left\{x^{2,0},y^{2,0},\eta_1^{2,0},\eta_2^{2,0}\right\}$$

when $t=0$.

Since $\zeta\in\mathrm{span}\left\{x^{2,0},y^{2,0}\right\}$, the issue boils down to its complementary vector $\xi\equiv\alpha\zeta$ in $\mathrm{span}\left\{x^{2,0},y^{2,0}\right\}$. Fortunately ξ does have a projection onto the tangent space to the orbits of $A^u\oplus A^d$. Combining this with the other requirements for zero planes it is argued in [Wilh2], that the points in S^4 over which there are 0 planes are those described in the previous theorem.

The actual zero planes have the form

Theorem 4.3. *If P is a plane with 0 curvature at a point where $(t, \sin 2\theta) \neq (0,0)$ and $\cos 2\theta \neq 0$, then P has the form*

$$P = \operatorname{span}\{\zeta, W\}$$

where

$$W \in V_1 \oplus V_2.$$

If ζ has the form

$$\zeta = x^{2,0}\cos\varphi + y^{2,0}\sin\varphi,$$

then W has the form

$$\cos\lambda\left(\frac{\mathfrak{v}}{\nu^2}, \frac{\mathfrak{v}}{\nu^2}\right) + \sin\lambda\left(\frac{\ddot{\vartheta}_1}{\nu^2}, \frac{\ddot{\vartheta}_1}{\nu^2}\cos\psi + \frac{\ddot{\vartheta}_2}{\nu^2}\sin\psi\right)$$

where

$$\psi = \pi - 2\varphi,$$

$\ddot{\vartheta}_1, \ddot{\vartheta}_2 \in \operatorname{span}\{\vartheta_1, \vartheta_2\}$, *correspond to spherical combinations $\ddot{\gamma}_1, \ddot{\gamma}_2$ of $\{\gamma_1, \gamma_2\}$ that satisfy $\alpha\ddot{\gamma}_1 = \ddot{\gamma}_2$, and $(\cos\lambda, \sin\lambda)$ is the point in the first quadrant of $\mathbb{R}^2$ that is on the unit circle and on the ellipse parameterized by*

$$\sigma \longmapsto \left(\frac{\cos\sigma}{2}, \frac{\sin\sigma}{L(t,\theta)}\right). \tag{4.4}$$

When $\cos 2\theta = 0$, there are zero planes of the form described above. In addition there are zero planes of the form

$$P = \operatorname{span}\left\{x^{2,0}, W\right\}$$

where

$$W = \left(\frac{\mathfrak{v}}{\nu^2}\frac{1}{2} + \frac{\vartheta}{\nu^2}\frac{\sqrt{3}}{2}, \frac{\mathfrak{v}}{\nu^2}\frac{1}{2} - \frac{\vartheta}{\nu^2}\frac{\sqrt{3}}{2}\right).$$

Remark 4.5. *There is a further conjugacy condition for a vector of the form of W to actually be horizontal for $q_{2,-1} : \Sigma^7 \longrightarrow S^4$. Because of this, in a given fiber of $\Sigma^7 \longrightarrow S^4$ over a point in $\mathcal{Z} \subset S^4$ most points do not in fact have zero curvatures, and at most points where there is a zero curvature, there is just one zero curvature. None of these issues will be important for us, so we will not review them.*

Remark 4.6. *The unit circle and the ellipse in question do not intersect when $L(t,\theta) > 1$. When this happens the corresponding Ws are not horizontal for the Gromoll-Meyer submersion.*

4.1. **Zero Curvatures at $t = 0$.** When $t = 0$, all points have positive curvature except for certain points with $\cos 2\theta \sin 2\theta = 0$. The lack of 0–planes in Σ^7 is caused by the zero planes of $Sp(2)$ not intersecting the horizontal distribution of the Gromoll–Meyer submersion. The reason for this is the fact that the unit circle and the ellipse in (4.4) do not intersect when $L(t,\theta) > 1$. So the corresponding Ws are not horizontal for the Gromoll-Meyer submersion. For example, if $t = 0$ and $\sin 2\theta \neq 0$, then $\zeta = -y^{2,0}$ and $L(0,\theta) = 2$. For $\operatorname{span}\left\{y^{2,0}, W\right\}$ to have 0 curvature, with respect to g_ν, W must have the form

$$\cos\lambda(\mathfrak{v}, \mathfrak{v}) + \sin\lambda(\vartheta, \vartheta) = (N_1\beta p, N_2\beta)$$

for some purely imaginary $\beta \in S^3 \subset \mathbb{H}$.

When $t = 0$, we have $V_h = V_{\tilde{h}}$ so none of the horizontal vectors

$$\left(N_1\beta p, -N_2\beta + p^{-1}_{V_h,V_{\tilde{h}}}\left(\bar{p}\beta p N_2\right)\right)$$

can have the required form

$$\left(N_1\beta p, N_2\beta\right).$$

When $(t,\theta) = (0,0)$ or $\left(0,\frac{\pi}{2}\right)$, the definition of ζ is ambiguous. The definition of $x^{2,0}$ is also ambiguous since the α coordinate is nonexistent at $t = 0$. In fact, the three vectors $x^{2,0}$, $\eta_1^{2,0}$, and $\eta_2^{2,0}$ project under $p_{2,-1}\circ q_{2,-1}$ to a basis for the normal space of $S^1_{\mathbb{R}} \subset S^4$. Declaring that a particular purely, imaginary unit quaternion is "α" amounts to declaring that a particular unit normal vector to $S^1_{\mathbb{R}}$ is "$x^{2,0}$". This choice is somewhat irrelevant since, on the level of S^4, the isometric action A^{h_2} fixes $S^1_{\mathbb{R}}$ and acts transitively on the normal space. Thus, to find 0 curvatures when $(\sin 2t, \sin 2\theta) = (0,0)$, we only need to consider planes of the form

$$P = \operatorname{span}\{z, W\}$$

where $z \in \operatorname{span}\left\{x^{,2,0}, y^{2,0}\right\}$ and $W \in V_{2,-1}$. There will of course be other 0–planes, but they are the images of these under A^{h_2}.

Since $L(0,\theta) \equiv 2$, when $\theta \neq 0, \frac{\pi}{2}$, there are no 0–curvatures when $t = 0$, provided θ is not $0, \frac{\pi}{4}, \frac{\pi}{2}$, or $\frac{3\pi}{4}$. The details can be found in [Wilh2], but the basic reason is contained in the remark above, when $\theta \neq 0, \frac{\pi}{2}$, then $\zeta = y^{2,0}$, and the Ws that together with y form 0 planes are not horizontal at $t = 0$.

The structure of the 0–planes when $(t,\theta) = (0,0)$ or $\left(0,\frac{\pi}{2}\right)$ was claimed in [Wilh2, p. 556] to be

Proposition 4.7. *Let*

$$\zeta_\varphi = x^{2,0}\cos\varphi + y^{2,0}\sin\varphi$$

for some $\varphi \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. *If* $(t,\theta) = (0,0)$ *or* $\left(0,\frac{\pi}{2}\right)$ *and* $|\sin\varphi| \leq \frac{1}{2}$, *then there are values of* p *for which* ζ_φ *is in* 0*–planes of the form*

$$P = \operatorname{span}\left\{\zeta_\varphi, W\right\}$$

where

$$W = \cos\lambda\left(\mathfrak{v},\mathfrak{v}\right) + \sin\lambda\left(\vartheta_1, \vartheta_1\cos\psi + \vartheta_2\sin\psi\right)$$

and

$$\psi = \pi - 2\varphi.$$

Any other 0*–plane is the image of one of these under* A^{h_2}.

The details of this were not given in [Wilh2], since it was not crucial to the goal of that paper. Since we will need to use it, we will prove it here.

The value of $\cos\lambda$ is determined by φ; the relationship can be inferred from our proof.

Proof. As explained in [Wilh2] it is enough to consider planes of the form

$$P = \operatorname{span}\{z, W\}$$

where z is horizontal for $p_{2,-1} : \Sigma^7 \longrightarrow S^4$ and $W \in V_1 \oplus V_2$ is horizontal for $q_{2,-1} : Sp(2) \longrightarrow \Sigma^7$. Since $t = 0$, we can use the isometries A^{h_2} to further reduce our consideration to planes with $z \in \operatorname{span}\left\{x^{2,-1}, y^{2,-1}\right\}$. In other words we may replace z with

$$\zeta_\varphi = x^{2,0}\cos\varphi + y^{2,0}\sin\varphi.$$

Using Proposition 4.6 in [Wilh2], this then forces W to have the form listed in the statement. It only remains to check what Ws are horizontal for $q_{2,-1}$ when $t = 0$.

We explained above that when $\zeta_\varphi = y^{2,0}$, the required W is not horizontal.

When $\zeta_\varphi = x^{2,0}$, the required W is

$$W = \left(\frac{\mathfrak{v}}{2} + \frac{\sqrt{3}}{2}\vartheta, \frac{\mathfrak{v}}{2} - \frac{\sqrt{3}}{2}\vartheta \right).$$

We see that W can be realized in the form

$$\left(N_1 \beta p, -N_2 \beta + p^{-1}_{V_h, V_{\tilde{h}}} \left(\bar{p} \beta p N_2 \right) \right)$$

by choosing

$$\beta = \left(\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\gamma \right)$$

and p so that

$$\bar{p}\beta p = \alpha.$$

Now we consider the general problem of realizing

$$W = \cos\lambda \left(\mathfrak{v}, \mathfrak{v} \right) + \sin\lambda \left(\vartheta_1, \vartheta_1 \cos\psi + \vartheta_2 \sin\psi \right)$$

in the form

$$\left(N_1 \beta p, -N_2 \beta + p^{-1}_{V_h, V_{\tilde{h}}} \left(\bar{p} \beta p N_2 \right) \right).$$

The first coordinate,

$$\mathfrak{v} \cos\lambda + \sin\lambda \vartheta_1,$$

of W forces us to set

$$\beta = \alpha \cos\lambda + \gamma_1 \sin\lambda.$$

The question then becomes whether there is a choice of p that will achieve the desired second coordinate. The second coordinate of W can be written

$$\mathfrak{v} \cos\lambda + \left(\vartheta_1 \cos\psi + \vartheta_2 \sin\psi \right) \sin\lambda = N_2 \alpha \cos\lambda + N_2 \gamma \sin\lambda. \tag{4.8}$$

for $\gamma = \gamma_1 \cos\psi + \gamma_2 \sin\psi$. On the other hand if we set

$$\bar{p}\beta p = \alpha \cos\sigma + \gamma' \sin\sigma$$

then

$$\begin{aligned} -N_2 \beta + p^{-1}_{V_h, V_{\tilde{h}}} \left(\bar{p} \beta p N_2 \right) &= -N_2 \left(\alpha \cos\lambda + \gamma_1 \sin\lambda \right) + \left(\bar{p}\beta p \right) N_2 \\ &= -N_2 \alpha \cos\lambda - N_2 \gamma_1 \sin\lambda + \left(\alpha \cos\sigma + \gamma' \sin\sigma \right) N_2 \\ &= N_2 \left(-\alpha \cos\lambda + \alpha \cos\sigma \right) + N_2 \left(-\gamma_1 \sin\lambda + \gamma' \sin\sigma \right) \end{aligned}$$

since N_2 is real and therefore commutes with all quaternions.

Equating this with 4.8 gives us the equations

$$\begin{aligned} \alpha \cos\lambda &= -\alpha \cos\lambda + \alpha \cos\sigma, \\ \gamma \sin\lambda &= -\gamma_1 \sin\lambda + \gamma' \sin\sigma \end{aligned}$$

or

$$\begin{aligned} \cos\sigma &= 2\cos\lambda, \\ \gamma' \sin\sigma &= \left(\gamma + \gamma_1 \right) \sin\lambda. \end{aligned}$$

We can always choose p so that γ' points in the direction of $\gamma+\gamma_1$. The issue is that since $\gamma+\gamma_1$ has a variable length, sometimes there are solutions and sometimes there are not. In fact, if we set

$$L = |\gamma + \gamma_1|,$$

then our equations become

$$\begin{aligned} \cos\lambda &= \frac{\cos\sigma}{2}, \\ \sin\lambda &= \frac{\sin\sigma}{L}. \end{aligned}$$

So the question becomes whether or not the unit circle $(\cos\lambda, \sin\lambda)$ intersects the ellipse whose parametrization is

$$\sigma \longmapsto \left(\frac{\cos\sigma}{2}, \frac{\sin\sigma}{L}\right).$$

Thus when $|L| \leq 1$ there are solutions and when $L > 1$ there are no solutions. So it remains to analyze how L depends on φ.

Since

$$\begin{aligned} \gamma &= \gamma_1 \cos\psi + \gamma_2 \sin\psi, \\ \psi &= \pi - 2\varphi, \end{aligned}$$

we have

$$\begin{aligned} \gamma &= \gamma_1 \cos(\pi - 2\varphi) + \gamma_2 \sin(\pi - 2\varphi) \\ &= -\gamma_1 \cos 2\varphi + \gamma_2 \sin 2\varphi. \end{aligned}$$

Thus

$$\begin{aligned} L^2 &= |\gamma_1 + \gamma|^2 \\ &= 1 - 2\cos 2\varphi + 1. \end{aligned}$$

So our condition, $L \leq 1$ for 0 curvature is

$$\begin{aligned} 2 - 2\cos 2\varphi &\leq 1 \\ -2\cos 2\varphi &\leq -1 \end{aligned}$$

or

$$\cos 2\varphi \geq \frac{1}{2}$$

Keeping in mind that $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we get

$$-\frac{\pi}{3} \leq 2\varphi \leq \frac{\pi}{3}$$

or

$$-\frac{1}{2} \leq \sin\varphi \leq \frac{1}{2}.$$

□

5. Further Summary

Having scaled the fibers of $\Sigma^7 \longrightarrow S^4$, we can get pointwise positive curvature along any *single* (formerly) flat torus via a conformal change. The idea is that the Hessian of the conformal factor needs to cancel the s^2 term in Equation 1.9,

$$\operatorname{curv}_{g_s}(X,W) = -s^2\left(D_X\left(|H_w|\, D_X\,|H_w|\right)\right) + s^4\left(D_X\,|H_w|\right)^2. \tag{5.1}$$

Unfortunately, there is no conformal change that will produce pointwise positive curvature along all of the tori simultaneously. The problem is that only the component of W that is horizontal for $\Sigma^7 \longrightarrow S^4$ appears in our curvature formula. Along any one torus, the vector H_w is a Killing field for the $SO(3)$–action on S^4, but the precise Killing field, and more importantly the ratio $\frac{|H_w|}{|W|}$ varies from torus to torus, so the size of the required Hessian varies as well.

This difficulty is overcome by using only a "partial" conformal change. The restriction of the metric to the distribution $\operatorname{span}\{(N\alpha p, N\alpha)\}$ is not modified. The metric only changes on the orthogonal complement of $\operatorname{span}\{(N\alpha p, N\alpha)\}$. The details are carried out in Section 10.

A further difficulty is created by the fact that the pieces of the zero locus with $L(t,\theta) \leq 1$ and $\cos 2\theta = 0$ intersect at certain points over points in S^4 where $t \geq \frac{\pi}{6}$ and $\cos 2\theta = 0$. A description of this intersection is given in [Wilh2], Theorem E(iv,v).

The difficulty this creates is that the natural choices of conformal factors do not agree on this intersection.

To circumvent this difficulty, in Sections 7, 8, and the appendix we analyze the effect on Equation 5.1 of running the h_2–Cheeger perturbation for a long time. If ν is the parameter of this perturbation, then it turns out that making ν small has the effect of concentrating all of the terms on the right hand side of equation 5.1, $-s^2\left(D_X\left(|H_w|\, D_X\,|H_w|\right)\right) + s^4\left(D_X\,|H_w|\right)^2$, around $t = 0$. We will make ν small enough so that we can choose a (partial) conformal factor that is constant near the intersection of the two pieces of the zero locus, and hence not have to worry about the conflict that the intersection creates.

The intersection of the two pieces of the zero locus, also creates a notational conflict. To simplify the exposition we will henceforth write explicitly only about the planes at points where $L(t,\theta) \leq 1$. With the obvious modifications in notation and a few simplifications, the argument simultaneously will give us positive curvature near the planes where $\cos 2\theta = 0$.

Unfortunately, to really move the support of the partial conformal change away from the intersection we have to make ν depend on s. In the end we will pick $\nu = O\left(s^{6/7}\right)$. This means that our ultimate metric is not obtained as an infinitesimal perturbation of any (known) metric with nonnegative curvature. This fact will further complicate our exposition. Before we can explain why, some further clarification is needed.

Imagine that we have a deformation in which all of the former zero curvature planes, $\operatorname{span}\{\zeta, W\}$, are positively curved. Next comes the daunting challenge of establishing that an entire neighborhood (of uniform size) of these planes in the Grassmannian is positively curved. We have to consider what happens when we move the base point of our plane and also when we move the plane with out moving the base point.

To deal with points that are close to, but not on the old zero locus, we expand our definition of W to include certain vectors in $TSp(2)$ that are close to, but not on the old zero locus.

At points with 0–curvature,

$$W = (N_1\beta p, N_2\delta),$$

is determined by the requirements that $\mathrm{curv}_{(Sp(2),g_{\nu,l})}(\zeta, W) = 0$ and that W be horizontal for the Gromoll-Meyer submersion. The points in $\left(\Sigma^7, g_{\nu,l}\right)$ with positive curvature are images of points in $Sp(2)$ at which no horizontal W solves

$$\mathrm{curv}_{\nu,l}(\zeta, W) = 0.$$

For the purpose of this discussion only, we require $|\beta| = |\delta| = 1$, and we let Z^4 be the set of (t, θ) for which there is some zero plane in Σ^7. For $(t, \theta) \in Z^4$ the size of the γ–component of δ depends only on (t, θ), and not on (α, p). At points in $Sp(2)$ with $(t, \theta) \in Z^4$, we let W be any vector in $TSp(2)$ proportional to such a $(N_1\beta p, N_2\delta)$, with the size of the γ–component of δ determined by (t, θ) and $\mathrm{curv}_{(Sp(2),g_{\nu,l})}(\zeta, W) = 0$. Note that such W are *not* required to be horizontal for the Gromoll-Meyer submersion, there are no such horizontal Ws when (α, p) is such that Σ^7 is positively curved at (t, θ, α, p), and W is of course multivalued.

At points in $Sp(2)$ with $(t, \theta) \notin Z^4$, we let W be any vector of the form $(N_1\beta p, N_2\delta)$ with $\mathrm{curv}_{(Sp(2),g_{\nu,l})}(\zeta, W) = 0$ and $\beta, \delta \in \mathrm{span}\{\gamma_1, \gamma_2\}$. Of course, when $(t, \theta) \notin Z^4$, W is *never* horizontal for the Gromoll–Meyer submersion.

In all of our subsequent statements we assume that $\mathrm{span}\{\zeta, W\}$ is any one of these planes, whether or not it corresponds to a zero plane in Σ^7. In this way we will only have to worry about deforming our planes within the fibers of the Grassmannian.

We get positive curvature on the Gromoll-Meyer sphere by proving

Theorem 5.2. *There is a neighborhood N of the set of all $\{\zeta, W\}$ in the Grassmannian of $Sp(2)$, a choice of (ν, l), and a deformation g_{new} of $g_{\nu,l}$, that is invariant under the Gromoll-Meyer action, so that*

$$\mathrm{curv}_{g_{new}}|_N > 0$$

and $\mathrm{curv}_{g_{new}} > 0$ *on horizontal planes in the complement of N. In particular,* $\mathrm{curv}_{g_{new}} > 0$ *on all horizontal planes.*

We can now explain why the fact that our deformation is not infinitesimal will further complicate our exposition.

If our deformation is infinitesimal, then we only have to understand the "quadratic perturbations" of our planes in the Grassmannian.

To explain what this means precisely, let $\{g_s\}$ be a C^∞ family of metrics on a compact manifold with g_0 having nonnegative curvature, and let any zero curvature plane with respect to g_0 be represented by $\mathrm{span}\{\zeta, W\}$. We represent a general plane near $\mathrm{span}\{\zeta, W\}$ in the form $P = \mathrm{span}\{\zeta + \sigma z, W + \tau V\}$ where $z \perp \zeta$, $V \perp W$, and $\sigma, \tau \in \mathbb{R}$. The curvature is then a quartic polynomial

$$P(\sigma, \tau) = \mathrm{curv}(\zeta + \sigma z, W + \tau V)$$

in σ and τ.

Let R^s be the curvature tensor of g_s, let R^{old} be the curvature tensor with respect to g_0, and let $R^{\text{diff,s}} = R^s - R^{\text{old}}$. Let P^{old} and $P^{\text{diff,s}}$ have the obvious meaning. It is not hard to see

Lemma 5.3. *At all points for which g_0 has some 0–curvatures, g_s is positively curved for all sufficiently small s provided for all 0–planes,* $\operatorname{span}\{\zeta, W\}$ *,with respect to g_0,*

$$\begin{aligned} \frac{\partial}{\partial s}\text{curv}^{\text{diff,s}}(\zeta, W)|_{s=0} &= 0 \\ \frac{\partial^2}{\partial s^2}\text{curv}^{\text{diff,s}}(\zeta, W)|_{s=0} &> 0, \\ P^{\text{old}}(\sigma, \tau) &> 0, \end{aligned}$$

for all $(\sigma, \tau) \neq (0, 0)$, and

$$\begin{aligned} & P_Q(\sigma, \tau) \equiv \text{curv}^{\text{diff,s}}(\zeta, W) + 2\sigma R^{\text{diff,s}}(\zeta, W, W, z) + 2\tau R^{\text{diff,s}}(W, \zeta, \zeta, V) \\ & +\sigma^2 \text{curv}^{\text{old}}(z, W) + 2\sigma\tau\left[R^{\text{old}}(\zeta, W, V, z) + R^{\text{old}}(\zeta, V, W, z)\right] \\ & +\tau^2 \text{curv}^{\text{old}}(\zeta, V) \\ > \ & 0 \end{aligned} \tag{5.4}$$

for all sufficiently small s and all $\sigma, \tau \in \mathbb{R}$.

Remark 5.5. *In the abstract setting of this lemma, we can not guarantee that the metrics become positively curved because we know nothing about points that are close to, but not on the point wise 0–curvature locus of g_0. This is not a concern for the Gromoll-Meyer sphere because we have explained how to extend* $\operatorname{span}\{\zeta, W\}$ *to a family of planes in $Sp(2)$ that includes all points in a neighborhood of the point wise 0–locus (and also includes planes that are not horizontal for $\Sigma^7 \longrightarrow S^4$).*

It should also be emphasized that we never establish the hypotheses of this Lemma for our deformation. This is because our deformation is not infinitesimal. We have never the less included the result because it suggests a reasonable frame work for our computations.

Proof. Since we do not use this, we give only a sketch of the proof.

The idea is that all of the other terms of $P(\sigma, \tau)$ are either positive, 0 or too small to matter. Since g_0 is nonnegatively curved, the constant and linear terms are 0 when $s = 0$. Since

$$\left|R^{\text{diff,s}}\right| = O(s)$$

the quadratic, cubic, and quartic terms of $P^{\text{diff,s}}$ are smaller than

$$sO\left(\sigma^2 + \sigma\tau + \tau^2 + \sigma\tau^2 + \sigma^2\tau + \sigma^2\tau^2\right)$$

and hence are smaller than the corresponding terms of P^{old}, if s is sufficiently small.

On the one hand, the minimum of 5.4 occurs in the region where

$$\max\{\sigma, \tau\} = O(s),$$

and the size of this minimum is $O\left(s^2\right)$, so in this region the cubic, and quartic terms of P are too small too matter. On the other hand, when $\max\{\sigma, \tau\} > O(s)$, the linear terms have order

$$O(s)\max\{\sigma, \tau\}$$

and since $P^{\text{old}}(\sigma,\tau) > 0$ our curvature has order

$$\geq O\left(\sigma^2+\tau^2\right),$$

so the linear terms are too small to matter. □

Since our deformation is not infinitesimal, we will need to understand the full polynomial $P(\sigma,\tau)$. In fact, all of the possible values of all of the possible $P(\sigma,\tau)$s only describe the curvatures of an open dense subset in the Grassmannian. The curvatures of the complement of this open dense set are described by quadratic sub-polynomials of the $P(\sigma,\tau)$ that are proportional to sums of quartic, cubic, and pure quadratic terms of the various $P(\sigma,\tau)$s.

We will establish positive curvature on the Gromoll-Meyer sphere by showing that all of these polynomials and sub-polynomials are positive.

Since $\nu = O\left(s^{6/7}\right)$, we still have that s is much smaller than ν. Morally this means that even though our deformation is not infinitesimal, it is still fairly short term. The upshot of this is that many of the higher order coefficients of P^{diff} will be too small to matter. Those that are large will turn out to be comparable (in a favorable way) to terms in P^{old}. We carry this out in sections 12 and 13.

It turns out that the metric we have outlined thus far is not actually positively curved. The problem is that we do not actually get inequality 5.4 everywhere. To correct this problem we make a further modification of the metric in section 6. We call this the "redistribution" perturbation, and the resulting metric is $g_{\nu,re}$.

Finally, there is one further Cheeger deformation that we use that was not used in the earlier papers. The diagonal of U and D, which we will call $\Delta(U,D)$. The purpose of this final Cheeger deformation is that coupled with the h_1–Cheeger deformation it will allow us to see that any plane whose projection onto the vertical space is nondegenerate is positively curved. Although none of the original zero planes have this feature, this observation will still be useful, since it will allow us to immediately see that many of the possible perturbations of $\text{span}\{\zeta, W\}$ are positively curved. Modulo an identification this diagonal perturbation is also used in [EschKer].

The positively curved metric that we obtain can probably be constructed via several orderings of our deformations. However, to make our construction unambiguous, we will adopt the following order:

(1): The $(h_1 \oplus h_2)$–Cheeger deformation
(2): The redistribution, described in section 6.
(3): The $(U \oplus D)$–Cheeger deformation.
(4): The scaling of the fibers.
(5): The partial conformal change.
(6): The $\Delta(U,D)$ Cheeger deformation and a further h_1–deformation.

We will accordingly discuss the redistribution perturbation next. Although this is the logical order, it is not entirely clear that this order of exposition is optimal. The real need for the redistribution only becomes clear after one has done the subsequent computations; moreover, the desired change in the curvature is also only clear after further computations have been carried out. The reader may therefore wish to skip the next section, until its need becomes clear. We have written the rest of the paper in a sufficiently abstract form so that with the exception of subsection 8.1 this should be possible. The exceptional subsection concerns an effect of the redistribution that is not discussed in section 6.

Since our deformation is fairly short term, we have divided our curvature computations into to those required to prove 5.4 and those required to understand the higher order terms of $P(\sigma,\tau)$. The part necessary to prove 5.4 is Sections 6–11, by the end of which we will have proven

Lemma 5.6. *(Main Lemma) Let $g_{\nu,re,l}$ be the metric obtained after carrying out the deformations* 1–3 *above. Let g_{new} be the metric obtained after carrying out the deformations* 1–5 *above. Set*

$$R^{\text{diff}} = R^{\text{new}} - R^{\nu,re,l}.$$

Then for any choice of V and z as above with z horizontal for $p_{2,-1} : \Sigma^7 \longrightarrow S^4$ and any $\sigma,\tau \in \mathbb{R}$,

$$\begin{aligned} & P_Q(\sigma,\tau) = \text{curv}^{\text{diff}}(\zeta,W) + 2\sigma R^{\text{diff}}(\zeta,W,W,z) + 2\tau R^{\text{diff}}(W,\zeta,\zeta,V) \\ & +\sigma^2 \text{curv}^{\nu,re,l}(z,W) + 2\sigma\tau\left[R^{\nu,re,l}(\zeta,W,V,z) + R^{\nu,re,l}(\zeta,V,W,z)\right] \\ & +\tau^2 \text{curv}^{\nu,re,l}(\zeta,V) \\ > \ & 0. \end{aligned}$$

Notation: We denote the metrics obtained following deformations 1–5 above g_ν, $g_{\nu,re}$, $g_{\nu,re,l}$, g_s, and g_{new} respectively. We let notation like $\text{curv}^{\nu,re,l}$ and R^s have the obviously meaning.

We will not discuss the role of deformation 6, any further. It is a Cheeger deformation, so its effect is well understood. In particular, it preserves nonnegative and positive curvatures, and for us the purpose is that it allows the a priori simplification of the polynomial $P(\sigma,\tau)$ that we discussed above, and was explained in detail in Proposition 0.1.

The notation $O(s)$ will (as usual) stand for a quantity that converges to 0 faster than a fixed constant times s. The notation O will stand for a quantity that is too small to effect whether or not our metric is positively curved.

6. The Redistribution

As we mentioned above the metric obtained by carrying out deformations (1) and (3)–(6) described above is not positively curved. It is not possible to fully explain why at this point, but as mentioned above, the Main Lemma does not hold, in particular, there are choices of τ and V so that

$$P_Q(0,\tau) = \text{curv}^{\text{diff}}(\zeta,W) + 2\tau R^{\text{diff}}(W,\zeta,\zeta,V) + \tau^2 \text{curv}^{\text{old}}(\zeta,V) < 0.$$

To fix this problem we discuss the redistribution deformation (2) here. The idea is that certain (positive) curvatures of the type, $\text{curv}^{\text{old}}(\zeta,V)$, are redistributed so that they become larger near $t=0$ and relatively smaller away from $t=0$. This is at least a reasonable goal, since (as we'll see in section 8) $\text{curv}^{\text{diff}}(\zeta,W)$ and $R^{\text{diff}}(W,\zeta,\zeta,V)$ are both concentrated near $t=0$.

Within $V_1 \oplus V_2$ there is a 3–dimensional subdistribution $\mathcal{Z}$, that has zero curvature with ζ. $\mathcal{Z}^\perp \subset V_1 \oplus V_2$ is therefore a three dimensional subdistribution, along which we redistribute the curvature with ζ by warping the metric by a function φ whose gradient is proportional to ζ.

We want to concentrate the curvature near $t=0$, so we choose φ to be concave down near $t=0$ and concave up away from $t=0$.

More specifically, using φ' for $D_\zeta(\varphi)$, we choose φ so that

$$\begin{aligned} \varphi'(0) &= 0 \\ -101 &< \frac{\varphi''}{\nu^2} < -100 \text{ on an interval of size } O(\nu) \text{ near } t=0 \end{aligned}$$

and

$$10,000\nu^3 < \varphi'' < 10,001\nu^3 \text{ on an interval that looks like } \left[O(\nu), \frac{1}{100}\right].$$

For this section only, we call the metric obtained by doing only the Cheeger perturbations (1), (3), and (6) described above g_{old}, and we call the metric obtained by doing deformations (1), (2), (3), and (6), g_{new}. Where by (2) we mean, that we multiply the restriction of the metric to $\mathcal{Z}^\perp$ by φ^2, and do not change the metric on the orthogonal complement of $\mathcal{Z}^\perp$.

Our choice of φ'' allows us to also have

$$\begin{aligned} |\varphi'| &\leq O\left(100\nu^3\right) \\ \left|\varphi^2 - 1\right| &\leq O\left(100\nu^3\right), \\ \varphi|_{\left[O\left(\frac{1}{100}\right), \frac{\pi}{4}\right]} &\equiv 1. \end{aligned}$$

Since we carry out this change on $Sp(2)$ before some of our Cheeger deformations we have to check that the resulting metric is still invariant under the various S^3–actions. To see this, simply note that they all leave $V_1 \oplus V_2$ and ζ invariant. From this it follows that they all leave $\mathcal{Z}$ and $\mathcal{Z}^\perp$ invariant, and hence they all leave g_{new}–invariant.

Remark 6.1. *The constants* 100, 101, 10,000*, ect. really just symbolize large constants that are independent of our choice of metric parameters. We have not verified that our whole argument actually works with these particular constants. This question is fairly subtle, but since it is merely academic we have only checked that the argument works with* some *fixed constants playing the role of* 100, 101, 10,000*, ect.*

It is not surprising that the effect of this change in metric is to redistribute $\text{curv}^{\text{old}}\left(\zeta, \mathcal{Z}^\perp\right)$ toward $t=0$. The fact that we can do this without changing other curvatures in a substantial way, is an amazing fact, that makes our whole argument work.

Theorem 6.2. *g_{new} induces a metric of nonnegative curvature on Σ^7 whose zero planes are identical to those of g_{old}. Moreover, for any $V \in \mathcal{Z}^\perp$*

$$\text{curv}^{\text{new}}(\zeta, V) = \text{curv}^{\text{old}}(\zeta, V) + \varphi\varphi''|V|^2_{\text{old}},$$

and all other curvatures satisfy

$$\text{curv}^{\text{new}}(z, u) \geq \text{curv}^{\text{old}}(z, u) + O(\nu)\,\text{curv}^{\text{old}}(z, u).$$

Remark 6.3. *Please note that we are not asserting the existence of a new non-negatively curved metric on $Sp(2)$, only on Σ^7. The difference is that we have a tighter control on the pre-existing 0–curvatures of Σ^7. The result is, nevertheless, surprising. For a quick explanation of why it holds, we point to the extreme amount of rigidity present. (Cf* [Tapp2]*.) For example, since the distribution $\mathcal{Z}$ is parallel*

along ζ, it follows that any vector v tangent to $\mathcal{Z}^\perp$ can be extended to a field V tangent to $\mathcal{Z}^\perp$ so that

$$(\nabla_\zeta V)^{V_1 \oplus V_2} = 0.$$

Within this section, we will call such a field "vertically parallel". Along a curve tangent to H, these fields look like $(N_1\beta, N_2\delta)$, and hence in the language of Lie groups are the left invariant fields determined by $\begin{pmatrix} \beta & 0 \\ 0 & \delta \end{pmatrix}$. In the directions tangent to $V_1 \oplus V_2$, the splitting $\mathcal{Z} \oplus \mathcal{Z}^\perp$ is right invariant, but not left invariant. So we will extend these "vertically parallel" to be right invariant along $V_1 \oplus V_2$. If U is such a field and Z is basic horizontal, then (as O'Neill observed)

$$[U, Z]^H = 0.$$

With respect to the biinvariant metric we have

$$\nabla_Z U \in H, \text{ and}$$

Since the orbits of $A^{h_1} \oplus A^{h_2}$ are totally geodesic, $\nabla_U Z$ is also in H, so $[U, Z]^{V_1 \oplus V_2} = 0$, and in fact

$$[U, Z] = 0.$$

Proposition 6.4. *For $P \in \mathcal{Z}^\perp$, vertically parallel along ζ,*

$$\nabla^{\nu,re}_\zeta P = \varphi^2 \nabla^\nu_\zeta P + \frac{\varphi'}{\varphi} P$$

$$\nabla^{\nu,re}_P P = \nabla^\nu_P P - \varphi\varphi' \left|P\right|^2_\nu \zeta$$

For $Z \in H$, perpendicular to ζ and basic horizontal for $h_1 \oplus h_2$

$$\nabla^{\nu,re}_P Z = \varphi^2 \nabla^\nu_P Z,$$

and for $U \in H$ and basic horizontal for $h_1 \oplus h_2$ and for Z basic horizontal for $h_1 \oplus h_2$ or for $Z \in \mathcal{Z}$ and vertically parallel

$$\nabla^{\nu,re}_U Z = \nabla^\nu_U Z.$$

Proof. For $P \in \mathcal{Z}^\perp$, and vertically parallel

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_\zeta P, P \right\rangle_{\nu,re} &= D_\zeta \left\langle P, P \right\rangle_{\nu,re} \\ &= 2\varphi\varphi' \left\langle P, P \right\rangle_\nu \\ &= 2\left\langle \nabla^\nu_\zeta P, P \right\rangle_\nu + 2\varphi\varphi' \left\langle P, P \right\rangle_\nu. \end{aligned}$$

For $Q \in \mathcal{Z}^\perp$ vertically parallel (with respect to g_ν) and perpendicular to P

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_\zeta P, Q \right\rangle_{\nu,re} &= 2\left\langle \nabla^\nu_\zeta P, Q \right\rangle_\nu \\ &= 0. \end{aligned}$$

For $Q \in \mathcal{Z}$, vertically parallel, we also know that $[\zeta, Q] = [\zeta, P] = 0$. So

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_\zeta P, Q \right\rangle_{\nu,re} &= \left\langle [\zeta, P], Q \right\rangle_{\nu,re} - \left\langle [\zeta, Q], P \right\rangle_{\nu,re} \\ &= 0 \\ &= 2\left\langle \nabla^\nu_\zeta P, Q \right\rangle_{\nu,re} \end{aligned}$$

For Z in the orthogonal complement H of $V_1 \oplus V_2$, and basic horizontal for $h_1 \oplus h_2$

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_{\zeta} P, Z\right\rangle_{\nu,re} &= -\left\langle [\zeta, Z], P\right\rangle_{\nu,re} \\ &= -\varphi^2 \left\langle [\zeta, Z], P\right\rangle_{\nu} \\ &= 2\varphi^2 \left\langle \nabla^{\nu}_{\zeta} P, Z\right\rangle_{\nu,re} \\ &= 2\varphi^2 \left\langle A^{\nu,h_1\oplus h_2}_{\zeta} P, Z\right\rangle_{\nu,re}. \end{aligned}$$

Combining equations gives us

$$\nabla^{\nu,re}_{\zeta} P = \varphi^2 \nabla^{\nu}_{\zeta} P + \frac{\varphi'}{\varphi} P$$

as claimed.

$$\begin{aligned} \left\langle \nabla^{\nu,re}_{P} P, \zeta\right\rangle_{\nu,re} &= -\left\langle \nabla^{\nu,re}_{\zeta} P, P\right\rangle_{\nu,re} \\ &= -\left\langle \nabla^{\nu}_{\zeta} P, P\right\rangle_{\nu} - \varphi\varphi' \left\langle P, P\right\rangle_{\nu} \\ &= \left\langle \nabla^{\nu}_{P} P, \zeta\right\rangle_{\nu,re} - \varphi\varphi' \left\langle P, P\right\rangle_{\nu} \end{aligned}$$

For $Z \in H$ and perpendicular to ζ

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_{P} P, Z\right\rangle_{\nu,re} &= -D_Z \left\langle P, P\right\rangle_{\nu,re} + 2\left\langle [Z, P], P\right\rangle_{\nu,re} \\ &= 2\varphi^2 \left\langle \nabla^{\nu}_{P} P, Z\right\rangle_{\nu} \end{aligned}$$

However, since $\left\langle \nabla^{\nu}_{P} P, Z\right\rangle_{\nu} = 0$, we conclude that

$$2\left\langle \nabla^{\nu,re}_{P} P, Z\right\rangle_{\nu,re} = \left\langle \nabla^{\nu}_{P} P, Z\right\rangle_{\nu} = 0$$

For $Q \in \mathcal{Z}^{\perp}$,

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_{P} P, Q\right\rangle_{\nu,re} &= 2\varphi^2 \left\langle \nabla^{\nu}_{P} P, Q\right\rangle_{\nu} \\ &= 2\left\langle \nabla^{\nu}_{P} P, Q\right\rangle_{\nu,re} \\ &= 0 \end{aligned}$$

For $Q \in \mathcal{Z}$

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_{P} P, Q\right\rangle_{\nu,re} &= 2\left\langle [Q, P], P\right\rangle_{\nu,re} \\ &= 2\varphi^2 \left\langle [Q, P], P\right\rangle_{\nu} \\ &= 2\varphi^2 \left\langle \nabla^{\nu}_{P} P, Q\right\rangle_{\nu} \\ &= 2\varphi^2 \left\langle \nabla^{\nu}_{P} P, Q\right\rangle_{\nu,re} \end{aligned}$$

However, since $\left\langle \nabla^{\nu}_{P} P, Q\right\rangle_{\nu} = 0$, both sides are again 0. Combining equations we have

$$\nabla^{\nu,re}_{P} P = \nabla^{\nu}_{P} P - \varphi\varphi' \left|P\right|^2_{\nu} \zeta.$$

For $Z, Y \in H$, basic horizontal and Z perpendicular to ζ

$$\begin{aligned} 2\left\langle \nabla^{\nu,re}_{P} Z, Y\right\rangle_{\nu,re} &= \left\langle [Y, Z], P\right\rangle_{\nu,re} \\ &= \varphi^2 \left\langle [Y, Z], P\right\rangle_{\nu} \\ &= 2\varphi^2 \left\langle \nabla^{\nu}_{P} Z, Y\right\rangle_{\nu,re} \end{aligned}$$

For $Z \in H$, perpendicular to ζ and for $Q \in \mathcal{Z}^{\perp}$

$$\begin{aligned} 2\left\langle \nabla_P^{\nu,re} Z, Q\right\rangle_{\nu,re} &= D_Z \left\langle P, Q\right\rangle_{\nu,re} - \left\langle \left[Z,P\right], Q\right\rangle_{\nu,re} - \left\langle \left[Z,Q\right], P\right\rangle_{\nu,re} \\ &= 2\varphi^2 \left\langle \nabla_P^{\nu} Z, Q\right\rangle_{\nu} \\ &= 2\left\langle \nabla_P^{\nu} Z, Q\right\rangle_{\nu,re} \end{aligned}$$

If Q is chosen to be one of our vertically parallel fields, then all three terms in the second expression are 0, so in fact

$$2\left\langle \nabla_P^{\nu,re} Z, Q\right\rangle_{\nu,re} = 2\left\langle \nabla_P^{\nu} Z, Q\right\rangle_{\nu,re} = 2\left\langle \nabla_P^{\nu} Z, Q\right\rangle_{\nu} = 0$$

Similarly for $Q \in \mathcal{Z}$ vertically parallel, we have

$$2\left\langle \nabla_P^{\nu,re} Z, Q\right\rangle_{\nu,re} = -\left\langle \left[Z,P\right], Q\right\rangle_{\nu,re} - \left\langle \left[Z,Q\right], P\right\rangle_{\nu,re}$$

However, since both terms are 0 we have

$$2\left\langle \nabla_P^{\nu,re} Z, Q\right\rangle_{\nu,re} = 2\left\langle \nabla_P^{\nu} Z, Q\right\rangle_{\nu,re} = 2\left\langle \nabla_P^{\nu} Z, Q\right\rangle_{\nu} = 0$$

Combining equations we have

$$\nabla_P^{\nu,re} Z = \varphi^2 \nabla_P^{\nu} Z.$$

Finally, the last equation

$$\nabla_U^{\nu,re} Z = \nabla_U^{\nu} Z,$$

follows from the Koszul formula. □

Proposition 6.5. *For $P \in \mathcal{Z}^{\perp}$*

$$R^{\nu,re}\left(P,\zeta\right)\zeta = \varphi^2 R^{\nu}\left(P,\zeta\right)\zeta - 3\varphi\varphi' A_{\zeta}^{h_1\oplus h_2} P - \frac{\varphi''}{\varphi} P$$

$$R^{\nu,re}\left(\zeta,P\right)P = \varphi^4 R^{\nu}\left(\zeta,P\right)P - \left(\varphi\varphi''\right)\left|P\right|_{\nu}^2 \zeta$$

Proof. For $P \in \mathcal{Z}^{\perp}$, vertically parallel, we know that $[\zeta, P] = 0$. Since ζ is a geodesic field

$$\begin{aligned} R^{\nu,re}\left(P,\zeta\right)\zeta &= -\nabla_{\zeta}^{\nu,re}\nabla_P^{\nu,re}\zeta \\ &= -\nabla_{\zeta}^{\nu,re}\left(\varphi^2\nabla_{\zeta}^{\nu}P + \frac{\varphi'}{\varphi}P\right) \\ &= -2\varphi\varphi' A_{\zeta}^{h_1\oplus h_2}P - \varphi^2\nabla_{\zeta}^{\nu,re}\nabla_{\zeta}^{\nu}P - \frac{\varphi\varphi'' - \left(\varphi'\right)^2}{\varphi^2}P - \frac{\varphi'}{\varphi}\nabla_{\zeta}^{\nu,re}P \\ &= -2\varphi\varphi' A_{\zeta}^{h_1\oplus h_2}P - \varphi^2\nabla_{\zeta}^{\nu,re}\nabla_{\zeta}^{\nu}P - \frac{\varphi\varphi'' - \left(\varphi'\right)^2}{\varphi^2}P - \frac{\left(\varphi'\right)^2}{\varphi^2}P - \varphi^2\frac{\varphi'}{\varphi}\nabla_{\zeta}^{\nu}P \\ &= \varphi^2 R^{\nu}\left(P,\zeta\right)\zeta - 3\varphi\varphi' A_{\zeta}^{h_1\oplus h_2}P - \frac{\varphi''}{\varphi}P \end{aligned}$$

$$\begin{aligned} R^{\nu,re}\left(\zeta,P\right)P &= \nabla_{\zeta}^{\nu,re}\nabla_P^{\nu,re}P - \nabla_P^{\nu,re}\nabla_{\zeta}^{\nu,re}P \\ &= \nabla_{\zeta}^{\nu,re}\left(\nabla_P^{\nu}P - \varphi\varphi'\left|P\right|_{\nu}^2\zeta\right) - \nabla_P^{\nu,re}\left(\varphi^2\nabla_{\zeta}^{\nu}P + \frac{\varphi'}{\varphi}P\right) \end{aligned}$$

Since

$$\begin{aligned} \nabla_P^{\nu}P &= 0 \\ \nabla_{\zeta}^{\nu,re}\nabla_P^{\nu}P &= \nabla_{\zeta}^{\nu}\nabla_P^{\nu}P = 0. \end{aligned}$$

We use the third equation of the previous proposition to get

$$\nabla_P^{\nu,re}\left(\varphi^2\nabla_\zeta^\nu P\right)=\varphi^4\nabla_P^\nu\nabla_\zeta^\nu P.$$

So

$$\begin{aligned}R^{\nu,re}(\zeta,P)P &= \varphi^4R^{\nu,re}(\zeta,P)P-\nabla_\zeta^{\nu,re}\left(\varphi\varphi'|P|_\nu^2\zeta\right)-\left(\frac{\varphi'}{\varphi}\nabla_P^{\nu,re}P\right)\\ &= \varphi^4R^{\nu,re}(\zeta,P)P-\left(\varphi\varphi''+(\varphi')^2\right)|P|_\nu^2\zeta+\frac{\varphi'}{\varphi}\varphi\varphi'|P|_\nu^2\zeta\\ &= \varphi^4R^{\nu,re}(\zeta,P)P-(\varphi\varphi'')|P|_\nu^2\zeta\end{aligned}$$

□

Proposition 6.6. *For $W\in\mathcal{Z}$, and vertically parallel along ζ with respect to g_ν*

$$\begin{aligned}\nabla_\zeta^{\nu,re}W &= \nabla_\zeta^\nu W=0\\ \nabla_W^{\nu,re}W &= \nabla_W^\nu W=0.\end{aligned}$$

Proof. For the first equation, the point is that for $V\in\mathcal{Z}^\perp$, and vertically parallel,

$$[\zeta,W]=[\zeta,V]=0.$$

For the second equation, when we compute the inner product with $P\in\mathcal{Z}^\perp$, we extend W and P to be invariant under $A^{h_1}\oplus A^{h_2}$, so we can compute their Lie bracket as though they are right invariant vector fields in S^3. In particular,

$$\langle[W,P],W\rangle_{\nu,re}=\langle[W,P],W\rangle_\nu=0$$

so

$$\langle\nabla_W^{\nu,re}W,P\rangle_{\nu,re}=\langle\nabla_W^{\nu,re}W,P\rangle_\nu=0.$$

□

Proposition 6.7.

$$\begin{aligned}R^{\nu,re}(W,\zeta)\zeta &= R^\nu(W,\zeta)\zeta=0\\ R^{\nu,re}(\zeta,W)W &= R^\nu(\zeta,W)W=0.\end{aligned}$$

Proof.

$$R^{\nu,re}(W,\zeta)\zeta=-\nabla_\zeta^{\nu,re}\nabla_W^{\nu,re}\zeta=0=R^\nu(W,\zeta)\zeta$$

$$\begin{aligned}R^{\nu,re}(\zeta,W)W &= \nabla_\zeta^{\nu,re}\nabla_W^{\nu,re}W-\nabla_W^{\nu,re}\nabla_\zeta^{\nu,re}W\\ &= 0=R^\nu(\zeta,W)W.\end{aligned}$$

□

Proposition 6.8. *For $z\in H$ perpendicular to ζ and $P\in\mathcal{Z}^\perp$*

$$R^{\nu,re}(z,P)P=\varphi^4R^\nu(z,P)P-\varphi\varphi'|P|_\nu^2\left(\nabla_z^\nu\zeta\right)^{P,\perp},$$

where the superscript $^{P,\perp}$ denotes the component perpendicular to P.

$$R^{\nu,re}(P,z)z=\varphi^2R^\nu(P,z)z+\mathrm{II}^\zeta(z,z)\frac{\varphi'}{\varphi}P,$$

where

$$\mathrm{II}^\zeta(z,z)=\langle\nabla_z^\nu z,\zeta\rangle.$$

Note that these give consistent answers for the sectional curvatures—

$$\begin{aligned}\left\langle R^{\nu,re}(z,P)P,z\right\rangle &= \varphi^4\left\langle R^{\nu}(z,P)P,z\right\rangle - \varphi\varphi'|P|_\nu^2\left\langle \nabla^\nu_z\zeta,z\right\rangle\\ \left\langle R^{\nu,re}(P,z)z,P\right\rangle &= \varphi^4\left\langle R^{\nu}(P,z)z,P\right\rangle + \mathrm{II}^\zeta(z,z)\frac{\varphi'}{\varphi}\left\langle P,P\right\rangle_{\nu,re}\end{aligned}$$

Proof. Choose P to be the vertically parallel extension, then

$$\begin{aligned}R^{\nu,re}(z,P)P &= \nabla^{\nu,re}_z\nabla^{\nu,re}_P P - \nabla^{\nu,re}_P\nabla^{\nu,re}_z P\\ &= \nabla^{\nu,re}_z\left(\nabla^\nu_P P - \varphi\varphi'|P|^2_\nu\,\zeta\right) - \nabla^{\nu,re}_P\left(\varphi^2\nabla^\nu_P z\right)\end{aligned}$$

Since $\nabla^\nu_P P = 0$

$$\nabla^{\nu,re}_z\left(\nabla^\nu_P P\right) = \nabla^\nu_z\left(\nabla^\nu_P P\right) = 0.$$

Also

$$\begin{aligned}\nabla^{\nu,re}_P\left(\varphi^2\nabla^\nu_P z\right) &= \varphi^4\nabla^\nu_P\left(\nabla^\nu_P z\right) + \varphi^2\frac{\varphi'}{\varphi}P\left\langle\nabla^\nu_P z,\zeta\right\rangle_\nu\\ &= \varphi^4\nabla^\nu_P\left(\nabla^\nu_P z\right) + \varphi\varphi'\left\langle\nabla^\nu_P z,\zeta\right\rangle_\nu P\end{aligned}$$

So

$$R^{\nu,re}(z,P)P = \varphi^4 R^\nu(z,P)P - \varphi\varphi'|P|^2_\nu\,\nabla^{\nu,re}_z\zeta - \varphi\varphi'\left\langle\nabla^\nu_P z,\zeta\right\rangle_\nu P.$$

The last term on the right does not seem to be correct since it is proportional to P. The formula is nevertheless correct since this term cancels with the P–component of the second term. Indeed

$$\begin{aligned}&-\varphi\varphi'|P|^2_\nu\left\langle\nabla^{\nu,re}_z\zeta,\frac{P}{|P|_{\nu,re}}\right\rangle_{\nu,re}\frac{P}{|P|_{\nu,re}} - \varphi\varphi'\left\langle\nabla^\nu_P z,\zeta\right\rangle_\nu P\\ =\;& -\varphi\varphi'\left\langle\nabla^\nu_z\zeta,P\right\rangle_\nu P - \varphi\varphi'\left\langle\nabla^\nu_P z,\zeta\right\rangle_\nu P\\ =\;& \varphi\varphi'\left\langle\zeta,\nabla^\nu_z P\right\rangle_\nu P - \varphi\varphi'\left\langle\nabla^\nu_P z,\zeta\right\rangle_\nu P\\ =\;& 0\end{aligned}$$

So

$$R^{\nu,re}(z,P)P = \varphi^4 R^\nu(z,P)P - \varphi\varphi'|P|^2_\nu\left(\nabla^\nu_z\zeta\right)^{P,\perp}.$$

as claimed.

Extend z so that its basic horizontal and tangent to an intrinsic geodesic for the metric spheres around $(t,\sin 2\theta) = (0,0)$. Then

$$\begin{aligned}R^{\nu,re}(P,z)z &= \nabla^{\nu,re}_P\nabla^{\nu,re}_z z - \nabla^{\nu,re}_z\nabla^{\nu,re}_P z\\ &= \nabla^{\nu,re}_P\nabla^\nu_z z - \nabla^{\nu,re}_z\varphi^2\nabla^\nu_P z\end{aligned}$$

Since $\nabla^\nu_z z$ is proportional to ζ write

$$\begin{aligned}\nabla^\nu_z z &= \mathrm{II}^\zeta(z,z)\,\zeta\\ \nabla^{\nu,re}_P\nabla^\nu_z z &= \mathrm{II}^\zeta(z,z)\left(\varphi^2\nabla^\nu_\zeta P + \frac{\varphi'}{\varphi}P\right)\\ &= \varphi^2\nabla^\nu_P\nabla^\nu_z z + \mathrm{II}^\zeta(z,z)\frac{\varphi'}{\varphi}P\end{aligned}$$

Since $\nabla^\nu_P z$ is horizontal and $z\perp\zeta$

$$\nabla^{\nu,re}_z\varphi^2\nabla^\nu_P z = \varphi^2\nabla^\nu_z\nabla^\nu_P z$$

So

$$R^{\nu,re}(P,z)\,z=\varphi^2R^{\nu}(P,z)\,z+\mathrm{II}^{\zeta}(z,z)\,\frac{\varphi'}{\varphi}P.$$

□

Proposition 6.9. *For $z\in H$ perpendicular to ζ and for $W\in\mathcal{Z}$, vertically parallel with respect to g_ν*

$$\nabla^{\nu,re}_zW=\nabla^{\nu}_zW\in H\cap\left(\operatorname{span}\{\zeta\}\right)^{\perp}.$$

If P and Q are in $\mathcal{Z}^{\perp}$ and invariant under $A^{h_1}\oplus A^{h_2}$ and W and U are in $\mathcal{Z}$ and invariant under $A^{h_1}\oplus A^{h_2}$, then

$$\begin{aligned}
\left(\nabla^{\nu,re}_WP\right)^H&=\left(\nabla^{\nu}_WP\right)^H=0,\\
\left(\nabla^{\nu,re}_PQ\right)^{V_1\oplus V_2}&=\left(\nabla^{\nu}_PQ\right)^{V_1\oplus V_2},\\
\left(\nabla^{\nu,re}_WU\right)^{V_1\oplus V_2}&=\left(\nabla^{\nu}_WU\right)^{V_1\oplus V_2},\\
\left(\nabla^{\nu,re}_PW\right)^{\mathcal{Z}^{\perp}}&=\varphi^{-2}\left(\nabla^{\nu}_PW\right)^{\mathcal{Z}^{\perp}},\\
\left(\nabla^{\nu,re}_WP\right)^{\mathcal{Z}}&=\varphi^{2}\left(\nabla^{\nu}_WP\right)^{\mathcal{Z}},\\
\left(\nabla^{\nu,re}_PW\right)^{\mathcal{Z}}&=O\left(1-\varphi^2\right)\left(\nabla^{\nu}_PW\right)^{\mathcal{Z}},\\
\left(\nabla^{\nu,re}_WP\right)^{\mathcal{Z}^{\perp}}&=O\left(1-\varphi^2\right)\left(\nabla^{\nu}_WP\right)^{\mathcal{Z}^{\perp}}
\end{aligned}$$

Proof. For $U\in\mathcal{Z}^{\perp}$ vertically parallel and z basic horizontal all terms in the Koszul formula for $\langle\nabla_zW,U\rangle$ are 0 with respect to both metrics. For U perpendicular to $\mathcal{Z}^{\perp}$, all terms in the Koszul formula for $\langle\nabla^{\nu,re}_zW,U\rangle$ are the same for both metrics, so

$$\nabla^{\nu,re}_zW=\nabla^{\nu}_zW\in H\cap\left(\operatorname{span}\{\zeta\}\right)^{\perp}.$$

If $Z\in H$ is basic horizontal, then all terms in the Koszul formulas for

$$\langle\nabla^{\nu,re}_WP,Z\rangle_{\nu,re}\ \text{ and }\ \langle\nabla^{\nu}_WP,Z\rangle_{\nu}$$

vanish, so $\left(\nabla^{\nu,re}_WP\right)^H=\left(\nabla^{\nu}_WP\right)^H=0$.

If P and Q are in $\mathcal{Z}^{\perp}$ and W is in $\mathcal{Z}$ and all three fields are invariant under $A^{h_1}\oplus A^{h_1}$, then

$$\begin{aligned}
2\left\langle\nabla^{\nu,re}_PQ,W\right\rangle_{\nu,re}&=\left\langle[P,Q],W\right\rangle_{\nu,re}-\left\langle[Q,W],P\right\rangle_{\nu,re}+\left\langle[W,P],Q\right\rangle_{\nu,re}\\
&=\left\langle[P,Q],W\right\rangle_{\nu}-\varphi^2\left\langle[Q,W],P\right\rangle_{\nu}+\varphi^2\left\langle[W,P],Q\right\rangle_{\nu}
\end{aligned}$$

We can compute these Lie brackets as though the fields are right invariant fields in S^3, so

$$-\varphi^2\left\langle[Q,W],P\right\rangle_{\nu}+\varphi^2\left\langle[W,P],Q\right\rangle_{\nu}=0$$

and

$$\begin{aligned}
2\left\langle\nabla^{\nu,re}_PQ,W\right\rangle&=\left\langle[P,Q],W\right\rangle_{\nu}\\
&=2\left\langle\nabla^{\nu}_PQ,W\right\rangle_{\nu,re}
\end{aligned}$$

If V is also in $\mathcal{Z}^{\perp}$, then

$$\left\langle\nabla^{\nu,re}_PQ,V\right\rangle_{\nu,re}=\varphi^2\left\langle\nabla^{\nu}_PQ,V\right\rangle_{\nu}=\left\langle\nabla^{\nu}_PQ,V\right\rangle_{\nu,re}$$

So

$$\left(\nabla^{\nu,re}_PQ\right)^{V_1\oplus V_2}=\left(\nabla^{\nu}_PQ\right)^{V_1\oplus V_2}$$

as claimed.

A similar argument give us

$$\left(\nabla_W^{\nu,re}U\right)^{V_1\oplus V_2} = \left(\nabla_W^{\nu}U\right)^{V_1\oplus V_2}$$

Now suppose Q is in $\mathcal{Z}^\perp$ and invariant under $A^{h_1}\oplus A^{h_2}$. Since $\mathcal{Z}$ and $\mathcal{Z}^\perp$ are invariant under $A^{h_1}\oplus A^{h_2}$

$$\left\langle \nabla_P^{\nu,re}W, Q\right\rangle_{\nu,re} = -\left\langle W, \nabla_P^{\nu,re}Q\right\rangle_{\nu,re} = -\left\langle W, \nabla_P^{\nu}Q\right\rangle_{\nu,re} = \left\langle \nabla_P^{\nu}W, Q\right\rangle_{\nu} = \varphi^{-2}\left\langle \nabla_P^{\nu}W, Q\right\rangle_{\nu,re}$$

proving the fifth equation.

Similarly, if U is in $\mathcal{Z}$ and invariant under $A^{h_1}\oplus A^{h_2}$,

$$\left\langle \nabla_W^{\nu,re}P, U\right\rangle_{\nu,re} = -\left\langle P, \nabla_W^{\nu,re}U\right\rangle_{\nu,re} = -\left\langle P, \nabla_W^{\nu}U\right\rangle_{\nu,re} = -\varphi^2\left\langle P, \nabla_W^{\nu}U\right\rangle_{\nu} = \varphi^2\left\langle \nabla_W^{\nu}P, U\right\rangle_{\nu,re}$$

proving the sixth equation.

The last two equations have similar proofs. The Koszul formulas only have Lie Bracket terms, only we must compare terms with multiplied by φ^2 with terms with no φ^2. This leads us to get only the approximate answers that we have asserted. □

Proposition 6.10. *For $z\in H$ perpendicular to ζ and for $W\in\mathcal{Z}$,*

$$R^{\nu,re}(z,W)W = R^{\nu}(z,W)W \in H\cap\left(\operatorname{span}\{\zeta\}\right)^{\perp}$$
$$R^{\nu,re}(W,z)z = R^{\nu}(W,z)z,$$
$$\left[R^{\nu,re}(W,\zeta)z\right]^{H} = \left[R^{\nu}(W,\zeta)z\right]^{H}.$$
$$\left[R^{\nu,re}(W,\zeta)z\right]^{\mathcal{Z}} = \varphi^2\left[R^{\nu}(W,\zeta)z\right]^{\mathcal{Z}}.$$
$$\left|\left[R^{\nu,re}(W,\zeta)z\right]^{\mathcal{Z}^\perp}\right| = O\left(1-\varphi^2\right)|z|\,|W|.$$

Proof. Choose z to be basic horizontal and W to be vertically parallel, then

$$R^{\nu,re}(z,W)W = \nabla_z^{\nu,re}\nabla_W^{\nu,re}W - \nabla_W^{\nu,re}\nabla_z^{\nu,re}W$$

Since

$$\begin{aligned}\nabla_W^{\nu,re}W &= \nabla_W^{\nu}W = 0,\\ \nabla_z^{\nu,re}\nabla_W^{\nu,re}W &= \nabla_z^{\nu}\nabla_W^{\nu}W = 0.\end{aligned}$$

On the other hand, using the previous proposition twice we have

$$\nabla_W^{\nu,re}\nabla_z^{\nu,re}W = \nabla_W^{\nu}\nabla_z^{\nu}W \in H\cap\left(\operatorname{span}\{\zeta\}\right)^{\perp}$$

So

$$R^{\nu,re}(z,W)W = R^{\nu}(z,W)W \in H\cap\left(\operatorname{span}\{\zeta\}\right)^{\perp}$$

Choose z to be basic horizontal and W to be vertically parallel, then

$$R^{\nu,re}(W,z)z = \nabla_W^{\nu,re}\nabla_z^{\nu,re}z - \nabla_z^{\nu,re}\nabla_W^{\nu,re}z$$

Since

$$\nabla_z^{\nu,re}z = \nabla_z^{\nu}z \in H,$$
$$\nabla_W^{\nu,re}\nabla_z^{\nu,re}z = \nabla_W^{\nu,re}\nabla_z^{\nu}z = \nabla_W^{\nu}\nabla_z^{\nu}z$$

where the last equality follows from the previous proposition and Proposition 6.6. As before we have

$$\nabla_W^{\nu,re}z = \nabla_W^{\nu}z \in H\cap\left(\operatorname{span}\{\zeta\}\right)^{\perp}.$$

So

$$\nabla_z^{\nu,re}\nabla_W^{\nu,re}z = \nabla_z^{\nu}\nabla_W^{\nu}z.$$

So

$$R^{\nu,re}(W,z)z = R^{\nu}(W,z)z.$$

To prove the final three equations we note that since $[W,\zeta]=0$,

$$R^{\nu,re}(W,\zeta)z=\nabla^{\nu,re}_W\nabla^{\nu,re}_\zeta z-\nabla^{\nu,re}_\zeta\nabla^{\nu,re}_W z.$$

Since

$$\begin{aligned}\nabla^{\nu,re}_W z &= \nabla^\nu_W z\in H\cap(\operatorname{span}\{\zeta\})^\perp\\ \nabla^{\nu,re}_\zeta\nabla^{\nu,re}_W z &= \nabla^\nu_\zeta\nabla^\nu_W z.\end{aligned}$$

On the other hand

$$\left(\nabla^{\nu,re}_\zeta z\right)^H$$

is basic horizontal, so

$$\nabla^{\nu,re}_W\left(\nabla^{\nu,re}_\zeta z\right)^H=\nabla^\nu_W\left(\nabla^\nu_\zeta z\right)^H\in H\cap(\operatorname{span}\{\zeta\})^\perp.$$

Since $\nabla^{\nu,re}_W\left(\nabla^{\nu,re}_\zeta z\right)^{V_1\oplus V_2}$ and $\nabla^\nu_W\left(\nabla^\nu_\zeta z\right)^{V_1\oplus V_2}$ are both in $V_1\oplus V_2$, it follows that

$$\left(\nabla^{\nu,re}_W\nabla^{\nu,re}_\zeta z\right)^H=\nabla^{\nu,re}_W\left(\nabla^{\nu,re}_\zeta z\right)^H=\nabla^\nu_W\left(\nabla^\nu_\zeta z\right)^H=\left(\nabla^\nu_W\nabla^\nu_\zeta z\right)^H.$$

So

$$\left[R^{\nu,re}(W,\zeta)z\right]^H=\left[R^\nu(W,\zeta)z\right]^H.$$

Since

$$\left(\nabla^{\nu,re}_\zeta z\right)^{V_1\oplus V_2}\in\mathcal{Z}^\perp,$$

it follows from the previous proposition that

$$\left(\nabla^{\nu,re}_W\nabla^{\nu,re}_\zeta z\right)^{\mathcal{Z}}=\varphi^2\left(\nabla^\nu_W\nabla^\nu_\zeta z\right)^{\mathcal{Z}}.$$

We also have $\nabla^{\nu,re}_\zeta\nabla^{\nu,re}_W z=\nabla^\nu_\zeta\nabla^\nu_W z$. However, since $\nabla^\nu_W z\in H\cap(\operatorname{span}\{\zeta\})^\perp$, we have $\left(\nabla^\nu_\zeta\nabla^\nu_W z\right)^{\mathcal{Z}}=0$. So

$$\left[R^{\nu,re}(W,\zeta)z\right]^{\mathcal{Z}}=\varphi^2\left[R^\nu(W,\zeta)z\right]^{\mathcal{Z}}.$$

On the other hand, we just have

$$\left(\nabla^{\nu,re}_W\nabla^{\nu,re}_\zeta z\right)^{\mathcal{Z}^\perp}=O\left(1-\varphi^2\right)\left(\nabla^\nu_W\nabla^\nu_\zeta z\right)^{\mathcal{Z}^\perp}$$

Combining this with

$$\begin{aligned}\nabla^{\nu,re}_\zeta\nabla^{\nu,re}_W z&=\nabla^\nu_\zeta\nabla^\nu_W z\\ \left[R^\nu(W,\zeta)z\right]^{\mathcal{Z}^\perp}&=0,\end{aligned}$$

we have

$$\left|\left[R^{\nu,re}(W,\zeta)z\right]^{\mathcal{Z}^\perp}\right|=O\left(1-\varphi^2\right)|z|\,|W|.$$

□

A very similar argument gives us

Proposition 6.11.

$$\begin{aligned}\left[R^{\nu,re}(W,z)\zeta\right]^H&=\left[R^\nu(W,z)\zeta\right]^H.\\ \left[R^{\nu,re}(W,z)\zeta\right]^{\mathcal{Z}}&=\varphi^2\left[R^\nu(W,z)\zeta\right]^{\mathcal{Z}}.\\ \left|\left[R^{\nu,re}(W,z)\zeta\right]^{\mathcal{Z}^\perp}\right|&=O\left(1-\varphi^2\right)|z|\,|W|.\end{aligned}$$

Proposition 6.12. *For $U, V, Q \in \mathcal{Z} \cup \mathcal{Z}^{\perp}$ and mutually perpendicular*

$$\left(R^{\nu,re}\left(U,V\right)Q\right)^{H} = O\left(\varphi'\right)|U||V||Q|\,\zeta.$$

Proof. Extend all three vectors in be invariant under $A^{h_1} \oplus A^{h_2}$. We have

$$\left(R^{\nu,re}\left(U,V\right)Q\right)^{H} = \left(\nabla^{\nu,re}_{U}\nabla^{\nu,re}_{V}Q - \nabla^{\nu,re}_{V}\nabla^{\nu,re}_{U}Q - \nabla^{\nu,re}_{[U,V]}Q\right)^{H}.$$

Our covariant derivative computations and our hypothesis about the three vectors being mutually perpendicular give us that the H–components of each of $\nabla^{\nu,re}_{V}Q$, $\nabla^{\nu,re}_{U}Q$, and $[U,V]$ are 0. Therefore using Propositions 6.4, 6.6, and 6.9 we have

$$\begin{aligned}
\left(\nabla^{\nu,re}_{U}\nabla^{\nu,re}_{V}Q\right)^{H} &= \left\langle U^{\mathcal{Z}^{\perp}}, \nabla^{\nu,re}_{V}Q\right\rangle \frac{\varphi'}{\varphi}\zeta, \\
\left(\nabla^{\nu,re}_{V}\nabla^{\nu,re}_{U}Q\right)^{H} &= \left\langle V^{\mathcal{Z}^{\perp}}, \nabla^{\nu,re}_{U}Q\right\rangle \frac{\varphi'}{\varphi}\zeta, \text{ and} \\
\left(\nabla^{\nu,re}_{[U,V]}Q\right)^{H} &= \left\langle Q^{\mathcal{Z}^{\perp}}, [U,V]\right\rangle \frac{\varphi'}{\varphi}\zeta.
\end{aligned}$$

So

$$\left(R^{\nu,re}\left(U,V\right)Q\right)^{H} = O\left(\varphi'\right)|U||V||Q|\,\zeta$$

as claimed. □

When all four vectors are tangent to $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ we have

Proposition 6.13. *For $u, v, w, z \in \mathcal{Z} \oplus \mathcal{Z}^{\perp}$,*

$$R^{\nu,re}\left(u,v,w,z\right) = O\left(1-\varphi^2\right)R^{\nu}\left(u,v,w,z\right) + O\left(1-\varphi^2\right)|u||v||w||z|$$

and

$$R^{\nu,re}\left(u,w,w,u\right) = O\left(1-\varphi^2\right)R^{\nu}\left(u,w,w,u\right)$$

Proof. If U, W, and Z are in either $\mathcal{Z}$ or $\mathcal{Z}^{\perp}$ and invariant under $A^{h_1} \oplus A^{h_2}$, then in the Koszul formula for $2\left\langle \nabla^{\nu,re}_{U}W, Z\right\rangle$, the derivative terms vanish The new Lie bracket terms can differ from the old ones by a multiplicative factor of $O\left(1-\varphi^2\right)$. Applying this principle several times yields the result. □

Finally mimicking the proof of O'Neill's horizontal curvature equation we have

Proposition 6.14. *If x, y, z, and u are in H,then*

$$R^{\nu,re}\left(x,y,z,u\right) = O\left(1-\varphi^2\right)R^{\nu}\left(x,y,z,u\right)$$

To complete the proof of Theorem 6.2 it remains to establish the assertion about nonnegative curvature.

A plane that is perpendicular to either ζ or W is positively curved, since such planes were uniformly positively curved before the redistribution, and the redistribution has a small effect on curvatures.

A plane that is not perpendicular to ζ and not perpendicular to W has the form $P = \operatorname{span}\left\{\zeta + \sigma z, W + \tau V\right\}$. Because of the Cheeger deformation (6) we may assume that z is in the horizontal space for the Gromoll-Meyer submersion $Sp\left(2\right) \longrightarrow S^4$.

Our curvature is a quartic polynomial

$$P\left(\sigma,\tau\right) = R\left(\zeta + \sigma z, W + \tau V, W + \tau V, \zeta + \sigma z\right).$$

We have seen that the constant and linear terms vanish with respect to g_{new}. So our polynomial is

$$\begin{aligned} P(\sigma,\tau) &= \sigma^2 R^{\text{new}}(z,W,W,z) + 2\sigma\tau R^{\text{new}}(\zeta,W,V,z) + 2\sigma\tau R^{\text{new}}(\zeta,V,W,z) + \tau^2 R^{\text{new}}(\zeta,V,V,\zeta) \\ &\quad +2\sigma^2\tau R^{\text{new}}(z,W,V,z) + 2\sigma\tau^2 R^{\text{new}}(\zeta,V,V,z) + \sigma^2\tau^2 R^{\text{new}}(z,V,V,z) \end{aligned}$$

combining our curvature computations with the fact

$$\begin{aligned} 1-\varphi^2 &= O\left(100\nu^3\right) \\ \varphi' &= O\left(100\nu^3\right) \\ -\varphi'' &\geq -\frac{\nu^2}{100} \end{aligned}$$

gives us that

$$P(\sigma,\tau) \geq \left(1-O\left(\nu^3\right)\right) P^{\text{old}}(\sigma,\tau) - \frac{\tau^2\nu^2}{100} R^{\text{old}}(\zeta,V,V,\zeta) + Q(\sigma,\tau). \tag{6.15}$$

Here $Q(\sigma,\tau)$ is a quartic polynomial that looks like

$$Q(\sigma,\tau) = C_{\sigma\tau}\sigma\tau + C_{\sigma^2\tau}\sigma^2\tau + C_{\sigma\tau^2}\sigma\tau^2,$$

whose coefficients $C_{\sigma\tau}, C_{\sigma^2\tau}$, and $C_{\sigma\tau^2}$ satisfy

$$\begin{aligned} C_{\sigma\tau} &\leq O(\nu)\sqrt{R^{\text{new}}(z,W,W,z)}\sqrt{R^{\text{new}}(\zeta,V,V,\zeta)} \\ C_{\sigma^2\tau} &\leq O(\nu)\sqrt{R^{\text{new}}(z,W,W,z)}\sqrt{R^{\text{new}}(z,V,V,z)} \\ C_{\sigma\tau^2} &\leq O(\nu)\sqrt{R^{\text{new}}(\zeta,V,V,\zeta)}\sqrt{R^{\text{new}}(z,V,V,z)}. \end{aligned}$$

These estimates imply that we can replace $Q(\sigma,\tau)$ in 6.15 with O. For example, the quadratic

$$\begin{aligned} & \sigma^2 R^{\text{new}}(z,W,W,z) + \sigma\tau C_{\sigma\tau}\sigma\tau + \tau^2 R^{\text{new}}(\zeta,V,V,\zeta) \\ \geq\ & \sigma^2\left(R^{\text{new}}(z,W,W,z) - \frac{O\left(\nu^2\right) R^{\text{new}}(z,W,W,z)\, R^{\text{new}}(\zeta,V,V,\zeta)}{R^{\text{new}}(\zeta,V,V,\zeta)}\right) \\ \geq\ & \sigma^2\left(R^{\text{new}}(z,W,W,z) - O\left(\nu^2\right) R^{\text{new}}(z,W,W,z)\right) \\ =\ & \sigma^2\left(R^{\text{new}}(z,W,W,z) + O\right) \end{aligned}$$

Similar arguments allow us to drop the $C_{\sigma^2\tau}\sigma^2\tau$ and $C_{\sigma\tau^2}\sigma\tau^2$ terms of $Q(\sigma,\tau)$. (Cf Theorem 12.1). So 6.15 becomes

$$P(\sigma,\tau) \geq \left(1-O\left(\nu^3\right)\right) P^{\text{old}}(\sigma,\tau) - \frac{\tau^2\nu^2}{100} R^{\text{old}}(\zeta,V,V,\zeta) + O. \tag{6.15}$$

We have an inequality instead of an equality because in many cases the curvature is much bigger. For example from Proposition 6.5 we have that for $P \in \mathcal{Z}^{\perp}$

$$\begin{aligned} \langle R^{\text{new}}(P,\zeta)\zeta, P\rangle &\geq \varphi^4\left\langle R^{\text{old}}(P,\zeta)\zeta, P\right\rangle - (\varphi\varphi'')|P|^2_{\text{old}} \\ &\geq \varphi^4\left\langle R^{\text{old}}(P,\zeta)\zeta, P\right\rangle - \frac{\nu^2}{100}|P|^2_{\text{old}} \end{aligned}$$

but in many places this curvature is larger. Similarly from Proposition 6.8 we have that for $z \in H$, perpendicular to ζ and for $P \in \mathcal{Z}^{\perp}$

$$\langle R^{\text{new}}(P,z)z, P\rangle = \varphi^4\left\langle R^{\text{old}}(z,P)P, z\right\rangle + \text{II}^{\zeta}(z,z)\frac{\varphi'}{\varphi}\langle P,P\rangle_{\text{new}}$$

The extra term here is nonnegative since both $\text{II}^{\zeta}(z,z)$ and φ' are nonpositive.

The theorem follows from inequality 6.15.

7. The Warping function induced by $Sp(2)$

As promised, in the next two sections we analyze the effect on Equation 5.1 of running the h_2–Cheeger perturbation for a long time. If ν is the parameter of this perturbation, then we will show that making ν small has the effect of concentrating all of the terms on the right hand side of equation 5.1,

$$\operatorname{curv}(\zeta, W) = -s^2 \left(D_\zeta \left(|H_w| D_\zeta |H_w|\right)\right) + s^4 \left(D_\zeta |H_w|\right)^2,$$

around $t = 0$. (In the Gromoll-Meyer sphere ζ plays the role of X.)

The advantage of doing this is that it will allow us to choose our "partial" conformal factor so that it is constant away from $t = 0$, thus avoiding an analysis of how the partial conformal change effects the intersection of the two pieces of the zero curvature locus.

Along any integral curve of ζ, $|H_w|$ is the length of a Killing field of our $SO(3)$–action on S^4. Since the principal orbits of this action on S^4 are two spheres and the action on these two spheres is standard, these two spheres are round.

So that our geometry is more easily comparable to the standard round S^4, we look at the Killing fields

$$\left(0, \frac{\vartheta}{2}\right)$$

on $Sp(2)$ and we set

$$\psi = \left| \left(0, \frac{\vartheta}{2}\right)^{\text{horiz}} \right|.$$

To understand the geometric meaning of ψ, think of the join decomposition described in the remark after Proposition 2.5,

$$S^4 = S^1_{\mathbb{R}} * S^2_{\text{Im}}.$$

The S^2s of the join decomposition are the principal orbits of the $SO(3)$–action and the intrinsic metric on them is ψ^2 times the unit metric.

Along any integral curve of ζ, H_w is a constant multiple of $\left(0, \frac{\vartheta}{2}\right)^{\text{horiz}}$ we call this multiple w_h, so

$$\begin{aligned} H_w &= w_h \left(0, \frac{\vartheta}{2}\right)^{\text{horiz}} \\ |H_w| &= w_h \psi, \text{ and} \\ w_h &= O\left(\frac{1}{\nu^2}\right). \end{aligned}$$

Remark 7.1. *The exact value of w_h depends on which integral curve of ζ we are on. The variation can be seen by noticing how $\sin\lambda$ varies in Proposition 4.7. It is for precisely this reason that we cannot use a regular conformal change to even out the curvature.*

Since $\left|\left(0, \frac{\vartheta}{2}\right)\right| = \frac{\nu}{2}$ and $\psi = \left|\left(0, \frac{\vartheta}{2}\right)^{\text{horiz}}\right|$, it is not hard to see that the effect of the h_2–Cheeger perturbation on the geometry of S^4 is to shrink the S^2s. More precisely the S^3s that are the join of S^2_{Im} and any $S^0 \subset S^1_{\mathbb{R}}$ become very thin "cigars". Unfortunately this coarse description is not sufficient for our purposes, since we need to understand the derivatives and second derivatives of ψ.

We will prove in subsection 8.1 that the redistribution described in the previous section has a minimal effect on ψ. Once this is established, it will be enough to know the effect of the two Cheeger parameters ν and l. For now we just focus on this.

When we want to emphasize the dependence of ψ on ν and l we will write, $\psi_{\nu,l}$.

To find $\psi_{\nu,l}$ we recall that the horizontal vectors that project to the S^2s look like

$$(\cos 2t)\,\eta^{2,0} = \left((\cos 2t)\,\eta, (\cos 2t)\,\eta + \sin 2t\frac{\vartheta}{\nu^2}\right),$$

here as always, the notational convention on page 17 is in effect. So

$$\begin{aligned}\psi_{\nu,l} &= \frac{1}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}}\left\langle\left(0,\frac{\vartheta}{2}\right),(\cos 2t)\,\eta^{2,0}\right\rangle \\ &= \frac{1}{2}\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}}.\end{aligned}$$

Using the formulas for the projections of $\eta^{2,0}$ onto the orbits of $A^u \times A^d$ from [Wilh2] we have

$$\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l} = \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + \frac{1}{2l^2}\left(1-\cos^2 2t\cos^2 2\theta\right)$$

and

Proposition 7.2.

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{\nu,l} &= \frac{\left(1+\frac{1}{2l^2}\sin^2 2\theta\right)\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}} \\ &= \frac{\left|x^{2,0}\right|^2_{\nu,l}\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}} \\ \frac{\partial}{\partial\theta}\psi_{\nu,l} &= -\frac{1}{4l^2}\frac{\sin 2t\cos^2 2t\sin 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\psi_{\nu,l} &= -\left|x^{2,0}\right|^2_{\nu,l}\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\left(-4\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t + \frac{2}{\nu_l^2} + 4\left(\frac{1}{\nu_l^2}\right)\cos^2 2t\right) \\ \frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\psi_{\nu,l} &= \frac{\cos 2t\sin 4\theta}{l^2\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\left(-\frac{1}{2}\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t + \frac{1}{\nu_l^2}\sin^2 2t\right)\end{aligned}$$

$$\frac{\partial^2}{\partial\theta^2}\psi_{\nu,l} = -\frac{\sin 2t\cos^2 2t}{l^2}\frac{\cos 4\theta\left(\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t + \frac{1}{\nu_l^2}\sin^2 2t\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}} + \frac{3}{2}\frac{\sin 2t\cos^4 2t}{4l^4}\frac{\sin^2 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}$$

The computations are long, but straightforward. Since the results are not qualitatively surprising, we have deferred giving the details until the appendix.

8. Concentrated Curvature Near $t = 0$

Plugging $\zeta = X$ and $|H_w| = w_h\psi$ into 1.9 gives us

$$\operatorname{curv}_{g_s}(\zeta, W) = -s^2 w_h^2\left(D_\zeta\left(\psi D_\zeta\psi\right)\right) + w_h^2 s^4\left(D_\zeta\psi\right)^2.$$

If z is the parameter of an integral curve of ζ, then the leading order, total derivative term, $-s^2 w_h^2 \left(D_\zeta \left(\psi D_\zeta \psi\right)\right)$, is negative near $z=0$, positive for large enough z, and integrates to 0. The effect of the ν perturbation is to concentrate this region of negativity, and the bulk of the region of positivity near $z=0$. Before proving this we need

Proposition 8.1. *Let n be the normalized gradient field for* $\operatorname{dist}\left(S^1_{\mathbb{R}}, \cdot\right)$ *on S^4 with respect to $g_{\nu,l}$. If*

$$\zeta = n\cos\varphi + y^{2,0}\sin\varphi,$$

then

$$\begin{aligned} D_\zeta\left(\cos\varphi\right) &= O\left(t\right) \\ D_\zeta\left(\sin\varphi\right) &= O\left(t\right). \end{aligned}$$

Proof. Let c_ζ be an integral curve of ζ starting at $(t,\theta) = (0,0)$. Consider the triangles, $\triangle_\theta$ whose sides are the geodesic with $t=0$, c_ζ, and the various geodesics that are integral curves of n starting at $(t,\theta) = (0,\theta)$.

Let φ_0 be the angle between $\dot{c}_\zeta\left(0\right)$ and n. Then the interior angles of $\triangle_\theta$ are $\frac{\pi}{2}, \frac{\pi}{2} - \varphi_0$, and φ. So

$$\varphi = \varphi_0 + \text{angle–excess}\left(\triangle_\theta\right).$$

Since $\operatorname{area}\left(\triangle_\theta\right) = O\left(\theta^2\right)$, the result follows. □

Proposition 8.2. *For $t > \frac{\nu}{2}$*

$$-s^2 w_h^2 \left(D_\zeta\left(\psi D_\zeta \psi\right)\right) > 0$$

and

$$\operatorname{curv}_s\left(\zeta, W\right)|_{\left[O\left(c^{3/4}\right), \frac{\pi}{4}\right]} \leq \int_{\gamma_\zeta} \operatorname{curv}_s\left(\zeta, W\right)$$

provided $c\nu = s^{6/7}$ and $l = O\left(\nu^{\frac{1}{3}}\right)$.

Remark 8.3. *Together these inequalities imply that all of the negative curvature of g_s occurs on the interval $[0,\nu]$ and the bulk of the positive curvature occurs on $[\nu, O\left(c\right)]$. In particular, g_s is positively curved for $t > \frac{\nu}{\sqrt{8}}$ and our partial warping can be carried out on $[0, O\left(c\right)]$.*

Remark 8.4. *Our proof relies on the computations of the various derivatives of ψ that are stated in previous section and proven in the Appendix. They are done in the Appendix with respect to the metric $g_{\nu,l}$, while to justify this proposition we will need to know them with respect to $g_{\nu,re,l}$. So technically this proposition is about an (as yet) undiscussed metric $g_{\nu,l,s}$. I.e. the metric obtained by scaling the fibers of $Sp\left(2\right) \longrightarrow S^4$ after performing the Cheeger deformation $A^u \times A^d \times A^{h_1} \times A^{h_2}$, but with out performing the redistribution. We will show in Subsection 8.1 (at the end of this section) that the effect of the redistribution on the various derivatives of ψ is sufficiently small so that this proposition remains valid for the actual metric g_s.*

Proof. From the previous section we have

$$\begin{aligned} \frac{\partial}{\partial t}\psi_{\nu,l} &= \frac{\left|x^{2,0}\right|^2_{\nu,l}\cos 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^3_{\nu,l}} \\ \frac{\partial}{\partial \theta}\psi_{\nu,l} &= -\frac{1}{4l^2}\frac{\sin 2t\cos^2 2t\sin 4\theta}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^3_{\nu,l}} \end{aligned}$$

Since the $\frac{\partial}{\partial\theta}$–direction is a linear combination of the vectors $y^{2,0}$ and $(-\mathfrak{v},\mathfrak{v})$ and $D_{(-\mathfrak{v},\mathfrak{v})}\psi_{\nu,l}=0$, we get

$$D_{y^{2,0}}\psi_{\nu,l}=-\frac{1}{4l^2}\frac{\sin 2t\cos^3 2t\sin 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}},$$

where the extra factor of $\cos 2t$ is $\left\langle \frac{\partial}{\partial\theta}, y^{2,0}\right\rangle$. So if

$$\zeta=n\cos\varphi+y^{2,0}\sin\varphi$$

$$D_\zeta\psi_{\nu,l}=\frac{\left|x^{2,0}\right|^2_{\nu,l}\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}}\cos\varphi-\frac{1}{4l^2}\frac{\sin 2t\cos^3 2t\sin 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}}\sin\varphi$$

So

$$\left(D_\zeta\left(\psi_{\nu,l}\right)\right)^2\leq 2\left(\frac{\left|x^{2,0}\right|^4_{\nu,l}\cos^2 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^6_{\nu,l}}\right)\cos^2\varphi+\frac{\sin^2 2t\cos^6 2t\sin^2 4\theta}{8\left|(\cos 2t)\,\eta^{2,0}\right|^6_{\nu,l}\,l^4}\sin^2\varphi.$$

We can also get an explicit formula for $-\psi_{\nu,l}D_\zeta D_\zeta\left(\psi_{\nu,l}\right)$, but its quite complicated, so its easier to estimate it. First notice that erasing various A–tensors we have

$$-\frac{D_\zeta D_\zeta\left(\psi_{\nu,l}\right)}{\psi_{\nu,l}}\geq \mathrm{curv}_{g_\nu}\left(\zeta,\eta^{2,0}_u\right),$$

where $\eta^{2,0}_u=\frac{\eta^{2,0}}{|\eta^{2,0}|}$. So

$$\begin{aligned}-\psi_{\nu,l}D_\zeta D_\zeta\left(\psi_{\nu,l}\right) &\geq \psi^2_{\nu,l}\mathrm{curv}_{g_\nu}\left(\zeta,\eta^{2,0}_u\right)\\ &= \frac{\psi^2_{\nu,l}}{\left|\cos 2t\eta^{2,0}\right|^2_{\nu,l}}\left(\cos^2 2t+\frac{1}{2}\sin^2 2t\right).\end{aligned}$$

So to determine where the total derivative is positive, it suffices to solve

$$\psi^2_{\nu,l}\left(\cos^2 2t\right)\geq 2\left(\frac{\left|x^{2,0}\right|^4_{\nu,l}\cos^2 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^4_{\nu,l}}\right)\cos^2\varphi+\frac{\sin^2 2t\cos^6 2t\sin^2 4\theta}{8\left|(\cos 2t)\,\eta^{2,0}\right|^4_{\nu,l}\,l^4}\sin^2\varphi$$

or

$$\frac{\sin^2 2t}{4\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}}\geq 2\left(\frac{\left|x^{2,0}\right|^4_{\nu,l}}{\left|(\cos 2t)\,\eta^{2,0}\right|^4_{\nu,l}}\right)\cos^2\varphi+\frac{\sin^2 2t\cos^4 2t\sin^2 4\theta}{8\left|(\cos 2t)\,\eta^{2,0}\right|^4_{\nu,l}\,l^4}\sin^2\varphi$$

or

$$\frac{\sin^2 2t}{4}\geq 2\left(\frac{\left|x^{2,0}\right|^4_{\nu,l}}{\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}}\right)\cos^2\varphi+\frac{\sin^2 2t\cos^4 2t\sin^2 4\theta}{8\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\,l^4}\sin^2\varphi$$

Since $l=O\left(\nu^{1/3}\right)$, and on the integral curves of ζ in the former 0–locus, $\sin 4\theta=O\left(\sin 2\theta\right)=O\left(\sin 2t\right)$, and from the appendix we have

$$\begin{aligned}\left|\cos 2t\eta^{2,0}\right|^2_{\nu,l} &= 1+\frac{\sin^2 2\theta}{2l^2}+\left(\frac{1}{\nu^2}+\frac{1}{2l^2}-\left(1+\frac{\sin^2 2\theta}{2l^2}\right)\right)\sin^2 2t\\ &\geq 1+\frac{\sin^2 2t}{\nu^2}+\frac{\sin^2 2t}{2l^2},\end{aligned}$$

the last term and the $\left|x^{2,0}\right|_{\nu,l}^4$ factor on the first term can be ignored. So (with a minor adjustment) our inequality is

$$\frac{\sin^2 2t}{4} \geq 2\left(\frac{1}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^2}\right)$$

or

$$t^2 \geq \frac{2}{1+\frac{\sin^2 2t}{\nu^2}} = \frac{2\nu^2}{\nu^2+\sin^2 2t},$$

which happens when $t \geq O\left(\nu^{1/2}\right)$, which is not good enough for our purposes.

However, assuming that $t \leq \nu^{1/2}$ allows us to greatly simplify our estimates for $-\psi_{\nu,l} D_\zeta D_\zeta\left(\psi_{\nu,l}\right)$. Indeed starting with

$$\zeta = n\cos\varphi + y\sin\varphi$$

we have

$$\begin{aligned} D_\zeta D_\zeta \psi_{\nu,l} &= \cos^2\varphi \frac{\partial^2}{\partial t^2}\psi_{\nu,l} + 2\cos\varphi\sin\varphi\cos 2t \frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\psi_{\nu,l} + \sin^2\varphi\cos^2 2t\frac{\partial^2}{\partial\theta^2}\psi_{\nu,l} \\ &\quad + \frac{\left|\hat{x}^{2,0}\right|_{\nu,l}^2 \cos 2t}{\left|(\cos 2t)\,\hat{\eta}^{2,0}\right|_{\nu,l}^3}\left(D_\zeta\cos\varphi\right) - \frac{1}{4l^2}\frac{\sin 2t\cos^3 2t\cos 2\theta\sin 2\theta}{\left|(\cos 2t)\,\hat{\eta}^{2,0}\right|_{\nu,l}^3}\left(D_\zeta\sin\varphi\right). \end{aligned}$$

When we consider our formulas for $\frac{\partial^2}{\partial t^2}\psi_{\nu,l}$, $\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\psi_{\nu,l}$, and $\frac{\partial^2}{\partial\theta^2}\psi_{\nu,l}$ from the appendix, and the fact that $\left(D_\zeta\sin\varphi\right) = O\left(t\right)$, we see that the second, third and last terms are dominated by the first term when $t \leq O\left(\nu^{1/2}\right)$.

The fourth term is positive (in $-D_\zeta D_\zeta\left(\psi_{\nu,l}\right)$), so dropping it gives us that for $t \leq \nu^{1/2}$

$$\begin{aligned} -D_\zeta D_\zeta\psi_{\nu,l} &\geq -\frac{9}{10}\cos^2\varphi\frac{\partial^2}{\partial t^2}\psi_{\nu,l} \\ &\geq \left|x^{2,0}\right|_{\nu,l}^2\left(\cos^2\varphi\right)\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^5}\frac{5}{\nu_l^2}. \end{aligned}$$

Similarly, when $t \leq \nu^{1/2}$ we have that the second term in our estimate for $\left(D_\zeta\left(\psi_{\nu,l}\right)\right)^2$ is overwhelmed by the first. So

$$\left(D_\zeta\left(\psi_{\nu,l}\right)\right)^2 \leq 2\left(\frac{\left|x^{2,0}\right|_{\nu,l}^4\cos^2 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^6}\right)\cos^2\varphi.$$

Thus the total derivative is positive when

$$\psi_{\nu,l}\left|x^{2,0}\right|_{\nu,l}^2\left(\cos^2\varphi\right)\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^5}\frac{5}{\nu_l^2} \geq 2\left(\frac{\left|x^{2,0}\right|_{\nu,l}^4\cos^2 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^6}\right)\cos^2\varphi.$$

Since $\psi_{\nu,l} = \frac{\sin 2t}{2\left|(\cos 2t)\eta^{2,0}\right|}$, this is equivalent to

$$\begin{aligned} \frac{1}{2}\sin^2 2t\frac{5}{\nu_l^2} &\geq 2\left(\left|x^{2,0}\right|_{\nu,l}^2\cos^2 2t\right) \text{ or} \\ \sin^2 2t &\geq \nu_l^2 \text{ or} \\ 4t^2 &\geq \nu_l^2 \end{aligned}$$

so its enough to have

$$t \geq \frac{1}{2}\nu_l.$$

To prove the integral inequality we first note that

$$\begin{aligned}\left(D_\zeta \psi_{\nu,l}\right)^2 &\geq \left(\frac{1+\frac{\sin^2 2\theta}{l^2}}{2\left(\cos 2t+\frac{\sin^2 2t}{\nu^2}\right)^3}\right)\\ &\geq \frac{1}{16} \text{ for } t\in\left[0,\frac{\nu}{2}\right]\end{aligned}$$

So

$$\begin{aligned}\int_{\gamma_\zeta} \mathrm{curv}_s\left(\zeta,W\right) &= \int_{\gamma_\zeta} w_h^2 s^4 \left(D_\zeta\psi\right)^2\\ &\geq O\left(w_h^2 s^4 \nu\right).\end{aligned}$$

On the other hand, we note that for $t>\nu$

$$\left|\mathrm{curv}_s\left(\zeta,W\right)\right| \leq 2s^2 w_h^2 \left|\psi_{\nu,l} D_\zeta D_\zeta\left(\psi_{\nu,l}\right)\right|$$

So we have to find the interval where

$$2s^2 w_h^2 \left|\psi_{\nu,l} D_\zeta D_\zeta\left(\psi_{\nu,l}\right)\right| \leq O\left(w_h^2 s^4 \nu\right),$$

or

$$\left|\psi_{\nu,l} D_\zeta D_\zeta\left(\psi_{\nu,l}\right)\right| \leq O\left(s^2\nu\right),$$

Since

$$\begin{aligned}D_\zeta D_\zeta \psi_{\nu,l} &= \cos^2\varphi \frac{\partial^2}{\partial t^2}\psi_{\nu,l} + 2\cos\varphi\sin\varphi\cos 2t \frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\psi_{\nu,l} + \sin^2\varphi\cos^2 2t\frac{\partial^2}{\partial\theta^2}\psi_{\nu,l}\\ &+\frac{\left|\hat{x}^{2,0}\right|^2_{\nu,l}\cos 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^3_{\nu,l}}\left(D_\zeta\cos\varphi\right) - \frac{1}{4l^2}\frac{\sin 2t\cos^3 2t\cos 2\theta\sin 2\theta}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^3_{\nu,l}}\left(D_\zeta\sin\varphi\right),\end{aligned}$$

we can use our formulas for $\left|\left(\cos 2t\right)\eta^{2,0}\right|^2_{\nu,l}$ and the second derivatives of ψ and from the appendix to get a formula for $\psi_{\nu,l}D_\zeta D_\zeta\left(\psi_{\nu,l}\right)$. So the only unknown quantities in this (complicated) formula are $\left(D_\zeta\cos\varphi\right)$ and $\left(D_\zeta\sin\varphi\right)$, whose order is $O\left(1\right)$. The important point is that for generic t, the largest terms in this formula for $\psi_{\nu,l}D_\zeta D_\zeta\left(\psi_{\nu,l}\right)$ are of order $\frac{\nu^4}{l^2}$. So we have that for sufficiently large t

$$\left|\psi_{\nu,l} D_\zeta D_\zeta\left(\psi_{\nu,l}\right)\right| = O\left(\frac{\nu^4}{l^2}\right)$$

using $l=O\left(\nu^{1/3}\right)$ and $\nu=O\left(s^{6/7}\right)$ we then get for t sufficiently large

$$\begin{aligned}\left|\psi_{\nu,l} D_\zeta D_\zeta\left(\psi_{\nu,l}\right)\right| &\leq O\left(\frac{\nu^4}{\nu^{2/3}}\right)\\ &= O\left(\nu\nu^{7/3}\right)\\ &= O\left(\nu\left(s^{6/7}\right)^{7/3}\right)\\ &= O\left(\nu s^2\right)\end{aligned}$$

as desired.

The interval where this holds is $\left[O\left(c\right), \frac{\pi}{4}\right]$, where c is the constant so that $c\nu = s^{6/7}$. □

Before leaving the subject of derivatives of ψ we establish the following estimate, which will be used in Section 11.

Lemma 8.5.

$$\left| \frac{\psi}{D_\zeta D_\zeta \psi} \left[D_\zeta \left(\psi D_\zeta \psi \right) \right] \right| \leq \frac{\nu_l^2}{4}. \tag{8.6}$$

Remark 8.7. *Since*

$$D_\zeta \left(\psi D_\zeta \psi \right) = \psi D_\zeta D_\zeta \psi + \left(D_\zeta \psi \right)^2$$

and the two terms have opposite sign, it suffices to show

$$\frac{\psi}{D_\zeta D_\zeta \psi} \max \left\{ \psi D_\zeta D_\zeta \psi, \left(D_\zeta \psi \right)^2 \right\} \leq \frac{\nu_l^2}{4}.$$

Since we prove this stronger estimate, we doubt that $\frac{1}{4}$ is the optimal constant in 8.6; it is, nevertheless, sufficient for our purposes.

Proof. We have

$$\frac{\psi}{D_\zeta D_\zeta \psi} \psi D_\zeta D_\zeta \psi = \psi^2,$$

and

$$\begin{aligned} \psi^2 &= \frac{1}{4} \frac{\sin^2 2t}{\left| \left(\cos 2t \right) \eta^{2,0} \right|_{\nu,l}^2} \\ &= \frac{1}{4} \frac{\sin^2 2t}{\left(\left| x^{2,0} \right|_{\nu,l}^2 \cos^2 2t + \frac{1}{\nu_l^2} \sin^2 2t \right)} \\ &= \frac{\nu_l^2}{4} \frac{\sin^2 2t}{\left(\nu_l^2 \left| x^{2,0} \right|_{\nu,l}^2 \cos^2 2t + \sin^2 2t \right)} \\ &\leq \frac{\nu_l^2}{4}. \end{aligned}$$

We saw above that

$$\psi D_\zeta D_\zeta \psi \geq \left(D_\zeta \psi \right)^2$$

when $t > \frac{\nu_l}{2}$, so we only have to establish

$$\frac{\psi}{D_\zeta D_\zeta \psi} \left(D_\zeta \psi \right)^2 \leq \frac{\nu_l^2}{4}$$

when $t < \frac{\nu_l}{2}$.

We saw in the previous proof that for $t < \frac{\nu_l}{2}$,

$$\left| D_\zeta D_\zeta \left(\psi_{\nu,l} \right) \right| \geq \left| x^{2,0} \right|_{\nu,l}^2 \left(\cos^2 \varphi \right) \frac{\sin 2t}{\left| \left(\cos 2t \right) \eta^{2,0} \right|_{\nu,l}^5} \frac{5}{\nu_l^2}$$

Similarly we have

$$\left(D_\zeta \left(\psi_{\nu,l} \right) \right)^2 \leq 1.1 \left(\frac{\left| \hat{x}^{2,0} \right|_{\nu,l}^4}{\left| \left(\cos 2t \right) \hat{\eta}^{2,0} \right|_{\nu,l}^6} \right) \cos^2 \varphi$$

for $t < \frac{\nu_l}{2}$.

So for $t < \frac{\nu_l}{2}$,

$$\begin{aligned}
& \left| \frac{\psi}{D_\zeta D_\zeta \psi} \left(D_\zeta \left(\psi_{\nu,l} \right) \right)^2 \right| \\
\leq & \frac{\frac{1}{2} \frac{\sin 2t}{\left| (\cos 2t) \tilde{\eta}^{2,0} \right|_{\nu,l}} \left[1.1 \left(\frac{\left| \hat{x}^{2,0} \right|^4_{\nu,l}}{\left| (\cos 2t) \tilde{\eta}^{2,0} \right|^6_{\nu,l}} \right) \cos^2 \varphi . \right]}{\left| \hat{x}^{2,0} \right|^2_{\nu,l} \frac{\sin 2t}{\left| (\cos 2t) \hat{\eta}^{2,0} \right|^5_{\nu,l}} \left(\frac{5}{\nu_l^2} \right) \cos^2 \varphi} \\
\leq & \frac{1.1 \left| \hat{x}^{2,0} \right|^2_{\nu,l} \nu_l^2}{10 \left| (\cos 2t) \, \hat{\eta}^{2,0} \right|^2_{\nu,l}} \\
\leq & \frac{\left| \hat{x}^{2,0} \right|^2_{\nu,l} \nu_l^2}{5 \left| (\cos 2t) \, \hat{\eta}^{2,0} \right|^2_{\nu,l}} \\
\leq & \frac{\nu_l^2}{4},
\end{aligned}$$

as desired. □

8.1. **Effect of Redistribution on ψ.**

Proposition 8.8. *Proposition 8.2 remains true after the redistribution.*

Proof. First we get a formula for ψ after the redistribution in terms of ψ before the redistribution. In other words, we will compare $\psi_{\nu,l}$ and $\psi_{\nu,re,l}$. For this proof only we call $\psi_{\nu,l}$, ψ_{old}, and all other quantities that are computed with respect to $g_{\nu,l}$ will have an "old" sub or superscript attached.

All of our derivatives of ψ in this proof will be in the ζ–direction so we write ψ' for $D_\zeta \psi$.

Keeping in mind that $\psi_{\nu,re,l}$ is the length of the horizontal part of the Killing field $\left(0, \frac{1}{2}\vartheta\right)$, we see that we just need to compute the inner product of $\left(0, \frac{1}{2}\vartheta\right)$ with the appropriate horizontal vector. Motivated by our computations of Cheeger perturbations we see that in fact

$$\psi_{\nu,re,l} = \frac{1}{2} \frac{\sin 2t}{\left| \cos 2t \tilde{\eta}^{2,0} \right|_{\nu,re,l}}$$

where $\tilde{\eta}^{2,0}$ is in the γ–part of the horizontal space. More specifically

$$\cos 2t \tilde{\eta}^{2,0} = \cos 2t \eta^{2,0} + \frac{\left(1 - \varphi^2\right)}{\varphi^2} \left(\cos 2t \eta^{2,0} \right)^{\mathcal{Z}^\perp}$$

Since the redistribution occurs before the (U, D)–Cheeger perturbation, the computation of $\left(\cos 2t \eta^{2,0} \right)^{\mathcal{Z}^\perp}$, can be viewed as happening with respect to the metric with $l = \infty$, or more formally it happens within the $Sp\left(2\right)$–factor of $\left(S^3\right)^2 \times Sp\left(2\right)$, where the product metric is the one that gives the (U, D)–Cheeger deformation.

To compute $\left(\cos 2t \eta^{2,0} \right)^{\mathcal{Z}^\perp}$ we need its direction within $\mathcal{Z}^\perp$. This direction looks like

$$\frac{1}{\sqrt{2}} \left(\frac{\vartheta_3}{\nu}, \frac{\vartheta}{\nu} \right),$$

there is a relationship between ϑ_3 and ϑ, but it will not be important here.

So

$$\begin{aligned}\left|\left(\cos 2t\eta^{2,0}\right)^{\mathcal{Z}^\perp}\right| &= \left|\left\langle \frac{1}{\sqrt{2}}\left(\frac{\vartheta_3}{\nu},\frac{\vartheta}{\nu}\right),\left(0,\sin 2t\frac{\vartheta}{\nu^2}\right)\right\rangle_\nu \frac{1}{\sqrt{2}}\left(\frac{\vartheta_3}{\nu},\frac{\vartheta}{\nu}\right)\right| \\ &= \left|\frac{1}{2}\frac{\sin 2t}{\nu}\left(\frac{\vartheta_3}{\nu},\frac{\vartheta}{\nu}\right)\right| \\ &= \frac{1}{2}\frac{\sin 2t}{\nu}\end{aligned}$$

and

$$\begin{aligned}\psi^2_{\nu,re,l} &= \frac{1}{4}\frac{\sin^2 2t}{\left|\cos 2t\tilde{\eta}^{2,0}\right|^2_{\nu,re,l}} \\ &= \frac{1}{4}\frac{\sin^2 2t}{\left|\cos 2t\eta^{2,0}\right|^2_{\text{old}} + 2\frac{(1-\varphi^2)}{\varphi^2}\left\langle \cos 2t\eta^{2,0},\left(\cos 2t\eta^{2,0}\right)^{\mathcal{Z}^\perp}\right\rangle + \left|\frac{(1-\varphi^2)}{\varphi^2}\left(\cos 2t\eta^{2,0}\right)^{\mathcal{Z}^\perp}\right|^2} \\ &= \frac{1}{4}\frac{\sin^2 2t}{\left|\cos 2t\eta^{2,0}\right|^2_{\text{old}} + \frac{1}{2}\frac{(1-\varphi^2)}{\varphi^2}\frac{\sin^2 2t}{\nu^2} + \frac{(1-\varphi^2)^2}{4\varphi^4}\frac{\sin^2 2t}{\nu^2}} \\ &= \frac{\sin^2 2t}{4\left|\cos 2t\eta^{2,0}\right|^2_{\text{old}}}\frac{1}{1+\frac{1}{2}\frac{\sin^2 2t}{\nu^2\left|\cos 2t\eta^{2,0}\right|^2_{\text{old}}}\frac{(1-\varphi^2)}{\varphi^2} + \frac{(1-\varphi^2)^2}{4\varphi^4}\frac{\sin^2 2t}{\nu^2\left|\cos 2t\eta^{2,0}\right|^2_{\text{old}}}} \\ &= \frac{\sin^2 2t}{4\left|\cos 2t\eta^{2,0}\right|^2_{\text{old}}}\frac{1}{1+2\frac{\psi^2_{\text{old}}}{\nu^2}\left(\frac{(1-\varphi^2)}{\varphi^2}+\frac{(1-\varphi^2)^2}{2\varphi^4}\right)} \\ &= \psi^2_{\text{old}}\left(1-2\frac{\psi^2_{\text{old}}}{\nu^2}\left(1-\varphi^2\right)\right)+O\end{aligned}$$

Since

$$\left(1-\varphi^2\right) = O\left(\nu^3\right)$$

We have

$$\psi^2_{\nu,re,l} = \psi^2_{\text{old}} + O$$

and

$$\left(\psi^2_{\nu,re,l}\right)' = \left(\psi^2_{\text{old}}\right)' + 8\frac{\psi^3_{\text{old}}}{\nu^2}\psi'_{\text{old}}\left(\varphi^2-1\right) + 4\frac{\psi^4_{\text{old}}}{\nu^2}\varphi\varphi' + O$$

Since we also have

$$\left(\psi^2_{\nu,re,l}\right)' = 2\psi_{\nu,re,l}\psi'_{\nu,re,l}$$

We get

$$\psi'_{\nu,re,l} = \frac{\frac{1}{2}\left(\psi^2_{\text{old}}\right)' + 4\frac{\psi^3_{\text{old}}}{\nu^2}\psi'_{\text{old}}\left(\varphi^2-1\right) + 2\frac{\psi^4_{\text{old}}}{\nu^2}\varphi\varphi'}{\psi_{\nu,re,l}} + O.$$

Using $\psi^2_{\nu,re,l} = \psi^2_{\text{old}} + O$, this becomes

$$\psi'_{\nu,re,l} = \psi'_{\text{old}} + 4\frac{\psi^2_{\text{old}}}{\nu^2}\psi'_{\text{old}}\left(\varphi^2-1\right) + 2\frac{\psi^3_{\text{old}}}{\nu^2}\varphi\varphi' + O$$

Since

$$\begin{aligned}\varphi' &= O\left(100\nu^3\right) \\ \varphi^2-1 &= O\left(100\nu^3\right)\end{aligned}$$

and

$$\psi'_{\text{old}} \geq O\left(\nu^3\right)\cos 2t$$

we get

$$\psi'_{\nu,re,l} = \psi'_{\text{old}} + O.$$

It is impossible to get a similar formula for $\left(\psi^2_{\nu,re,l}\right)''$ in terms of $\left(\psi^2_{\text{old}}\right)''$, since $\left(\psi^2_{\text{old}}\right)''$ has a 0 around $O(\nu)$. Instead we will show that the difference $\left|\left(\psi^2_{\nu,re,l}\right)'' - \left(\psi^2_{\text{old}}\right)''\right|$ is pointwise much smaller than $\max\left\{\left(\psi'_{\text{old}}\right)^2, \left|\psi_{\text{old}}\psi''_{\text{old}}\right|\right\}$.

Combining this with our estimate $\psi'_{\text{redistr}} = \psi'_{\text{old}} + O$ gives us the proposition.

Starting with

$$\left(\psi^2_{\nu,re,l}\right)' = \left(\psi^2_{\text{old}}\right)' + 8\frac{\psi^3_{\text{old}}}{\nu^2}\psi'_{\text{old}}\left(\varphi^2 - 1\right) + 4\frac{\psi^4_{\text{old}}}{\nu^2}\varphi\varphi' + O$$

we have

$$\begin{aligned}\left(\psi^2_{\nu,re,l}\right)'' &= \left(\psi^2_{\text{old}}\right)'' + 24\frac{\psi^2_{\text{old}}}{\nu^2}\left(\psi'_{\text{old}}\right)^2\left(\varphi^2-1\right) + 8\frac{\psi^3_{\text{old}}}{\nu^2}\psi''_{\text{old}}\left(\varphi^2-1\right) + 32\frac{\psi^3_{\text{old}}}{\nu^2}\psi'_{\text{old}}\varphi\varphi' \\ &\quad + 4\frac{\psi^4_{\text{old}}}{\nu^2}\left(\varphi'\right)^2 + 4\frac{\psi^4_{\text{old}}}{\nu^2}\varphi\varphi''\end{aligned}$$

The second term is everywhere much smaller than $\left(\psi'_{\text{old}}\right)^2$. Similarly we can bound the third term by

$$\left|8\frac{\psi^3_{\text{old}}}{\nu^2}\psi''_{\text{old}}\left(\varphi^2-1\right)\right| \leq \left|800\nu\psi^3_{\text{old}}\psi''_{\text{old}}\right|$$

which is much smaller than $\psi_{\text{old}}\psi''_{\text{old}}$. The fourth term is

$$\left|32\frac{\psi^3_{\text{old}}}{\nu^2}\psi'_{\text{old}}\varphi\varphi'\right| \leq 3200\nu\psi^3_{\text{old}}\psi'_{\text{old}}$$

and hence is much smaller than $\left(\psi'_{\text{old}}\right)^2$ in the region where $t \leq O(c)$ that matters.

The fifth term is smaller than $O\left(\nu^8\right)$ and 0 at $t = 0$ and hence smaller than both $\left(\psi'_{\text{old}}\right)^2$ and $\psi_{\text{old}}\psi''_{\text{old}}$ everywhere $t \leq O(c)$.

The last term

$$\left|4\frac{\psi^4_{\text{old}}}{\nu^2}\varphi\varphi''\right| \leq 400\psi^4_{\text{old}}$$

and hence is smaller than both $\left(\psi'_{\text{old}}\right)^2$ and $\psi_{\text{old}}\psi''_{\text{old}}$ on $(0, 100\nu)$. On the other hand, on $\left(50\nu, \frac{\pi}{4}\right)$,

$$\left|4\frac{\psi^4_{\text{old}}}{\nu^2}\varphi\varphi''\right| \leq 40,000\nu\psi^4_{\text{old}}$$

and hence is much smaller than $\psi_{\text{old}}\psi''_{\text{old}}$. □

9. Concrete A–Tensor Estimates

In this section we refine our formulas for the two key $(1,3)$–curvature tensors

$$R^s(\zeta, W)W \text{ and}$$
$$R^s(W, \zeta)\zeta$$

after the fibers are shrunk. We have to go beyond the abstract situation of section 1, to compute the iterated A–tensors of $\Sigma^7 \longrightarrow S^4$. Substituting

$$\begin{aligned} \zeta &= X \\ w_h k_\gamma &= H_w \end{aligned}$$

into Lemma 1.10 we have

Lemma 9.1.

$$R^{g_s}(W,\zeta)\,\zeta = -s^2 w_h \left(\frac{D_\zeta D_\zeta \psi}{\psi}\right) k_\gamma - s^2 \left[w_h \frac{D_\zeta \psi}{\psi} A_\zeta k_\gamma\right]$$

$$\left(R^{g_s}(\zeta,W)\,W\right)^{\mathcal{H}} = -s^2 w_h^2 \psi \nabla_\zeta (\operatorname{grad}\psi) - \left(1-s^2\right) s^2 w_h \frac{D_\zeta \psi}{\psi} A_{k_\gamma} W^{\mathcal{V}}$$

The possibilities for the iterated A–tensors in the curvature formulas above are a bit daunting. We can nevertheless get estimates. First let $(V_1 \oplus V_2)^{GM}$ denote the intersection $V_1 \oplus V_2$ with the horizontal space for the Gromoll–Meyer submersion $q_{2,-1} : Sp(2) \longrightarrow \Sigma^7$, and let $V_{2,-1}$ be the horizontal lift to $TSp(2)$ of the vertical space of $p_{2,-1} : \Sigma^7 \longrightarrow S^4$. Then away from $t = \frac{\pi}{4}$, the orthogonal projection onto the vertical space $V_{2,-1}$ restricts to an isomorphism $p_{\text{orthog}} : (V_1 \oplus V_2)^{GM} \longrightarrow V_{2,-1}$. Therefore the following lemma will give us all of the data that we need.

Lemma 9.2. *Let* II *denote the second fundamental form of the S^2s in S^4, and let S denote the shape operator.*

For $U \in V_1 \oplus V_2$, extend U to be a Killing field for the $(h_1 \oplus h_2)$–action. Then for $z \in \operatorname{span}\left\{x^{2,0}, y^{2,0}\right\}$, and $k_\gamma = \psi \eta_u^{2,0}$, with $\left|\eta_u^{2,0}\right| = 1$

$$\begin{aligned} A_z U^{\mathcal{V}} &= \left(\nabla_z^{\nu,re,l} U\right)^{\mathcal{H}} - S_z\left(U^{\mathcal{H}}\right), \\ A_{k_\gamma} U^{\mathcal{V}} &= \frac{\psi}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla_{(\eta,\eta)}^{\nu,re,l} U\right)^{\mathcal{H}} - \operatorname{II}\left(k_\gamma, U^{\mathcal{H}}\right) + 4\frac{\psi^3}{\nu^3}\left|U^\alpha\right|_{h_2} \eta_{u,4}^{2,0} + O \end{aligned}$$

where $\eta_{u,4}^{2,0}$ is the vector in span$\left\{\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right\}$ that is perpendicular to k_γ and U^α denotes the α–part of U.

Proof. To prove the first equation extend U to be a Killing field for the $V_1 \oplus V_2$ action. Then

$$\begin{aligned} A_z U^{\mathcal{V}} &= \left[\nabla_z^{\nu,re,l}\left(U - U^{\mathcal{H}}\right)\right]^{\mathcal{H}} \\ &= \left(\nabla_z^{\nu,re,l} U\right)^{\mathcal{H}} - \left(\nabla_z^{\nu,re,l} U^{\mathcal{H}}\right)^{\mathcal{H}}. \end{aligned}$$

Since $U^{\mathcal{H}}$ is a Killing field for the h_2–action on S^4, if we extend z to be a constant linear combination of $x^{2,0}$ and $y^{2,0}$, then $\left(\left[z, U^{\mathcal{H}}\right]\right)^{\mathcal{H}} = 0$. So

$$A_z U^{\mathcal{V}} = \left(\nabla_z^{\nu,re,l} U\right)^{\mathcal{H}} - S_z\left(U^{\mathcal{H}}\right)$$

as claimed.

For the second equation we again extend U to be a Killing field for the $V_1 \oplus V_2$ action. As before

$$A_{k_\gamma} U^{\mathcal{V}} = \left(\nabla_{k_\gamma}^{\nu,re,l}\left(U - U^{\mathcal{H}}\right)\right)^{\mathcal{H}}$$

Now

$$\begin{aligned}\left(\nabla^{\nu,re,l}_{k_\gamma}U\right)^{\mathcal{H}} &= \psi\left(\nabla^{\nu,re,l}_{\eta^{2,0}_u}(U)\right)^{\mathcal{H}}\\ &= \frac{\psi}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)}U\right)^{\mathcal{H}}+(0,V)^{\mathcal{H}}\end{aligned}$$

where we have split $\eta^{2,0}_u$ into its horizontal and vertical parts for $h_1\oplus h_2$. Thus V is a vector tangent to the h_2 orbits and perpendicular to U. It comes from differentiating U in the direction of the V_2–part of k_γ. Since we are taking the horizontal part of V, only the α–component of U makes a contribution. Since $k_\gamma=\psi\eta^{2,0}_u$, and the V_2–part of $\eta^{2,0}_u$ is $\left(0,2\psi\frac{\vartheta}{\nu_2^2}\right)$, we have

$$(0,V)^{\mathcal{H}}=\psi\left(\nabla^{\nu,re,l}_{\left(0,2\psi\frac{\vartheta}{\nu_2^2}\right)}(0,U^\alpha)\right)^{\mathcal{H}}.$$

If, for example, $(0,U^\alpha)=\left(0,\frac{N\alpha}{\nu^2}\right)$, then

$$\begin{aligned}(0,V) &= 2\psi^2\left(0,\frac{N\gamma_4}{\nu^4}\right),\text{ and}\\ \left|(0,V)^{\mathcal{H}}\right| &= \left|2\left\langle\psi^2\left(0,\frac{N\gamma_4}{\nu^4}\right),\eta^{2,0}_{u,4}\right\rangle\right|\end{aligned}$$

where $\left(0,\frac{N\gamma_4}{\nu^4}\right)$ and $\eta^{2,0}_{u,4}$ are perpendicular to k_γ. Thus

$$\begin{aligned}\left|(0,V)^{\mathcal{H}}\right| &= 4\frac{\psi^3}{\nu^4}\\ &= 4\frac{\psi^3}{\nu^3}\left|U^\alpha\right|_{h_2},\end{aligned}$$

and

$$(0,V)^{\mathcal{H}}=\left(4\frac{\psi^3}{\nu^3}\left|U^\alpha\right|_{h_2}\right)\eta^{2,0}_{u,4}+O$$

The "O" is present because we did not take the effect of the (U,D)–deformation into account. The computation is very similar, but since $l=O\left(\nu^{1/3}\right)$, the terms we get do not play a significant role.

For the other term, since U is a Killing field for the h_2–action

$$\left[\nabla^{\nu,re,l}_{k_\gamma}U^{\mathcal{H}}\right]^{\mathcal{H}}=\mathrm{II}\left(k_\gamma,U^{\mathcal{H}}\right).$$

So combining equations yields the claim. □

Combining the previous two results gives us

Proposition 9.3. *For $U\in\mathcal{H}_{p_{2,-1}}$,*

$$\left\langle R^s(W,\zeta)\zeta,U\right\rangle=-s^2w_h\left(\frac{D_\zeta D_\zeta\psi}{\psi}\right)\left\langle k_\gamma,U\right\rangle$$

For $U \in V_1 \oplus V_2, U$ extend U to be a Killing field for the $(h_1 \oplus h_2)$–action. Then

$$\begin{aligned}\langle R^s(W,\zeta)\zeta, U\rangle &= -s^2 w_h \left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)\langle k_\gamma, U\rangle - s^2\left(1-s^2\right) w_h \frac{D_\zeta \psi}{\psi}\left\langle k_\gamma, S_\zeta\left(U^{\mathcal{H}}\right)\right\rangle \\ &\quad + s^2\left(1-s^2\right) w_h \frac{D_\zeta \psi}{\psi}\left\langle k_\gamma, \nabla^{\nu,re,l}_\zeta U\right\rangle.\end{aligned}$$

Let $\eta^{2,0}_{u,W}$ be the unit vector in $\operatorname{span}\left\{\eta^{2,0}_{u,1}, \eta^{2,0}_{u,2}\right\}$ that is proportional to the projection of W onto $\operatorname{span}\left\{\eta^{2,0}_{u,1}, \eta^{2,0}_{u,2}\right\}$, and let $\eta^{2,0}_{u,W^\perp}$ be perpendicular to $\eta^{2,0}_{u,W}$. Then

$$\begin{aligned}\left(R^s(\zeta,W)W\right)^{\mathcal{H}} &= -s^2 w_h^2 \nabla_\zeta\left(\psi \operatorname{grad}\psi\right) + s^4 w_h^2 \left(D_\zeta \psi\right)\left(\operatorname{grad}\psi\right) + \\ &\quad -s^2 w_h \frac{D_\zeta \psi}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} + 4 w_h s^2 \left[D_\zeta \psi\right]\frac{\psi^2}{\nu^3}\left|W^\alpha\right|_{h_2}\eta^{2,0}_{u,W^\perp} + O\end{aligned}$$

Proof. From Lemma 9.1 we have

$$R^{g_s}(W,\zeta)\zeta = -s^2 w_h\left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)k_\gamma - s^2\left[w_h \frac{D_\zeta \psi}{\psi} A_\zeta k_\gamma\right]$$

So for $U \in \mathcal{H}_{p_{2,-1}}$,

$$\langle R^s(W,\zeta)\zeta, U\rangle = -s^2 w_h\left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)\langle k_\gamma, U\rangle,$$

and for $U \in V_1 \oplus V_2$

$$\begin{aligned}\langle R^s(W,\zeta)\zeta, U\rangle_s &= -s^2 w_h\left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)\langle k_\gamma, U\rangle_s - s^2 w_h \frac{D_\zeta \psi}{\psi}\langle A_\zeta k_\gamma, U\rangle_s \\ &= -s^2 w_h\left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)\langle k_\gamma, U\rangle_{\nu,re,l} + s^2\left(1-s^2\right) w_h \frac{D_\zeta \psi}{\psi}\left\langle k_\gamma, A_\zeta U^{\mathcal{V}}\right\rangle_{\nu,re,l}\end{aligned}$$

Applying Lemma 9.2

$$\begin{aligned}\langle R^s(W,\zeta)\zeta, U\rangle &= -s^2 w_h\left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)\langle k_\gamma, U\rangle - s^2\left(1-s^2\right) w_h \frac{D_\zeta \psi}{\psi}\left\langle k_\gamma, S_\zeta\left(U^{\mathcal{H}}\right)\right\rangle \\ &\quad + s^2\left(1-s^2\right) w_h \frac{D_\zeta \psi}{\psi}\left\langle k_\gamma, \nabla^{\nu,re,l}_\zeta U\right\rangle\end{aligned}$$

From Lemma 9.1

$$\left(R^{g_s}(\zeta,W)W\right)^{\mathcal{H}} = -s^2 w_h^2 \psi \nabla_\zeta\left(\operatorname{grad}\psi\right) - \left(1-s^2\right) s^2 w_h \frac{D_\zeta \psi}{\psi} A_{k_\gamma} W^{\mathcal{V}}$$

Applying Lemma 9.2

$$\begin{aligned}\left(R^s(\zeta,W)W\right)^{\mathcal{H}} &= -s^2 w_h^2 \psi \nabla_\zeta\left(\operatorname{grad}\psi\right) + \left(1-s^2\right) s^2 w_h \frac{D_\zeta \psi}{\psi}\mathrm{II}\left(k_\gamma, W^{\mathcal{H}}\right) \\ &\quad - \left(1-s^2\right) s^2 w_h \frac{D_\zeta \psi}{\psi}\frac{\psi}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} + \\ &\quad + 4 w_h s^2\left(1-s^2\right)\frac{D_\zeta \psi}{\psi}\frac{\psi^3}{\nu^3}\left|W^\alpha\right|_{h_2}\eta^{2,0}_{u,W^\perp} + O\end{aligned}$$

where $\eta^{2,0}_{u,W^\perp}$ is the unit vector in span $\left\{\eta^{2,0}_{1,u}, \eta^{2,0}_{2,u}\right\}$ that is perpendicular to $W^{\mathcal{H}}$. Thus

$$\begin{aligned}
\left(R^s\left(\zeta, W\right)W\right)^{\mathcal{H}} &= -s^2 w_h^2 \psi \nabla_\zeta\left(\operatorname{grad}\psi\right) - \left(1-s^2\right)s^2 w_h^2 \left(D_\zeta\psi\right)\left(\operatorname{grad}\psi\right) + \\
&\quad -\frac{s^2 w_h D_\zeta\psi}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)}W\right)^{\mathcal{H}} + 4w_h s^2\left[D_\zeta\psi\right]\frac{\psi^2}{\nu^3}\left|W^\alpha\right|_{h_2}\eta^{2,0}_{u,W^\perp} + O \\
&= -s^2 w_h^2 \nabla_\zeta\left(\psi\operatorname{grad}\psi\right) + s^4 w_h^2\left(D_\zeta\psi\right)\left(\operatorname{grad}\psi\right) + \\
&\quad -\frac{s^2 w_h D_\zeta\psi}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)}W\right)^{\mathcal{H}} + 4w_h s^2\left[D_\zeta\psi\right]\frac{\psi^2}{\nu^3}\left|W^\alpha\right|_{h_2}\eta^{2,0}_{u,W^\perp} + O
\end{aligned}$$

□

Corollary 9.4.

$$\left\langle\left(R^{g_s}\left(\zeta, W\right)W\right), \zeta\right\rangle = -\left(s^2 w_h^2\right)D_\zeta\left(\psi D_\zeta\psi\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2$$

Proof. For redundancy we compute $\left\langle\left(R^{g_s}\left(\zeta, W\right)W\right), \zeta\right\rangle$ twice, using each of the last two formulas of the previous proposition. Since

$$\nabla^{\text{redistr}}_\zeta W \equiv 0,$$

the second formula gives us

$$\begin{aligned}
\left\langle R^{\text{redistr}}\left(W, \zeta\right)\zeta, W\right\rangle &= -s^2 w_h\left(\frac{D_\zeta D_\zeta\psi}{\psi}\right)\left\langle k_\gamma, W\right\rangle - s^2\left(1-s^2\right)w_h\frac{D_\zeta\psi}{\psi}\left\langle k_\gamma, S_\zeta\left(W^{\mathcal{H}}\right)\right\rangle \\
&= -s^2 w_h^2 \psi D_\zeta D_\zeta\psi - s^2\left(1-s^2\right)w_h^2\left(D_\zeta\psi\right)^2 \\
&= -s^2 w_h^2\left(\psi D_\zeta D_\zeta\psi + \left(D_\zeta\psi\right)^2\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2 \\
&= -\left(s^2 w_h^2\right)D_\zeta\left(\psi D\zeta\psi\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2
\end{aligned}$$

Computing the other way we get

$$\begin{aligned}
\left\langle R^{g_s}\left(\zeta, W\right)W, \zeta\right\rangle &= -s^2 w_h^2\left\langle\nabla_\zeta\left(\psi\operatorname{grad}\psi\right), \zeta\right\rangle + s^4 w_h^2\left(D_\zeta\psi\right)^2 \\
&= -s^2 w_h^2\left(\left(D_\zeta\psi\right)^2 + \psi\left\langle\nabla_\zeta\operatorname{grad}\psi, \zeta\right\rangle\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2 \\
&= -s^2 w_h^2\left(\psi D_\zeta D_\zeta\psi + \left(D_\zeta\psi\right)^2\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2 \\
&= -\left(s^2 w_h^2\right)D_\zeta\left(\psi D\zeta\psi\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2
\end{aligned}$$

□

In the remainder of this section we record the effect of the s–deformation on some key covariant derivatives that we will need later.

Proposition 9.5.

$$\nabla^s_W W = -s^2 w_h^2 \psi\operatorname{grad}\,\psi,$$

$$\begin{aligned}
\nabla^s_W W^\gamma &= -s^2 w_h^2 \psi\operatorname{grad}\,\psi, \text{ and} \\
\nabla^s_{W^\gamma} W^\gamma &= -s^2 w_h^2 \psi\operatorname{grad}\,\psi
\end{aligned}$$

where W^γ is the γ–part of W.

Proof. Since W is a Killing field on $Sp(2)$

$$\begin{aligned}\langle \nabla^s_W W, Z\rangle &= -\langle \nabla^s_Z W, W\rangle \\ &= -\frac{1}{2} D_Z \langle W, W\rangle \\ &= -\frac{1}{2} D_Z |W|^2 \\ &= -|W| D_Z |W| \\ &= -\langle |W|_s \operatorname{grad} |W|_s, Z\rangle\end{aligned}$$

Since

$$\begin{aligned}|W| &= \sqrt{(1-s^2)|W^v|^2_{\nu,re,l} + |W^h|^2_{\nu,re,l}}, \text{ and} \\ &\quad |W|_{\nu,re,l} \text{ is constant}\end{aligned}$$

$$\begin{aligned}D_Z |W|_s &= \frac{1}{2}\left((1-s^2)|W^v|^2_{\nu,re,l} + \left|W^h\right|^2_{\nu,re,l}\right)^{-1/2}\left(-s^2 D_Z |W^v|^2_{\nu,re,l}\right) \\ &= \frac{1}{2}\frac{s^2}{|W|_s} D_Z\left(\left|W^h\right|^2_{\nu,re,l}\right) \\ &= \frac{s^2}{|W|_s}\left|W^h\right|_{\nu,re,l} D_Z\left(\left|W^h\right|_{\nu,re,l}\right) \\ &= \frac{s^2}{|W|_s} w_h \psi D_Z(w_h \psi) \\ &= \frac{s^2}{|W|_s} w_h^2 \psi D_Z(\psi)\end{aligned}$$

Thus

$$\begin{aligned}\langle \nabla^s_W W, Z\rangle &= -|W|_s D_Z |W|_s \\ &= -s^2 w_h^2 \psi D_Z(\psi) \\ &= -s^2 w_h^2 \langle Z, \psi \operatorname{grad} \psi\rangle\end{aligned}$$

So

$$\nabla^s_W W = -s^2 w_h^2 \psi \operatorname{grad} \psi$$

as claimed.

Since W^γ is also a Killing field we have

$$\begin{aligned}\langle \nabla^s_W W^\gamma, Z\rangle &= -\langle \nabla^s_Z W^\gamma, W\rangle \\ &= -\frac{1}{2} D_Z \langle W^\gamma, W\rangle .\end{aligned}$$

But $D_Z \langle W^\gamma, W\rangle = D_Z \langle W, W\rangle$ so $\nabla^s_W W^\gamma = -s^2 w_h^2 \psi \operatorname{grad} \psi$. Similarly $\nabla^s_{W^\gamma} W^\gamma = -s^2 w_h^2 \psi \operatorname{grad} \psi$ □

Proposition 9.6.

$$\nabla^s_\zeta W = \nabla^s_\zeta W^\gamma = s^2 \frac{D_\zeta \psi}{\psi} H_w.$$

Proof. For any vertical field U with respect to $p_{2,-1} : \Sigma \longrightarrow S^4$ we have $D_U \langle W, \zeta \rangle = D_W \langle U, \zeta \rangle = \langle [U, W], \zeta \rangle = [U, \zeta]^{horiz} = 0$. So the Koszul formula gives us

$$\begin{aligned} 2 \left\langle \nabla^s_\zeta W, U \right\rangle_s &= D_\zeta \langle W, U \rangle_s + \langle [\zeta, W], U \rangle_s + \langle [U, \zeta], W \rangle_s \\ &= 2\left(1 - s^2\right) \left\langle \nabla^{\nu,re,l}_\zeta W, U \right\rangle_{\nu,re,l} \\ &= 0. \end{aligned}$$

Breaking W into its horizontal and vertical parts we have

$$\begin{aligned} 0 &= \left(\nabla^{\nu,re,l}_\zeta W\right)^{\mathcal{H}} \\ &= \left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}} + \left(\nabla^{\nu,re,l}_\zeta H_w\right)^{\mathcal{H}} \end{aligned}$$

On the one hand, $\left(\nabla^{\nu,re,l}_\zeta H_w\right)^{\mathcal{H}} = \left(\nabla^s_\zeta H_w\right)^{\mathcal{H}}$. On the other hand, for any basic horizontal field Z

$$\begin{aligned} 2 \left\langle \nabla^s_\zeta V, Z \right\rangle_s &= -\langle [\zeta, Z], V \rangle_s \\ &= -\left(1 - s^2\right) \langle [\zeta, Z], V \rangle_0 \\ &= \left(1 - s^2\right) 2 \left\langle \nabla^{\nu,re,l}_\zeta V, Z \right\rangle_s \end{aligned}$$

So

$$\left(\nabla^s_\zeta V\right)^{\mathcal{H}} = \left(1 - s^2\right) \left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}}$$

and

$$\begin{aligned} \nabla^s_\zeta W &= \left(\nabla^s_\zeta W\right)^{\mathcal{H}} \\ &= \left(\nabla^s_\zeta V\right)^{\mathcal{H}} + \left(\nabla^s_\zeta H_w\right)^{\mathcal{H}} \\ &= \left(\nabla^{\nu,re,l}_\zeta V + \nabla^{\nu,re,l}_\zeta H_w\right)^{\mathcal{H}} - s^2 \left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}} \\ &= -s^2 \left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}} \\ &= s^2 \frac{D_\zeta |H_w|}{|H_w|} H_w \\ &= s^2 \frac{D_\zeta \psi}{\psi} H_w \end{aligned}$$

where for the next to last equality we have used Lemma 1.8. A similar argument gives us $\nabla^s_\zeta W^\gamma = s^2 \frac{D_\zeta \psi}{\psi} H_w$. □

10. Partial Conformal Change

Having carried out deformations (1)–(4), we have apparently made things worse. Indeed, from Corollary 9.4, we see that near $t = 0$, some of the planes that used to have 0–curvature now have negative curvature. The ray of hope is that, as we discussed in section 1, the integral of the curvatures over the old zero locus is now positive. In this section, we will even it out to make it positive everywhere. The metric that we obtain is in fact positively curved; however, after this section we will only know that it is positively curved along the former zero locus. In the final three sections we check that the curvature is positive everywhere.

Consider the 1–dimensional subdistribution

$$\Delta(\alpha) = \operatorname{span}\{(N\alpha p, N\alpha)\}.$$

We change the metric on $Sp(2)$ by multiplying the restriction to the orthogonal complement of $\Delta(\alpha)$ by a function e^{2f}. We leave $\Delta(\alpha)$ and its orthogonal complement perpendicular to each other, and we leave the metric restricted to $\Delta(\alpha)$ unchanged.

In each S^7–factor of $Sp(2) \subset S^7 \times S^7$, our distribution $\Delta(\alpha)$ is the intersection of the vertical spaces of the two Hopf fibrations h and $\tilde{h}$. Since the two Hopf actions are by symmetries of each other, our distribution $\Delta(\alpha)$ is invariant under the Gromoll-Meyer action of $\left(S^3 \times S^3\right)$ on $Sp(2)$, and also under the symmetry action of S^3. So our new metric will be invariant under all of these actions. In particular, it induces a metric on Σ^7.

Our notational convention of writing vectors before the $\left(A^u \times A^d\right)$–Cheeger deformation doesn't matter much when we talk about $\Delta(\alpha)$, since its invariant under the "Cheeger parameterization". On the other hand, the orthogonal complement of $\Delta(\alpha)$ is not invariant, and we continue with our convention of page 17.

We choose

$$f = C - \frac{s^2}{2\nu^2}\psi^2 + E,$$

where C is a constant that is a little larger than 1 and E is a function $Sp(2) \longrightarrow \mathbb{R}$ that is much smaller than $\frac{s^2}{\nu^2}\psi_{\nu,l}^2$ in the C^2–topology. The function E has the form

$$E = I \circ \operatorname{dist}_{S^4}((0,0),\cdot) \circ p_{GM}$$

where

$$p_{GM} : Sp(2) \longrightarrow S^4$$

is the Gromoll-Meyer submersion, $(0,0)$ one of the two points in S^4 with $(\sin 2t, \sin 2\theta) = (0,0)$, and $I : \mathbb{R} \longrightarrow \mathbb{R}$ is a function that satisfies

$$\begin{aligned} I'(0) &= 0, \\ I'|_{[O(c),\frac{\pi}{4}]} &\equiv 0, \\ I'' &= O\left(\frac{s^4}{\nu^2}\right). \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{grad} f &= -\frac{s^2}{\nu^2}\psi \operatorname{grad}\psi + \operatorname{grad} E \\ &= -\frac{s^2}{\nu^2}\psi \operatorname{grad}\psi + I'\zeta. \end{aligned}$$

Remark 10.1. *There is a minor problem with our partial conformal change. Our distribution, $\Delta(\alpha)$, is three dimensional at $t = 0$, and one dimensional everywhere else. We circumvent this by having our conformal change be a standard conformal change in a very, very small neighborhood of $t = 0$, and then flattening out the $\Delta(\alpha)$ portion. Since we can do this on an arbitrarily small neighborhood of $t = 0$, the effect on curvatures can be made to be irrelevant.*

Lemma 10.2. *Let ∇^{old} and ∇^{new} denote the covariant derivative before and after the partial conformal change. If x, y are fields that are orthogonal to $\Delta(\alpha)$, then*

$$\begin{aligned}(10.3)\qquad \nabla^{\text{new}}_x y &= O\left(e^{2f}-1\right)\left(\nabla^{\text{old}}_x y\right)^{\Delta(\alpha)} \\ &\quad + \left(\nabla^{\text{old}}_x y\right)^{\Delta(\alpha),\perp} + (D_x f)\, y + (D_y f)\, x - \langle x, y\rangle \nabla f,\end{aligned}$$

where the superscripts $^{\Delta(\alpha)}$ and $^{\Delta(\alpha),\perp}$ denote the components tangent and perpendicular to $\Delta(\alpha)$.

Proof. If we replace the first two terms on the right hand side of equation 10.3 with $\nabla^{\text{old}}_x y$, then we get the formula for the covariant derivative after an actual conformal change. It can be found in exercise 5a on page 90 in [Pet]. The three derivative terms come from the three derivative terms in the Koszul formula.

When we test $\nabla^{\text{new}}_x y$ by taking its inner product with a vector in the orthogonal complement of $\Delta(\alpha)$, the Koszul formula looks precisely like the one for a standard conformal change, and so we certainly have that the component of $\nabla^{\text{new}}_x y$ that is perpendicular to $\Delta(\alpha)$ is given by 10.3.

Finding the component tangent to $\Delta(\alpha)$ takes more care. The important point is that there is no standard field that is tangent to $\Delta(\alpha)$. Indeed, "α" changes in the directions $\text{span}\{(\eta_1,\eta_1),(\eta_2,\eta_2)\}$. So even though we can compute the precise formula for the $\Delta(\alpha)$–component in many cases, we can't get a general formula that is much better than equation 10.3. □

To deal with covariant derivatives involving vectors in $\Delta(\alpha)$ we prove

Lemma 10.4. **(i):** *For x and U fields with*

$$x \in H \cup V_1 \oplus V_2 \text{ and } U \in \text{span}\{(N\alpha p, N\alpha)\}$$
$$\nabla^{\text{new}}_x U = O\left(e^{2f}-1\right)\nabla^{\text{old}}_x U, \text{ and}$$
$$\nabla^{\text{new}}_U x = O\left(e^{2f}-1\right)\nabla^{\text{old}}_U x.$$

(ii): *For $U = (N\alpha p, N\alpha)$*

$$\nabla^{\text{new}}_U U = \nabla^{\text{old}}_U U.$$

Proof. Since at least one of our fields is in $\Delta(\alpha)$, the three derivative terms from equation 10.3 are not present. For (i) the three Lie bracket terms of the Koszul formula can be a bit complicated, so again we can't get general formulas that are much better than the two we have asserted.

For (ii) the key point is that for Z perpendicular to $\Delta(\alpha)$, the Koszul formula gives us

$$\begin{aligned}2\left\langle \nabla^{\text{new}}_U U, Z\right\rangle_{\text{new}} &= -D_Z\left\langle U,U\right\rangle_{\text{new}} + 2\left\langle [Z,U], U\right\rangle_{\text{new}} \\ &= -D_Z\left\langle U,U\right\rangle_{\text{old}} + 2\left\langle [Z,U], U\right\rangle_{\text{old}} \\ &= 2\left\langle \nabla^{\text{old}}_U U, Z\right\rangle_{\text{old}}\end{aligned}$$

Similarly $\left\langle \nabla^{\text{new}}_U U, U\right\rangle_{\text{new}} = \left\langle \nabla^{\text{old}}_U U, U\right\rangle_{\text{old}}$. □

For us the really important curvatures are

$$\left(R(\zeta, W)\, W\right)^{\mathcal{H}} \text{ and}$$
$$R(W,\zeta)\,\zeta.$$

Fortunately we can get precise formulas for the required covariant derivatives.

Note that W is typically neither tangent nor perpendicular to $\Delta(\alpha)$. We let W^γ denote the component of W that is perpendicular to $\Delta(\alpha)$. With this we have

Lemma 10.5.

$$\begin{aligned}\nabla^{\text{new}}_W W &= \nabla^{\text{old}}_W W - \langle W^\gamma, W^\gamma\rangle \nabla f\\ \nabla^{\text{new}}_\zeta W &= \nabla^{\text{old}}_\zeta W + (D_\zeta f) W^\gamma, \text{ and}\\ \nabla^{\text{new}}_\zeta \zeta &= \nabla^{\text{old}}_\zeta \zeta + 2(D_\zeta f)\zeta - \nabla f.\end{aligned}$$

Remark 10.6. *In other words, if we replace W with W^γ then the formulas for the three covariant derivatives are precisely the same as that of a standard conformal change.*

Proof. If Z is any standard field that is either ζ, W, or initially perpendicular to $\operatorname{span}\{\zeta, W\}$, then all three Lie bracket terms in the three Koszul formulas for

$$\langle \nabla^{\text{new}}_W W, Z\rangle,\ \langle \nabla^{\text{new}}_\zeta W, Z\rangle, \text{ and } \langle \nabla^{\text{new}}_\zeta \zeta, Z\rangle$$

vanish. So the only change in the Koszul formula comes from the three derivative terms, and only the γ–component of W effects these terms. □

To get the two key curvature formulas, we will also have to check the $\Delta(\alpha)$–components of the various iterated covariant derivatives. Since $\Delta(\alpha)$ is contained in the vertical space of $Sp(2) \longrightarrow S^4$, we do not need to worry about the $\Delta(\alpha)$–component of

$$(R(\zeta, W)W)^{\mathcal{H}}$$

Thus it suffices to check the following.

Proposition 10.7. *Before and after the partial conformal change the $\Delta(\alpha)$–components of*

$$\begin{aligned}&\nabla_\zeta \nabla^{\text{new}}_\zeta W, \text{ and}\\ &\nabla_W \nabla^{\text{new}}_\zeta \zeta\end{aligned}$$

are 0.

Proof. The bottom line is that all of the Lie Bracket terms in all of the relevant Koszul formulas are 0. Because of the importance of the result we check this.

Let $\mathcal{V}$ be a unit field in $\operatorname{span}\{\Delta(\alpha)\}$. Since the partial conformal change occurs after the (U, D)–Cheeger deformations, we will have to consider all of these computations as occurring in $\left(S^3\right)^2 \times Sp(2)$.

For $\langle \nabla_\zeta \nabla^{\text{new}}_\zeta W, \mathcal{V}\rangle$ we first note that $\nabla^{\text{new}}_\zeta W = \nabla^{\text{old}}_\zeta W + (D_\zeta f) W^\gamma$ and $\nabla^{\text{old}}_\zeta W \in \operatorname{span}\{H_w\}$. Next we point out that in both the $Sp(2)$ and the $\left(S^3\right)^2$–factors, $[\zeta, \mathcal{V}] = 0$. It remains to compute each of

$$\begin{aligned}&\langle [\zeta, H_w], \mathcal{V}\rangle,\\ &\langle [\mathcal{V}, H_w], \zeta\rangle,\\ &\langle [\zeta, W^\gamma], \mathcal{V}\rangle,\\ &\langle [\mathcal{V}, W^\gamma], \zeta\rangle\end{aligned}$$

These are all 0 in both the $Sp(2)$ and the $\left(S^3\right)^2$–components because in each case one of the vectors in the inner product is an α–vector and one of the vectors is a γ–vector.

For $\left\langle \nabla_W \nabla_\zeta^{\text{new}} \zeta, \mathcal{V} \right\rangle$, we note that

$$\nabla_\zeta^{\text{new}} \zeta = \nabla_\zeta^{\text{old}} \zeta + 2 \left(D_\zeta f \right) \zeta - \nabla f$$

and $\nabla_\zeta^{\text{old}} \zeta = 0$. The terms

$$\begin{array}{l} \left\langle \left[\mathcal{V}, W \right], \zeta \right\rangle \text{ and} \\ \left\langle \left[\mathcal{V}, W \right], \nabla f \right\rangle \end{array}$$

are 0 since $[W, \mathcal{V}]$ is a γ–vector and both ζ and ∇f are α–vectors.

The computations that gave us these 0–planes in the first place yield that each of

$$\begin{array}{l} \left[W, \zeta \right], \\ \left[\mathcal{V}, \zeta \right], \\ \left[\mathcal{V}, \nabla f \right] \end{array}$$

is 0.

The inner product

$$\left\langle \left[W, \nabla f \right], \mathcal{V} \right\rangle$$

is 0 in the $Sp\left(2\right)$ factor since W is vertically parallel. In the $\left(S^3\right)^2$–factor, we point out that $[W, \nabla f]$ is a γ–vector so

$$\left\langle \left[W, \nabla f \right], \mathcal{V} \right\rangle = 0.$$

□

Combining the previous two Lemmas we see that our two key curvature tensors

$$\begin{array}{l} R^{\text{new}} \left(W, \zeta \right) \zeta \text{ and} \\ \left(R^{\text{new}} \left(\zeta, W \right) W \right)^{\mathcal{H}} \end{array}$$

are obtained from R^{old}, from the familiar conformal change formulas (cf exercise 5B on page 90 in [Pet]) with W replaced by W^γ.

Proposition 10.8. *For any vector U*

$$\begin{array}{rcl} e^{-2f} \left\langle R^{\text{new}} \left(W, \zeta \right) \zeta, U \right\rangle & = & \left\langle R^{\text{old}} \left(W, \zeta \right) \zeta, U \right\rangle \\ & & -g \left(W^\gamma, U \right) \text{Hess}_f \left(\zeta, \zeta \right) - g \left(\zeta, \zeta \right) \text{Hess}_f \left(W^\gamma, U \right) + g \left(\zeta, U \right) \text{Hess}_f \left(W^\gamma, \zeta \right) \\ & & +g \left(W^\gamma, U \right) D_\zeta f D_\zeta f - g \left(\zeta, \zeta \right) g \left(W^\gamma, U \right) \left| \text{grad} f \right|^2 \end{array}$$

For any vector $Z \in H^{2,-1}$

$$\begin{array}{l} e^{-2f} \left\langle R^{\text{new}} \left(\zeta, W \right) W, Z \right\rangle = \left\langle R^{\text{old}} \left(\zeta, W \right) W, Z \right\rangle \\ -g \left(\zeta, Z \right) \text{Hess}_f \left(W^\gamma, W^\gamma \right) - g \left(W^\gamma, W^\gamma \right) \text{Hess}_f \left(\zeta, Z \right) + g \left(W^\gamma, Z \right) \text{Hess}_f \left(\zeta, W^\gamma \right) \\ +g \left(W^\gamma, W^\gamma \right) D_\zeta f D_Z f - g \left(W^\gamma, W^\gamma \right) g \left(\zeta, Z \right) \left| \text{grad} f \right|^2 \end{array}$$

Since our deformation is not infinitesimal, this result is not enough. By combining our first two lemmas on the covariant derivatives of the almost conformal change we have

Proposition 10.9. *For arbitrary $X, Y, Z,$ and U*

$$\begin{aligned} e^{-2f} R^{\text{new}}(X,Y,Z,U) &= R^{\text{old}}(X,Y,Z,U) \\ &\quad -g(X,U)\,\text{Hess}_f(Y,Z) - g(Y,Z)\,\text{Hess}_f(X,U) \\ &\quad +g(X,Z)\,\text{Hess}_f(Y,U) + g(Y,U)\,\text{Hess}_f(X,Z) \\ &\quad +g(X,U)\,D_Y f D_Z f + g(Y,Z)\,D_X f D_U f \\ &\quad -g(Y,U)\,D_X f D_Z f - g(X,Z)\,D_Y f D_U f \\ &\quad -g(Y,Z)\,g(X,U)\,|\text{grad} f|^2 + g(X,Z)\,g(Y,U)\,|\text{grad} f|^2, \\ &\quad +O\left(e^{2f}-1, |\text{grad} f|\right) \max\left\{R^{\text{old}}(X,Y,Z,U), |X||Y||Z||U|\right\} \end{aligned}$$

To evaluate curvatures we need to compute the Hessian of f. Recall that ξ is the vector in $\text{span}\left\{x^{2,0}, y^{2,0}\right\}$ that is perpendicular to ζ. Some of the formulas below are redundant. We include the redundancy for later convenience.

Proposition 10.10.

$$\text{Hess}_f(\zeta,\zeta) = -\frac{s^2}{\nu^2} D_\zeta(\psi D_\zeta \psi) + I''$$

$$\text{Hess}_f(\zeta,\xi) = \frac{s^2}{\nu^2}\left(D_\zeta(\psi)\,D_\xi(\psi) + \psi D_\zeta\left[D_\xi(\psi)\right] - \psi D_\xi(\psi)\,O\left(\frac{t}{l^2}\right)\right)$$

$$\text{Hess}_f\left(\zeta, y^{2,0}\right) = -\frac{s^2}{\nu^2}\left(D_\zeta(\psi)\,D_{y^{2,0}}(\psi) + \psi D_\zeta D_{y^{2,0}}(\psi) - \psi\,|\text{grad}\,\psi|\,O\left(\frac{t}{l^2}\right)\right) + I''\left\langle \zeta, y^{2,0}\right\rangle$$

$$\nabla^s_{W^\gamma} \text{grad} f = -\frac{s^4}{\nu^2}\,|\text{grad}\,\psi|^2\,H_w + \frac{s^2}{\nu^2}\psi\,(D_\xi\psi)\,\nabla^{\nu,re,l}_W \xi + O$$

$$\text{Hess}_f(W^\gamma, W^\gamma) = -s^4 \frac{w_h^2}{\nu^2}\psi^2\,|\text{grad}\,\psi|^2 + O$$

Proof. Since

$$\text{grad}\, f = -\frac{s^2}{\nu^2}\psi \text{grad}\,\psi + I'\zeta.$$

we have

$$\begin{aligned} \text{Hess}_f(\zeta,\zeta) &= -\frac{s^2}{\nu^2}\left\langle \nabla_\zeta(\psi\text{grad}\,\psi), \zeta\right\rangle + \left\langle \nabla_\zeta(I'\zeta), \zeta\right\rangle \\ &= -\frac{s^2}{\nu^2}\left((D_\zeta\psi)^2 + \psi\left\langle \nabla_\zeta(\text{grad}\psi), \zeta\right\rangle\right) + I'' \\ &= -\frac{s^2}{\nu^2}\left((D_\zeta\psi)^2 + \psi D_\zeta D_\zeta \psi\right) + I'' \\ &= -\frac{s^2}{\nu^2} D_\zeta(\psi D_\zeta\psi) + I'' \end{aligned}$$

$$\begin{aligned}
\mathrm{Hess}_f(\zeta,\xi) &= \langle \nabla_\zeta \mathrm{grad} f, \xi\rangle \\
&= -\frac{s^2}{\nu^2}\left\langle \nabla_\zeta\left(\psi \mathrm{grad}\,\psi\right),\xi\right\rangle + \left\langle \nabla_\zeta\left(I'\zeta\right),\xi\right\rangle \\
&= -\frac{s^2}{\nu^2}\left(D_\zeta(\psi)\left\langle \mathrm{grad}\,\psi,\xi\right\rangle + \psi\left\langle \nabla_\zeta\left(\mathrm{grad}\,\psi\right),\xi\right\rangle\right) \\
&= -\frac{s^2}{\nu^2}\left(D_\zeta(\psi) D_\xi(\psi) + \psi D_\zeta\left\langle\left(\mathrm{grad}\,\psi\right),\xi\right\rangle - \psi\left\langle\left(\mathrm{grad}\,\psi\right),\nabla_\zeta\xi\right\rangle\right) \\
&= -\frac{s^2}{\nu^2}\left(D_\zeta(\psi) D_\xi(\psi) + \psi D_\zeta D_\xi(\psi) - \psi\left\langle\left(\mathrm{grad}\,\psi\right),\xi\right\rangle O\left(\frac{t}{l^2}\right)\right) \\
&= -\frac{s^2}{\nu^2}\left(D_\zeta(\psi) D_\xi(\psi) + \psi D_\zeta\left[D_\xi(\psi)\right] - \psi D_\xi(\psi) O\left(\frac{t}{l^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hess}_f\left(\zeta, y^{2,0}\right) &= \left\langle \nabla_\zeta \mathrm{grad} f, y^{2,0}\right\rangle \\
&= -\frac{s^2}{\nu^2}\left\langle \nabla_\zeta\left(\psi \mathrm{grad}\,\psi\right), y^{2,0}\right\rangle + \left\langle \nabla_\zeta\left(I'\zeta\right), y^{2,0}\right\rangle \\
&= -\frac{s^2}{\nu^2}\left(D_\zeta(\psi)\left\langle \mathrm{grad}\,\psi, y^{2,0}\right\rangle + \psi\left\langle \nabla_\zeta\left(\mathrm{grad}\,\psi\right), y^{2,0}\right\rangle\right) + O \\
&= -\frac{s^2}{\nu^2}\left(D_\zeta(\psi) D_{y^{2,0}}(\psi) + \psi D_\zeta\left\langle\left(\mathrm{grad}\,\psi\right), y^{2,0}\right\rangle - \psi\left\langle\left(\mathrm{grad}\,\psi\right),\nabla_\zeta y^{2,0}\right\rangle\right) + O
\end{aligned}$$

To evaluate the next to last term

$$\begin{aligned}
\left\langle \mathrm{grad}\,\psi, \nabla_\zeta y^{2,0}\right\rangle &= \cos\varphi\left\langle \mathrm{grad}\,\psi, \nabla_{x^{20}} y^{2,0}\right\rangle + \sin\varphi\left\langle \mathrm{grad}\,\psi, \nabla_{y^{20}} y^{2,0}\right\rangle \\
&= \left|\mathrm{grad}\,\psi\right| O\left(\frac{t}{l^2}\right)
\end{aligned}$$

So

$$\mathrm{Hess}_f\left(\zeta, y^{2,0}\right) = -\frac{s^2}{\nu^2}\left(D_\zeta(\psi) D_{y^{2,0}}(\psi) + \psi D_\zeta D_{y^{2,0}}(\psi) - \psi\left|\mathrm{grad}\,\psi\right| O\left(\frac{t}{l^2}\right)\right) + O$$

To find $\nabla_{W^\gamma}\mathrm{grad} f$ we note that since $\mathrm{grad} f \in \mathrm{span}\left\{x^{2,0}, y^{2,0}\right\}$, and $\nabla^{\nu,re,l}_{W^\gamma}\zeta = 0$, we should think of $\mathrm{grad} f$ as a linear combination of ζ and ξ. Since this combination is constant in the W direction we have

$$\begin{aligned}
\nabla^{\nu,re,l}_{W^\gamma}\mathrm{grad} f &= \langle \mathrm{grad} f, \xi\rangle \nabla^{\nu,re,l}_{W^\gamma}\xi \\
&= \frac{s^2}{\nu^2}\psi\left(D_\xi\psi\right)\nabla^{\nu,re,l}_{W^\gamma}\xi
\end{aligned}$$

We proved in Proposition 9.6 that

$$\begin{aligned}
\nabla^s_\zeta W^\gamma &= s^2\frac{D_\zeta\psi}{\psi}H_w \\
&= \nabla^{\nu,re,l}_\zeta W^\gamma + s^2\frac{D_\zeta\psi}{\psi}H_w.
\end{aligned}$$

A similar argument gives us

$$\nabla^s_{W^\gamma}\mathrm{grad} f = \nabla^{\nu,re,l}_{W^\gamma}\mathrm{grad} f + s^2\frac{D_{\mathrm{grad} f}\psi}{\psi}H_w$$

Substituting we get

$$\nabla^s_{W^\gamma}\mathrm{grad} f = \frac{s^2}{\nu^2}\psi\left(D_\xi\psi\right)\nabla^{\nu,re,l}_{W^\gamma}\xi + s^2\frac{\langle \mathrm{grad} f, \mathrm{grad}\,\psi\rangle}{\psi}H_w$$

Since $\operatorname{grad} f = -\frac{s^2}{\nu^2}\psi \operatorname{grad}\psi + I'\zeta$ and $I' = O\left(\frac{s^4}{\nu^2}\right)$, we get

$$\nabla^s_{W^\gamma} \operatorname{grad} f = \frac{s^2}{\nu^2}\psi\left(D_\xi \psi\right)\nabla^{\nu,re,l}_W \xi - \frac{s^4}{\nu^2}\left|\operatorname{grad}\psi\right|^2 H_w + O$$

as claimed.

For redundancy we compute

$$\begin{aligned}
-\operatorname{Hess}_f\left(W^\gamma, W^\gamma\right) &= -\left\langle \nabla_{W^\gamma}\operatorname{grad} f, W^\gamma\right\rangle \\
&= +\left\langle \operatorname{grad} f, \nabla_{W^\gamma} W^\gamma\right\rangle \\
&= \left\langle -\frac{s^2}{\nu^2}\psi\operatorname{grad}\psi, -s^2 w_h^2 \psi \operatorname{grad}\,\psi\right\rangle + \left\langle -I'\zeta, -s^2 w_h^2 \psi \operatorname{grad}\,\psi\right\rangle \\
&= s^4 \frac{w_h^2}{\nu^2}\psi^2\left|\operatorname{grad}\psi\right|^2 + O
\end{aligned}$$

□

We can now compute $\operatorname{curv}(\zeta, W)$

Proposition 10.11.

$$e^{-2f}\left\langle R^{\text{new}}\left(\zeta, W\right)W, \zeta\right\rangle_{\text{new}} = s^4 w_h^2\left(D_\zeta\psi\right)^2 + s^4\frac{w_h^2}{\nu^2}\psi^2\left\langle\operatorname{grad}\psi, \zeta\right\rangle^2 + \iota + O,$$

where

$$\iota \equiv -\left|W^\gamma\right|^2 I''.$$

In particular, we can choose ι so that the zero planes with respect to $g_{\nu,l}$ have positive curvature with respect to g_{new}.

Proof. Our partial conformal change formula gives us

$$\begin{aligned}
e^{-2f}\left\langle R^{\text{new}}\left(\zeta, W\right)W, \zeta\right\rangle_{\text{new}} &= \left\langle R^s\left(\zeta, W\right)W, \zeta\right\rangle_s - \operatorname{Hess}_f\left(\zeta,\zeta\right)\left|W^\gamma\right|^2 - \operatorname{Hess}_f\left(W^\gamma, W^\gamma\right)\left|\zeta\right|^2 \\
&\quad + \left(D_\zeta f\right)^2\left|W^\gamma\right|^2 - \left|\nabla f\right|^2\left|W^\gamma\right|^2\left|\zeta\right|^2
\end{aligned}$$

To evaluate this we combine

$$\left\langle\left(R^s\left(\zeta, W\right)W\right), \zeta\right\rangle_s = -\left(s^2 w_h^2\right) D_\zeta\left(\psi D_\zeta\psi\right) + s^4 w_h^2\left(D_\zeta\psi\right)^2 + O$$

$$\begin{aligned}
\left|W^\gamma\right|^2\operatorname{Hess}_f\left(\zeta,\zeta\right) &= -\left|W^\gamma\right|^2\frac{s^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right) + \left|W^\gamma\right|^2 I'' \\
&= -\nu^2 w_h^2\frac{s^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right) - \iota \\
&= -w_h^2 s^2 D_\zeta\left(\psi D_\zeta\psi\right) - \iota
\end{aligned}$$

$$\operatorname{Hess}_f\left(W^\gamma, W^\gamma\right) = -s^4\frac{w_h^2}{\nu^2}\psi^2\left|\operatorname{grad}\psi\right|^2 + O$$

and

$$\begin{aligned}
-\left|W^\gamma\right|^2\left|\operatorname{grad} f\right|^2 + \left|W^\gamma\right|^2\left(D_\zeta f\right)^2 &= -\left|W^\gamma\right|^2\frac{s^4}{\nu^4}\psi^2\left|\operatorname{grad}\psi\right|^2 + \left|W^\gamma\right|^2\frac{s^4}{\nu^4}\psi^2\left\langle\operatorname{grad}\psi,\zeta\right\rangle^2 + O \\
&= -s^4\frac{w_h^2}{\nu^2}\psi^2\left|\operatorname{grad}\psi\right|^2 + s^4\frac{w_h^2}{\nu^2}\psi^2\left\langle\operatorname{grad}\psi,\zeta\right\rangle^2 + O,
\end{aligned}$$

to get

$$e^{-2f}\left\langle R^{\text{new}}\left(\zeta, W\right)W, \zeta\right\rangle_{\text{new}} = s^4 w_h^2\left(D_\zeta\psi\right)^2 + s^4\frac{w_h^2}{\nu^2}\psi^2\left\langle\operatorname{grad}\psi,\zeta\right\rangle^2 + \iota + O$$

as desired. □

Proposition 10.12.

$$e^{-2f}R^{\text{new}}(\zeta,W,W,\xi)=-s^2w_h\frac{D_\zeta\psi}{|\cos 2t\eta^{2,0}|}\left\langle\nabla^{\nu,re,l}_{(\eta,\eta)}W,\xi\right\rangle+O$$

Proof. From Proposition 10.8 we have

$$e^{-2f}R^{\text{new}}(\zeta,W,W,\xi)=R^{\text{old}}(\zeta,W,W,\xi)-g(W^\gamma,W^\gamma)\,\text{Hess}_f(\zeta,\xi)+g(W^\gamma,W^\gamma)\,D_\zeta f D_\xi f$$

From Proposition 9.3 we have

$$\begin{aligned}\left\langle\left(R^{\text{old}}(\zeta,W)W\right)^{\mathcal{H}},\xi\right\rangle &= -s^2w_h^2\left\langle\nabla_\zeta(\psi\text{grad}\,\psi),\xi\right\rangle+s^4w_h^2(D_\zeta\psi)\langle\text{grad}\,\psi,\xi\rangle\\ &\quad -s^2w_h\frac{D_\zeta\psi}{|\cos 2t\eta^{2,0}|}\left\langle\nabla^{\nu,re,l}_{(\eta,\eta)}W,\xi\right\rangle\end{aligned}$$

Since

$$\text{grad}\,f=-\frac{s^2}{\nu^2}\psi\text{grad}\,\psi+I'\zeta,$$

$$\begin{aligned}e^{-2f}R^{\text{new}}(\zeta,W,W,\xi) &= -s^2w_h^2\left\langle\nabla_\zeta(\psi\text{grad}\,\psi),\xi\right\rangle+s^4w_h^2(D_\zeta\psi)(\langle\text{grad}\,\psi,\xi\rangle)\\ &\quad -s^2w_h\frac{D_\zeta\psi}{|\cos 2t\eta^{2,0}|}\left\langle\nabla^{\nu,re,l}_{(\eta,\eta)}W,\xi\right\rangle\\ &\quad +\nu^2w_h^2\frac{s^2}{\nu^2}\left\langle\nabla_\zeta(\psi\text{grad}\,\psi),\xi\right\rangle+g(W^\gamma,W^\gamma)D_\zeta f D_\xi f+O\\ &= s^4w_h^2(D_\zeta\psi)(D_\xi\psi)+\nu^2w_h^2\frac{s^4}{\nu^4}\psi^2(D_\zeta\psi)(D_\xi\psi)\\ &\quad -s^2w_h\frac{D_\zeta\psi}{|\cos 2t\eta^{2,0}|}\left\langle\nabla^{\nu,re,l}_{(\eta,\eta)}W,\xi\right\rangle+O\\ &= -s^2w_h\frac{D_\zeta\psi}{|\cos 2t\eta^{2,0}|}\left\langle\nabla^{\nu,re,l}_{(\eta,\eta)}W,\xi\right\rangle+O\end{aligned}$$

□

Let $\eta^{2,0}_{u,W}$ be the unit vector in span $\left\{\eta^{2,0}_{u,1},\eta^{2,0}_{u,2}\right\}$ that is proportional to the projection of W onto span $\left\{\eta^{2,0}_{u,1},\eta^{2,0}_{u,2}\right\}$,and let $\eta^{2,0}_{u,W^\perp}$ be perpendicular to $\eta^{2,0}_{u,W}$.

Proposition 10.13.

$$e^{-2f}R^{\text{new}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right)=-s^2w_h(D_\zeta D_\zeta\psi)+w_h\psi\frac{s^2}{\nu^2}D_\zeta(\psi D_\zeta\psi)+O$$

Proof. Indeed for $U=\eta^{2,0}_{u,W}$ we have

$$\begin{aligned}e^{-2f}R^{\text{new}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right) &= R^{\text{old}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right)\\ &\quad -\left\langle W,\eta^{2,0}_{u,W}\right\rangle\text{Hess}_f(\zeta,\zeta)-\text{Hess}_f\left(W,\eta^{2,0}_{u,W}\right)\\ &\quad +\left\langle W,\eta^{2,0}_{u,W}\right\rangle(D_\zeta f)^2+\\ &\quad -\left\langle W,\eta^{2,0}_{u,W}\right\rangle|\text{grad}f|^2.\end{aligned}$$

Using Propositions 9.3 and 10.10 this becomes

$$\begin{aligned} e^{-2f} R^{\text{new}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right) &= -s^2 w_h \left(\frac{D_\zeta D_\zeta \psi}{\psi}\right)\left\langle k_\gamma, \eta^{2,0}_{u,W}\right\rangle \\ &\quad + \left\langle W, \eta^{2,0}_{u,W}\right\rangle \frac{s^2}{\nu^2} D_\zeta\left(\psi D_\zeta \psi\right) \\ &\quad + \frac{s^4}{\nu^2}\left|\operatorname{grad}\psi\right|^2 \left\langle H_w, \eta^{2,0}_{u,W}\right\rangle + O\left(\frac{s^2}{\nu^2}\psi\left(D_\xi\psi\right)\right) \\ &\quad + O\left(\frac{s^4}{\nu^4}\left|\operatorname{grad}\psi\right|^2\psi^2\right)\left\langle H_w, \eta^{2,0}_{u,W}\right\rangle + O \end{aligned}$$

So

$$\begin{aligned} e^{-2f} R^{\text{new}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right) &= -s^2 w_h\left(D_\zeta D_\zeta\psi\right) + w_h\psi\frac{s^2}{\nu^2} D_\zeta\left(\psi D_\zeta\psi\right) \\ &\quad + \frac{s^4}{\nu^2} w_h\left|\operatorname{grad}\psi\right|^2\psi + O\left(s^2 w_h\psi\left(D_\xi\psi\right)\right) + O\left(\frac{s^4}{\nu^4} w_h\left|\operatorname{grad}\psi\right|^2\psi^3\right) + O \\ &= -s^2 w_h\left(D_\zeta D_\zeta\psi\right) + w_h\psi\frac{s^2}{\nu^2} D_\zeta\left(\psi D_\zeta\psi\right) + O \end{aligned}$$

□

Proposition 10.14.

$$e^{-2f}\left\langle R^{\text{new}}\left(\zeta, W\right) W, \eta^{2,0}_{u,W^\perp}\right\rangle = 4 w_h s^2 D_\zeta\psi\frac{\psi^2}{\nu^3}\left|W_\alpha\right|_{h_2} + O$$

Proof. The partial conformal change has no effect here. So this is just what comes from Proposition 9.3. □

Proposition 10.15. *For U perpendicular to span$\left\{W, \eta^{2,0}_{u,W}\right\}$.*

(i): *If $U \in \mathcal{H}_{p_{2,-1}}$*

$$\left\langle R^{\text{old}}\left(W,\zeta\right)\zeta, U\right\rangle = 0$$

(ii): *If $U \in V_1 \oplus V_2$,*

$$\left\langle R^{\text{old}}\left(W,\zeta\right)\zeta, U\right\rangle = s^2 w_h D_\zeta\psi\left\langle \eta^{2,0}_u, \nabla^{\nu,re,l}_\zeta U\right\rangle + O$$

Proof. For $U \in \mathcal{H}_{p_{2,-1}}$,

$$\left\langle R^{\text{old}}\left(W,\zeta\right)\zeta, U\right\rangle = -s^2 w_h\left(\frac{D_\zeta D_\zeta\psi}{\psi}\right)\left\langle k_\gamma, U\right\rangle,$$

and this is 0, if U is also perpendicular to span$\left\{W, \eta^{2,0}_{u,W}\right\}$.

For $U \in V_1 \oplus V_2$, extend U to be a Killing field for the $(h_1 \oplus h_2)$–action. Then

$$\begin{aligned} \left\langle R^{\text{old}}\left(W,\zeta\right)\zeta, U\right\rangle &= -s^2 w_h\left(\frac{D_\zeta D_\zeta\psi}{\psi}\right)\left\langle k_\gamma, U\right\rangle - s^2\left(1-s^2\right) w_h\frac{D_\zeta\psi}{\psi}\left\langle k_\gamma, S_\zeta\left(U^{\mathcal{H}}\right)\right\rangle \\ &\quad + s^2\left(1-s^2\right) w_h\frac{D_\zeta\psi}{\psi}\left\langle k_\gamma, \nabla^{\nu,re,l}_\zeta U\right\rangle \\ &= s^2 w_h D_\zeta\psi\left\langle \eta^{2,0}_u, \nabla^{\nu,re,l}_\zeta U\right\rangle + O \end{aligned}$$

since U is also perpendicular to span$\left\{W, \eta^{2,0}_{u,W}\right\}$. □

Corollary 10.16. *For U perpendicular to span$\left\{W, \eta_{u,W}^{2,0}\right\}$*

(i): *If $U \in \mathcal{H}_{p_{2,-1}}$*

$$\langle R^{\text{new}}(W,\zeta)\,\zeta, U\rangle = O$$

(ii): *If $U \in V_1 \oplus V_2$ and a Killing field for the $(h_1 \oplus h_2)$–action*

$$\langle R^{\text{new}}(W,\zeta)\,\zeta, U\rangle = -e^{2f}s^2 w_h D_\zeta \psi \left\langle \eta_u^{2,0}, \nabla_\zeta^{\nu,re,l} U\right\rangle + O$$

Proof. The partial conformal change does contribute some nonzero terms here, but they are too small to matter. □

11. Quadratic Perturbations of Planes

Having established that the planes span $\{\zeta, W\}$ are now positively curved, we are left with the daunting problem of establishing that an entire neighborhood of these planes in the Grassmannian is positively curved. I.e. proving Theorem 5.2. Our first task will be to prove the main lemma (5.6), which we do in this section.

Accordingly, we represent a general plane near span $\{\zeta, W\}$ in the form $P = \text{span}\{\zeta + \sigma z, W + \tau V\}$ where $z \perp \zeta$, $V \perp W$. The curvature is then a quartic polynomial

$$P(\sigma,\tau) = \text{curv}(\zeta + \sigma z, W + \tau V)$$

in σ and τ. As we mentioned in section 5, running the Cheeger perturbations by h_1 and $\Delta(U,D)$ for a long time, allows us to reduce to the case $z \in H_{p_{2,-1}}$.

Our first task is to analyze the "quadratic perturbation", i.e. to prove the main lemma, that is we will show that for all $\sigma, \tau \in \mathbb{R}$ and for all possible choice of z and V,

$$\begin{aligned} & P_Q(\sigma,\tau) = \text{curv}^{\text{diff}}(\zeta, W) + 2\sigma R^{\text{diff}}(\zeta, W, W, z) + 2\tau R^{\text{diff}}(W, \zeta, \zeta, V) \\ & +\sigma^2 \text{curv}^{\nu,re,l}(z, W) + 2\sigma\tau\left[R^{\nu,re,l}(\zeta, W, V, z) + R^{\nu,re,l}(\zeta, V, W, z)\right] \\ & +\tau^2 \text{curv}^{\nu,re,l}(\zeta, V) \\ > \;& 0, \end{aligned}$$

where

$$\begin{aligned} R^{\text{diff}} &= R^{\text{new}} - R^{\nu,re,l} \text{ and} \\ \text{curv}^{\text{diff}} &= \text{curv}^{\text{new}} - \text{curv}^{\nu,re,l}. \end{aligned}$$

Because of the e^{2f}–factor in the partial conformal change curvature formulas, we will ultimately want this to also hold with all of the (ν, re, l)–curvature terms multiplied by e^{2f}. This is actually easier to prove, and is in fact what we will do. Because e^{2f} is pretty close to 1, our argument also gives the main lemma, but this is just an academic point.

We have already established that $\text{curv}^{\text{diff}}(\zeta, W) = \text{curv}^{\text{new}}(\zeta, W) > 0$. By combining

- $P^{\nu,re,l}(\sigma,\tau) > 0$ for all $\sigma, \tau \in \mathbb{R}$, and
- The constant and linear terms of $P^{\nu,re,l}$ are 0

we see that

$$\sigma^2 \text{curv}^{\nu,re,l}(z, W) + 2\sigma\tau\left[R^{\nu,re,l}(\zeta, W, V, z) + R^{\nu,re,l}(\zeta, V, W, z)\right] + \tau^2 \text{curv}^{\nu,re,l}(\zeta, V) > 0$$

for all $\sigma, \tau \in \mathbb{R}$.

Therefore we only need to focus on the cases where the two linear coefficients $R^{\text{diff}}(\zeta, W, W, z)$ and $R^{\text{diff}}(W, \zeta, \zeta, V)$ are large enough so that they could possibly cause a negative curvature. By combining our formulas for the curvature of the partial conformal change with Propositions 9.3, 10.10, 10.12, 10.13, 10.14, and Corollary 10.16 we see that these are

- $V = U \in V_1 \oplus V_2$ is perpendicular to $\operatorname{span}\left\{W, \eta^{2,0}_{u,W}\right\}$.
- $z = \xi$
- $V = \eta^{2,0}_{u,W}$
- $z = \eta^{2,0}_{u,W^\perp}$.

In the first two cases we will show that the linear terms are not even close to being large enough to create negative curvature. Because this turns out to be the case, to dispense with the first two possibilities, it will be enough to consider just the single variable quadratics corresponding to the perturbations $\operatorname{span}\{\zeta, W + \tau U\}$ and $\operatorname{span}\{\zeta + \sigma\xi, W\}$.

In the first case we consider the single variable quadratic polynomial

$$P(\tau) = \operatorname{curv}^{\text{diff}}(\zeta, W) + 2\tau R^{\text{diff}}(W, \zeta, \zeta, U) + \tau^2 e^{2f} \operatorname{curv}^{\nu,re,l}(\zeta, U).$$

The minimum of this quadratic polynomial is

$$\operatorname{curv}^{\text{new}}(\zeta, W) - \frac{\left\langle R^{\text{diff}}(W, \zeta)\zeta, U\right\rangle^2}{e^{2f}\operatorname{curv}^{\nu,re,l}(\zeta, U)}$$

Combining Proposition 10.11 and Corollary 10.16 we get that

(11.1)

$$P(\tau) \geq e^{2f}\left(s^4 w_h^2 (D_\zeta\psi)^2 + s^4 \frac{w_h^2}{\nu^2}\psi^2 \langle \operatorname{grad}\psi, \zeta\rangle^2 + \iota\right) - e^{2f} s^4 w_h^2 (D_\zeta\psi)^2 \frac{\left\langle \eta^{2,0}_u, \nabla^{\nu,re,l}_\zeta U\right\rangle^2}{\operatorname{curv}^{\nu,re,l}(\zeta, U)}$$

Using Theorem 6.2 we will prove

Proposition 11.2. *For any constant $c > O(\nu)$, there a choice of metric $g_{\nu,re,l}$ so that with respect to $g_{\nu,re,l}$*

$$\int_\mu (D_\zeta\psi)^2 \frac{\left\langle \eta^{2,0}_u, \nabla^{\nu,re,l}_\zeta U\right\rangle^2}{\operatorname{curv}^{\nu,re,l}(\zeta, U)} \leq c\int_\mu (D_\zeta\psi)^2,$$

where μ is any of the geodesics of length $\frac{\pi}{4}$, tangent to ζ along the old zero locus, starting over either of the two points in S^4 with $(t, \sin 2\theta) = (0, 0)$.

Moreover, for any constant $c > O(\nu)$, there is a choice of $g_{\nu,re,l}$ and a choice of ι so that with respect to $g_{\nu,re,l}$

(11.3)

$$c\left[e^{2f}\left(s^4 w_h^2 (D_\zeta\psi)^2 + s^4 \frac{w_h^2}{\nu^2}\psi^2 \langle \operatorname{grad}\psi, \zeta\rangle^2 + \iota\right)\right] \geq e^{2f} s^4 w_h^2 (D_\zeta\psi)^2 \frac{\left\langle \eta^{2,0}_u, \nabla^{\nu,re,l}_\zeta U\right\rangle^2}{\operatorname{curv}^{\nu,re,l}(\zeta, U)}.$$

In particular, $P(\tau) > 0$.

Remark 11.4. *At this point we can begin to appreciate the need for the redistribution metric. It allows us to make the negative term in 11.1 as small as we like. It will become clear after we have considered the case when $V = \eta^{2,0}_{u,W}$ that without this redistribution there would in fact be some negative curvatures.*

Proof. From Proposition 6.4 we see that the redistribution has very little effect on $\left\langle \eta_u^{2,0}, \nabla_\zeta U \right\rangle^2$. To compute this quantity with respect to $g_{\nu,l}$, we must again consider $Sp(2) \times \left(S^3\right)^2$. So for the purpose of this proof we suspend the notational convention on page 17, and revert to the "$\hat{}$" notation for discussing Cheeger deformations.

$$\left\langle \hat{\eta}_u^{2,0}, \nabla_\zeta^{\nu,l} \hat{U} \right\rangle^2 = \left\langle \eta_u^{2,0}, \nabla_\zeta^{\nu} U \right\rangle_{Sp(2)}^2 + \left\langle \left(\hat{\eta}_u^{2,0}\right)^{\left(S^3\right)^2}, \nabla_\zeta \hat{U} \right\rangle_{(S^3)^2}^2.$$

The $Sp(2)$ derivative is given by quaternion multiplication and lives in the orthogonal complement H of $V_1 \oplus V_2$. So

$$\begin{aligned} \left\langle \eta_u^{2,0}, \nabla_\zeta^{\nu} U \right\rangle_{Sp(2)}^2 &= \frac{\cos^2 2t}{\left|\cos 2t\eta^{2,0}\right|^2} \left\langle (\eta, \eta), \nabla_\zeta^{\nu} U \right\rangle_\nu^2 \\ &\leq \frac{1}{\left|\cos 2t\eta^{2,0}\right|^2} \mathrm{curv}^\nu (\zeta, U) \end{aligned}$$

Our estimates for the $\left(S^3\right)^2$–portion will be efficient, but not optimal. First notice that if $|U|_\nu = O\left(\frac{1}{\nu}\right)$, then

$$\left| \left(\nabla_\zeta \hat{U}\right)^{S^3} \right|_l = O\left(\frac{1}{l}\right),$$

since

$$\left| \left(\hat{\eta}_u^{2,0}\right)^{\left(S^3\right)^2} \right|_l = \frac{1}{\left|\cos 2t\eta^{2,0}\right|^2} O\left(\frac{t}{l}\right)$$

we get

$$\left\langle \left(\hat{\eta}_u^{2,0}\right)^{\left(S^3\right)^2}, \nabla_\zeta \hat{U} \right\rangle_{(S^3)^2}^2 \leq \frac{1}{\left|\cos 2t\eta^{2,0}\right|^2} O\left(\frac{t^2}{l^2}\right)$$

combining estimates we have

$$\begin{aligned} \left\langle \hat{\eta}_u^{2,0}, \nabla_\zeta^{\nu,l} \hat{U} \right\rangle^2 &\leq \frac{1}{\left|\cos 2t\eta^{2,0}\right|^2} \left(\mathrm{curv}^\nu (\zeta, U) + O\left(\frac{t^2}{l^2}\right) \right) \\ &\leq 1.1 \mathrm{curv}^{\nu,l} (\zeta, U) \end{aligned}$$

From Proposition 6.4 we see that (with an irrelevant adjustment), this estimate also holds with $\left\langle \hat{\eta}_u^{2,0}, \nabla_\zeta^{\nu,l} \hat{U} \right\rangle^2$ replaced by $\left\langle \hat{\eta}_u^{2,0}, \nabla_\zeta^{\nu,re,l} \hat{U} \right\rangle^2$. On the other hand, we see from Theorem 6.2 and O'Neill's horizontal curvature equation that

$$\mathrm{curv}^{\nu,re,l} (\zeta, U) \geq \mathrm{curv}^\nu (\zeta, U) + \varphi\varphi'' |U|_\nu^2,$$

Since $(D_\zeta \psi)^2$ is concentrated on a set that looks like $[0, O(\nu)]$, we can redistribute the ratio

$$\frac{\left\langle \eta_u^{2,0}, \nabla_\zeta^{\nu,re,l} U \right\rangle^2}{e^{2f} \mathrm{curv}^{\nu,re,l} (\zeta, U)}$$

so that it is small where $(D_\zeta \psi)^2$ is large, and large where $(D_\zeta \psi)^2$ is small. The choice of φ'' that we made at the beginning of section 6 will give us the desired integral inequality (perhaps with an adjustment of the constants $100, 10,000, \ldots$).

To get 11.3 we combine the integral inequality with the fact that we have yet to impose any conditions on ι except,

$$\begin{aligned} \iota &= O\left(s^4 w_h^2\right), \text{ and} \\ \int_\mu \iota &= 0. \end{aligned}$$

To get 11.3 we must now require that ι be positive and (relatively large) on a region that looks like $\left[\text{const}\,\nu, \frac{\pi}{4}\right]$. The quantity $\text{const}\,\nu$ is near the inflection point of the redistribution function φ. □

In the case where $z = \xi$ we will again see that linear term is overwhelmingly dominated. Consider the quadratic

$$P(\sigma) = \text{curv}^{\text{diff}}(\zeta, W) + 2\sigma R^{\text{diff}}(\zeta, W, W, \xi) + \sigma^2 e^{2f} \text{curv}^{\text{old}}(\xi, W)$$

The minimum is

$$\text{curv}^{\text{diff}}(\zeta, W) - \frac{R^{\text{diff}}(\zeta, W, W, \xi)^2}{e^{2f} \text{curv}^{\text{old}}(\xi, W)}$$

Combining Propositions 10.11 and 10.12 we see that this is

$$e^{2f}\left(s^4 w_h^2 (D_\zeta \psi)^2 + s^4 \frac{w_h^2}{\nu^2} \psi^2 \langle \text{grad}\,\psi, \zeta\rangle^2 + \iota\right) - e^{2f} \frac{s^4 w_h^2 (D_\zeta \psi)^2 \left\langle \nabla^{\nu,re,l}_{(\eta,\eta)} W, \xi \right\rangle^2}{\left|\cos 2t\eta^{2,0}\right|^2 \text{curv}^{\nu,re,l}(\xi, W)}.$$

(11.5)

Since

$$\begin{aligned} \text{curv}^{\nu,re,l}(\xi, W) &\geq \left\langle \nabla^\nu_{(\eta,\eta)} W, \xi \right\rangle^2_\nu + O\left(\frac{t^2}{l^6}\right), \\ \left\langle \nabla^{\nu,re,l}_{(\eta,\eta)} W, \xi \right\rangle^2 &\leq \left\langle \nabla^\nu_{(\eta,\eta)} W, \xi \right\rangle^2_\nu + O\left(\frac{t^2}{l^4}\right) + O\left(\frac{t^4}{l^8}\right), \\ \left\langle \nabla^\nu_{(\eta,\eta)} W, \xi \right\rangle^2_\nu &\leq 1, \\ \frac{1}{\left|\cos 2t\eta^{2,0}\right|^2} &\leq \frac{1}{\left(\cos^2 2t + \frac{\sin^2 2t}{\nu^2}\right)} \\ &= \frac{\nu^2}{\left(\nu^2 \cos^2 2t + \sin^2 2t\right)}, \end{aligned}$$

and

$$l = O\left(\nu^{1/3}\right)$$

we see that the negative term in 11.5 is much smaller than the positive term, provided the constant c so that $l = c\nu^{1/3}$ is relatively large.

In the final two cases, $V = \eta^{2,0}_{u,W}$ and $z = \eta^{2,0}_{u,W^\perp}$, the linear terms can be a substantial fraction of the total, so we will have to be more careful. In particular, we will have to consider the entire polynomial $P_Q(\sigma, \tau)$. We start by analyzing the two mixed quadratic coefficients

$$R^{\nu,re,l}\left(\zeta, W, \eta^{2,0}_{u,W}, \eta^{2,0}_{u,W^\perp}\right), R^{\nu,re,l}\left(\zeta, \eta^{2,0}_{u,W}, W, \eta^{2,0}_{u,W^\perp}\right).$$

First notice that they are 0 if our only deformations of the biinvariant metric are the h_1 and h_2 Cheeger perturbations. We track the effect of the U and D perturbations by considering the corresponding submersion $S^3 \times S^3 \times Sp(2) \longrightarrow$

$Sp(2)$. As we have observed the components of $R^{\nu,re,l}\left(\zeta, W, \eta^{2,0}_{u,W}, \eta^{2,0}_{u,W^\perp}\right)$ and $R^{\nu,re,l}\left(\zeta, \eta^{2,0}_{u,W}, W, \eta^{2,0}_{u,W^\perp}\right)$ that come from the $Sp(2)$–factor of $\left(S^3\right)^2 \times Sp(2)$ are 0. For similar reasons the components that come from the S^3–factor are 0. The A–tensors of $S^3 \times S^3 \times Sp(2) \longrightarrow Sp(2)$ and $Sp(2) \longrightarrow \Sigma^7$ might make a nonzero contribution, but its contribution to the curvature of the entire plane

$$\mathrm{curv}^{\nu,re,l}\left(\zeta + \sigma\eta^{2,0}_{u,W^\perp}, W + \tau\eta^{2,0}_{u,W}\right)$$

is nonnegative so we may drop it. (As long as we drop it from all curvatures.) Finally we saw in section 6 that the redistribution deformation only has a large effect on curvatures that have ζ in two variables. So in the end we see that these two mixed terms are too small to matter.

Although this simplifies matters considerably, we still have to consider the rest of $P_Q(\sigma,\tau)$ as a whole. More specifically we have to verify that

$$\mathrm{curv}^{\mathrm{diff}}(\zeta, W) - \frac{R^{\mathrm{diff}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right)^2}{e^{2f}\mathrm{curv}^{\nu,re,l}\left(\zeta,\eta^{2,0}_{u,W}\right)} - \frac{R^{\mathrm{diff}}\left(\zeta,W,W,\eta^{2,0}_{u,W^\perp}\right)^2}{e^{2f}\mathrm{curv}^{\nu,re,l}\left(\eta^{2,0}_{u,W^\perp},W\right)} > 0.$$

To simplify the exposition we compute the sum of the first two terms and then the last term. Using Propositions 10.11 and 10.13 and the fact that

$$\mathrm{curv}^{\nu,re,l}\left(\zeta,\eta^{2,0}_{u,W}\right) = -\frac{D_\zeta D_\zeta \psi}{\psi} + O,$$

we find

$$\mathrm{curv}^{\mathrm{diff}}(\zeta, W) - \frac{R^{\mathrm{diff}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right)^2}{e^{2f}\mathrm{curv}^{\nu,re,l}\left(\zeta,\eta^{2,0}_{u,W}\right)} + O$$

$$\begin{aligned}
&= e^{2f}\left(s^4 w_h^2 \left(D_\zeta\psi\right)^2 + s^4\frac{w_h^2}{\nu^2}\psi^2\left\langle \mathrm{grad}\,\psi,\zeta\right\rangle^2 + \iota\right) - \frac{e^{4f}\left(-s^2 w_h\left(D_\zeta D_\zeta\psi\right) + w_h\psi\frac{s^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right)\right)^2}{-e^{2f}\frac{D_\zeta D_\zeta\psi}{\psi}} \\
&= e^{2f}s^4 w_h^2\left(\left(D_\zeta\psi\right)^2 + \frac{\psi^2}{\nu^2}\left\langle\mathrm{grad}\,\psi,\zeta\right\rangle^2\right) + e^{2f}\iota \\
&\quad + e^{2f}s^4 w_h^2\left[\frac{\psi}{D_\zeta D_\zeta\psi}\left(\left(D_\zeta D_\zeta\psi\right)^2 - \frac{2}{\nu^2}\left(D_\zeta D_\zeta\psi\right)\psi D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi^2}{\nu^4}\left[D_\zeta\left(\psi D_\zeta\psi\right)\right]^2\right)\right] \\
&= e^{2f}s^4 w_h^2\left(\left(D_\zeta\psi\right)^2 + \frac{\psi^2}{\nu^2}\left\langle\mathrm{grad}\,\psi,\zeta\right\rangle^2\right) + e^{2f}\iota \\
&\quad + e^{2f}s^4 w_h^2\left(\psi\left(D_\zeta D_\zeta\psi\right) - 2\frac{\psi^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi}{D_\zeta D_\zeta\psi}\frac{\psi^2}{\nu^4}\left[D_\zeta\left(\psi D_\zeta\psi\right)\right]^2\right) \\
&= e^{2f}s^4 w_h^2\left(D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi^2}{\nu^2}\left(D_\zeta\psi\right)^2 - 2\frac{\psi^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi}{D_\zeta D_\zeta\psi}\frac{\psi^2}{\nu^4}\left[D_\zeta\left(\psi D_\zeta\psi\right)\right]^2\right) + e^{2f}\iota
\end{aligned}$$

The integral the first term is 0. The integral of the second term is positive and the integral of the third term is positive as well, since the total derivative is positive where ψ is small and negative where ψ is larger.

The next to last term has a negative integral, but in Lemma 8.5 we showed

$$\left|\frac{\psi}{D_\zeta D_\zeta \psi}\left[D_\zeta\left(\psi D_\zeta \psi\right)\right]\right| \leq \frac{\nu^2}{4}.$$

so

$$\left|\frac{\psi}{D_\zeta D_\zeta \psi}\frac{\psi^2}{\nu^4}\left[D_\zeta\left(\psi D_\zeta \psi\right)\right]^2\right| \leq \frac{\psi^2}{4\nu^2}\left|D_\zeta\left(\psi D_\zeta \psi\right)\right|$$

This is an eighth of the third term so

$$\begin{aligned}
&\operatorname{curv}^{\mathrm{diff}}(\zeta, W) - \frac{R^{\mathrm{diff}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right)^2}{e^{2f}\operatorname{curv}^{\nu,re,l}\left(\zeta,\eta^{2,0}_{u,W}\right)} + O \\
\geq\ & e^{2f}s^4 w_h^2\left(D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi^2}{\nu^2}\left(D_\zeta\psi\right)^2 - 2\frac{\psi^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi^2}{4\nu^2}\left|D_\zeta\left(\psi D_\zeta\psi\right)\right|\right) + e^{2f}\iota \\
=\ & e^{2f}s^4 w_h^2\left(D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi^2}{\nu^2}\left(D_\zeta\psi\right)^2 - \frac{7}{4}\frac{\psi^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right)\right) + e^{2f}\iota
\end{aligned} \tag{11.6}$$

Finally using Proposition 10.14

$$\begin{aligned}
\frac{R^{\mathrm{diff}}\left(\zeta,W,W,\eta^{2,0}_{u,W^\perp}\right)^2}{e^{2f}\operatorname{curv}^{\nu,re,l}\left(\eta^{2,0}_{u,W^\perp},W\right)} &= s^4 w_h^2\left(D_\zeta\psi\right)^2\frac{e^{4f}\left(4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|\right)^2}{e^{2f}\operatorname{curv}^{\nu,re,l}\left(\eta^{2,0}_{u,W^\perp},W\right)} \\
&\leq e^{2f}s^4 w_h^2\left(D_\zeta\psi\right)^2\frac{\left(4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|\right)^2}{\left|\cos 2t\eta^{2,0}\right|^{-2} + 4\frac{\psi^2}{\nu^6}},
\end{aligned}$$

where the factor of 4 in the denominator comes from the fact that $\psi = \frac{1}{2}\frac{\sin 2t}{\left|\cos 2t\eta^{2,0}\right|}$. Since $\left|W_\alpha\right|^2 \leq \frac{1}{4\nu^2}$, we get another factor of 4. So

$$\frac{R^{\mathrm{diff}}\left(\zeta,W,W,\eta^{2,0}_{u,W^\perp}\right)^2}{e^{2f}\operatorname{curv}^{\nu,re,l}\left(\eta^{2,0}_{u,W^\perp},W\right)} \leq e^{2f}s^4 w_h^2\left(D_\zeta\psi\right)^2\frac{\psi^2}{\nu^2}$$

Combining the displays we get

$$\begin{aligned}
&\operatorname{curv}^{\mathrm{diff}}(\zeta, W) - \frac{R^{\mathrm{diff}}\left(W,\zeta,\zeta,\eta^{2,0}_{u,W}\right)^2}{e^{2f}\operatorname{curv}^{\nu,re,l}\left(\zeta,\eta^{2,0}_{u,W}\right)} - \frac{R^{\mathrm{diff}}\left(\zeta,W,W,\eta^{2,0}_{u,W^\perp}\right)^2}{e^{2f}\operatorname{curv}^{\nu,re,l}\left(\eta^{2,0}_{u,W^\perp},W\right)} \geq \\
&e^{2f}s^4 w_h^2\left(D_\zeta\left(\psi D_\zeta\psi\right) + \frac{\psi^2}{\nu^2}\left(D_\zeta\psi\right)^2 - \frac{7}{4}\frac{\psi^2}{\nu^2}D_\zeta\left(\psi D_\zeta\psi\right) - \frac{\psi^2}{\nu^2}\left(D_\zeta\psi\right)^2\right) + e^{2f}\iota + O \\
=\ & e^{2f}s^4 w_h^2\left(D_\zeta\left(\psi D_\zeta\psi\right) - \frac{7}{4}\frac{\psi^2}{\nu^2}D_\zeta\left[\psi D_\zeta\psi\right]\right) + \iota e^{2f} + O.
\end{aligned}$$

So we can choose ι so that the right hand side is point wise positive. With some moments of reflection we see that this choice of ι can be consistent with the choice required for the proof of Proposition 11.2.

Remark 11.7. *With a careful review of the estimates in this section one can appreciate the necessity of the redistribution. Indeed without the redistribution, we can't do much better in Proposition 11.2 than*

$$\frac{\left\langle \eta_u^{2,0}, \nabla_\zeta^{\nu,re,l} U \right\rangle^2}{\text{curv}^{\nu,re,l}(\zeta, U)} \leq 1.$$

Tracing through the rest of our estimates one can then see that there can be a vector in $V \in \text{span}\left\{V_1 \oplus V_2, \eta_{u,W}^{2,0}\right\}$ *so that the single variable polynomial*

$$P(\tau) = \text{curv}(\zeta, W + \tau V)$$

has some negative values. To see this one must also observe that the integrals of

$$\frac{\psi^2}{\nu^2}(\psi D_\zeta D_\zeta \psi) \text{ and } \frac{\psi^2}{\nu^2} D_\zeta [\psi D_\zeta \psi]$$

are something like $O\left(\frac{1}{100}\right)$ *times the integral of*

$$\psi D_\zeta D_\zeta \psi.$$

12. Higher Order Terms

To prove that the Gromoll-Meyer sphere is now positively curved it remains to show that the higher order terms in the curvature polynomial

$$P(\sigma, \tau) = \text{curv}(\zeta + \sigma z, W + \tau V),$$

do not change enough under our deformations to create a nonpositive curvature.

Recall that it is enough to consider the case when $z \in H^{2,-1}$. For computational convenience, we choose z and V so that their components in $\text{span}\left\{x^{2,0}, y^{2,0}\right\}$ are proportional to $y^{2,0}$. In addition, we choose V so that its component in V_2 is perpendicular to the γ-part of W. We further assume that z and V are normalized so that they are spherical combinations of our standard vectors.

The curvature of P is a quartic polynomial

$$P(\sigma, \tau) = R(\zeta + \sigma z, W + \tau V, W + \tau V, \zeta + \sigma z)$$

in σ and τ.

In addition we must verify the positivity of the quadratic subpolynomials

$$\begin{aligned} Q_\zeta(\sigma) &= \text{curv}(\zeta + \sigma z, V) \text{ and} \\ .Q_W(\tau) &= \text{curv}(z, W + \tau V). \end{aligned}$$

We let $\varkappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ stand for a function so that $\lim_{s \to 0} \varkappa(s) = 0$.

Set

$$\begin{aligned} &H^{\text{diff}}(\sigma, \tau) \equiv \tau^2 R^{\text{diff}}(\zeta, V, V, \zeta) + 2\sigma\tau R^{\text{diff}}(\zeta, W, V, z) + 2\sigma\tau R^{\text{diff}}(\zeta, V, W, z) + \sigma^2 R^{\text{diff}}(z, W, W, z) \\ &+2\sigma\tau^2 R^{\text{diff}}(\zeta, V, V, z) + 2\sigma^2\tau R^{\text{diff}}(z, W, V, z) + \sigma^2\tau^2 \ R^{\text{diff}}(z, V, V, z), \end{aligned}$$

and let $P^{\nu,re,l}(\sigma, \tau)$ be the curvature polynomial for $g_{\nu,re,l}$.

Theorem 12.1. *To verify that* $P(\sigma, \tau) > 0$, $Q_\zeta(\sigma) > 0$, *and* $Q_W(\tau) > 0$ *for all* σ, τ, *we may ignore*

(a): *Any term in a coefficient of* H^{diff} *that is smaller than* $\varkappa(s)$ *times the corresponding coefficient of* $P^{\nu,re,l}$.

(b): *Any term in the $(\sigma\tau)$–coefficient of H^{diff} that is smaller than*

$$\chi(s)\sqrt{\text{curv}^{\nu,re,l}(\zeta,V)}\sqrt{\text{curv}^{\nu,re,l}(z,W)}$$

(c): *Any term in the $\left(\sigma^2\tau\right)$–coefficient of H^{diff} that is smaller than*

$$\chi(s)\sqrt{\text{curv}^{\nu,re,l}(z,W)}\sqrt{\text{curv}^{\nu,re,l}(z,V)}$$

(d): *Any term in the $\left(\sigma\tau^2\right)$–coefficient of H^{diff} that is smaller than*

$$\chi(s)\sqrt{\text{curv}^{\nu,re,l}(\zeta,V)}\sqrt{\text{curv}^{\nu,re,l}(z,V)}.$$

Proof. Part (a) follows from the main lemma and the fact that $P^{\nu,re,l}(\sigma,\tau)>0$, $Q_{\zeta}^{\nu,re,l}(\sigma)>0$, and $Q_{W}^{\nu,re,l}(\tau)>0$ for all $\sigma,\tau>0$.

To prove part (b) we fix τ and view the τ^2 and σ^2 terms of $P^{\nu,re,l}$ together with the term in the $(\sigma\tau)$–coefficient that is smaller than

$$\chi(s)\sqrt{\text{curv}^{\nu,re,l}(\zeta,V)}\sqrt{\text{curv}^{\nu,re,l}(z,W)}.$$

as a quadratic in σ. The minimum is smaller than

$$\begin{aligned}
&\tau^2\left(\text{curv}^{\nu,re,l}(\zeta,V)-\chi(s)^2\frac{\left(\sqrt{\text{curv}^{\nu,re,l}(\zeta,V)}\sqrt{\text{curv}^{\nu,re,l}(z,W)}\right)^2}{\text{curv}^{\nu,re,l}(z,W)}\right)\\
=\;&\tau^2\left(\text{curv}^{\nu,re,l}(\zeta,V)-\chi(s)^2\,\text{curv}^{\nu,re,l}(\zeta,V)\right)\\
=\;&\tau^2\text{curv}^{\nu,re,l}(\zeta,V)-O.
\end{aligned}$$

Parts (c) and (d) are proven with similar arguments. For part (c), we dominate the portion of the $\left(\sigma^2\tau\right)$–coefficient of H^{diff} in question with the σ^2 and $\sigma^2\tau^2$–coefficients of $P^{\nu,re,l}(\sigma,\tau)$. For part (d), we dominate the portion of the $\left(\sigma\tau^2\right)$–coefficient of H^{diff} in question with the τ^2 and $\sigma^2\tau^2$–coefficients of $P^{\nu,re,l}(\sigma,\tau)$.

We do not have to consider the $Q_{\zeta}(\sigma)$s and $Q_{W}(\tau)$s for part (b). The proofs of parts (c) and (d) for the $Q_{\zeta}(\sigma)$s and $Q_{W}(\tau)$s are essentially the same as the proofs for $P(\sigma,\tau)$. □

Remark 12.2. *Since many of the possible coefficients of $P^{\nu,re,l}$ can be large, many of the terms that this theorem allows us to ignore are in fact large. Its just that their effect is swamped by certain terms of $P^{\nu,re,l}$.*

We let

$$R^{\text{diff,big}}$$

denote the terms of R^{diff} that can not be thrown out using the previous theorem.

Theorem 12.3. *If z and V are as above and normalized as in our standard basis, then $\text{curv}^{\text{diff,big}}(z,V)$ and $\text{curv}^{\text{diff,big}}(z,W)$, are nonnegative and*

$$\left|R^{\text{diff,big}}(z,V,W,z)\right|\leq\sqrt{\text{curv}^{\text{diff,big}}(z,V)}\sqrt{\text{curv}^{\text{diff,big}}(z,W)}$$

All other coefficients of $R^{\text{diff,big}}$ are 0, unless our perturbation bivector (z,V) has a nonzero inner product with either the case when $z=y^{2,0}$ and $V=\eta_{u,W}^{2,0}$ or with the case when $z=\eta_{u,W}^{2,0}$ and $V=y^{2,0}$.

Theorem 12.4. *If $z = y^{2,0}$ and $V = \eta_{u,W}^{2,0}$ or $z = \eta_{u,W}^{2,0}$ and $V = y^{2,0}$, then*

$$\begin{aligned}
\left\langle R^{\text{diff}}\left(W, y^{2,0}\right) y^{2,0}, W\right\rangle &= e^{2f} s^2 w_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right) + \varkappa(s) \\
\left|R^{\text{diff,big}}\left(\zeta, \eta_{u,W}^{2,0}, W, y^{2,0}\right)\right| &= \left|R^{\text{diff,big}}\left(\zeta, W, \eta_{u,W}^{2,0}, y^{2,0}\right)\right| \\
&= e^{2f} s^2 w_h \psi \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right)\left\langle y^{2,0}, \zeta\right\rangle + \varkappa(s) \\
\left|R^{\text{diff,big}}\left(W, y^{2,0} y^{2,0}, \eta_{u,W}^{2,0}\right)\right| &= e^{2f} s^2 w_h \psi \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right) + \varkappa(s)
\end{aligned}$$

and all other coefficients of $R^{\text{diff,big}}$ are 0.

Before discussing the proofs, we show how these two theorems gives us that $P(\sigma,\tau) > 0, Q_\zeta(\sigma) > 0$, and $Q_W(\tau) > 0$ for all $\sigma, \tau \in \mathbb{R}$, and hence that the Gromoll-Meyer sphere is positively curved. The proofs that $Q_\zeta(\sigma) > 0$, and $Q_W(\tau) > 0$ are strictly contained in the proof that $P(\sigma,\tau) > 0$,so we only write out the details that $P(\sigma,\tau) > 0$.

We discuss the case of Theorem 12.3 and then those of Theorem 12.4.

From our proof of the main lemma, we have that in the case of Theorem 12.3

$$\begin{aligned}
P(\sigma,\tau) \geq\ & O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) + \sigma^2 \text{curv}^{\text{diff,big}}(z,W) \\
& + 2\sigma^2\tau\sqrt{\text{curv}^{\text{diff,big}}(z,V)}\sqrt{\text{curv}^{\text{diff,big}}(z,W)} + \sigma^2\tau^2 \text{curv}^{\text{diff,big}}(z,V) + O.
\end{aligned}$$

The sum

$$\begin{aligned}
& \sigma^2 \text{curv}^{\text{diff,big}}(z,W) + 2\sigma^2\tau\sqrt{\text{curv}^{\text{diff,big}}(z,V)}\sqrt{\text{curv}^{\text{diff,big}}(z,W)} + \sigma^2\tau^2 \text{curv}^{\text{diff,big}}(z,V) \\
= \ & \sigma^2\left(\sqrt{\text{curv}^{\text{diff,big}}(z,W)} + \tau\sqrt{\text{curv}^{\text{diff,big}}(z,V)}\right)^2
\end{aligned}$$

is nonnegative so we may drop it.

Thus

$$P(\sigma,\tau) \geq O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) + O,$$

and hence is positive.

In the case of Theorem 12.4, when $z = y^{2,0}$ and $V = \eta_{u,W}^{2,0}$

$$\begin{aligned}
P(\sigma,\tau) \geq\ & O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) + \sigma^2 \text{curv}^{\text{diff,big}}\left(y^{2,0}, W\right) \\
& + 2\sigma\tau R^{\text{diff,big}}\left(\zeta, W, \eta_u^{2,0}, y^{2,0}\right) + 2\sigma\tau R^{\text{diff,big}}\left(\zeta, \eta_u^{2,0}, W, y^{2,0}\right) + \\
& + 2\sigma^2\tau R^{\text{diff,big}}\left(y^{2,0}, W, \eta_u^{2,0}, y^{2,0}\right) + 2\sigma\tau^2 R^{\text{diff,big}}\left(\zeta, \eta_u^{2,0}, \eta_u^{2,0}, y^{2,0}\right) + O.
\end{aligned}$$

Plugging in our curvature estimates we get

$$\begin{aligned}
P(\sigma,\tau) \geq\ & O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) + \sigma^2 e^{2f} s^2 w_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right) \\
& + 4\sigma\tau e^{2f} s^2 w_h \psi \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right)\left\langle y^{2,0}, \zeta\right\rangle \\
& + 2\sigma^2\tau e^{2f} s^2 w_h \psi \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right)
\end{aligned}$$

For fixed τ, we can view the σ^2 and τ^2 terms of $P^{\nu,re,l}(\sigma,\tau)$ together with

$$\sigma^2 e^{2f} s^2 w_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right) \text{ and } 4\sigma\tau e^{2f} s^2 w_h \psi \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right)\left\langle y^{2,0}, \zeta\right\rangle$$

as a quadratic in σ. Since $\left|\left\langle y^{2,0},\zeta\right\rangle\right| \leq \frac{1}{2} + O(t)$, the minimum is

$$\begin{aligned}
&\tau^2\left(\mathrm{curv}^{\nu,re,l}\left(\zeta,\eta_u^{2,0}\right) - \frac{\left[s^2 w_h \psi \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)\right]^2}{\mathrm{curv}^{\nu,re,l}\left(y^{2,0},W\right) + s^2 w_h^2 \psi^2 \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)}\right) + O \\
\geq\ & \tau^2 \mathrm{curv}^{\nu,re,l}\left(\zeta,\eta_u^{2,0}\right) + O.
\end{aligned}$$

Thus we may replace the mixed quadratic $\sigma\tau$ term with O, and our estimate becomes

$$\begin{aligned}
P(\sigma,\tau) \geq\ & O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) + \sigma^2 e^{2f} s^2 w_h^2 \psi^2 \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right) \\
& + 2\sigma^2 \tau e^{2f} s^2 w_h \psi \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right) + O.
\end{aligned}$$

For fixed σ, we view the $\sigma^2 e^{2f} s^2 w_h^2 \psi^2 \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)$ term, the $2\sigma^2\tau e^{2f} s^2 w_h \psi \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)$ term and the $\sigma^2\tau^2$ term of $P^{\nu,re,l}(\sigma,\tau)$ as a quadratic in τ. The minimum is

$$\sigma^2 e^{2f}\left(s^2 w_h^2 \psi^2 \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right) - s^4 w_h^2 \psi^2 \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)\right) + O = \sigma^2 s^2 w_h^2 \psi^2 \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right) + O$$

So we again have

$$\begin{aligned}
P(\sigma,\tau) &\geq O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) + O \\
&> 0.
\end{aligned}$$

Finally, in the exceptional case when $z = \eta_{u,W}^{2,0}$ and $V = y^{2,0}$ we plug in our curvature estimates and get

$$\begin{aligned}
P(\sigma,\tau) \equiv\ & O\left(s^4 w_h^2 \nu\right) + P^{\nu,re,l}(\sigma,\tau) \\
& + 2\sigma\tau e^{2f} s^2 w_h \psi \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)\left\langle y^{2,0},\zeta\right\rangle + O
\end{aligned}$$

When $t \geq O\left(\nu^{1/2}\right)$, the $(\sigma\tau)$–term is dominated by the $\sigma^2 \mathrm{curv}\left(\eta_{u,W}^{2,0},W\right)$ and $\tau^2 \mathrm{curv}\left(y^{2,0},\zeta\right)$ terms of $P^{\nu,re,l}(\sigma,\tau)$. So we may assume that $t \leq O\left(\nu^{1/2}\right)$.

In this case, we view the $\sigma\tau$, and $\sigma^2\tau^2$ terms of $P(\sigma,\tau)$ as a quadratic in $\sigma\tau$. The minimum of this quadratic is

$$-e^{2f}\frac{\left(2s^2 w_h \psi \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)\left\langle y^{2,0},\zeta\right\rangle\right)^2}{4\mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)} + O$$

Since $\left|\left\langle y^{2,0},\zeta\right\rangle\right| \leq \frac{1}{2} + O(t)$ our minimum is

$$\begin{aligned}
&\geq -e^{2f} s^4 w_h^2 \psi^2\left(\frac{1}{4}\mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right) + \mathrm{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right) t\right) \\
&= -e^{2f}\frac{1}{4} s^4 w_h^2 \psi^2\left[\left(\frac{D_\zeta D_\zeta \psi}{\psi} + \frac{t^2\psi^2}{l^6}\right)\right] + O \\
&= -e^{2f}\frac{1}{4} s^4 w_h^2 \psi D_\zeta D_\zeta \psi + O \qquad (12.4)
\end{aligned}$$

This is of the order of our constant coefficient

$$\mathrm{curv}^{new}(\zeta,W) = e^{2f} s^4 w_h^2 \left(D_\zeta \psi\right)^2\left[1 + \frac{\psi^2}{\nu^2}\right] + e^{2f}\iota.$$

So we will have to be careful here.

Notice that the minimum occurs when

$$\sigma\tau = O\left(s^2 w_h \psi\right),$$

and we have not used the two positive quadratic terms

$$\sigma^2 \text{curv}\left(\eta_{u,W}^{2,0}, W\right) + \tau^2 \text{curv}\left(y^{2,0}, \zeta\right).$$

It will be sufficient to show that near the minimum this sum is much larger than $O\left(s^4 w_h^2 \nu\right)$ .We will actually show that this holds except for $t \in \left[0, s^2 w_h \nu\right]$. We will then argue that with a very minor adjustment in ι, we can easily dominate the negative term 12.4 on the exceptional region.

Thus we have positive curvature except possibly if

$$\sigma^2 \text{curv}\left(\eta_{u,W}^{2,0}, W\right) \leq O\left(s^4 w_h^2 \nu\right)$$

or

$$\begin{aligned}
\sigma^2\left(1 + \frac{\psi^2}{l^6}\right) &\leq O\left(s^4 w_h^2 \nu\right) \text{ or} \\
\sigma^2 &\leq \frac{O\left(s^4 w_h^2 \nu\right)}{\left(1 + \frac{\psi^2}{l^6}\right)} \text{ or} \\
\frac{1}{\sigma} &\geq \frac{\sqrt{1 + \frac{\psi^2}{l^6}}}{O\left(s^2 w_h \nu^{1/2}\right)}
\end{aligned}$$

Since we also have

$$\sigma\tau = O\left(s^2 w_h \psi\right),$$

we get

$$\begin{aligned}
\tau &\geq \frac{O\left(s^2 w_h \psi\right)}{\sigma} \\
&\geq O\left(s^2 w_h \psi\right) \frac{\sqrt{1 + \frac{\psi^2}{l^6}}}{O\left(s^2 w_h \nu^{1/2}\right)} \\
&\geq O\left(\frac{\psi}{\nu^{1/2}}\right) \sqrt{1 + \frac{\psi^2}{l^6}}
\end{aligned}$$

Thus our quadratic term

$$\tau^2 \text{curv}\left(y^{2,0}, \zeta\right) \geq O\left(\frac{\psi^2}{\nu}\right)\left(1 + \frac{\psi^2}{l^6}\right).$$

This is much larger than $O\left(s^4 w_h^2 \nu\right)$, except if

$$\begin{aligned}
\psi^2 &\leq O\left(s^4 w_h^2 \nu^2\right), \text{ or} \\
\psi &\leq O\left(s^2 w_h \nu\right)
\end{aligned}$$

Since

$$\frac{\partial}{\partial t}\psi = O\left(1\right),$$

on $\left[0, O\left(\nu\right)\right]$, the exceptional region is when $t \in \left[0, O\left(s^2 w_h \nu\right)\right]$.

On this region, we see from Proposition 14.2 that

$$\begin{aligned} |\psi D_\zeta D_\zeta \psi| &\leq 7\frac{\sin^2 2t}{\nu^2} \\ &\leq \frac{O\left(s^2 w_h \nu\right)^2}{\nu^2} \\ &= O\left(s^4 w_h^2\right) \end{aligned}$$

So the absolute value of our minimum in (12.4) is

$$\leq O\left(s^4 w_h^2\right)^2 = O\left(s^8 w_h^4\right)$$

and the integral of our minimum over this exceptional region is

$$O\left(s^{10} w_h^5 \nu\right).$$

So with an extremely small adjustment to ι, we can dominate the negative term 12.4 even on this exceptional region.

13. Higher order computations

In this section we prove Theorems 12.3 and 12.4, and so (modulo the appendix) complete the proof that the Gromoll-Meyer sphere admits positive curvature. To do this we think of the lift of $T\Sigma^7$ to $TSp(2)$ as split into

$$\operatorname{span}\{\zeta\} \oplus \operatorname{span}\left\{y^{2,0}\right\} \oplus \operatorname{span}\left\{\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right\} \oplus \operatorname{span}\{W\} \oplus \operatorname{span}\left\{(V_1 \oplus V_2)^{\perp,W}\right\}.$$

Since $z \in H_{2,-1}$, it can only be in either the second or the third factor, whereas the perturbation vector V can be in any but the ζ or W factors.

We divide our computations accordingly. So we have five cases to consider

$$\begin{aligned} z &= y^{2,0},\ V \in (V_1 \oplus V_2)^{\perp,W} \\ z, V &\in \operatorname{span}\left\{\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right\} \\ z &= y^{2,0},\ V \in \operatorname{span}\left\{\eta_1^{2,0}, \eta_2^{2,0}\right\}, \\ z &\in \operatorname{span}\left\{\eta_1^{2,0}, \eta_2^{2,0}\right\},\ V = y^{2,0} \\ z &\in \operatorname{span}\left\{\eta_1^{2,0}, \eta_2^{2,0}\right\},\ V \in (V_1 \oplus V_2)^{\perp,W}. \end{aligned}$$

Some sectional curvature terms occur in more than one of these cases. So to simplify the accounting we handle the possible sectional curvatures in the first subsection. These are those that occur as quadratic or quartic coefficients of $P^{\text{diff,big}}(\sigma, \tau)$ in each of these five cases.

We also need the $\sigma\tau$, $\sigma\tau^2$, and $\sigma^2\tau$ coefficients of $P^{\text{diff,big}}(\sigma, \tau)$. These are computed on a case by case basis in the last four subsections. (The third and fourth case are handled as one in the third subsection.)

13.1. **Sectional Curvatures.** Letting $\mathcal{V}$ be a vector in $(V_1 \oplus V_2)^{\perp,W}$, the (unnormalized) sectional curvatures that we need are

$$\begin{aligned} &\operatorname{curv}^{\text{diff}}(\zeta, \mathcal{V}), \operatorname{curv}^{\text{diff}}\left(\zeta, \eta_u^{2,0}\right), \operatorname{curv}^{\text{diff}}\left(\zeta, y^{2,0}\right), \operatorname{curv}^{\text{diff}}\left(W, \eta_u^{2,0}\right), \operatorname{curv}^{\text{diff}}\left(W, y^{2,0}\right), \\ &\operatorname{curv}^{\text{diff}}\left(\eta_u^{2,0}, \mathcal{V}\right), \operatorname{curv}^{\text{diff}}\left(\eta_u^{2,0}, y^{2,0}\right), \operatorname{curv}^{\text{diff}}\left(\mathcal{V}, y^{2,0}\right), \operatorname{curv}^{\text{diff}}\left(\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right). \end{aligned}$$

The A–tensor term in 1.2 does not appear in the curvatures of two horizontal vectors, and the $s^2\text{curv}^{S^4}$ term and all of the terms of the partial conformal change are also small on these planes. Thus

Proposition 13.1. *The curvatures*

$$\text{curv}^{\text{diff,big}}\left(\zeta, y^{2,0}\right), \text{curv}^{\text{diff,big}}\left(\zeta, \eta_u^{2,0}\right), \text{curv}^{\text{diff,big}}\left(\eta_u^{2,0}, y^{2,0}\right), \text{curv}^{\text{diff,big}}\left(\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right)$$

are 0.

Proposition 13.2. *For* $\mathcal{V} \in (V_1 \oplus V_2)^{\perp,W}$

$$\begin{aligned}
\text{curv}^{\text{diff,big}}\left(W, y^{2,0}\right) &= s^2 w_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right) + O, \\
\text{curv}^{\text{diff,big}}\left(\mathcal{V}, y^{2,0}\right) &= s^2 v_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,V}^{2,0}\right)
\end{aligned}$$

Proof. The iterated A–tensor term is small because $S_y\left(k_\gamma\right)$ is small. Similarly, the partial conformal change is small because the various y–derivatives of ψ are small. The S^4–term gives the leading contribution so

$$\begin{aligned}
\text{curv}^{\text{diff,big}}\left(W, y^{2,0}\right) &= s^2 w_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,W}^{2,0}\right) + O \text{ and} \\
\text{curv}^{\text{diff,big}}\left(\mathcal{V}, y^{2,0}\right) &= s^2 w_h^2 \psi^2 \text{curv}^{S^4}\left(y^{2,0}, \eta_{u,V}^{2,0}\right) + O.
\end{aligned}$$

□

Proposition 13.3.

$$\text{curv}^{\text{diff,big}}\left(\zeta, \mathcal{V}\right) = 0$$

Proof. This computation looks like the computation of $\text{curv}(\zeta, W)$. The A–tensor, S^4–curvature, and $-|V|^2 \text{hess}_f\left(\zeta, \zeta\right)$ terms can all be large, but to leading order they cancel each other out. □

Proposition 13.4. *For* $V \in V_1 \oplus V_2$

$$R^{\text{diff,big}}\left(\eta_u^{2,0}, W, W, \eta_u^{2,0}\right) \geq e^{2f} w_h^2 s^2 \left|\text{grad}\,\psi\right|^2 \left(1 - \left\langle \eta_{u,W}^{2,0}, \eta_u^{2,0} \right\rangle^2\right)$$

$$R^{\text{diff,big}}\left(\eta_u^{2,0}, V, V, \eta_u^{2,0}\right) \geq e^{2f} w_h^2 s^2 \left|\text{grad}\,\psi\right|^2 \left(1 - \left\langle \eta_{u,V}^{2,0}, \eta_u^{2,0} \right\rangle^2\right)$$

Proof. The two inequalities have similar proofs, so we just focus on the first.

$$R^s\left(\eta_u^{2,0}, W, W, \eta_u^{2,0}\right) = R^{\nu,re,l}\left(\eta_u^{2,0}, W, W, \eta_u^{2,0}\right) + s^2 R^{S^4}\left(\eta_u^{2,0}, H, H, \eta_u^{2,0}\right) - s^2\left(1 - s^2\right)\left|A_{\eta_u^{2,0}} W^v\right|^2$$

We have

$$\begin{aligned}
A_{\eta_u^{2,0}} W^v &= \frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} - \text{II}\left(\eta_u^{2,0}, W^{\mathcal{H}}\right) + 4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|\left(\eta_u^{2,0}\right)^{\perp} \\
&= \frac{O\left(1 + \frac{t}{l^2} + \frac{t^2}{l^6}\right)}{\left|\cos 2t\eta^{2,0}\right|} + w_h \text{grad}\psi \left\langle \eta_{u,W}^{2,0}, \eta_u^{2,0} \right\rangle + 4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|_{h_2}\left(\eta_u^{2,0}\right)^{\perp}
\end{aligned}$$

So

$$s^2\left|A_{\eta_u^{2,0}} W^v\right|^2 = s^2 O\left(\frac{\psi^4}{\nu^6}\left|W_\alpha\right|_{h_2}^2\right) + s^2 w_h^2 \left|\text{grad}\psi\right|^2 \left\langle \eta_{u,W}^{2,0}, \eta_u^{2,0} \right\rangle^2 + O$$

Since

$$\operatorname{curv}\left(\eta_u^{2,0}, W\right) \geq \frac{\psi^2}{\nu^4}\left|W_\alpha\right|_{h_2}^2,$$

we can bound the first term of the A–tensor by

$$s^2\frac{\psi^2}{\nu^2}\operatorname{curv}\left(\eta_u^{2,0}, W\right) \geq s^2 O\left(\frac{\psi^4}{\nu^6}\left|W_\alpha\right|_{h_2}^2\right).$$

The second term in our expression for $s^2\left|A_{\eta_u^{2,0}}W^v\right|^2$ compares well with the $-\left|W\right|^2\operatorname{Hess}_f\left(\eta_u^{2,0},\eta_u^{2,0}\right)$ term from the partial conformal change indeed

$$\begin{aligned}
-\left|W\right|^2\operatorname{Hess}_f\left(\eta_u^{2,0},\eta_u^{2,0}\right) &= -\left|W\right|^2\left\langle\nabla_{\eta_u^{2,0}}\operatorname{grad} f, \eta_u^{2,0}\right\rangle \\
&= \left|W\right|^2\frac{s^2}{\nu^2}\left\langle\nabla_{\eta_u^{2,0}}\left(\psi\operatorname{grad}\psi\right), \eta_u^{2,0}\right\rangle + O \\
&= -\left|W\right|^2\frac{s^2}{\nu^2}\left\langle\left(\psi\operatorname{grad}\psi\right), \nabla_{\eta_u^{2,0}}\eta_u^{2,0}\right\rangle + O \\
&= \left|W\right|^2\frac{s^2}{\nu^2}\left|\operatorname{grad}\psi\right|^2 + O \\
&= w_h^2 s^2\left|\operatorname{grad}\psi\right|^2 + O,
\end{aligned}$$

So combining displays we have

$$\begin{aligned}
R^{\text{new}}\left(\eta_u^{2,0}, W, W, \eta_u^{2,0}\right) \geq\ & R^{\nu,re,l}\left(\eta_u^{2,0}, W, W, \eta_u^{2,0}\right) + s^2 R^{S^4}\left(\eta_u^{2,0}, H, H, \eta_u^{2,0}\right) - s^2\operatorname{curv}\left(\eta_u^{2,0}, W\right) \\
&+ w_h^2 s^2\left|\operatorname{grad}\psi\right|^2\left(1 - \left\langle\eta_{u,W}^{2,0}, \eta_u^{2,0}\right\rangle^2\right) + O
\end{aligned}$$

so

$$R^{\text{diff,big}}\left(\eta_u^{2,0}, W, W, \eta_u^{2,0}\right) \geq w_h^2 s^2\left|\operatorname{grad}\psi\right|^2\left(1 - \left\langle\eta_{u,W}^{2,0}, \eta_u^{2,0}\right\rangle^2\right)$$

as claimed. □

13.2. $z = y^{2,0}$, $V \in V_1 \oplus V_2$.

Proposition 13.5. *If V is in $(V_1 \oplus V_2)$ and the h_2–part of V is perpendicular to W_γ, then*

$$\begin{aligned}
\left|\left\langle R^{\text{diff}}\left(W,\zeta\right)y^{2,0}, V\right\rangle\right| &\leq D_{y^{2,0}}\left(\psi\right)O\left(s^2 w_h\right) + D_\zeta\left(\psi\right)O\left(s^2 v_h\right) + O\left(\frac{s^2}{l^3}\right) \\
&\leq \varkappa\left(s\right) \\
\left|\left\langle R^{\text{diff}}\left(W, y^{2,0}\right)\zeta, V\right\rangle\right| &\leq D_\zeta\left(\psi\right)O\left(s^2 w_h\right) \leq \varkappa\left(s\right)
\end{aligned}$$

and

$$\begin{aligned}
\left|\left\langle R^{\text{diff}}\left(W, y^{2,0}\right)y^{2,0}, V\right\rangle\right| &\leq D_{y^{2,0}}\left(\psi\right)O\left(s^2\left(w_h + v_h\right)\right) + O\left(\frac{s^2}{l^3}\right) \\
&= \varkappa\left(s\right)
\end{aligned}$$

$$\left|\left\langle R^{\text{diff}}\left(V, y^{2,0}\right)\zeta, V\right\rangle\right| \leq \varkappa\left(s\right)$$

In particular, for all four curvatures $R^{\text{diff,big}} = 0$.

Proof. To find the effect of shrinking the fibers we use equations 1.2 and get

$$\begin{aligned}
R^s\left(W,\zeta\right)y^{2,0} &= R^s\left(W^{\mathcal{V}},\zeta\right)y^{2,0}+R^s\left(W^{\mathcal{H}},\zeta\right)y^{2,0}\\
&= \left(1-s^2\right)R^{\nu,re,l}(W^{\mathcal{V}},\zeta)y^{2,0}+s^2\left(R^{\nu,re,l}(W^{\mathcal{V}},\zeta)y^{2,0}\right)^{\mathcal{V}}+s^2A_{\zeta}A_{y^{2,0}}W^{\mathcal{V}}\\
&\quad+\left(1-s^2\right)R^{\nu,re,l}(W^{\mathcal{H}},\zeta)y^{2,0}+s^2\left(R^{\nu,re,l}(W^{\mathcal{H}},\zeta)y^{2,0}\right)^{\mathcal{V}}+s^2R^{S^4}(W^{\mathcal{H}},\zeta)y^{2,0}\\
&= \left(1-s^2\right)\left(R^{\nu,re,l}\left(W,\zeta\right)y^{2,0}\right)^{\mathcal{H}}+\left(R^{\nu,re,l}\left(W,\zeta\right)y^{2,0}\right)^{\mathcal{V}}\\
&\quad+s^2A_{\zeta}A_{y^{2,0}}W^{\mathcal{V}}+s^2R^{S^4}(W^{\mathcal{H}},\zeta)y^{2,0}
\end{aligned}$$

Similarly

$$\begin{aligned}
R^s\left(W,y^{2,0}\right)\zeta &= \left(1-s^2\right)\left(R^{\nu,re,l}\left(W,y^{2,0}\right)\zeta\right)^{\mathcal{H}}+\left(R^{\nu,re,l}\left(W,y^{2,0}\right)\zeta\right)^{\mathcal{V}}\\
&\quad+s^2A_{y^{2,0}}A_{\zeta}W^{\mathcal{V}}+s^2R^{S^4}(W^{\mathcal{H}},y^{2,0})\zeta,\text{ and}
\end{aligned}$$

$$\begin{aligned}
R^s\left(W,y^{2,0}\right)y^{2,0} &= \left(1-s^2\right)\left(R^{\nu,re,l}\left(W,y^{2,0}\right)y^{2,0}\right)^{\mathcal{H}}+\left(R^{\nu,re,l}\left(W,y^{2,0}\right)y^{2,0}\right)^{\mathcal{V}}\\
&\quad+s^2A_{y^{2,0}}A_{y^{2,0}}W^{\mathcal{V}}+s^2R^{S^4}\left(W^{\mathcal{H}},y^{2,0}\right)y^{2,0}.
\end{aligned}$$

Since A^{h_2}–induces an $SO\left(3\right)$–action on S^4 that is standard on the S^2_{Im}s and leaves ζ and $y^{2,0}$ invariant, the restriction to TS^2_{Im} of the compositions of orthogonal projection to TS^2_{Im} with any of $R^{S^4}\left(\cdot,y^{2,0}\right)y^{2,0}$, $R^{S^4}(\cdot,\zeta)y^{2,0}$, or $R^{S^4}(\cdot,y^{2,0})\zeta$ are homotheties. In particular, for V in $(V_1\oplus V_2)\cap H^{GM}$ with the h_2–part of V perpendicular to W_γ, we have

$$\begin{aligned}
\left\langle R^{S^4}\left(W^{\mathcal{H}},y^{2,0}\right)y^{2,0},V^{\mathcal{H}}\right\rangle &= 0,\\
\left\langle R^{S^4}(W^{\mathcal{H}},\zeta)y^{2,0},V^{\mathcal{H}}\right\rangle &= 0,\text{ and}\\
\left\langle R^{S^4}(W^{\mathcal{H}},y^{2,0})\zeta,V^{\mathcal{H}}\right\rangle &= 0.
\end{aligned}$$

For V in $(V_1\oplus V_2)$ we use Lemma 9.2 to see that

$$\begin{aligned}
\left|s^2\left\langle A_{\zeta}A_{y^{2,0}}W^{\mathcal{V}},V\right\rangle\right| &= \left|s^2\left\langle A_{y^{2,0}}W^{\mathcal{V}},A_{\zeta}V\right\rangle\right|\\
&= \left|s^2\left\langle\left(\nabla^{\nu,re,l}_{y^{2,0}}W\right)^{\mathcal{H}}-S_{y^{2,0}}\left(W^{\mathcal{H}}\right),\left(\nabla^{\nu,re,l}_{\zeta}V\right)^{\mathcal{H}}-S_{\zeta}\left(V^{\mathcal{H}}\right)\right\rangle\right|\\
&= \left|s^2\left\langle\left(\nabla^{\nu,re,l}_{y^{2,0}}W\right)^{\mathcal{H}}-w_hD_{y^{2,0}}\left(\psi\right)\frac{k_{\gamma,W}}{\psi},\left(\nabla^{\nu,re,l}_{\zeta}V\right)^{\mathcal{H}}-v_hD_{\zeta}\left(\psi\right)\frac{k_{\gamma,V}}{\psi}\right\rangle\right|
\end{aligned}$$

Since the γ–part of V is perpendicular to the γ–part of W,

$$\begin{aligned}
\left|s^2\left\langle A_{\zeta}A_{y^{2,0}}W^{\mathcal{V}},V\right\rangle\right| &\leq s^2\left(\left|\left\langle\left(\nabla^{\nu,re,l}_{y^{2,0}}W\right)^{\mathcal{H}},v_hD_{\zeta}\left(\psi\right)\frac{k_{\gamma,V}}{\psi}\right\rangle\right|+\left|\left\langle w_hD_{y^{2,0}}\left(\psi\right)\frac{k_{\gamma,W}}{\psi},\left(\nabla^{\nu,re,l}_{\zeta}V\right)^{\mathcal{H}}\right\rangle\right|\right)\\
&\quad+s^2\left|\left\langle\left(\nabla^{\nu,re,l}_{y^{2,0}}W\right)^{\mathcal{H}},\left(\nabla^{\nu,re,l}_{\zeta}V\right)^{\mathcal{H}}\right\rangle\right|\\
&\leq D_{y^{2,0}}\left(\psi\right)O\left(s^2w_h\right)+D_{\zeta}\left(\psi\right)O\left(s^2v_h\right)+O\left(\frac{s^2}{l^4}\right)\\
&\leq \varkappa\left(s\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}\left|s^2\left\langle A_{y^{2,0}}A_\zeta W^{\mathcal{V}},V\right\rangle\right| &\leq D_\zeta(\psi)\,O\left(s^2w_h\right)+O\\ &\leq \varkappa(s)\,,\text{ and}\end{aligned}$$

$$\begin{aligned}\left|s^2\left\langle A_{y^{2,0}}A_{y^{2,0}}W^{\mathcal{V}},V\right\rangle\right| &\leq D_{y^{2,0}}(\psi)\,O\left(s^2w_h\right)+D_{y^{2,0}}(\psi)\,O\left(s^2v_h\right)+O\\ &\leq \varkappa(s)\,.\end{aligned}$$

(There are fewer terms in the estimate for $\left|s^2\left\langle A_{y^{2,0}}A_\zeta W^{\mathcal{V}},V\right\rangle\right|$ since $\nabla^{\nu,re,l}_\zeta W=0$.)

These three A–tensor inequalities give the first three inequalities after the fibers have been shrunken.

Combining this with our partial conformal change and Hessian formulas yields the first three results.

The final curvature is also small, but this fact is much subtler.

The A–tensor part give us

$$\begin{aligned}s^2\left\langle A_{y^{2,0}}A_\zeta V^{\mathcal{V}},V\right\rangle &= -s^2\left\langle A_\zeta V^{\mathcal{V}},A_{y^{2,0}}V^{\mathcal{V}}\right\rangle\\ &= -s^2\left\langle\left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}}-S_\zeta\left(V^{\mathcal{H}}\right),\left(\nabla^{\nu,re,l}_{y^{2,0}}V\right)^{\mathcal{H}}-S_{y^{2,0}}\left(V^{\mathcal{H}}\right)\right\rangle\\ &= -s^2\left\langle\left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}}-v_hD_\zeta(\psi)\frac{k_{\gamma,V}}{\psi},\left(\nabla^{\nu,re,l}_y V\right)^{\mathcal{H}}-v_hD_{y^{2,0}}(\psi)\frac{k_{\gamma,V}}{\psi}\right\rangle\\ &= -s^2v_h^2D_\zeta(\psi)\,D_{y^{2,0}}(\psi)-s^2v_h\left(D_\zeta(\psi)+D_{y^{2,0}}(\psi)\right)+O\\ &= -s^2v_h^2D_\zeta(\psi)\,D_{y^{2,0}}(\psi)+O\end{aligned}$$

The S^4–curvature gives us

$$\begin{aligned}s^2R^{S^4}\left(\zeta,V^{horiz},V^{horiz},y^{2,0}\right) &= -s^2\left|V^{horiz}\right|\left\langle\nabla_\zeta\mathrm{grad}\left|V^{horiz}\right|,y^{2,0}\right\rangle\\ &= -s^2v_h^2\psi\left\langle\nabla_\zeta\mathrm{grad}\psi,y^{2,0}\right\rangle\end{aligned}$$

Adding we get

$$\begin{aligned}R^{\mathrm{diff},s}\left(\zeta,V^{horiz},V^{horiz},y^{2,0}\right) &= -s^2v_h^2\left\langle\nabla_\zeta\psi\mathrm{grad}\psi,y^{2,0}\right\rangle\\ &= \left|V^\gamma\right|^2\mathrm{hess}\left(\zeta,y^{2,0}\right)+O\end{aligned}$$

So this cancels with a hessian term from the partial conformal change. The other terms of the partial conformal change are small, so the result follows. □

13.3. $z,V\in\mathrm{span}\left\{\eta^{2,0}_{u,1},\eta^{2,0}_{u,2}\right\}$.

Proposition 13.6. *If $z,V\in\mathrm{span}\left\{\eta^{2,0}_{u,1},\eta^{2,0}_{u,2}\right\}$ and $|z|=|V|=1$, then*

$$R^{\mathrm{diff}}(\zeta,W,V,z)\,,\ R^{\mathrm{diff}}(\zeta,V,W,z)\,,\ \text{and } R^{\mathrm{diff}}(\zeta,V,V,z)$$

are all 0.

$$R^{\mathrm{diff}}(z,V,W,z)=\langle V,W\rangle\left\langle z,\eta^{2,0}_{u,W^\perp}\right\rangle\left(s^2\mathrm{curv}^{S^4}\left(\eta^{2,0}_{W^\perp},\eta^{2,0}_W\right)-\frac{s^2}{\nu^2}\left|\mathrm{grad}\psi\right|^2\right)+O$$

Remark 13.7. *For generic t, $curv^{\nu,re,l}\left(\zeta,\eta^{2,0}_u\right)=O\left(\nu^2\right)$, so it is important to have pretty tight estimates $R^{\mathrm{diff}}(\zeta,W,V,z)$, $R^{\mathrm{diff}}(\zeta,V,W,z)$, and $R^{\mathrm{diff}}(\zeta,V,V,z)$.*

Proof. In all four cases the A–tensor term is 0 because at least three of the vectors are horizontal.

In the first three cases all other terms of R^{diff} are also 0.

The S^4–curvature term is 0 because of the fact that three of the vectors are in $\text{span}\left\{\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right\}$ and one of the vectors is ζ. Hessian terms are all 0 because the hessian of ζ with each of the other three vectors is 0 and ζ is perpendicular to each of the other three vectors. The derivative terms of the partial conformal change are all 0 because the directional derivative of f in each of the directions W, V, and z is 0 and because ζ is perpendicular to each of the other three vectors.

The error component

$$+O\left(e^{2f}-1, |\text{grad} f|\right) \max\left\{R^{\text{old}}\left(X,Y,Z,U\right), |X|\,|Y|\,|Z|\,|U|\right\}$$

of Proposition 10.9 is also 0.

This is because

the Lie brackets of all of $z, V,$ and W with ζ have no $\Delta\left(\alpha\right)-$component,

the Lie brackets of $\left(N\alpha p, N\alpha\right)$ with each of $z, V,$ and W have no ζ–component, and

the Lie bracket of ζ and $\left(N\alpha p, N\alpha\right)$ is 0.

For the last curvature, we note that only the components of z that are perpendicular to V and W can make a contribution.

The first term comes from the S^4–curvature via the s–perturbation and the second term comes from

$$\begin{aligned}
-\left\langle V,W\right\rangle \text{hess}\left(z,z\right) &= -\left\langle V,W\right\rangle \text{hess}\left(\eta_{u,W^{\perp}}^{2,0}, \eta_{u,W^{\perp}}^{2,0}\right) \\
&= -\left\langle V,W\right\rangle \frac{s^2}{\nu^2}\left|\text{grad}\psi\right|\psi\frac{\left|\text{grad}\psi\right|}{\psi}+O \\
&= -\left\langle V,W\right\rangle \frac{s^2}{\nu^2}\left|\text{grad}\psi\right|^2+O
\end{aligned}$$

There are other nonzero terms that come from the partial conformal change, but they are much smaller. □

Since

$$\begin{aligned}
\text{curv}\left(\eta_{W^{\perp}}^{2,0}, W\right) &= \frac{1}{\left|\cos 2t\eta^{2,0}\right|^2}+\frac{\psi^2}{\nu^6} \\
\text{curv}\left(\eta_{W^{\perp}}^{2,0}, \eta_{W}^{2,0}\right) &= \frac{1}{\left|\cos 2t\eta^{2,0}\right|^4}+\frac{\psi^4}{\nu^6}
\end{aligned}$$

we have in any case that

Proposition 13.8. *If* $z, V \in \text{span}\left\{\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right\}$, *then*

$$R^{\text{diff,big}}\left(\zeta,W,V,z\right) = R^{\text{diff,big}}\left(\zeta,V,W,z\right) = R^{\text{diff,big}}\left(\zeta,V,V,z\right) = R^{\text{diff,big}}\left(z,V,W,z\right) = 0$$

13.4. $z = y^{2,0}$, $V \in \text{span}\left\{\eta_1^{2,0}, \eta_2^{2,0}\right\}$ **or** $z \in \text{span}\left\{\eta_1^{2,0}, \eta_2^{2,0}\right\}$, $V = y^{2,0}$**.**

Proposition 13.9.

$$\begin{aligned}
R^{\text{diff,big}}\left(\zeta,\eta_u^{2,0},W,y^{2,0}\right) &= R^{\text{diff,big}}\left(\zeta,W,\eta_u^{2,0},y^{2,0}\right)\\
&= s^2 w_h\psi\text{curv}^{S^4}\left(y^{2,0},\eta_u^{2,0}\right)\left\langle y^{2,0},\zeta\right\rangle\left\langle \eta_{u,W}^{2,0},\eta_u^{2,0}\right\rangle + O\\
R^{\text{diff,big}}\left(\zeta,y^{2,0},W,\eta_u^{2,0}\right) &= 0\\
\left\langle R^{\text{diff,big}}\left(W,y^{2,0}\right)y^{2,0},\eta_u^{2,0}\right\rangle &= s^2 w_h\psi\text{curv}^{S^4}\left(y^{2,0},\eta_{u,W}^{2,0}\right)\left\langle \eta_{u,W}^{2,0},\eta_u^{2,0}\right\rangle\\
\left\langle R^{\text{diff}}\left(\eta_u^{2,0},\zeta\right)y^{2,0},\eta_u^{2,0}\right\rangle &= \left\langle R^{\text{diff}}\left(W,\eta_u^{2,0},\eta_u^{2,0}\right),y^{2,0}\right\rangle = \left\langle R^{\text{diff}}\left(y^{2,0},\zeta,\eta_u^{2,0}\right),y^{2,0}\right\rangle = 0
\end{aligned}$$

Proof. Each of the curvatures involves at most one vector that is not horizontal, so the A–tensor contribution from the s–perturbation is 0. For the first two curvatures, the S^4 term gives us

$$\begin{aligned}
\left\langle R^{\text{s}}\left(W,\zeta\right)y^{2,0},\eta_u^{2,0}\right\rangle &= \left\langle R^{\text{s}}\left(W,y^{2,0}\right)\zeta,\eta_u^{2,0}\right\rangle\\
&= s^2\left\langle W,\eta_u^{2,0}\right\rangle\text{curv}^{S^4}\left(y^{2,0},\eta_u^{2,0}\right)\left\langle y^{2,0},\zeta\right\rangle\\
&= s^2 w_h\psi\text{curv}^{S^4}\left(y^{2,0},\eta_u^{2,0}\right)\left\langle y^{2,0},\zeta\right\rangle\left\langle \eta_{u,W}^{2,0},\eta_u^{2,0}\right\rangle.
\end{aligned}$$

For the third curvature $R^{S^4}\left(\zeta,y^{2,0},W,\eta_u^{2,0}\right)=0$. So $R^{s}\left(\zeta,y^{2,0},W,\eta_u^{2,0}\right)=0$. Similarly

$$\begin{aligned}
\left\langle R^{\text{s}}\left(W,y^{2,0}\right)y^{2,0},\eta_u^{2,0}\right\rangle &= \left\langle R^{\nu,re,l}\left(W,y^{2,0}\right)y^{2,0},\eta_u^{2,0}\right\rangle + s^2 w_h\psi\text{curv}^{S^4}\left(y^{2,0},\eta_u^{2,0}\right)\left\langle \eta_{u,W}^{2,0},\eta_u^{2,0}\right\rangle, \text{ and}\\
\left\langle R^{\text{s}}\left(\eta_u^{2,0},\zeta\right)y^{2,0},\eta_u^{2,0}\right\rangle &= s^2\text{curv}^{S^4}\left(y^{2,0},\eta_u^{2,0}\right)\left\langle y^{2,0},\zeta\right\rangle = O, \text{ and}\\
\left\langle R^{\text{s}}\left(W,\eta_u^{2,0},\eta_u^{2,0}\right),y^{2,0}\right\rangle &= \left\langle R^{\text{s}}\left(y^{2,0},\zeta,\eta_u^{2,0}\right),y^{2,0}\right\rangle = 0
\end{aligned}$$

Combining these computations with our partial conformal change and Hessian formulas yields the result. □

13.5. $z\in\text{span}\left\{\eta_1^{2,0},\eta_2^{2,0}\right\}$, $V\in V_1\oplus V_2$.

Proposition 13.10. *For $V\in V_1\oplus V_2$ with V_2–component perpendicular to the γ–part of W and normalized so that $|V|_{h_2}=O\left(\frac{1}{\nu}\right)$*

$$\left|\left\langle R^{s}\left(W,\zeta\right)\eta_u^{2,0},V\right\rangle\right| \le s^2 v_h\left|\text{grad}\psi\right|\frac{\psi}{\nu}\sqrt{\text{curv}^{\nu,l}\left(\eta_u^{2,0},W_\alpha\right)} + O$$

$$\left|\left\langle R^{s}\left(W,\eta_u^{2,0}\right)\zeta,V\right\rangle\right| \le s^2 w_h D_\zeta\left[\psi\right]O\left(\frac{\psi}{\nu}\sqrt{\text{curv}^{\nu,l}\left(\eta_u^{2,0},W\right)}\right) + O$$

Proof.

$$\begin{aligned}
R^{s}\left(W,\zeta\right)\eta_u^{2,0} &= R^{s}\left(W^{\mathcal{V}},\zeta\right)\eta_u^{2,0} + R^{s}\left(W^{\mathcal{H}},\zeta\right)\eta_u^{2,0}\\
&= \left(1-s^2\right)R^{\nu,re,l}(W^{\mathcal{V}},\zeta)\eta_u^{2,0} + s^2\left(R^{\nu,re,l}(W^{\mathcal{V}},\zeta)\eta_u^{2,0}\right)^{\mathcal{V}} + s^2 A_\zeta A_{\eta_u^{2,0}}W^{\mathcal{V}}\\
&\quad + \left(1-s^2\right)R^{\nu,re,l}(W^{\mathcal{H}},\zeta)\eta_u^{2,0} + s^2\left(R^{\nu,re,l}(W^{\mathcal{H}},\zeta)\eta_u^{2,0}\right)^{\mathcal{V}} + s^2 R^{S^4}(W^{\mathcal{H}},\zeta)\eta_u^{2,0}\\
&= \left(1-s^2\right)\left(R^{\nu,re,l}\left(W,\zeta\right)\eta_u^{2,0}\right)^{\mathcal{H}} + \left(R^{\nu,re,l}\left(W,\zeta\right)\eta_u^{2,0}\right)^{\mathcal{V}}\\
&\quad + s^2 A_\zeta A_{\eta_u^{2,0}}W^{\mathcal{V}} + s^2 R^{S^4}(W^{\mathcal{H}},\zeta)\eta_u^{2,0}
\end{aligned}$$

As before we have

$$\left\langle R^{S^4}(W^{\mathcal{H}},\zeta)\eta_u^{2,0},V\right\rangle = 0.$$

$$\begin{aligned} A_{\eta_u^{2,0}} W^{\mathcal{V}} &= -\mathrm{II}\left(\eta_u^{2,0}, W^{\mathcal{H}}\right) + \frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} + 4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|_{h_2}\left(\eta_u^{2,0}\right)^{\perp} \\ &= w_h \mathrm{grad}\psi \left\langle \eta_u^{2,0}, \frac{W^{\mathcal{H}}}{\left|W^{\mathcal{H}}\right|} \right\rangle + \frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} + 4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|_{h_2}\left(\eta_u^{2,0}\right)^{\perp} \end{aligned}$$

where $\left(\eta_u^{2,0}\right)^{\perp}$ is the spherical combination of $\mathrm{span}\left\{\eta_{u,1}^{2,0}, \eta_{u,2}^{2,0}\right\}$ that's perpendicular to $\eta_u^{2,0}$.

To estimate the last term note

$$\begin{aligned} \mathrm{curv}^{\nu,re,l}\left(W, \eta_u^{2,0}\right) &\geq 4\frac{\psi^2}{\nu^4}\left|W_\alpha\right|_{h_2}^2, \\ 2\frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(W, \eta_u^{2,0}\right)} &\geq \frac{\psi}{\nu} 4\frac{\psi}{\nu^2}\left|W_\alpha\right|_{h_2} \\ &= 4\frac{\psi^2}{\nu^3}\left|W_\alpha\right|_{h_2}. \end{aligned}$$

We estimate the middle term as

$$\frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left|\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}}\right| = \frac{1}{\left|\cos 2t\eta^{2,0}\right|} O\left(1 + \frac{t}{l^2} + \frac{t^2}{l^4}\right).$$

The $\frac{t}{l^2}$ and $\frac{t^2}{l^4}$ terms come from differentiating the S^3–factor of $\hat{W}$ in $\left(S^3\right)^2 \times Sp(2)$. The $\frac{t}{l^2}$ comes from the derivative in the $Sp(2)$ direction. The factor of t, is present because we are taking the horizontal part of the answer, and the entire horizontal space is perpendicular to the orbits of the (U, D)–action when $(\sin 2t, \sin 2\theta) = (0, 0)$. The $\frac{t^2}{l^4}$–factor comes from taking the derivative in the S^3–direction. The extra factor of t comes the fact that (η, η) is perpendicular to the orbits of the (U, D)–action when $(\sin 2t, \sin 2\theta) = (0, 0)$.

On the other hand,

$$A_\zeta V^{\mathcal{V}} = -v_h \frac{D_\zeta[\psi]}{\psi} k_V + \left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}}.$$

and if V has the usual normalization, then

$$\left|\left(\nabla^{\nu,re,l}_\zeta V\right)^{\mathcal{H}}\right| = O\left(1 + \frac{t}{l^2}\right).$$

So

$$\begin{aligned} & \left|s^2 \left\langle A_\zeta A_{\eta_u^{2,0}} W^{\mathcal{V}}, V^{\mathcal{V}} \right\rangle\right| \\ \leq\ & 2s^2 v_h \left|\mathrm{grad}\psi\right| \frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(W, \eta_u^{2,0}\right)} \\ & + s^2 v_h \frac{D_\zeta[\psi]}{\left|\cos 2t\eta^{2,0}\right|^2} O\left(1 + \frac{t}{l^2} + \frac{t^2}{l^4}\right) \\ & + s^2 w_h \mathrm{grad}\psi O\left(1 + \frac{t}{l^2}\right) + \frac{s^2}{\left|\cos 2t\eta^{2,0}\right|} O\left(1 + \frac{t}{l^2}\right) O\left(1 + \frac{t}{l^2} + \frac{t^2}{l^4}\right) \\ & + 2\frac{s^2}{\left|\cos 2t\eta^{2,0}\right|} O\left(1 + \frac{t}{l^2}\right) \frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(W, \eta_u^{2,0}\right)} \\ \leq\ & s^2 v_h \left|\mathrm{grad}\psi\right| \frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(W, \eta_u^{2,0}\right)} + \varkappa(s) + O \end{aligned}$$

It follows that

$$\left|\left\langle R^s\left(W,\zeta\right)\eta_u^{2,0},V\right\rangle\right| \leq s^2 v_h \left|\mathrm{grad}\psi\right| \frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0},W_\alpha\right)} + O$$

Similarly, since

$$A_\zeta W^{\mathcal{V}} = -w_h \frac{D_\zeta\left[\psi\right]}{\psi} k_\gamma$$

and

$$\begin{aligned} A_{\eta_u^{2,0}} V^{\mathcal{V}} &= \frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)}V\right)^{\mathcal{H}} - \mathrm{II}\left(\eta_u^{2,0},V^{\mathcal{H}}\right) + 4\frac{\psi^2}{\nu^3}\left|V_\alpha\right|_{h_2}\left(\eta_u^{2,0}\right)^\perp \\ &= \frac{1}{\left|\cos 2t\eta^{2,0}\right|} O\left(1+\frac{t}{l^2}+\frac{t^2}{l^4}\right) + v_h \mathrm{grad}\psi\left\langle \eta_u^{2,0}, \frac{V^{\mathcal{H}}}{\left|V^{\mathcal{H}}\right|}\right\rangle + 4\frac{\psi^2}{\nu^3}\left|V_\alpha\right|_{h_2}\left(\eta_u^{2,0}\right)^\perp \end{aligned}$$

where $\left(\eta_u^{2,0}\right)^\perp$ is the spherical combination of $\mathrm{span}\left\{\eta_{u,1}^{2,0},\eta_{u,2}^{2,0}\right\}$ that's perpendicular to $\eta_u^{2,0}$.

It follows that

$$\begin{aligned} &\left|s^2\left\langle A_{\eta_u^{2,0}} A_\zeta W^{\mathcal{V}}, V\right\rangle\right| \\ = &\left|s^2 w_h D_\zeta\left[\psi\right]\left(\frac{O\left(1+\frac{t}{l^2}+\frac{t^2}{l^4}\right)}{\left|\cos 2t\eta^{2,0}\right|} + 4\frac{\psi^2}{\nu^3}\left|V_\alpha\right|_{h_2}\left\langle \eta_{u,W}^{2,0},\left(\eta_u^{2,0}\right)^\perp\right\rangle\right)\right| \end{aligned}$$

The first term is too small to matter. It is natural to control the second term in terms of $\mathrm{curv}^{\nu,re,l}\left(V^{h_2,\alpha},\eta_u^{2,0}\right)$, however, since this a mixed quadratic term, it is much nicer subsequently if we can control it in terms of $\mathrm{curv}^{\nu,re,l}\left(W,\eta_u^{2,0}\right)$. To do this we need to use our normalization $\left|V\right|_{h_2} = O\left(\frac{1}{\nu}\right)$.

Since

$$\mathrm{curv}^{\nu,re,l}\left(W,\eta_u^{2,0}\right) \geq 4\frac{\psi^2}{\nu^6}\left\langle \eta_{u,W}^{2,0},\left(\eta_u^{2,0}\right)^\perp\right\rangle,$$

we have

$$\begin{aligned} \frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(W,\eta_u^{2,0}\right)} &\geq \frac{\psi}{\nu} 2\left(\frac{\psi}{\nu^3}\left\langle \eta_{u,W}^{2,0},\left(\eta_u^{2,0}\right)^\perp\right\rangle^{1/2}\right) \\ &= O\left(\frac{\psi^2}{\nu^3}\left|V_\alpha\right|_{h_2}\right)\left\langle \eta_{u,W}^{2,0},\left(\eta_u^{2,0}\right)^\perp\right\rangle^{1/2} \end{aligned}$$

It follows that

$$\left|\left\langle R^s\left(W,\eta_u^{2,0}\right)\zeta,V\right\rangle\right| \leq s^2 w_h D_\zeta\left[\psi\right] O\left(\frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(W,\eta_u^{2,0}\right)}\right) + O$$

□

Corollary 13.11.

$$\begin{aligned} \left|\left\langle R^{\mathrm{diff,big}}\left(W,\zeta\right)\eta_u^{2,0},V\right\rangle\right| &= 0 \\ \left|\left\langle R^{\mathrm{diff,big}}\left(W,\eta_u^{2,0}\right)\zeta,V\right\rangle\right| &= 0. \end{aligned}$$

Proposition 13.12. *For $V \in V_1 \oplus V_2$ with the V_2 part of V perpendicular to the γ–part of W*

$$\left|R^{\mathrm{diff,big}}\left(\zeta,V,V,\eta_u^{2,0}\right)\right| = 0$$

$$\left|R^{\text{diff,big}}\left(\eta_u^{2,0},W,V,\eta_u^{2,0}\right)\right|\leq\sqrt{\text{curv}^{\text{diff,big}}\left(\eta_u^{2,0},V\right)}\sqrt{\text{curv}^{\text{diff,big}}\left(\eta_u^{2,0},W\right)}$$

Proof. We will use

$$A_\zeta V=-v_h\frac{D_\zeta(\psi)}{\psi}k_{\gamma,V}+\left(\nabla_\zeta^{\nu,re,l}V\right)^{\mathcal{H}}$$

and

$$A_{\eta_u^{2,0}}V=\frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left(\nabla_{(\eta,\eta)}^{\nu,re,l}V\right)^{\mathcal{H}}-\text{II}\left(\eta_u^{2,0},V^{\mathcal{H}}\right)+O\left(\frac{\psi^2}{\nu^3}\left|V_\alpha\right|\right)\left(\eta_u^{2,0}\right)^\perp.$$

If V has the usual normalization, and we estimate as in the previous proof we get

$$\left(\nabla_\zeta^{\nu,re,l}V\right)^{\mathcal{H}}=O\left(1+\frac{t}{l^2}\right)\text{ and}$$

$$\frac{1}{\left|\cos 2t\eta^{2,0}\right|}\left|\left(\nabla_{(\eta,\eta)}^{\nu,re,l}V\right)^{\mathcal{H}}\right|=\frac{1}{\left|\cos 2t\eta^{2,0}\right|}O\left(1+\frac{t}{l^2}+\frac{t^2}{l^4}\right).$$

So

$$\begin{aligned}&\left|s^2\left\langle A_\zeta V_\gamma,A_{\eta_u^{2,0}}V_\gamma\right\rangle\right|\\ \leq\;& s^2v_hD_\zeta(\psi)\frac{O\left(1+\frac{t}{l^2}+\frac{t^2}{l^4}\right)}{\left|\cos 2t\eta^{2,0}\right|}+s^2v_hD_\zeta(\psi)O\left(\frac{\psi^2}{\nu^3}\left|V_\alpha\right|\right)\left\langle\eta_{u,V}^{2,0},\left(\eta_u^{2,0}\right)^\perp\right\rangle\\ &+s^2v_h\left|\text{grad}\psi\right|O\left(1+\frac{t}{l^2}\right)+s^2O\left(\frac{\psi^2}{\nu^3}\left|V_\alpha\right|\right)O\left(1+\frac{t}{l^2\left|\cos 2t\eta^{2,0}\right|}\right)\\ &+\frac{s^2}{\left|\cos 2t\eta^{2,0}\right|}O\left(1+\frac{t}{l^2}+\frac{t^2}{l^4}\right)O\left(1+\frac{t}{l^2}\right)\end{aligned}$$

where the extra factor of $\frac{1}{\left|\cos 2t\eta^{2,0}\right|}$ in the $\frac{t}{l^2\left|\cos 2t\eta^{2,0}\right|}$ part of the fourth term comes from the fact that we would be taking the component of $\nabla_\zeta^{\nu,re,l}V$ in span $\left\{\eta_{1,u}^{2,0},\eta_{2,u}^{2,0}\right\}$. It follows that

$$\left|s^2\left\langle A_\zeta V_\gamma,A_{\eta_u^{2,0}}V_\gamma\right\rangle\right|\leq s^2v_hD_\zeta(\psi)O\left(\frac{\psi^2}{\nu^3}\left|V_\alpha\right|\right)\left\langle\eta_{u,V}^{2,0},\left(\eta_u^{2,0}\right)^\perp\right\rangle+O.$$

Since V has the usual normalization,

$$\text{curv}^{\nu,re,l}\left(\eta_u^{2,0},V\right)\geq\frac{\psi^2}{\nu^4}\left|V_\alpha\right|_{h_2}^2$$

So

$$\begin{aligned}s^2v_h\frac{\psi}{\nu}\sqrt{\text{curv}^{\nu,re,l}\left(\eta_u^{2,0},V\right)}&\geq\left(s^2v_h\frac{\psi}{\nu}\right)\frac{\psi}{\nu^2}\left|V_\alpha\right|_{h_2}\\ &=s^2v_h\frac{\psi^2}{\nu^3}\left|V_\alpha\right|\\ &\geq\left|s^2\left\langle A_\zeta V_\gamma,A_{\eta_u^{2,0}}V_\gamma\right\rangle\right|\end{aligned}$$

So

$$\left|R^{\text{diff},s}\left(\zeta,V,V,\eta_u^{2,0}\right)\right|\leq\chi(s)\sqrt{\text{curv}^{\nu,re,l}\left(\eta_u^{2,0},V\right)}+O$$

where

$$R^{\text{diff},s}=R^s-R^{\nu,re,l}.$$

The partial conformal change does not add anything nearly this large so we have proven the first statement.

To find $R^{\text{diff},s}\left(\eta_u^{2,0}, W, V, \eta_u^{2,0}\right)$ we use

$$\begin{aligned} A_{\eta_u^{2,0}} W &= \frac{1}{|\cos 2t\eta^{2,0}|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} - \mathrm{II}\left(\eta_u^{2,0}, W^{\mathcal{H}}\right) + O\left(\frac{\psi^2}{\nu^3}|W_\alpha|_{h_2}\right)\left(\eta_u^{2,0}\right)^{\perp} \\ &= \frac{1}{|\cos 2t\eta^{2,0}|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} W\right)^{\mathcal{H}} + \frac{\mathrm{grad}\psi}{\psi}\left\langle W, \eta_u^{2,0}\right\rangle + O\left(\frac{\psi^2}{\nu^3}|W_\alpha|_{h_2}\right)\left(\eta_u^{2,0}\right)^{\perp} \end{aligned}$$

and

$$\begin{aligned} A_{\eta_u^{2,0}} V &= \frac{1}{|\cos 2t\eta^{2,0}|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} V\right)^{\mathcal{H}} - \mathrm{II}\left(\eta_u^{2,0}, V^{\mathcal{H}}\right) + O\left(\frac{\psi^2}{\nu^3}|V_\alpha|_{h_2}\right)\left(\eta_u^{2,0}\right)^{\perp} \\ &= \frac{1}{|\cos 2t\eta^{2,0}|}\left(\nabla^{\nu,re,l}_{(\eta,\eta)} V\right)^{\mathcal{H}} + \frac{\mathrm{grad}\psi}{\psi}\left\langle V, \eta_u^{2,0}\right\rangle + O\left(\frac{\psi^2}{\nu^3}|V_\alpha|_{h_2}\right)\left(\eta_u^{2,0}\right)^{\perp}. \end{aligned}$$

Letting γ_1, γ_2, and γ_3 be the unit γ–quaternions corresponding to W, V, and $\eta_u^{2,0}$,we have

$$\begin{aligned} \left|s^2\left\langle A_{\eta_u^{2,0}} W, A_{\eta_u^{2,0}} V\right\rangle\right| &= s^2\left(w_h v_h\right)|\mathrm{grad}\psi|^2\left\langle\gamma_1,\gamma_3\right\rangle\left\langle\gamma_2,\gamma_3\right\rangle \\ &\quad + s^2\left(w_h + v_h\right)\mathrm{grad}\psi\frac{O\left(1+\frac{t}{l^2}+\frac{t^2}{l^4}\right)}{|\cos 2t\eta^{2,0}|} \\ &\quad + s^2 O\left(\frac{\psi^2}{\nu^3}\left(|W_\alpha|_{h_2} + |V_\alpha|_{h_2}\right)\right)\frac{O\left(1+\frac{\psi}{l^2}+\frac{\psi^2}{l^4}\right)}{|\cos 2t\eta^{2,0}|} \\ &\quad + s^2 O\left(\frac{\psi^2}{\nu^3}|W_\alpha|\frac{\psi^2}{\nu^3}|V_\alpha|\right) \end{aligned}$$

We again give V the usual normalization. So

$$\begin{aligned} \mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0}, V\right) &\geq \frac{\psi^2}{\nu^4}|V_\alpha|^2 \\ \frac{\psi}{\nu}\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0}, V\right)} &\geq \frac{\psi}{\nu}\left(\frac{\psi}{\nu^2}|V_\alpha|\right) \\ &= \frac{\psi^2}{\nu^3}|V_\alpha| \end{aligned}$$

So the third term is bounded by

$$s^2 O\left(\frac{\psi^2}{\nu^3}\left(|W_\alpha| + |V_\alpha|\right)\right)\frac{O\left(1+\frac{\psi}{l^2}+\frac{\psi^2}{l^4}\right)}{|\cos 2t\eta^{2,0}|} \leq O\left(s^2\frac{\psi}{\nu}\right)\left(\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0}, V\right)} + \sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0}, W\right)}\right)$$

and the last term is bounded by

$$s^2 O\left(\frac{\psi^2}{\nu^3}|W_\alpha|\frac{\psi^2}{\nu^3}|V_\alpha|\right) \leq s^2\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0}, V\right)}\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0}, W\right)}$$

Combining inequalities we have

$$\begin{aligned}\left|s^2\left\langle A_{\eta_u^{2,0}}W, A_{\eta_u^{2,0}}V\right\rangle\right| &\leq s^2\left(w_h v_h\right)\left|\mathrm{grad}\psi\right|^2\left\langle\gamma_1,\gamma_3\right\rangle\left\langle\gamma_2,\gamma_3\right\rangle \\ &\quad +O\left(s^2\right)\left(\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0},V\right)}+\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0},W\right)}\right) \\ &\quad +s^2\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0},V\right)}\sqrt{\mathrm{curv}^{\nu,re,l}\left(\eta_u^{2,0},W\right)}+O \qquad (13.13)\end{aligned}$$

To dominate the first term, $s^2\left(w_h v_h\right)\left|\mathrm{grad}\psi\right|^2\left\langle\gamma_1,\gamma_3\right\rangle\left\langle\gamma_2,\gamma_3\right\rangle$, we use the Proposition 13.4 to get

$$\begin{aligned}R^{\mathrm{diff,big}}\left(\eta_u^{2,0},W,W,\eta_u^{2,0}\right) &\geq e^{2f}w_h^2 s^2\left|\mathrm{grad}\,\psi\right|^2\left(1-\left\langle\gamma_1,\gamma_3\right\rangle^2\right) \\ &= e^{2f}w_h^2 s^2\left|\mathrm{grad}\,\psi\right|^2\left(\left\langle\gamma_2,\gamma_3\right\rangle^2\right)\end{aligned}$$

and

$$R^{\mathrm{diff,big}}\left(\eta_u^{2,0},V,V,\eta_u^{2,0}\right)\geq e^{2f}v_h^2 s^2\left|\mathrm{grad}\,\psi\right|^2\left(\left\langle\gamma_1,\gamma_3\right\rangle^2\right)$$

Combining the previous two displays gives us

$$e^{2f}s^2\left(w_h v_h\right)\left|\mathrm{grad}\psi\right|^2\left\langle\gamma_1,\gamma_3\right\rangle\left\langle\gamma_2,\gamma_3\right\rangle\leq\sqrt{\mathrm{curv}^{\mathrm{diff,big}}\left(\eta_{u,3}^{2,0},V\right)}\sqrt{\mathrm{curv}^{\mathrm{diff,big}}\left(\eta_{u,3}^{2,0},W\right)}$$

Plugging this into 13.13 gives us

$$\begin{aligned}&e^{2f}\left|s^2\left\langle A_{\eta_u^{2,0}}W, A_{\eta_{u,3}^{2,0}}V\right\rangle\right| \\ \leq\ &\sqrt{\mathrm{curv}^{\mathrm{diff,big}}\left(\eta_{u,3}^{2,0},V\right)}\sqrt{\mathrm{curv}^{\mathrm{diff,big}}\left(\eta_{u,3}^{2,0},W\right)}+ \\ &+O\left(s^2\right)\left(\sqrt{\mathrm{curv}\left(\eta_u^{2,0},V\right)}+\sqrt{\mathrm{curv}\left(\eta_u^{2,0},w\right)}\right) \\ &+s^2\sqrt{\mathrm{curv}\left(\eta_u^{2,0},V\right)}\sqrt{\mathrm{curv}\left(\eta_u^{2,0},W\right)}+O\end{aligned}$$

Arguing as before this gives us

$$\left|R^{\mathrm{diff,big}}\left(\eta_u^{2,0},W,V,\eta_u^{2,0}\right)\right|\leq\sqrt{\mathrm{curv}^{\mathrm{diff,big}}\left(\eta_{u,3}^{2,0},V\right)}\sqrt{\mathrm{curv}^{\mathrm{diff,big}}\left(\eta_{u,3}^{2,0},W\right)}$$

□

14. Appendix

This appendix contains the calculations we omitted in section 7.

$$
\begin{aligned}
\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l} &= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + 2l^{-2}\left(-\frac{1}{2}\sin 2t\cos 2t + \sin 2t\left(-\cos^2\theta\sin^2 t + \sin^2\theta\cos^2 t\right)\right)^2 \\
&\quad + \frac{2}{l^2}\frac{\sin^2 2\theta}{4}\left(\cos^2 2t + \sin^2 2t\right)^2 \\
&= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + 2l^{-2}\left(-\frac{1}{2}\sin 2t\cos 2t + \sin 2t\left(\sin^2\theta - \sin^2 t\right)\right)^2 \\
&\quad + \frac{\sin^2 2\theta}{2l^2} \\
&= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + 2l^{-2}\sin^2 2t\left(-\frac{1}{2}\cos 2t + \left(\sin^2\theta - \sin^2 t\right)\right)^2 + \frac{1}{l^2}\frac{\sin^2 2\theta}{2} \\
&= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + 2l^{-2}\sin^2 2t\left(-\frac{1}{2} + \sin^2\theta\right)^2 + \frac{1}{l^2}\frac{\sin^2 2\theta}{2} \\
&= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + 2l^{-2}\sin^2 2t\left(-\frac{1}{2}\cos 2\theta\right)^2 + \frac{1}{l^2}\frac{\sin^2 2\theta}{2} \\
&= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + \frac{1}{2l^2}\left(\sin^2 2t\left(\cos^2 2\theta\right) + \sin^2 2\theta\right) \\
&= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + \frac{1}{2l^2}\left(1 - \cos^2 2t\cos^2 2\theta\right)
\end{aligned}
$$

So we can now prove

Proposition 14.1.

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi_{\nu,l} &= \frac{\left(1 + \frac{1}{2l^2}\sin^2 2\theta\right)\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}} \\
&= \frac{\left|x^{2,0}\right|^2_{\nu,l}\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}} \\
\frac{\partial}{\partial\theta}\psi_{\nu,l} &= -\frac{1}{4l^2}\frac{\sin 2t\cos^2 2t\sin 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}}
\end{aligned}
$$

Proof. We first rearrange the terms in $\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}$ as follows

$$
\begin{aligned}
\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l} &= \cos^2 2t + \frac{\sin^2 2t}{\nu^2} + \frac{1}{2l^2}\left(1 - \cos^2 2t\cos^2 2\theta\right) \\
&= 1 - \sin^2 2t + \frac{\sin^2 2t}{\nu^2} + \frac{1}{2l^2} - \frac{1}{2l^2}\cos^2 2\theta + \frac{1}{2l^2}\sin^2 2t\cos^2 2\theta \\
&= 1 + \frac{1}{2l^2}\sin^2 2\theta + \frac{\sin^2 2t}{\nu^2} - \sin^2 2t + \frac{1}{2l^2}\sin^2 2t\cos^2 2\theta \\
&= 1 + \frac{1}{2l^2}\sin^2 2\theta + \frac{\sin^2 2t}{\nu^2} - \sin^2 2t + \frac{1}{2l^2}\sin^2 2t - \frac{1}{2l^2}\sin^2 2t\sin^2 2\theta \\
&= 1 + \frac{\sin^2 2\theta}{2l^2} + \left(\frac{1}{\nu^2} + \frac{1}{2l^2} - \left(1 + \frac{\sin^2 2\theta}{2l^2}\right)\right)\sin^2 2t
\end{aligned}
$$

Setting

$$\frac{1}{\nu_l^2} = \frac{1}{\nu^2} + \frac{1}{2l^2},$$

and using the fact that

$$\left|x^{2,0}\right|_{\nu,l}^2 = 1 + \frac{\sin^2 2\theta}{2l^2}$$

we get

$$\begin{aligned}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^2 &= \left|x^{2,0}\right|_{\nu,l}^2 + \left(\frac{1}{\nu_l^2} - \left|x^{2,0}\right|_{\nu,l}^2\right)\sin^2 2t \\ &= \left|x^{2,0}\right|_{\nu,l}^2 \cos^2 2t + \frac{1}{\nu_l^2}\sin^2 2t\end{aligned}$$

This gives us

$$\frac{\partial}{\partial t}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^2 = \left(\frac{1}{\nu_l^2} - \left|x^{2,0}\right|_{\nu,l}^2\right) 4\sin 2t\cos 2t$$

and using

$$\begin{aligned}\frac{\partial}{\partial\theta}\left|x^{2,0}\right|_{\nu,l}^2 &= \frac{\partial}{\partial\theta}\left(1 + \frac{\sin^2 2\theta}{2l^2}\right) \\ &= \frac{2\sin 2\theta\cos 2\theta}{l^2} \\ &= \frac{\sin 4\theta}{l^2}\end{aligned}$$

we get

$$\begin{aligned}\frac{\partial}{\partial\theta}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^2 &= \frac{\partial}{\partial\theta}\left(\left|x^{2,0}\right|_{\nu,l}^2 + \left(\frac{1}{\nu_l^2} - \left|x^{2,0}\right|_{\nu,l}^2\right)\sin^2 2t\right) \\ &= \frac{\sin 4\theta}{l^2} - \frac{\sin 4\theta}{l^2}\sin^2 2t \\ &= \frac{\sin 4\theta\cos^2 2t}{l^2}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{\nu,l} &= \frac{\partial}{\partial t}\frac{1}{2}\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}} \\ &= \frac{1}{2}\frac{2\cos 2t\left(\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^2\right) - \frac{1}{2}\sin 2t\left(\frac{\partial}{\partial t}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^2\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^3} \\ &= \frac{1}{2}\frac{2\cos 2t\left(\left|x^{2,0}\right|_{\nu,l}^2 + \left(\frac{1}{\nu_l^2} - \left|x^{2,0}\right|_{\nu,l}^2\right)\sin^2 2t\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^3} \\ &\quad -\frac{1}{2}\frac{\frac{1}{2}\sin 2t\left(\left(\frac{1}{\nu_l^2} - \left|x^{2,0}\right|_{\nu,l}^2\right)4\sin 2t\cos 2t\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^3} \\ &= \frac{\left|x^{2,0}\right|_{\nu,l}^2\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^3}\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial}{\partial\theta}\psi_{\nu,l} &= \frac{\partial}{\partial\theta}\frac{1}{2}\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}} \\
&= -\frac{1}{2}\frac{\sin 2t\left(\frac{\partial}{\partial\theta}\left[\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right]^{1/2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}} \\
&= -\frac{1}{4}\frac{\sin 2t\left[\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right]^{-\frac{1}{2}}\left(\frac{\partial}{\partial\theta}\left[\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right]\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}} \\
&= -\frac{1}{4}\frac{\sin 2t\left(\frac{\partial}{\partial\theta}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{3}} \\
&= -\frac{1}{4}\frac{\sin 2t\left(\frac{\sin 4\theta\cos^2 2t}{l^2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{3}} \\
&= -\frac{1}{4l^2}\frac{\sin 2t\cos^2 2t\sin 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{3}}.
\end{aligned}$$

□

Proposition 14.2.

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\psi_{\nu,l} &= -\left|x^{2,0}\right|_{\nu,l}^{2}\frac{\sin 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{5}}\left(-4\left|x^{2,0}\right|_{\nu,l}^{2}\cos^2 2t+\frac{2}{\nu_l^2}+4\left(\frac{1}{\nu_l^2}\right)\cos^2 2t\right) \\
\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\psi_{\nu,l} &= \frac{\cos 2t\sin 4\theta}{l^2\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{5}}\left(-\frac{1}{2}\left|x^{2,0}\right|_{\nu,l}^{2}\cos^2 2t+\frac{1}{\nu_l^2}\sin^2 2t\right)
\end{aligned}$$

$$\frac{\partial^2}{\partial\theta^2}\psi_{\nu,l}=-\frac{\sin 2t\cos^2 2t}{l^2}\frac{\cos 4\theta\left(\left|x^{2,0}\right|_{\nu,l}^{2}\cos^2 2t+\frac{1}{\nu_l^2}\sin^2 2t\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{5}}+\frac{3}{2}\frac{\sin 2t\cos^4 2t}{4l^4}\frac{\sin^2 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{5}}$$

Proof.

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\psi_{\nu,l} &= \left|x^{2,0}\right|_{\nu,l}^{2}\frac{\partial}{\partial t}\frac{\cos 2t}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{3}} \\
&= \left|x^{2,0}\right|_{\nu,l}^{2}\frac{-2\sin 2t\left(\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{3}\right)-\cos 2t\left[\frac{\partial}{\partial t}\left(\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right)^{3/2}\right]}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{6}} \\
&= \left|x^{2,0}\right|_{\nu,l}^{2}\frac{-2\sin 2t\left(\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{3}\right)-\frac{3}{2}\left(\left|(\cos 2t)\,\eta^{2,0}\right|^{2}\right)^{1/2}\cos 2t\left(\frac{\partial}{\partial t}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{6}} \\
&= \left|x^{2,0}\right|_{\nu,l}^{2}\frac{-2\sin 2t\left(\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right)-\frac{3}{2}\cos 2t\left(\frac{\partial}{\partial t}\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|_{\nu,l}^{5}}
\end{aligned}$$

$$
\begin{aligned}
= & -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{2\sin 2t\left(\left|x^{2,0}\right|_{\nu,l}^{2}+\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)\sin^2 2t\right)}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}} \\
& -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{\frac{3}{2}\cos 2t\left(\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)4\sin 2t\cos 2t\right)}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}}
\end{aligned}
$$

$$
\begin{aligned}
= & -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{\sin 2t\left(2\left|x^{2,0}\right|_{\nu,l}^{2}+2\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)\sin^2 2t\right)}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}} \\
& -\left|x^{2,0}\right|_{\nu,l}^{2} \sin 2t \frac{6\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)\cos^2 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}}
\end{aligned}
$$

$$
\begin{aligned}
= & -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{\sin 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}}\left(2\left|x^{2,0}\right|_{\nu,l}^{2}+2\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)\sin^2 2t+6\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)\cos^2 2t\right) \\
= & -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{\sin 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}}\left(2\left|x^{2,0}\right|_{\nu,l}^{2}\cos^2 2t+\frac{2\sin^2 2t}{\nu_l^2}+6\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|_{\nu,l}^{2}\right)\cos^2 2t\right) \\
= & -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{\sin 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}}\left(-4\left|x^{2,0}\right|_{\nu,l}^{2}\cos^2 2t+\frac{2\sin^2 2t}{\nu_l^2}+6\left(\frac{1}{\nu_l^2}\right)\cos^2 2t\right), \\
= & -\left|x^{2,0}\right|_{\nu,l}^{2} \frac{\sin 2t}{\left|\left(\cos 2t\right)\eta^{2,0}\right|_{\nu,l}^{5}}\left(-4\left|x^{2,0}\right|_{\nu,l}^{2}\cos^2 2t+\frac{2}{\nu_l^2}+4\left(\frac{1}{\nu_l^2}\right)\cos^2 2t\right)
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\psi_{\nu,l} &= \cos 2t\frac{\partial}{\partial\theta}\frac{\left|x^{2,0}\right|^2_{\nu,l}}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}}\\
&= \cos 2t\frac{\left(\frac{\sin 4\theta}{l^2}\right)\left(\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}} - \cos 2t\frac{\left|x^{2,0}\right|^2_{\nu,l}\left(\frac{\partial}{\partial\theta}\left[\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right]^{3/2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^6_{\nu,l}}\\
&= \cos 2t\frac{\left(\frac{\sin 4\theta}{l^2}\right)\left(\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\\
&\quad -\frac{3}{2}\cos 2t\frac{\left|x^{2,0}\right|^2_{\nu,l}\left[\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right]^{1/2}\left(\frac{\partial}{\partial\theta}\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^6_{\nu,l}}\\
&= \cos 2t\frac{\left(\frac{\sin 4\theta}{l^2}\right)\left(\left|x^{2,0}\right|^2_{\nu,l}+\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|^2_{\nu,l}\right)\sin^2 2t\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\\
&\quad -\frac{3}{2}\cos 2t\frac{\left|x^{2,0}\right|^2_{\nu,l}\frac{\sin 4\theta\cos^2 2t}{l^2}}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\\
&= \cos 2t\frac{\left(\frac{\sin 4\theta}{l^2}\right)\left(\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t+\frac{1}{\nu_l^2}\sin^2 2t\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\\
&\quad -\frac{3}{2}\cos 2t\frac{\left|x^{2,0}\right|^2_{\nu,l}\frac{\sin 4\theta\cos^2 2t}{l^2}}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\\
&= \frac{\cos 2t\sin 4\theta}{l^2\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\left(\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t+\frac{1}{\nu_l^2}\sin^2 2t-\frac{3}{2}\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t\right)\\
&= \frac{\cos 2t\sin 4\theta}{l^2\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\left(-\frac{1}{2}\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t+\frac{1}{\nu_l^2}\sin^2 2t\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2}\psi_{\nu,l} &= -\frac{\sin 2t\cos^2 2t}{4l^2}\frac{\partial}{\partial\theta}\frac{\sin 4\theta}{\left|(\cos 2t)\,\eta^{2,0}\right|^3_{\nu,l}}\\
&= -\frac{\sin 2t\cos^2 2t}{4l^2}\frac{4\cos 4\theta\left(\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}+\frac{\sin 2t\cos^2 2t}{4l^2}\frac{\sin 4\theta\left(\frac{\partial}{\partial\theta}\left[\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right]^{3/2}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^6_{\nu,l}}\\
&= -\frac{\sin 2t\cos^2 2t}{4l^2}\frac{4\cos 4\theta\left(\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^5_{\nu,l}}\\
&\quad +\frac{3}{2}\frac{\sin 2t\cos^2 2t}{4l^2}\frac{\sin 4\theta\left[\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right]^{1/2}\left(\frac{\partial}{\partial\theta}\left|(\cos 2t)\,\eta^{2,0}\right|^2_{\nu,l}\right)}{\left|(\cos 2t)\,\eta^{2,0}\right|^6_{\nu,l}}
\end{aligned}$$

$$
\begin{aligned}
= & -\frac{\sin 2t\cos^2 2t}{l^2}\frac{\cos 4\theta\left(\left|x^{2,0}\right|^2_{\nu,l}+\left(\frac{1}{\nu_l^2}-\left|x^{2,0}\right|^2_{\nu,l}\right)\sin^2 2t\right)}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^5_{\nu,l}} \\
& +\frac{3}{2}\frac{\sin 2t\cos^2 2t}{4l^2}\frac{\sin 4\theta\left(\cos^2 2t\frac{\sin 4\theta}{l^2}\right)}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^5_{\nu,l}} \\
= & -\frac{\sin 2t\cos^2 2t}{l^2}\frac{\cos 4\theta\left(\left|x^{2,0}\right|^2_{\nu,l}\cos^2 2t+\frac{1}{\nu_l^2}\sin^2 2t\right)}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^5_{\nu,l}} \\
& +\frac{3}{2}\frac{\sin 2t\cos^4 2t}{4l^4}\frac{\sin^2 4\theta}{\left|\left(\cos 2t\right)\eta^{2,0}\right|^5_{\nu,l}}
\end{aligned}
$$

□

References

[Aub] T. Aubin, *Métriques riemanniennes et courbure*. J. Differential Geometry 4 1970 383–424.

[AbrMey] U. Abresch and W. Meyer, *Pinching below 1/4, injectivity radius and conjugate radius,* J. Differential Geometry **40** (1994) 643-691.

[Berg1] M. Berger, *Les variétés riemanniennes à courbure positive*, Bull. Soc. Math. Belg. 10 (1958) 88-104.

[Berg2] M. Berger, *Trois remarques sur les variétés riemanniennes à courbure positive*. C. R. Acad. Sci. Paris Sér. A-B 263 1966 A76–A78.

[Berg3] M. Berger, *On the Diameter of some Riemannian manifolds*, preprint, UC Berkeley, 1962.

[Berg4] M. Berger, *Sur les variétés riemaniennes pincées juste au-dessous de* $\frac{1}{4}$, Ann. Inst. Fourier, Genoble **33** (1983) 135-150.

[Bes] A. L. Besse, *Einstein manifolds*, Spinger verlag, 1987.

[BohmWilk] C. Böhm and B. Wilking, *Manifolds with positive curvature operators are space forms*. http://front.math.ucdavis.edu/math.DG/0606187.

[BourDesSent] J.-P. Bourguignon, A. Deschamps and P. Sentenac, *Quelques variations particulières d'un produit de métriques*. Ann. Sci. École Norm. Sup. (4) 6 (1973), 1–16.

[Cheeg] J. Cheeger, *Some examples of manifolds of nonnegative curvature*. J. Differential Geometry 8 (1973), 623–628.

[Ehrl] P. Ehrlich, *Deformations of scalar curvature*. Geometriae Dedicata 5 (1976), no. 1, 25–26.

[Esch1] J.-H. Eschenburg, *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*. Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie [Series of the Mathematical Institute of the University of Münster, Series 2], 32. Universität Münster, Mathematisches Institut, Münster, 1984. vii+177 pp.

[EschKer] J.-H. Eschenburg and M. Kerin, *Almost positive curvature on the Gromoll-Meyer 7-sphere*, preprint

[Ham] R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Diff. Geo. Vol. 2, International Press (1995) 7-136.

[Hit] N. Hitchin, *Harmonic spinors*. Advances in Math. 14 (1974), 1–55.

[GalMey] S. Gallot and D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d'une variété riemannianne*, J. Math. Pures. Appl. 54 (1975), 259-284.

[GluWarZil] H. Gluck, F. Warner and W. Ziller, *The geometry of the Hopf fibrations*. Enseign. Math. (2) 32 (1986), no. 3-4, 173–198.

[Grom] D. Gromoll, *Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären*. Math. Ann. 164 1966 353–371.

[GromDur] D. Gromoll and C. Duran, *Fairly General Warping,* Unpublished Lecture Notes by Carlos Duran based on lectures of Detlef Gromoll.

[GromMey] D. Gromoll and W. Meyer, *An exotic sphere with nonnegative sectional curvature.* Ann. of Math. (2) 100 (1974), 401–406.

[GromWals] D. Gromoll and G. Walschap, *Riemannian submersions,* Birkhäuser, to appear.

[GroLaw] M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds.* Inst. Hautes Études Sci. Publ. Math. No. 58 (1983), 83–196 (1984).

[GrovShio] K. Grove and K. Shiohama, *A generalized sphere theorem*, Ann. of Math. **106**, (1977), 201-211.

[GrovWilh] K. Grove and F. Wilhelm, *Metric constraints on exotic spheres via Alexandrov geometry.* J. Reine Angew. Math. 487 (1997), 201–217.

[Kling] W. Klingenberg, *Über Mannigfaltigheiten mit positiver Krümmung*, Comm. Math. Helv. 35 (1961), 47-54.

[Lich] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acd. Scien. Paris, 257 (1963), 7-9.

[MicMo] M.J. Micallef and J.D. Moore, *Minimal 2-spheres and the topology of manifolds with positive curvature on totally isotropic 2-planes*, Ann. of Math. 127 (1988), 199-227.

[Miln] J. Milnor, *On manifolds homeomorphic to the 7-sphere.* Ann. of Math. (2) 64 (1956), 399–405.

[Mok] N. Mok, *The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bi-sectional curvature*, J. Diff. Geo. 27 (1988), 179-214.

[On] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. 13 (1966), 459–469.

[Per] G. Perel'man, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Diff. Geo. 40 (1994), 209-212.

[Pet] P. Petersen, *Riemannian Geometry* 2^{nd} *Ed.* New York: Springer Verlag, 2006.

[PetWilh] P. Petersen and F. Wilhelm, *Examples of Riemannian manifolds with positive curvature almost everywhere.* Geom. Topol. 3 (1999), 331–367

[Poor] W. A. Poor, *Some exotic spheres with positive Ricci curvature.* Math. Ann. 216 (1975), no. 3, 245–252.

[Rau] H.E. Rauch, *A contribution to differential geometry in the large*, Ann. Math. 54 (1951), 38-55.

[SchYau] R. Schoen, and S. T. Yau, *On the structure of manifolds with positive scalar curvature.* Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[Shik] Y. Shikata, *On the differentiable pinching problem.* Osaka J. Math. 4 1967 279–287.

[Stol] S. Stolz, *Simply connected manifolds of positive scalar curvature.* Ann. of Math. (2) 136 (1992), no. 3, 511–540.

[Stra] M. Strake, *Curvature increasing metric variations.* Math. Ann. 276 (1987), no. 4, 633–641.

[Suy] Y. Suyama, *Differentiable sphere theorem by curvature pinching.* J. Math. Soc. Japan 43 (1991), no. 3, 527–553.

[Tapp1] K. Tapp, *Quasi-positive curvature on homogeneous bundles.* J. Differential Geom. 65 (2003), no. 2, 273–287.

[Tapp2] K. Tapp, *Flats in Riemannian submersions from Lie groups,* preprint.

[Wal] N. Wallach, *Compact homogeneous riemannian manifolds with strictly positive curvature*, Ann. of Math. **96** (1972) 277-295.

[Weis] M. Weiss, *Pinching and concordance theory.* J. Differential Geom. 38 (1993), no. 2, 387–416.

[Wilh1] F. Wilhelm, *Exotic spheres with lots of positive curvatures.* J. Geom. Anal. 11 (2001), no. 1, 161–186.

[Wilh2] F. Wilhelm, *An exotic sphere with positive curvature almost everywhere.* J. Geom. Anal. 11 (2001), no. 3, 519–560.

[Wilk] B. Wilking, *Manifolds with positive sectional curvature almost everywhere.* Invent. Math. 148 (2002), no. 1, 117–141.

[Wrai] D. Wraith, *Exotic spheres with positive Ricci curvature.* J. Differential Geom. 45 (1997), no. 3, 638–649.

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