# An exotic sphere with positive sectional curvature

joint work with Peter Petersen

October 17, 2008

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- Is the conclusion of the Sphere Theorem optimal?

# A long history of partial answers

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- Brendle-Schon (2007)—NO —(using the Ricci flow).

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- It was possible that all of this effort was about a vacuous subject since (until now)
- there has not been a single example of an exotic sphere with positive sectional curvature.

# The Gromoll-Meyer Sphere

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- positive curvature almost everywhere is referred to as *almost positive curvature*.

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- The 2008 survey in the Bulletin of the AMS by Joachim and Wraith attributes the Ricci curvature deformation theorem to Ehrlich.

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- **Theorem** *(Böhm and Wilking 2006) Any complete metric with positive curvature operator flows to one with constant positive curvature via the Ricci flow.*

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- W.–The Gromoll-Meyer sphere admits almost positive curvature.
- Eschenburgh–Kerin (2007) Much shorter proof that the Gromoll-Meyer sphere admits almost positive curvature.

# An exotic sphere with positive curvature

- **Theorem** *The Gromoll-Meyer exotic sphere admits positive sectional curvature.*

# The Gromoll-Meyer Construction

- The Lie Group $Sp(2)$ can be viewed as the set of $(2 \times 2)$–matrices with quaternion entries so that

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- O'Neill—Riemannian submersions do not decrease the curvature of horizontal planes.
- A Lie group with a biinvariant metric is nonnegatively curved.
- Combining this with O'Neill's principle we see that the Gromoll-Meyer sphere is nonnegatively curved.

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- In 2002, Wilking showed that all of the zero planes in $\Sigma^7$ (and most if not all) known examples with nonnegative sectional curvature are contained in totally geodesic, flat tori (also Tapp 2006).

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- So if $G = S^3$, any plane whose projection to the orbits of $G$ is nondegenerate becomes positively curved.
- We go from Quasi-positive curvature to almost positive curvature on the GM-sphere via a Cheeger deformation with $\left(S^3\right)^4$.

# Petersen-W. 1999

- We get almost positive curvature on the unit tangent bundle of $S^4$ by deforming the bi-invariant metric on $Sp(2)$ using Cheeger's method and the $S^3 \times S^3 \times S^3 \times S^3$ action induced by the commuting $S^3$-actions

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- We get almost positive curvature on $\Sigma^7$ as a quotient of the same metric on $Sp(2)$ (W. 2001).

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6. The $\Delta(U, D)$ Cheeger deformation and a further $h_1$–deformation.

# Notation

- We let $g_1$, $g_{1,2}$, $g_{1,2,3}$, ect. be the metrics obtained after doing deformations (1), (1) and (2), or (1), (2), and (3) respectively.

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6. The $\Delta(U, D)$ Cheeger deformation and a further $h_1$–deformation—$g_{1,2,3,4,5,6}$.

- It also makes sense to talk about metrics like $g_{1,3}$, i.e. the metric obtained from doing just deformations (1) and (3) without deformation (2).

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- Some specific positive curvatures of $g_{1,3}$ are redistributed in $g_{1,2,3}$. The reasons for this are technical, but as far as we can tell without deformation (2) our methods will not produce positive curvature.
- It does not seem likely that either $g_{1,2}$ or $g_{1,2,3}$ are nonnegatively curved on $Sp(2)$ , but we have not verified this.

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- In fact this integral is positive over any of the flat tori of $g_{1,3}$.
- The role of deformation (5) is to even out the positive integral.

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- Deformation (4), scaling the fibers of $Sp(2) \longrightarrow S^4$, does not preserve nonnegative curvature.
- Nevertheless it is the key step in the argument.
- The integral of the curvatures of the zero planes of $g_{1,3}$ is positive.
- In fact this integral is positive over any of the flat tori of $g_{1,3}$.
- The role of deformation (5) is to even out the positive integral.
- The curvatures of the flat tori of $g_{1,3}$ are pointwise positive with respect to $g_{1,2,3,4,5}$.

# Simplifying the curvature polynomial

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- We'll focus only on what happens when the plane moves within the fibers of the Grassmannian.

# Moving within the Grassmannian fibers

- To do this we consider planes of the form

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- The role of the Cheeger deformations in 6 is that any fixed plane with a nondegenerate projection to the vertical space of $\Sigma^7 \longrightarrow S^4$ becomes positively curved, provided these deformations are carried out for a sufficiently long time.
- Although the zero planes $P = \mathrm{span}\{\zeta, W\}$ all have degenerate projections to the vertical space of $\Sigma^7 \longrightarrow S^4$, there are of course nearby planes whose projections are nondegenerate. Exploiting this idea we get

# Simplifying the curvature polynomial–II

- **Proposition:** *If all curvature polynomials whose corresponding planes have degenerate projection onto the vertical space of* $\Sigma^7 \longrightarrow S^4$ *are positive on* $\left(\Sigma^7, g_{1,2,3,4,5}\right)$, *then* $\left(\Sigma^7 g_{1,2,3,4,5,6}\right)$ *is positively curved, provided the Cheeger deformations in (6) are carried out for a sufficiently long time.*

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- *Since Cheeger deformations preserve positive curvature* $g_{1,2,3,4,5,6}$ *is also positively curved on* $N$. *So* $g_{1,2,3,4,5,6}$ *is positively curved.* □
- Thus the deformations in (6) allow us the computational convenience of assuming that the vector "$z$" is in the horizontal space of $\Sigma^7 \longrightarrow S^4$.

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- $$4 + \varepsilon \geq \sec \geq 1 \text{ (Petersen-Tao, 2008)}$$

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- There are a lot more metrics with $\sec \geq 1$ and diameter $> \frac{\pi}{2}$ than with

$$\frac{1}{4} \leq \sec < 1.$$