

SOME PRINCIPLES FOR DEFORMING NONNEGATIVE CURVATURE

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Dedicated to Karsten Grove

CONTENTS

1. Introduction	1
2. Deformations Preserving Totally Geodesic Families	4
3. Orthogonal Partial Conformal Change	7
4. Integrally Positive Curvature	11
4.1. Positive Curvature on a Single Initially Flat Torus	15
5. Tangential Partial Conformal Change	21
6. Long Term Cheeger Principle	25
6.1. Quadratic Nondegeneracy and Cheeger Deformations	28
7. Curvature Compression Principle	29
8. Synergy	37
References	41

1. INTRODUCTION

Let \mathcal{N} be the class of closed simply connected, smooth n -manifolds admitting nonnegative sectional curvature and $\mathcal{P} \subset \mathcal{N}$ the corresponding class for positive curvature. Known examples suggest that \mathcal{N} ought to be much larger than \mathcal{P} . On the other hand, there is no known obstruction that distinguishes between the two classes. So it is actually possible that $\mathcal{N} = \mathcal{P}$.

In [PetWilh2] we will give a deformation of the nonnegatively curved metric on the Gromoll-Meyer sphere [GromMey] to a positively curved metric. The purpose of this note is to elucidate a few abstract principles that will be used in this deformation, and possibly could be helpful for other deformations to positive curvature.

Besides a few exceptions, [Cheeg], [Dear2], [GrovVerdZil], [GrovZil1], [GrovZil2], [Guij] all known examples of compact nonnegatively curved manifolds are constructed as Riemannian submersions of compact Lie groups. A result in [Tapp2] then implies that the zero curvature planes of the nonexceptional examples are contained in totally geodesic 2-dimensional flats. As far as we are aware, the exceptional examples also have totally geodesic flats, provided of course that they have any zero curvature planes at all ([Dear2] and [GrovVerdZil]).

All known examples with nonnegative curvature, some zero curvatures, and positive curvature at a point, are the images of Riemannian submersions of compact Lie groups and hence have all zero planes contained in totally geodesic flats ([EschKer], [Esch], [GromMey], [Ker1], [Ker2], [PetWilh1], [Tapp1], [Wilh], and [Wilk].) So in

most cases, any attempt to put positive curvature on a known nonnegatively curved example must confront the issue of how to put positive curvature on a neighborhood of a totally geodesic flat torus.

More than 20 years ago Strake observed that the presence of a totally geodesic flat torus in a nonnegatively curved manifold means that there can be no deformation that is positive to first order. In principle, a first order deformation should be much easier to construct and verify than a higher order one. In fact, if $\{g_t\}_{t \in \mathbb{R}}$ is C^∞ family of metrics with g_0 a metric of nonnegative curvature, and if

$$\left. \frac{\partial}{\partial t} \sec_{g_t} P \right|_{t=0} > 0$$

for all planes P so that $\sec_{g_0} P = 0$, then g_t has positive curvature for all sufficiently small $t > 0$.

On the other hand, if for all planes P with $\sec_{g_0} P = 0$ we have

$$\begin{aligned} \left. \frac{\partial}{\partial t} \sec_{g_t} P \right|_{t=0} &= 0 \text{ and} \\ \left. \frac{\partial^2}{\partial t^2} \sec_{g_t} P \right|_{t=0} &> 0, \end{aligned}$$

then, without more information, we can not make any conclusion about obtaining positive curvature. For instance, if Φ_t is a flow that moves the zero planes to positive curvature, then the variation $(\Phi_t)^* g$ can satisfy the conditions above, yet clearly each of the metrics $(\Phi_t)^* g$ are isometric to g .

The obvious problem with such a gauge transformation is that it only moves zero planes to new places. Unfortunately, the discussion above illustrates that any attempt to put positive curvature on a (generic) known nonnegatively curved example must confront this issue. It is not enough to consider the effect of a deformation on the set, Z , of zero planes of the original metric. Instead we to have check that the curvature becomes positive in an entire neighborhood of Z .

To this bleak reality we offer the following ray of hope—

The very rigidity of totally geodesic flats can be exploited in attempts to deform them.

The rigidity of a totally geodesic flat within a fixed nonnegatively curved manifold is of course well known and well understood. Here we have in mind a different sort of rigidity. We will look at certain types of deformations that preserve totally geodesic flats, and other types of deformations that preserve aspects of the rigidity of totally geodesic flats. The tremendous advantage of this rigidity is that it will allow us to change one component of the curvature tensor while controlling the change in other components. Since the problem of prescribing the curvature tensor is highly over determined, in general, this is an entirely unreasonable thing to expect; nevertheless, the rigidity of totally geodesic flats will allow us to do this in certain narrowly constrained situations.

Besides Cheeger deformations, the metric changes that we use to go from the Gromoll-Meyer metric to our positively curved metric are

- a deformation that we call the *Orthogonal Partial Conformal Change*
- scaling of the fibers of the Riemannian submersion $Sp(2) \rightarrow S^4$, to create integrally positive curvature, and
- another deformation that we call the *Tangential Partial Conformal Change*

To describe a general *Partial Conformal Change* we start with a distribution $\mathcal{D} \subset TM$, and decompose our original metric as

$$g = g_{\mathcal{D}} + g_{\mathcal{D}^{\perp}}.$$

We then conformally change $g_{\mathcal{D}}$ while fixing $g_{\mathcal{D}^{\perp}}$.

Our use of the terms “*Orthogonal*” and “*Tangential*” is meant to convey that our changes will be relative to distributions that are either orthogonal or contain the original zero curvature planes respectively.

An abstraction of the orthogonal partial conformal change is discussed in Sections 2 and 3. It preserves nonnegative curvature, the zero curvature locus, and has the effect of redistributing certain positive curvatures along the initial zero curvature locus. Having a broader class of nonnegatively curved metrics could certainly be an advantage. In fact, if we were to perform our other deformations without doing the orthogonal partial conformal change we could make the old zero planes positively curved, but as far as we can tell would not get positive curvature. The idea that such a change is possible goes back at least to [Wals].

The fiber scaling is the central idea of the deformation to positive curvature on the Gromoll–Meyer sphere, Σ^7 . In section 4, we prove an abstract theorem about fiber scaling. This result implies that if we start with the metric from [Wilh] and scale the fibers of $Sp(2) \rightarrow S^4$, then we get integrally positive curvature over the sections that have zero curvature in [Wilh]. More precisely, the zero locus in [Wilh] consists of a (large) family of totally geodesic 2-dimensional tori. We will show that after scaling the fibers of $Sp(2) \rightarrow S^4$, the integral of the curvature over any of these tori becomes positive. The computation is fairly abstract, and the argument is made in these abstract terms, so no knowledge of the metric of [Wilh] is required.

In addition to proving that fiber scaling creates integrally positive curvature, our argument in section 4 will provide a precise formula for what happens to the curvature of each of the old zero curvature planes. The leading order term has both signs, so the metric with the fibers scaled has curvatures of both signs. On the other hand, the leading order term is also the Hessian of a function and along any *one* of our originally flat tori it can be canceled by a conformal change of metric. The details are carried out in subsection 4.1. Thus by reading section 4 the reader can get a quick impression of what the entire deformation does to the curvature of a single torus that is initially totally geodesic and flat.

Unfortunately, the conformal factor required to cancel the Hessian term from fiber scaling varies from torus to torus. Our actual deformation includes a partial conformal change for which the distribution \mathcal{D} contains the old zero curvature planes. This is our Tangential Partial Conformal Change. In Section 5, we describe an abstract set up for our tangential partial conformal deformation and show that the important curvatures change as though we had performed an actual conformal change. Combining the results of this section with our fiber scaling and conformal change calculations provides a method to obtain positive curvature on the initially flat planes of the Gromoll–Meyer sphere.

Section 6 is the first place in the paper where totally geodesic flats do not play a prominent role. Instead we detail an observation that Cheeger deformations can be used to create positive curvature even when the initial metric has curvatures of both signs. Modulo the so called “Cheeger Reparametrization” of the Grassmannian, Cheeger deformations preserve positive curvatures. In addition, any plane whose projection to the orbits “corresponds” to a positively curved plane will become

positively curved provided the deformation is carried out for a sufficiently long period.

We do not imagine that we are the first to make this observation, and in fact, took for granted that this idea was well understood when we wrote the first draft of [PetWilh2]. We have subsequently become aware that these ideas are not as well known as we originally assumed, so we have included them for the sake of completeness.

The curvatures of the zero planes of [Wilh] are not affected by Cheeger deformations, but most nearby planes feel the effect. Part of the role of long term Cheeger deformations is to simplify the problem of estimating the curvatures in a neighborhood of the original zero curvature locus.

Sections 7 and 8 are also part of our strategy to solve this problem, and are the sections that are most dependent on the others. While this paper is an attempt to divide some of our deformation of the Gromoll–Meyer sphere into digestible, abstract pieces, the reader should be aware that in at least one respect the argument is an intertwined whole.

In Section 7, we analyze the effect of certain Cheeger deformations on our formula for the curvatures of our tori after fiber scaling. We will show that Cheeger deformations have the effect of compressing the bulk of these curvatures into a small set, T_0 . Because T_0 is small the orthogonal partial conformal change will allow us to make certain curvatures much larger on T_0 , and “pay” with only a small decrease in curvature outside of T_0 . This synergy makes the problem of verifying positive curvature more tractable, is crucial to our whole argument, and explained in greater detail in Section 8.

It is natural to speculate on the extent to which some (or all) of these ideas might be useful in other deformations to positive curvature. For example there are non simply connected examples with nonnegative curvature that according to Synge’s Theorem can not admit positive curvature, so it is natural to ask where our methods break down in these examples. While we have not made an exhaustive study of this question, we can point out that if a totally geodesic flat is *vertical* for the submersion whose fibers are scaled, then our curvature formula shows that it will continue to be flat. This is the case for the metrics on $\mathbb{R}P^3 \times \mathbb{R}P^2$ and $S^3 \times S^2$ in [Wilh], with respect to the isometric $SO(3)$ -action of that paper. Since our total argument in [PetWilh2] is very long, there are many obstructions to using it in general. It seems more likely that individual pieces will find other applications.

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2. DEFORMATIONS PRESERVING TOTALLY GEODESIC FAMILIES

In the next two sections we describe an abstract framework for our orthogonal partial conformal change. Our exposition will be by “bootstrapping”, starting with some more general metric changes.

The problem of prescribing the curvature tensor of a Riemannian n -manifold with $n \geq 4$ is highly overdetermined. In particular, it is unreasonable to expect to change one component of the curvature tensor while holding other components

fixed. We should also not expect to change one component of the curvature tensor while keeping the change in other components small compared to the change in the desired component.

In the next two sections we explore exceptions to this principle that can be traced to the rigidity of totally geodesic flat tori in nonnegatively curved manifolds.

We begin by recalling,

Exercise 2.1. (5.4 in [Pet]) *Let γ be a geodesic in (M, g) . Let \tilde{g} be another metric on M which satisfies*

$$g(\dot{\gamma}, \cdot) = \tilde{g}(\dot{\gamma}, \cdot) : TM \longrightarrow \mathbb{R}.$$

Then γ is also a geodesic with respect to \tilde{g} .

A straightforward generalization is

Proposition 2.2. *Let \mathcal{S} be a family of totally geodesic submanifolds of (M, g) . Let \tilde{g} be another metric on M which satisfies*

$$g(X, \cdot) = \tilde{g}(X, \cdot) : TM \longrightarrow \mathbb{R}$$

for all vectors X tangent to a totally geodesic submanifold in \mathcal{S} , then \mathcal{S} is also a family of totally geodesic submanifolds of (M, \tilde{g}) .

Proof. If γ is any geodesic in $S \in \mathcal{S}$ with respect to g , then by the preceding exercise, γ is a geodesic of (M, \tilde{g}) . \square

Corollary 2.3. *If the totally geodesic family \mathcal{S} of the preceding proposition consists of totally geodesic flat submanifolds for (M, g) , then it also consists of totally geodesic flat submanifolds for (M, \tilde{g}) .*

Proof. The intrinsic metric on members of \mathcal{S} does not change. In particular, totally geodesic flats are preserved. \square

Throughout the paper we set

$$\text{curv}(X, W) \equiv R(X, W, W, X).$$

If

$$\text{span}\{X, W\}$$

is a 0-curvature plane of g , then a nearby plane has the form

$$\Pi_{X, W, Z, V}^{\sigma, \tau} \equiv \text{span}\{X + \sigma Z, W + \tau V\}$$

for some tangent vectors Z and V and some real numbers σ, τ . For each choice of $\{X, W, Z, V\}$ we then get a quartic polynomial

$$P(\sigma, \tau) = \text{curv}(X + \sigma Z, W + \tau V),$$

in σ and τ . A neighborhood of the zero planes (at the zero curvature points) can be described as

$$\left\{ \Pi_{X, W, Z, V}^{\sigma, \tau} \mid \text{curv}(X, W) = 0, (\sigma, \tau) \in [0, \varepsilon] \times [0, \varepsilon] \right\}.$$

Assuming that M is compact and ε is sufficiently small, we can arrange this representation so that all of the polynomials $P(\sigma, \tau)$ are positive on $[0, \varepsilon] \times [0, \varepsilon]$, except at $(\sigma, \tau) = (0, 0)$.

It is much easier to deform the metric within nonnegative curvature if, in addition, the total quadratic term of $P(\sigma, \tau)$ satisfies the following nondegeneracy condition

$\sigma^2 \text{curv}(Z, W) + 2\sigma\tau(R(X, W, V, Z) + R(X, V, W, Z)) + \tau^2 \text{curv}(X, V) > 0$ for all $(\sigma, \tau) \in S^1$.

We call this the *Quadratic Nondegeneracy Condition*.

Theorem 2.4. *Suppose that (M, g) is compact and nonnegatively curved and all of its zero planes are contained in a family \mathcal{S} of totally geodesic flat submanifolds, and satisfy the quadratic nondegeneracy condition.*

Let \tilde{g} be obtained from g as in the preceding proposition.

Then (M, \tilde{g}) is nonnegatively curved along the union of the family \mathcal{S} with precisely the same 0 curvature planes as g , provided \tilde{g} is sufficiently close to g in the C^2 -topology.

Remark 2.5. *This result can be viewed as an abstraction of Theorem 2.1 in [Wals].*

Proof. Since g and \tilde{g} are C^2 -close any 0-curvature planes of \tilde{g} must be close in the Grassmannian to 0-curvature planes of g .

We must show that

$$\tilde{P}(\sigma, \tau) = \text{curv}^{\tilde{g}}(X + \sigma Z, W + \tau V)$$

is also nonnegative on $[0, \varepsilon] \times [0, \varepsilon]$ and that it only vanishes when $\sigma = \tau = 0$.

Because X and W are tangent to a 0-curvature plane in a nonnegatively curved manifold

$$R^g(X, W)W = R^g(W, X)X = 0.$$

Since they are also tangent to a totally geodesic flat that is preserved under our deformation we have

$$R^{\tilde{g}}(X, W)W = R^{\tilde{g}}(W, X)X = 0.$$

So the constant and linear terms of $P(\sigma, \tau)$ and $\tilde{P}(\sigma, \tau)$ vanish.

Combining the quadratic nondegeneracy condition with the fact that

$$P(\sigma, \tau) \equiv \text{curv}^g(X + \sigma Z, W + \tau V) \geq 0,$$

and only vanishes within $[0, \varepsilon] \times [0, \varepsilon]$ when $\sigma = \tau = 0$, we conclude that $\tilde{P}(\sigma, \tau)$ is nonnegative and only vanishes within $[0, \varepsilon] \times [0, \varepsilon]$ when $\sigma = \tau = 0$, provided the coefficients of P and \tilde{P} are sufficiently close.

Thus (M, \tilde{g}) is nonnegatively curved on the union of the members of \mathcal{S} if \tilde{g} is sufficiently close to g in the C^2 -topology. \square

A problem with this theorem is that it does not tell us about the curvature of points in (M, \tilde{g}) that are not at a 0-curvature point of (M, g) . Of course there are various reasons why we might or might not know about these curvatures. In [PetWilh2], we will apply the following idea.

Corollary 2.6. *Let (M, g) be nonnegatively curved. Suppose $\pi : (M, g) \rightarrow \Sigma$ is a Riemannian submersion. Suppose also that the lifts of 0-planes of Σ are tangent to a family \mathcal{S} of totally geodesic flat submanifolds of M , and that the image*

$$\cup_{S \in \mathcal{S}} \pi(S)$$

contains a neighborhood U of all of the points of Σ that have 0-curvatures. Suppose that the quadratic nondegeneracy condition is satisfied on horizontal planes.

Let \tilde{g} be C^2 -close to g and satisfy

$$g(X, \cdot) = \tilde{g}(X, \cdot)$$

for all vectors tangent to a totally geodesic submanifold in \mathcal{S} .

If $\pi : (M, \tilde{g}) \rightarrow \Sigma$ is a Riemannian submersion, then the metric induced on Σ is nonnegatively curved with precisely the same 0 curvature planes as g .

3. ORTHOGONAL PARTIAL CONFORMAL CHANGE

With a few qualifications, the Orthogonal Partial Conformal Change in [PetWilh2] fits into the basic set up of the preceding corollary for the submersion $Sp(2) \rightarrow \Sigma^7$. The main deficiency is that the deformation of [PetWilh2] is only C^1 -small.

Although the preceding corollary is false for arbitrary C^1 -small deformations, there is a narrowly constrained situation where it holds.

The main tool is proven using Cartan formalism ([Spiv], Chap 7).

Lemma 3.1. *Suppose that $\{E_i\}$ is an orthonormal frame for g with dual coframe $\{\theta^i\}$. Suppose that $\tilde{\theta}^i = \phi^i \theta^i$ is an orthonormal coframe for \tilde{g} , where ϕ^i are smooth functions on M . Assume that*

$$d\phi^i = \psi^i \theta^1$$

and that

$$d\psi^i = \lambda^i \theta^1$$

for some other smooth functions ψ^i and λ^i . If the functions ϕ^i are close to 1 in the C^1 -topology, then the only components of $R^{\tilde{g}}(\tilde{E}_i, \tilde{E}_j, \tilde{E}_k, \tilde{E}_l)$ that are not close to R^g are the terms that up to symmetries of the curvature tensor can be reduced to $R^{\tilde{g}}(\tilde{E}_1, \tilde{E}_i, \tilde{E}_i, \tilde{E}_1)$.

Remark 3.2. *Note that the meaning of “close” depends on g .*

Proof. Following ([Spiv], Chap 7) we define $\{b_{jk}^i\}$, $\{a_{jk}^i\}$, and $\{\omega_j^i\}$ by

$$d\theta^i = \frac{1}{2} \sum_{j,k=1}^n b_{jk}^i \theta^j \wedge \theta^k,$$

$$a_{jk}^i = \frac{1}{2} (b_{jk}^i + b_{ki}^j - b_{ij}^k)$$

$$\omega_j^i = \sum_{k=1}^n a_{jk}^i \theta^k.$$

It then follows ([Spiv], Chap 7) that

$$b_{jk}^i = -b_{kj}^i \text{ and}$$

$$a_{jk}^i = -a_{ik}^j$$

The forms Ω_j^i

$$\Omega_j^i \equiv d\omega_j^i + \sum_{k=1}^n \omega_k^i \wedge \omega_j^k$$

are then curvatures. Specifically

$$g(R(X, Y) E_j, E_i) = \Omega_j^i(X, Y).$$

We now check how these functions get changed for the new frame.

$$\begin{aligned}
d\tilde{\theta}^i &= d(\phi^i\theta^i) \\
&= d\phi^i \wedge \theta^i + \frac{1}{2} \sum_{j,k=1}^n \phi^i b_{jk}^i \theta^j \wedge \theta^k \\
&= \psi^i \theta^1 \wedge \theta^i + \frac{1}{2} \sum_{j,k=1}^n \phi^i b_{jk}^i \theta^j \wedge \theta^k \\
&= \frac{1}{2} \frac{\psi^i}{\phi^1 \phi^i} \tilde{\theta}^1 \wedge \tilde{\theta}^i - \frac{1}{2} \frac{\psi^i}{\phi^1 \phi^i} \tilde{\theta}^i \wedge \tilde{\theta}^1 \\
&\quad + \frac{1}{2} \sum_{j,k=1}^n \frac{\phi^i}{\phi^j \phi^k} b_{jk}^i \tilde{\theta}^j \wedge \tilde{\theta}^k.
\end{aligned}$$

So the only \tilde{b}_{jk}^i s that depend on ψ are

$$\tilde{b}_{1i}^i = -\tilde{b}_{i1}^i = \frac{\psi^i}{\phi^1 \phi^i} + \frac{\phi^i}{\phi^1 \phi^i} b_{1i}^i.$$

So among the

$$\tilde{a}_{jk}^i = \frac{1}{2} (\tilde{b}_{jk}^i + \tilde{b}_{ki}^j - \tilde{b}_{ij}^k)$$

the ones potentially affected by ψ are $\tilde{a}_{1i}^i, \tilde{a}_{i,1}^i$, and $\tilde{a}_{i,i}^1$. However, the antisymmetry $\tilde{a}_{jk}^i = -\tilde{a}_{kj}^i$ implies that $\tilde{a}_{i,1}^i = 0$, and that $\tilde{a}_{1i}^i = -\tilde{a}_{i,i}^1$. The antisymmetry of the b s then gives us

$$\tilde{a}_{1i}^i = -\tilde{a}_{i,i}^1 = \tilde{b}_{1i}^i.$$

So in fact, the \tilde{a} s that depend on ψ are

$$\begin{aligned}
\tilde{a}_{1i}^i &= -\tilde{a}_{i,i}^1 \\
&= \frac{1}{2} (\tilde{b}_{1i}^i + \tilde{b}_{ii}^1 - \tilde{b}_{i1}^i) \\
&= \left(\frac{\psi^i}{\phi^1 \phi^i} + \frac{\phi^i}{\phi^1 \phi^i} b_{1i}^i \right) \\
&= \left(\frac{\psi^i}{\phi^1 \phi^i} + \frac{\phi^i}{\phi^1 \phi^i} a_{1i}^i \right)
\end{aligned}$$

Thus the only ω_j^i s that depend on ψ are

$$\begin{aligned}
\tilde{\omega}_1^i &= -\tilde{\omega}_i^1 \\
&= \tilde{a}_{1i}^i \tilde{\theta}^i + \sum_{k \neq i} \tilde{a}_{1k}^i \tilde{\theta}^k \\
&= \frac{\psi^i}{\phi^1 \phi^i} \tilde{\theta}^i + \sum_k a_{1k}^i \theta^k + O(C^0)
\end{aligned}$$

where by $O(C^0)$ we mean $O(\max\{1 - \phi^i\}) \sum_k \theta^k$.

It follows that the only Ω_j^i s that depend on the λ s are

$$\Omega_1^i = -\Omega_i^1 = d\tilde{\omega}_1^i + \sum_{k=1}^n \omega_k^i \wedge \omega_1^k.$$

Only the first term feels this “ C^2 -effect”. It is

$$d\tilde{\omega}_1^i = d\left(\frac{\psi^i}{\phi^1 \phi^i} \tilde{\theta}^i\right) + d\omega_1^i + O(C^1),$$

where by $O(C^1)$ we mean

$$O(\max\{1 - \phi^i, \psi^i\}) \sum_k d\theta^k + d\left(O(\max\{1 - \phi^i\}) \sum_k \theta^k\right).$$

We conclude that

$$d\tilde{\omega}_1^i = \lambda^i \theta^1 \wedge \tilde{\theta}^i + d\omega_1^i + O(C^1).$$

So we note that the only curvatures affected by the C^2 change are the sectional curvature spanned by E_1 and E_i \square

In our applications we will also need to know something more specific about how the other components of the curvature tensor change with such a deformation.

Corollary 3.3. *If at most one of the indices $\{i, j, k, l\}$ is 1, then*

$$\left| R^{\tilde{g}}(\tilde{E}_i, \tilde{E}_j, \tilde{E}_k, \tilde{E}_l) - R(E_i, E_j, E_k, E_l) \right| \leq O(\max\{\psi^i, 1 - \phi^i\}) O\left(\max_{i,j,k} \{|\omega_j^i|, |da_{jk}^i|\}\right)$$

Proof. We have

$$\left| \sum_{p=1}^n \tilde{\omega}_p^i \wedge \tilde{\omega}_j^p(\tilde{E}_l, \tilde{E}_k) - \sum_{p=1}^n \omega_p^i \wedge \omega_j^p(E_l, E_k) \right| \leq (O(\max\{1 - \phi^i, \psi^i\}) \max\{b_{jk}^i\})^2,$$

and

$$\begin{aligned} d\tilde{\omega}_j^i(\tilde{E}_l, \tilde{E}_p) &= d\left[\sum_{k=1}^n \tilde{a}_{jk}^i \tilde{\theta}^k\right](\tilde{E}_l, \tilde{E}_p) \\ &= \left[\sum_{k=1}^n d\tilde{a}_{jk}^i \wedge \tilde{\theta}^k + \tilde{a}_{jk}^i d\tilde{\theta}^k\right](\tilde{E}_l, \tilde{E}_p) \\ &= d\omega_j^i(E_l, E_p) \\ &\quad + O(\max\{\psi^i, 1 - \phi^i\}) \sum_{k=1}^n da_{jk}^i \wedge \theta^k(E_l, E_p) + O(\max\{\psi^i, 1 - \phi^i\}) \sum_{k=1}^n a_{jk}^i d\theta^k(E_l, E_p) \end{aligned}$$

Combining these formulas yields

$$\left| R^{\tilde{g}} - R \right| \leq O(C^1) O\left(\max_{i,j,k} \{|\omega_j^i|, |b_{jk}^i|, |da_{jk}^i|\}\right)$$

Since $O(\max_{i,j,k} \{b_{jk}^i\}) \leq O(\max_{i,j} |\omega_j^i|)$ the result follows. \square

In addition to the general set up of Corollary 2.6 we also assume,

- 1:** There is a smooth distance function r defined on a neighborhood of \mathcal{S} whose gradient we call X .

2: X is tangent to all the flats in \mathcal{S} .

Let \mathcal{O} be a distribution that is normal to each $S \in \mathcal{S}$, and let

$$\varphi \equiv f \circ r$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$, and is constant outside of a compact interval.

Let \tilde{g} be obtained from g by multiplying the lengths of all vectors in \mathcal{O} by φ , while keeping the orthogonal complement and the metric on the orthogonal complement of \mathcal{O} fixed. That is, \tilde{g} is obtained from g by doing a partial conformal change with distribution $\mathcal{D} = \mathcal{O}$ and conformal factor φ^2 .

Now let $\{E_i\}$ be an orthonormal frame for g with $X = E_1$, and $\text{span}\{E_2, \dots, E_p\} = \mathcal{O}$. Setting $\phi^i = \varphi$ for $i = 2, \dots, p$, and $\phi^i \equiv 1$ otherwise, then gives an example of the above lemma.

Applying the last formula in the proof of the preceding lemma to our situation yields.

Proposition 3.4. *For $V \in \mathcal{O}$*

$$R^{\tilde{g}}(V, X, X, V) = R^g(V, X, X, V) - \varphi'' |V|_g^2 |X|_g^2 + O(C^1)$$

where $\varphi'' = D_X D_X(\varphi) = f'' \circ r$.

Combining this with the previous lemma and the proofs of Theorem 2.4 and Corollary 2.6 gives us.

Theorem 3.5. *There is an $\varepsilon > 0$ so that (Σ, \tilde{g}) is nonnegatively curved provided,*

- φ is sufficiently close to 1 in the C^1 -topology, and
- For any $V \in \mathcal{O}$

$$\varepsilon \langle R^g(X, V) V, X \rangle - \varphi'' |V|_g^2 |X|_g^2 > 0$$

Moreover \tilde{g} has precisely the same 0 curvature planes as g .

Remark 3.6. *There are three further complications when the ideas of the previous two sections are applied in [PetWilh2].*

First, the parameter that describes the orthogonal partial conformal change is related to the parameter that describes one of the Cheeger deformations, and hence we can not merely use Lemma 3.1, but also must use Corollary 3.3. This is not a difficult point, but it is better to address it concretely.

Second, the actual field, X , that is used has a singularity along a set where it is multi-valued. This causes some of the connection forms for the original metric to blow up near these points. This means that the C^1 -change in φ can actually have a large effect on some curvatures. The set up is such that this effect only increases curvatures. This is also not a difficult point, but it is one that is best addressed concretely.

Lastly, we must verify that the quadratic nondegeneracy condition of Theorem 2.4 holds on Σ^7 . In the end we shall see that it only holds generically. This problem and its resolution turn out to be related to the second problem about the blow up of certain connection forms, as the places where the nondegeneracy condition fails are precisely at the poles of X .

In subsection 6.1, we provide a tool that shows how Cheeger deformations can be useful in simplifying the problem of verifying the nondegeneracy condition. We apply this tool in [PetWilh2] to show that Σ^7 satisfies the nondegeneracy condition except at the poles of X .

This situation is far from ideal since it means that we almost have quadratic degeneracy as we approach a pole of X . We will show that the orthogonal partial conformal change actually improves this situation, and makes it as nice as it can be.

This is accomplished by proving a further result that shows that the orthogonal partial conformal change gives us quadratic nondegeneracy condition, in a quantitative sense, as close as we please to the poles of X . This result exploits the blow up of the connection forms near the poles of X , and is also proven in [PetWilh2].

4. INTEGRALLY POSITIVE CURVATURE

Here we give abstract criteria that are sufficient to create integrally positive curvature on a totally geodesic flat, when the fibers of a Riemannian submersion are scaled. The only application of the theorem that we are aware of is to the Gromoll-Meyer sphere with the metric from [Wilh]. The issue of why the metric of [Wilh] satisfies the hypotheses of this theorem will be addressed in [PetWilh2].

Throughout this section let (M, g_0) be a Riemannian manifold with nonnegative sectional curvature and

$$\pi : (M, g_0) \longrightarrow B$$

a Riemannian submersion. Let g_s be the metric obtained from g_0 by scaling the lengths of the fibers of π by

$$\sqrt{1 - s^2}.$$

As usual we use the superscripts \mathcal{H} and \mathcal{V} to denote the horizontal and vertical parts of the vectors, R and A are the curvature and A -tensors for the unperturbed metric g , R^{g_s} denotes the new curvature tensor of g_s , and R^B is the curvature tensor of the base. We use the term ‘‘geodesic field’’ for any field X so that $\nabla_X X = 0$.

Theorem 4.1. *Let $T \subset M$ be a totally geodesic, flat torus spanned by commuting, orthogonal, geodesic fields X and W such that X is horizontal for π and $D\pi(W)$ is a Jacobi field along the integral curves of $D\pi(X)$.*

Then

$$(4.1) \quad R^{g_s}(X, W, W, X) = -\frac{s^2}{2} \left(D_X \left(D_X |W^{\mathcal{H}}|^2 \right) \right) + s^4 |A_X W^{\mathcal{V}}|^2.$$

In particular, if c is an integral curve of $D\pi(X)$ from a zero of $|W^{\mathcal{H}}|$ to a maximum of $|W^{\mathcal{H}}|$ along c , then

$$\int_c \text{curv}_{g_s}(X, W) = s^4 \int_c |A_X W^{\mathcal{V}}|^2.$$

So the curvature of $\text{span}\{X, W\}$ is integrally positive along c , provided $|A_X W^{\mathcal{V}}|^2$ is not identically 0 along c .

The reader should note that the above curvature formula is as important as the fact that the integral is positive. Since X is a geodesic field, the larger term $-\frac{s^2}{2} \left(D_X \left(D_X |W^{\mathcal{H}}|^2 \right) \right)$ is the Hessian, $\text{Hess}f(X, X)$, of the function

$$f = -\frac{s^2}{2} |W^{\mathcal{H}}|^2.$$

Therefore, we can cancel it with a conformal change involving f . Such a conformal change will create other terms of order s^4 in our expression for $\text{curv}_{g_s}(X, W)$. To

compare these terms with $s^4 |A_X W^\mathcal{V}|^2$, we will evaluate $A_X W^\mathcal{V}$ in the presence of some additional hypotheses, after we prove the theorem above. These additional hypotheses will also allow us to obtain formulas for the $(1, 3)$ -tensor, $R^{g_s}(W, X)X$ and the horizontal part of the $(1, 3)$ -tensor, $R^{g_s}(X, W)W$. To actually put positive curvature on the Gromoll–Meyer sphere, or indeed to perturb a neighborhood of any totally geodesic flat to positive curvature, these formulas will of course be necessary.

After refining our formula for $\text{curv}_{g_s}(X, W)$, we will explain in the next subsection precisely how to combine fiber scaling and a conformal change to put positive curvature on a single initially flat torus, subject to a few additional hypotheses.

Scaling the fibers of a Riemannian submersion was dubbed the “canonical variation” in [Bes]. One can find formulas for how curvature changes under the canonical variation in any of [Bes], [Dear1], [GromDur], or [GromWals]. To ultimately get positive curvature on the Gromoll–Meyer sphere, we have to control the curvature tensor in an entire neighborhood in the Grassmannian, so we will need several of these formulas. In fact, since the particular “ W ” that we have in mind is neither horizontal nor vertical for π , we need multiple formulas just to find $\text{curv}(X, W)$.

Given vertical vectors $U, V \in \mathcal{V}$ and horizontal vectors $X, Y, Z \in \mathcal{H}$, for $\pi : M \rightarrow B$ we have

$$\begin{aligned} (R^{g_s}(X, V)U)^\mathcal{H} &= (1 - s^2)(R(X, V)U)^\mathcal{H} + (1 - s^2)s^2 A_{A_X} U V \\ R^{g_s}(V, X)Y &= (1 - s^2)R(V, X)Y + s^2(R(V, X)Y)^\mathcal{V} + s^2 A_X A_Y V \\ (4.2) \quad R^{g_s}(X, Y)Z &= (1 - s^2)R(X, Y)Z + s^2(R(X, Y)Z)^\mathcal{V} + s^2 R^B(X, Y)Z \end{aligned}$$

To eventually understand the curvature in a neighborhood of the Gromoll–Meyer 0-locus, we will need formulas for

$$R^{g_s}(W, X)X \text{ and } (R^{g_s}(X, W)W)^\mathcal{H}$$

where X is as above and W is an arbitrary vector in TM .

Splitting W into horizontal and vertical parts and applying the formulas above we obtain the following.

Lemma 4.3. *Let X be a horizontal vector for π and let W be an arbitrary vector in TM . Then*

$$\begin{aligned} R^{g_s}(W, X)X &= (1 - s^2)R(W, X)X + s^2(R(W, X)X)^\mathcal{V} \\ &\quad + s^2 R^B(W^\mathcal{H}, X)X + s^2 A_X A_X W^\mathcal{V} \\ (R^{g_s}(X, W)W)^\mathcal{H} &= (1 - s^2)(R(X, W)W)^\mathcal{H} \\ &\quad + (1 - s^2)s^2 A_{A_X W^\mathcal{V}} W^\mathcal{V} + s^2 R^B(X, W^\mathcal{H})W^\mathcal{H} \end{aligned}$$

Remark 4.4. *Notice that the first curvature terms vanish in both formulas on the totally geodesic flat tori.*

Using the fact that $\text{curv}^{g_0}(X, W) = 0$ and either of the formulas for $R^{g_s}(W, X)X$ or $(R^{g_s}(X, W)W)^\mathcal{H}$ we have

$$(4.4) \quad \text{curv}^{g_s}(X, W) = s^2 \text{curv}^B(X, W^\mathcal{H}) - (1 - s^2)s^2 |A_X W^\mathcal{V}|^2.$$

Since $D\pi(W^{\mathcal{H}})$ is a Jacobi field along c , and writing $W^{\mathcal{H}}$ for $D\pi(W^{\mathcal{H}})$ we have

$$\begin{aligned} \text{curv}^B(X, W^{\mathcal{H}}) &= -\langle \nabla_X^B \nabla_X^B W^{\mathcal{H}}, W^{\mathcal{H}} \rangle \\ &= -D_X \langle \nabla_X^B W^{\mathcal{H}}, W^{\mathcal{H}} \rangle + \langle \nabla_X^B W^{\mathcal{H}}, \nabla_X^B W^{\mathcal{H}} \rangle \\ &= -\frac{1}{2} D_X D_X \langle W^{\mathcal{H}}, W^{\mathcal{H}} \rangle + \langle \nabla_X^B W^{\mathcal{H}}, \nabla_X^B W^{\mathcal{H}} \rangle \end{aligned}$$

Since $\nabla_X W \equiv 0$ we have

$$\begin{aligned} 0 &\equiv \nabla_X W \\ &= \nabla_X W^{\mathcal{H}} + \nabla_X W^{\mathcal{V}}. \end{aligned}$$

The horizontal part of this equation gives us

$$A_X W^{\mathcal{V}} = -(\nabla_X W^{\mathcal{H}})^{\mathcal{H}}.$$

Identifying $(\nabla_X W^{\mathcal{H}})^{\mathcal{H}}$ with $\nabla_X^B W^{\mathcal{H}}$ and substituting into the formula for $\text{curv}^B(X, W^{\mathcal{H}})$ we obtain

$$\text{curv}^B(X, W^{\mathcal{H}}) = -\frac{1}{2} D_X D_X |W^{\mathcal{H}}|^2 + |A_X W^{\mathcal{V}}|^2$$

Substituting this into our formula for $\text{curv}^{g_s}(X, W)$ yields

$$\begin{aligned} \text{curv}^{g_s}(X, W) &= -s^2 \frac{1}{2} D_X D_X |W^{\mathcal{H}}|^2 + s^2 |A_X W^{\mathcal{V}}|^2 - (1-s^2) s^2 |A_X W^{\mathcal{V}}|^2 \\ &= -s^2 \frac{1}{2} D_X D_X |W^{\mathcal{H}}|^2 + s^4 |A_X W^{\mathcal{V}}|^2, \end{aligned}$$

proving Theorem 4.1.

To help evaluate $A_X W^{\mathcal{V}}$, we add some general assumptions about the Riemannian submersion $\pi : (M, g_0) \rightarrow B$.

- There is an isometric action by G on M that is by symmetries of π .
- The intrinsic metrics on the principal orbits of G in B are homotheties of each other.
- The normal distribution to the orbits of G on B is integrable.

In addition we add some specific conditions to the hypotheses of Theorem 4.1:

- $W^{\mathcal{H}}$ is a Killing field for the G -action on B .
- $D\pi(X)$ is invariant under the action that G induces on B .
- $D\pi(X)$ is orthogonal to the orbits of G .

Since the normal distribution to the orbits of G on B is integrable we can extend any normal vector Z to a G -invariant normal field Z . Writing X for $D\pi(X)$ it then follows that all terms in the Koszul formula for

$$\langle \nabla_{W^{\mathcal{H}}}^B X, Z \rangle$$

vanish. In particular, $\nabla_{W^{\mathcal{H}}}^B X$ is tangent to the orbits of G .

If K is another Killing field for G , then X commutes with K as well as $W^{\mathcal{H}}$, thus $[K, W^{\mathcal{H}}]$ is perpendicular to X as it is again a Killing field. Combining this with our hypothesis that the intrinsic metrics on the principal orbits of G in B are homotheties of each other, we see from Koszul's formula that $\nabla_{W^{\mathcal{H}}} X$ is proportional

to $W^{\mathcal{H}}$ and can be calculated by

$$\begin{aligned} \langle \nabla_{W^{\mathcal{H}}} X, W^{\mathcal{H}} \rangle &= \langle \nabla_X W^{\mathcal{H}}, W^{\mathcal{H}} \rangle \\ &= \frac{1}{2} D_X |W^{\mathcal{H}}|^2 \\ &= |W^{\mathcal{H}}| D_X |W^{\mathcal{H}}|, \text{ so} \\ \nabla_{W^{\mathcal{H}}} X &= \frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} W^{\mathcal{H}}. \end{aligned}$$

Since

$$A_X W^{\mathcal{V}} = -(\nabla_X W^{\mathcal{H}})^{\mathcal{H}}$$

we conclude

Lemma 4.5. *With the additional hypotheses mentioned above*

$$A_X W^{\mathcal{V}} = -\frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} W^{\mathcal{H}}.$$

Plugging this into our curvature formula we get

$$(4.5) \quad \text{curv}^{gs}(X, W) = -s^2 \frac{1}{2} D_X D_X |W^{\mathcal{H}}|^2 + s^4 |D_X |W^{\mathcal{H}}||^2.$$

As we've mentioned, to get positive curvature on the Gromoll-Meyer sphere, we will have to understand certain other components of the $(1, 3)$ curvature tensor.

Lemma 4.6. *Using $W^{\mathcal{H}}$ for $d\pi(W)$ and X for $d\pi(X)$*

$$R^B(W^{\mathcal{H}}, X) X = -\left(\frac{D_X D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} \right) W^{\mathcal{H}}$$

Proof. Since X is a geodesic field and $W^{\mathcal{H}}$ is a Jacobi field along the integral curves of X

$$R^B(W^{\mathcal{H}}, X) X = -\nabla_X \nabla_X W^{\mathcal{H}}.$$

We discovered above that

$$\nabla_X W^{\mathcal{H}} = \nabla_{W^{\mathcal{H}}} X = \frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} W^{\mathcal{H}}.$$

Thus

$$\begin{aligned} R^B(W^{\mathcal{H}}, X) X &= -\nabla_X \left(\frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} W^{\mathcal{H}} \right) \\ &= -D_X \left(\frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} \right) W^{\mathcal{H}} - \left(\frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} \nabla_X W^{\mathcal{H}} \right) \\ &= -\left(\frac{|W^{\mathcal{H}}| D_X D_X |W^{\mathcal{H}}| - (D_X |W^{\mathcal{H}}|)^2}{|W^{\mathcal{H}}|^2} \right) W^{\mathcal{H}} - \left(\frac{D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} \right)^2 W^{\mathcal{H}} \\ &= -\left(\frac{D_X D_X |W^{\mathcal{H}}|}{|W^{\mathcal{H}}|} \right) W^{\mathcal{H}}. \end{aligned}$$

□

Lemma 4.7. *Using $W^{\mathcal{H}}$ for $d\pi(W)$ and X for $d\pi(X)$*

$$R^B(X, W^{\mathcal{H}})W^{\mathcal{H}} = -|W^{\mathcal{H}}|\nabla_X(\text{grad}|W^{\mathcal{H}}|).$$

Proof. Let Z be any vector field. Using that $W^{\mathcal{H}}$ is a Killing field we get

$$\begin{aligned} \langle \nabla_{W^{\mathcal{H}}}W^{\mathcal{H}}, Z \rangle &= -\langle \nabla_ZW^{\mathcal{H}}, W^{\mathcal{H}} \rangle \\ &= -\frac{1}{2}D_Z\langle W^{\mathcal{H}}, W^{\mathcal{H}} \rangle \\ &= -\frac{1}{2}D_Z|W^{\mathcal{H}}|^2 \\ &= -|W^{\mathcal{H}}|D_Z|W^{\mathcal{H}}| \\ &= -\langle |W^{\mathcal{H}}|\text{grad}|W^{\mathcal{H}}|, Z \rangle \end{aligned}$$

showing that

$$\nabla_{W^{\mathcal{H}}}W^{\mathcal{H}} = -|W^{\mathcal{H}}|\text{grad}|W^{\mathcal{H}}|.$$

Thus

$$\begin{aligned} R^B(X, W^{\mathcal{H}})W^{\mathcal{H}} &= \nabla_X\nabla_{W^{\mathcal{H}}}W^{\mathcal{H}} - \nabla_{W^{\mathcal{H}}}\nabla_XW^{\mathcal{H}} \\ &= -\nabla_X(|W^{\mathcal{H}}|\text{grad}|W^{\mathcal{H}}|) - \nabla_{W^{\mathcal{H}}}\left(\frac{D_X|W^{\mathcal{H}}|}{|W^{\mathcal{H}}|}W^{\mathcal{H}}\right) \\ &= -(D_X|W^{\mathcal{H}}|)\text{grad}|W^{\mathcal{H}}| - (|W^{\mathcal{H}}|\nabla_X\text{grad}|W^{\mathcal{H}}|) - \frac{D_X|W^{\mathcal{H}}|}{|W^{\mathcal{H}}|}\nabla_{W^{\mathcal{H}}}W^{\mathcal{H}} \\ &= -(D_X|W^{\mathcal{H}}|)\text{grad}|W^{\mathcal{H}}| - (|W^{\mathcal{H}}|\nabla_X\text{grad}|W^{\mathcal{H}}|) + \frac{D_X|W^{\mathcal{H}}|}{|W^{\mathcal{H}}|}|W^{\mathcal{H}}|\text{grad}|W^{\mathcal{H}}| \\ &= -(|W^{\mathcal{H}}|\nabla_X\text{grad}|W^{\mathcal{H}}|) \end{aligned}$$

□

Combining the calculations above we have

Lemma 4.8. *Let X and W be as in Theorem 4.1. Then*

$$\begin{aligned} R^{g_s}(W, X)X &= -s^2\left(\frac{D_XD_X|W^{\mathcal{H}}|}{|W^{\mathcal{H}}|}\right)W^{\mathcal{H}} - s^2\frac{D_X|W^{\mathcal{H}}|}{|W^{\mathcal{H}}|}A_XW^{\mathcal{H}} \\ (R^{g_s}(X, W)W)^{\mathcal{H}} &= -(1-s^2)s^2\frac{D_X|W^{\mathcal{H}}|}{|W^{\mathcal{H}}|}A_{W^{\mathcal{H}}}W^{\mathcal{V}} - s^2|W^{\mathcal{H}}|\nabla_X(\text{grad}|W^{\mathcal{H}}|). \end{aligned}$$

Remark 4.9. *The two A -tensors $A_XW^{\mathcal{H}}$ and $A_{W^{\mathcal{H}}}W^{\mathcal{V}}$ involve derivatives of vectors that are not tangent or normal to the totally geodesic tori. They cannot be determined abstractly, and are in fact dependent on the particular geometry. We give estimates for them in the case of the Gromoll-Meyer sphere in the section of [PetWilh2] called “Concrete A -tensor estimates”.*

4.1. Positive Curvature on a Single Initially Flat Torus. In this subsection we will explain how our fiber scaling calculations can be combined with a conformal change to put positive curvature on a *single* flat torus, T , that satisfies the hypotheses of the previous section as well as a few other mild hypotheses. This fact may seem reassuring, however, we emphasize that for the following reasons it is not sufficient to get positive curvature on the Gromoll–Meyer sphere.

- We will not learn (much) about the curvatures of nearby planes,

- The Gromoll–Meyer sphere with the metric of [Wilh] has many totally geodesic flat tori. For reasons that we shall make explicit in the next section, fiber scaling combined with a conformal change can not be used to put positive curvature on all of these tori simultaneously.

In Section 4 we discuss an abstract situation that allows for a certain type of partial conformal change to affect certain curvatures in the same way as an actual conformal change. By combining the results of that section and this one we will have a method that puts positive curvature on all of the totally geodesic flats of the Gromoll–Meyer sphere simultaneously, modulo the question of verifying that the Gromoll–Meyer sphere satisfies all of the necessary hypotheses. This last question is resolved in [PetWilh2], as well as the issue of actually verifying positive curvature.

Imagine that $T \subset M$ is a totally geodesic flat torus spanned by geodesic fields X and W satisfying all of the hypotheses of the previous section. Let

$$\tilde{T} : [0, \pi] \times [0, l] \longrightarrow M$$

be a parameterization of T with X a unit field whose integral curves, c_{s_0} , are

$$c_{s_0} : t \longmapsto \tilde{T}(t, s_0).$$

In particular, the integral curves of X are periodic with minimal period π .

We also assume that along each c_{s_0} the key function $|W^{\mathcal{H}}|$

- is periodic in the first variable with period $\frac{\pi}{2}$, i.e. $|W^{\mathcal{H}}|_{\tilde{T}(t, s_0)} = |W^{\mathcal{H}}|_{\tilde{T}(t+\frac{\pi}{2}, s_0)}$,
- has zeros only when t is $0, \frac{\pi}{2}, \pi, \dots$,
- and maxima only when t is $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$

We also assume that

$$\text{dist}\left(\tilde{T}(\{0\} \times [0, l]), \cdot\right)$$

is smooth on $\tilde{T}\left(\left(0, \frac{\pi}{4}\right) \times [0, l]\right)$ with gradient X .

To simplify notation we set

$$\psi = |W^{\mathcal{H}}|.$$

So after scaling the fibers of π by $\sqrt{1-s^2}$ we have from 4.5

$$(4.9) \quad \text{curv}_{g_s}(X, W) = -s^2(D_X(\psi D_X \psi)) + s^4(D_X \psi)^2.$$

We remind the reader that after the conformal change $\tilde{g} = e^{2f} g_s$ we will have

$$\begin{aligned} e^{-2f} \text{curv}_{\tilde{g}}(X, W) &= \text{curv}_{g_s}(X, W) - |W|_{g_s}^2 \text{Hess}f(X, X) - \text{Hess}f(W, W) \\ &\quad + (D_X f)^2 |W|_{g_s}^2 - |\text{grad}f|^2 |W|_{g_s}^2, \end{aligned}$$

provided X is unit and W is perpendicular to $\text{grad}f$ (cf [Pet] Exercise 3.5)

Our choice of conformal factor will look like

$$f = -\frac{s^2}{2(1-s^2)} \frac{\psi^2}{|W|^2} + \text{a much smaller term.}$$

The first conformal term $-|W|_{g_s}^2 \text{Hess}f(X, X)$ will nearly cancel with the leading term $-s^2(D_X(\psi D_X \psi))$ in $\text{curv}_{g_s}(X, W)$. For our initial metric $\nabla_W W \equiv 0$, so $\text{Hess}f(W, W)$ has order s^4 , as do the other two conformal terms, $(D_X f)^2 |W|_{g_s}^2$ and $|\nabla f|^2 |W|_{g_s}^2$. In the remainder of this section we will see more precisely what these terms actually are.

To do this we name the ‘‘much smaller term’’, E . The function E has the form

$$E = s^4 I \circ \text{dist} \left(\tilde{T}(\{0\} \times [0, l]), \cdot \right)$$

where $I : \mathbb{R} \rightarrow \mathbb{R}$ is a function that satisfies

$$I'(0) = I'\left(\frac{\pi}{4}\right) = 0,$$

Thus

$$\text{grad } f = -\frac{s^2}{(1-s^2)|W|^2} \psi \text{grad} \psi + s^4 I' X$$

To understand the effect that this conformal change has on our curvatures we will need to know the Hessian of f , and hence a covariant derivative that we have yet to compute.

Proposition 4.10.

$$\nabla_W^{g_s} W = -s^2 \psi \text{grad} \psi,$$

Proof. Before the fiber scaling $\nabla_W W = 0$. Breaking W into horizontal and vertical parts and using the Koszul formula we get

$$\begin{aligned} \nabla_W^{g_s} W &= \nabla_{W^\nu}^{g_s} W^\nu + \nabla_{W^\nu}^{g_s} W^\mathcal{H} + \nabla_{W^\mathcal{H}}^{g_s} W^\nu + \nabla_{W^\mathcal{H}}^{g_s} W^\mathcal{H} \\ &= (\nabla_{W^\nu} W^\nu)^\nu + (1-s^2) (\nabla_{W^\nu} W^\nu)^\mathcal{H} \\ &\quad + (\nabla_{W^\nu} W^\mathcal{H})^\nu + (1-s^2) (\nabla_{W^\nu} W^\mathcal{H})^\mathcal{H} \\ &\quad + (\nabla_{W^\mathcal{H}} W^\nu)^\nu + (1-s^2) (\nabla_{W^\mathcal{H}} W^\nu)^\mathcal{H} \\ &\quad + \nabla_{W^\mathcal{H}} W^\mathcal{H} \end{aligned}$$

Rearranging terms and using the fact that $\nabla_W W = 0$ yields

$$\nabla_W^{g_s} W = -s^2 [(\nabla_{W^\nu} W^\nu) + \nabla_{W^\nu} W^\mathcal{H} + \nabla_{W^\mathcal{H}} W^\nu]^\mathcal{H}.$$

We also have $(\nabla_W W)^\mathcal{H} = 0$, so

$$[\nabla_{W^\nu} W^\nu + \nabla_{W^\nu} W^\mathcal{H} + \nabla_{W^\mathcal{H}} W^\nu + \nabla_{W^\mathcal{H}} W^\mathcal{H}]^\mathcal{H} = 0.$$

Thus

$$\begin{aligned} \nabla_W^{g_s} W &= -s^2 [(\nabla_{W^\nu} W^\nu) + \nabla_{W^\nu} W^\mathcal{H} + \nabla_{W^\mathcal{H}} W^\nu]^\mathcal{H} \\ &= s^2 \nabla_{W^\mathcal{H}} W^\mathcal{H} \\ &= -s^2 |W^\mathcal{H}| \text{grad} |W^\mathcal{H}| \\ &= -s^2 \psi \text{grad} \psi, \end{aligned}$$

where we have used an equation in the proof of Lemma 4.7 for the next to last inequality. \square

Proposition 4.11.

$$\begin{aligned} \text{Hess } f(X, X) &= -\frac{s^2}{(1-s^2)|W|^2} D_X(\psi D_X \psi) + s^4 I'' \\ \text{Hess } f(W, W) &= -\frac{s^4}{(1-s^2)|W|^2} \psi^2 |\text{grad} \psi|^2 + O(s^6) \end{aligned}$$

Proof. Since

$$\operatorname{grad} f = -\frac{s^2}{(1-s^2)|W|^2} \psi \operatorname{grad} \psi + s^4 I' X$$

we have

$$\begin{aligned} \operatorname{Hess} f(X, X) &= -\frac{s^2}{(1-s^2)|W|^2} \langle \nabla_X (\psi \operatorname{grad} \psi), X \rangle + s^4 \langle \nabla_X (I' X), X \rangle \\ &= -\frac{s^2}{(1-s^2)|W|^2} \left((D_X \psi)^2 + \psi \langle \nabla_X (\operatorname{grad} \psi), X \rangle \right) + s^4 I'' \\ &= -\frac{s^2}{(1-s^2)|W|^2} \left((D_X \psi)^2 + \psi D_X D_X \psi \right) + s^4 I'' \\ &= -\frac{s^2}{(1-s^2)|W|^2} D_X (\psi D_X \psi) + s^4 I'' \end{aligned}$$

Since W is perpendicular to $\operatorname{grad} f$ we have

$$\begin{aligned} \operatorname{Hess} f(W, W) &= \langle \nabla_W \operatorname{grad} f, W \rangle \\ &= -\langle \operatorname{grad} f, \nabla_W W \rangle \end{aligned}$$

Using the previous proposition this gives us

$$\begin{aligned} \operatorname{Hess} f(W, W) &= -\left\langle -\frac{s^2}{(1-s^2)|W|^2} \psi \operatorname{grad} \psi, -s^2 \psi \operatorname{grad} \psi \right\rangle - s^4 \langle I' X, -s^2 \psi \operatorname{grad} \psi \rangle \\ &= -\frac{s^4}{(1-s^2)|W|^2} \psi^2 |\operatorname{grad} \psi|^2 + O(s^6) \end{aligned}$$

□

Proposition 4.12. *After fiber scaling and the conformal change we have*

$$e^{-2f} \operatorname{curv}(X, W) = s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6)$$

Remark 4.13. *We pick I'' so that the first four terms are $O(s^4)$. The first two are positive except at $t = \frac{\pi}{4}$. The third can have either sign, and since the integral of I'' over an integral curve of X is 0, the term $s^4 I'' |W|^2$ also has both signs. After proving the proposition we will argue that the integral $e^{-2f} \operatorname{curv}(X, W)$ is positive, and hence that an appropriate choice of I'' will give us point wise positive curvature.*

Proof. Combining $|X| \equiv 1$, equation 4.9, the formula for the curvature of a conformal change ([Pet], exercise 3.5), and the fact that W is perpendicular to $\operatorname{grad} f$ we have

$$\begin{aligned} e^{-2f} \operatorname{curv}_{\bar{g}}(X, W) &= -s^2 (D_X (\psi D_X \psi)) + s^4 (D_X \psi)^2 - |W|_{g_s}^2 \operatorname{Hess} f(X, X) - \operatorname{Hess} f(W, W) \\ &\quad + (D_X f)^2 |W|_{g_s}^2 - |\operatorname{grad} f|^2 |W|_{g_s}^2. \end{aligned}$$

To evaluate this we will need

$$\begin{aligned}
 |W|_{g_s}^2 &= (1-s^2) |W^\nu|^2 + |W^\mathcal{H}|^2 \\
 &= |W|^2 - s^2 |(W)^\nu|^2 \\
 &= |W|^2 - s^2 (|W|^2 - |W^\mathcal{H}|^2) \\
 &= (1-s^2) |W|^2 + s^2 |W^\mathcal{H}|^2 \\
 &= (1-s^2) |W|^2 + s^2 \psi^2.
 \end{aligned}$$

Combining this with the previous proposition we see that the sum of the first and third term is

$$\begin{aligned}
 &-s^2 (D_X (\psi D_X \psi)) - |W|_s^2 \text{Hess} f (X, X) \\
 = &-s^2 (D_X (\psi D_X \psi)) + (1-s^2) |W|^2 \frac{s^2}{(1-s^2) |W|^2} D_X (\psi D_X \psi) \\
 &+ s^2 \psi^2 \frac{s^2}{(1-s^2) |W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|_{g_s}^2 \\
 = &s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6).
 \end{aligned}$$

The sum of the fourth and last terms is

$$\begin{aligned}
 -\text{Hess} f (W, W) - |\text{grad} f|^2 |W|_{g_s}^2 &= \frac{s^4}{(1-s^2) |W|^2} \psi^2 |\text{grad} \psi|^2 - \frac{s^4}{(1-s^2)^2 |W|^4} |\psi \text{grad} \psi|^2 |W|_s^2 + O(s^6) \\
 &= O(s^6).
 \end{aligned}$$

The fifth term of our curvature formula is

$$(D_X f)^2 |W|_{g_s}^2 = s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + O(s^6).$$

Combining equations we obtain

$$e^{-2f} \text{curv} (X, W) = s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6)$$

as desired. \square

To understand the sign of the above formula we will need to understand some relationships between the integrals of the first three terms.

Proposition 4.14. *Let $\gamma : [0, \frac{\pi}{4}] \rightarrow M$ be an integral curve of X with $\gamma(0) \in \tilde{T}(\{0\} \times [0, l])$. Then using ψ' for $D_X \psi$*

$$\begin{aligned}
 \int_{\gamma} \psi^2 (\psi')^2 dt &= -\frac{1}{3} \int_{\gamma} \psi^2 (\psi \psi'') dt \\
 \int_{\gamma} \psi^2 (\psi \psi')' dt &= -2 \int_{\gamma} \psi^2 (\psi')^2 dt
 \end{aligned}$$

Proof. The first equation follows from integration by parts

$$\begin{aligned} \int_{\gamma} \psi^2 (\psi')^2 dt &= \int_{\gamma} \psi' (\psi^2 \psi') dt \\ &= \psi' \frac{1}{3} \psi^3 \Big|_0^{\frac{\pi}{4}} - \int_{\gamma} \psi'' \frac{1}{3} \psi^3 dt \\ &= -\frac{1}{3} \int_{\gamma} \psi'' \psi^3 dt \end{aligned}$$

So

$$\begin{aligned} \int_{\gamma} \psi^2 (\psi \psi')' dt &= \int_{\gamma} \psi^2 \{ (\psi')^2 + \psi \psi'' \} dt \\ &= \int_{\gamma} \psi^2 \{ (\psi')^2 - 3 (\psi')^2 \} dt \\ &= -2 \int_{\gamma} \psi^2 (\psi')^2 dt \end{aligned}$$

□

Using the second equation of the previous proposition we can re-write the integral of our curvature over γ as

$$\begin{aligned} \int_{\gamma} e^{-2f} \text{curv}(X, W) &= \int_{\gamma} s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6) \\ &= \int_{\gamma} s^4 (D_X \psi)^2 - s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 - s^4 I'' |W|^2 + O(s^6). \end{aligned}$$

Since $\psi^2 = |W^{\mathcal{H}}|^2$ we always have

$$\frac{\psi^2}{|W|^2} \leq 1.$$

Since we also have $\psi^2(0) = 0$, the inequality is strict at least for a while. It follows that the integral

$$\int_{\gamma} s^4 (D_X \psi)^2 - s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 \geq O(s^4) > 0$$

Since $I'(0) = I'(\frac{\pi}{4}) = 0$,

$$\int_{\gamma} I'' = 0,$$

so we also have

$$\int_{\gamma} e^{-2f} \text{curv}(X, W) > O(s^4) > 0.$$

However, the quantity

$$s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi)$$

can have some negative values, but by choosing I'' to be sufficiently negative in the region where

$$s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) < 0$$

we can make $e^{-2f} \text{curv}(X, W)$ positive in this region. We will have to pay for this by having I'' be nonnegative on the rest of $[0, \frac{\pi}{4}]$. Since

$$\int_{\gamma} s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) > 0$$

this can be achieved while keeping $e^{-2f} \text{curv}(X, W) > 0$ point wise.

5. TANGENTIAL PARTIAL CONFORMAL CHANGE

There are two basic reasons why the combination of fiber scaling and a conformal change as outlined above can not produce positive curvature on all of the initially flat tori in the Gromoll–Meyer sphere. Before stating them we recall that there are two families of initially flat tori in the Gromoll–Meyer sphere, \mathcal{F}_ζ and \mathcal{F}_ξ that intersect orthogonally.

1: For one of the two families, \mathcal{F}_ζ , the function

$$\frac{|W^{\mathcal{H}}|^2}{|W|^2} = \frac{\psi^2}{|W|^2}$$

varies from torus to torus. The required conformal factor is e^{2f} where

$$f = -\frac{s^2}{2(1-s^2)} \frac{\psi^2}{|W|^2} + s^4 E$$

and hence varies from torus to torus. A particular choice of f will give us positive curvature on *some* of our tori, but for the others the leading terms $-s^2 (D_X (\psi D_X \psi))$, and $-|W|_{g_s}^2 \text{Hess} f(X, X)$ will not cancel; so these tori would have curvatures of both signs. There is no **one** conformal factor that will simultaneously make all of the tori in \mathcal{F}_ζ positively curved.

2: The conformal factor required to make \mathcal{F}_ξ positively curved is different from all of the conformal factors required to make \mathcal{F}_ζ positively curved.

Note that either of these reasons is sufficient to see that a conformal change can not be combined with fiber scaling to put positive curvature on all of the totally geodesic flats of the Gromoll–Meyer sphere. We have mentioned both since both difficulties will have to be overcome.

In this section, we shall see that despite the problems mentioned above, the results of the previous section are at least *morally* correct. The tangential partial conformal change that we describe will have the same effect on the curvatures of the initially flat tori as an actual conformal change—with the correct conformal factor for each torus.

Although the key function $\frac{|W^{\mathcal{H}}|^2}{|W|^2}$ varies from torus to torus on the Gromoll–Meyer sphere, the way in which this ratio varies is rather special. In fact, W has an orthogonal decomposition

$$W = W^\alpha + W^\gamma$$

where

- W^α is vertical for $Sp(2) \rightarrow S^4$,
- and X and W^α span a totally geodesic flat.

Although W^γ is perpendicular to W^α it is neither vertical nor horizontal for $Sp(2) \rightarrow S^4$; however, because W^α is vertical, $W^{\mathcal{H}} = (W^\gamma)^{\mathcal{H}}$. In particular the ratio

$$\frac{|W^{\mathcal{H}}|^2}{|W^\gamma|^2}$$

is constant on the family of tori \mathcal{F}_ζ .

Exploiting this structure and the principle

totally geodesic flats are preserved when the metric is changed orthogonally to the flat,

we will resolve the first problem by choosing the partial conformal change to leave $g(W^\alpha, \cdot)$ unchanged.

We show here that such a change will have the same effect on $\text{curv}(X, W)$ as a conformal change, with W^γ playing the role of W .

The resolution of the second problem also exploits the principle that totally geodesic flats are preserved when the metric is changed orthogonally to the flat.

In the end we will make two partial conformal changes using f_ζ and f_ξ . The f_ζ change will leave the metric on \mathcal{F}_ξ unchanged, and the f_ξ change will leave the metric on \mathcal{F}_ζ unchanged. Since the two families of tori intersect orthogonally, we will be able to argue that the f_ζ change does not have any effect on the curvature of \mathcal{F}_ξ , and the f_ξ change does not have any effect on the curvature of \mathcal{F}_ζ .

The setup for our tangential partial conformal change is as follows. There are mutually orthogonal distributions \mathcal{X} , \mathcal{A} , and \mathcal{G} with the properties

- 1: \mathcal{X} is integrable and totally geodesic.
- 2: Any pair of vectors $Z \in \mathcal{X}$ and $U \in \mathcal{A}$ span a totally geodesic flat torus.
- 3: $[\mathcal{X}, \mathcal{A}^\perp] \subset \mathcal{A}^\perp$.
- 4: There is a function f whose gradient lies in \mathcal{X} .
- 5: There is a geodesic field $X \in \mathcal{X}$.
- 6: $[X, \mathcal{G}] \subset \mathcal{G}$.

We change the metric by multiplying the lengths of all vectors in the distribution $\text{span}\{X, \mathcal{G}\}$ by e^{2f} , while keeping the orthogonal complement of $\text{span}\{X, \mathcal{G}\}$ fixed. In particular, $g(\mathcal{A}, \cdot)$ is unchanged.

In the concrete situation we will have two functions f_ζ and f_ξ . To accommodate this here we also assume that there is a C^0 -small, but unspecified change to the orthogonal complement of $\text{span}\{X, \mathcal{G}, \mathcal{A}\}$. We call the resulting metric \tilde{g} .

In the concrete setting the splitting $W = W^\alpha + W^\gamma$ mentioned above satisfies $W^\alpha \in \mathcal{A}$ and $W^\gamma \in \mathcal{G}$.

We analyze here the effect of such a change on $R(W, X)X$ and $R(X, W^\gamma)W^\gamma$. Since $\text{span}\{X, W\}$ is an abstraction of the zero planes in the Gromoll–Meyer sphere, we would ideally also have formulas for $R(X, W)W$; however, we have not succeeded in making a satisfactory abstraction of this calculation, and so have deferred it to the concrete setting.

We use the indices z, α , to denote components of the θ s, ω s and Ω s corresponding to $z \in \mathcal{X}$ and $U^\alpha \in \mathcal{A}$. We use $\tilde{\cdot}$ to denote the metric quantities with respect to \tilde{g} , and “bar” to denote the quantities with respect to the metric obtained from g with respect to an actual conformal change with conformal factor e^{2f} , e.g. \bar{g} and $\bar{\omega}$.

Proposition 5.1. *For any vector $z \in \mathcal{X}$ and $U^\alpha \in \mathcal{A}$*

$$\begin{aligned}\tilde{\omega}_z^\alpha &\equiv 0, \\ \tilde{\omega}_z^i(U^\alpha) &= \tilde{\omega}_\alpha^i(z) = 0\end{aligned}$$

for all i .

Proof. The last two equations are equivalent to the statement that any $z \in \mathcal{X}$ and any $U^\alpha \in \mathcal{A}$ have extensions with $\tilde{\nabla}_{U^\alpha} Z = \tilde{\nabla}_Z U^\alpha = 0$. The presence of the flat tori give us this result for ∇ . For $N = Z, U^\alpha$ or normal to both Z and U^α we have

$$\begin{aligned}0 &= 2g(\nabla_{U^\alpha} Z, N) \\ &= -g([Z, N], U^\alpha) + g([N, U^\alpha], Z)\end{aligned}$$

Hypothesis 4 gives us that $g([N, U^\alpha], Z) = 0$. So it follows that $g([Z, N], U^\alpha) = 0$. This gives us

$$\begin{aligned}2\tilde{g}(\nabla_{U^\alpha} Z, N) &= -\tilde{g}([Z, N], U^\alpha) + \tilde{g}([N, U^\alpha], Z) \\ &= 0\end{aligned}$$

as desired.

The first equation is equivalent to $\langle \tilde{\nabla}_N Z, U^\alpha \rangle = 0$. This follows for the same reasons. \square

Proposition 5.2. *For X as above and $W \in \text{span}\{\mathcal{A}, \mathcal{G}\}$*

$$\begin{aligned}\tilde{\omega}_X^{E_k}(X) &= \bar{\omega}_X^{E_k}(X), \\ \tilde{\omega}_X^{E_k}(W) &= \bar{\omega}_X^{E_k}(W^\gamma), \\ \tilde{\omega}_W^{E_k}(X) &= \bar{\omega}_{W^\gamma}^{E_k}(X), \\ \tilde{\omega}_{W^\gamma}^{E_k}(W^\gamma) &= \bar{\omega}_{W^\gamma}^{E_k}(W^\gamma),\end{aligned}$$

where W^γ denotes the component of W in \mathcal{G} .

Proof. By the previous proposition we have

$$\tilde{\omega}_X^{E_k}(W) = \tilde{\omega}_X^{E_k}(W^\alpha) + \tilde{\omega}_X^{E_k}(W^\gamma) = \tilde{\omega}_X^{E_k}(W^\gamma).$$

So the second equation reduces to

$$\tilde{\omega}_X^{E_k}(W^\gamma) = \bar{\omega}_X^{E_k}(W^\gamma).$$

Similarly, the third equation reduces to

$$\tilde{\omega}_W^{E_k}(X) = \bar{\omega}_{W^\gamma}^{E_k}(X).$$

The proofs of each of these and the first and fourth equations are essentially the same, and boil down to the facts that

$$\begin{aligned}[X, X] &= [W^\gamma, W^\gamma] = 0, \text{ and} \\ [X, W^\gamma] &\in \mathcal{G}.\end{aligned}$$

For $\tilde{\omega}_X^{E_k}(W^\gamma) = \bar{\omega}_X^{E_k}(W^\gamma)$ the details are

$$\begin{aligned}\tilde{\omega}_{W^\gamma}^{E_k} X &= \tilde{g}\left(\tilde{\nabla}_X W^\gamma, E_k\right) \\ &= \frac{1}{2}\left(D_X \tilde{g}(W^\gamma, E_k) + D_{W^\gamma} \tilde{g}(X, E_k) - D_{E_k} \tilde{g}(X, W^\gamma)\right. \\ &\quad \left.+ \tilde{g}([X, W^\gamma], E_k) - \tilde{g}([W^\gamma, E_k], X) - \tilde{g}([X, E_k], W^\gamma)\right).\end{aligned}$$

For the fourth term we have $\tilde{g}([X, W^\gamma], E_k) = \bar{g}([X, W^\gamma], E_k)$. This is because of our hypothesis that $[X, W^\gamma] \in \mathcal{G}$. For the other terms we can also change \tilde{g} to \bar{g} for the same reason—that one of the vectors in the inner product is in $\text{span}\{X, \mathcal{G}\}$. Thus

$$\begin{aligned} \tilde{\omega}_{W^\gamma}^{E_k} X &= \frac{1}{2} (D_X \bar{g}(W^\gamma, E_k) + D_{W^\gamma} \bar{g}(X, E_k) - D_{E_k} \bar{g}(X, W^\gamma) \\ &\quad + \bar{g}([X, W^\gamma], E_k) - \bar{g}([W^\gamma, E_k], X) + \bar{g}([X, E_k], W^\gamma)) \\ &= \bar{g}(\bar{\nabla}_X W^\gamma, E_k) \\ &= \bar{\omega}_{W^\gamma}^{E_k} X. \end{aligned}$$

□

Proposition 5.3. *For any unit vector U*

$$\begin{aligned} \tilde{R}(W, X, X, W) &= \bar{R}(W^\gamma, X, X, W^\gamma) \\ \tilde{R}(U, X, X, W) &= \bar{R}(U, X, X, W^\gamma) + O(C^0) \max\{|\bar{\omega}_X^k(X)|, |\bar{\omega}_X^k(W^\gamma)|\}, \text{ and} \\ \tilde{R}(U, W^\gamma, W^\gamma, X) &= \bar{R}(U, W^\gamma, W^\gamma, X) + O(C^0) \max\{|\bar{\omega}_{W^\gamma}^k(W^\gamma)|, |\bar{\omega}_X^k(W^\gamma)|\} \end{aligned}$$

where $O(C^0)$ represents a quantity that is smaller than a constant times the difference in the C^0 norms of \tilde{g} and \bar{g} .

Proof. Using the previous two propositions we have

$$\begin{aligned} \tilde{R}(W, X, X, U) &= d\tilde{\omega}_X^U(W, X) + \sum \tilde{\omega}_k^U \wedge \tilde{\omega}_X^k(W, X) \\ &= d\tilde{\omega}_X^U(W, X) + \sum \tilde{\omega}_k^U(W) \tilde{\omega}_X^k(X) - \tilde{\omega}_k^U(X) \tilde{\omega}_X^k(W^\gamma) \end{aligned}$$

Since X and W^α initially span a totally geodesic flat, we can choose our extension of W so that $[W^\alpha, X] = 0$. Using this, the previous proposition and the hypothesis $[W^\gamma, X] \in \mathcal{G}$ we have

$$\begin{aligned} d\tilde{\omega}_X^U(W, X) &= D_W \tilde{\omega}_X^U(X) - D_X \tilde{\omega}_X^U(W) - \tilde{\omega}_X^U[W^\gamma, X] \\ &= D_{W^\alpha} \tilde{\omega}_X^U(X) + D_{W^\gamma} \tilde{\omega}_X^U(X) - D_X \tilde{\omega}_X^U(W^\gamma) - \tilde{\omega}_X^U[W^\gamma, X]. \end{aligned}$$

Since X is initially a geodesic field and W^α is perpendicular to the gradient of f , $D_{W^\alpha} \tilde{\omega}_X^U(X) = 0$, as long as U makes a constant angle with $\text{grad}f$. So with such a choice of U we have

$$\begin{aligned} d\tilde{\omega}_X^U(W, X) &= D_{W^\gamma} \tilde{\omega}_X^U(X) - D_X \tilde{\omega}_X^U(W^\gamma) - \tilde{\omega}_X^U[W^\gamma, X] \\ &= d\tilde{\omega}_X^U(W^\gamma, X). \end{aligned}$$

Since X is initially a geodesic field, $\bar{\nabla}_X X \in \text{span}\{X, \nabla f\}$. Thus $\bar{\omega}_X^k(X)$ is only nonzero for $E_k \in \mathcal{X}$. By the first proposition of this section, we have that for such E_k , $\tilde{\omega}_k^U(W^\alpha) = 0$, and hence

$$\tilde{\omega}_k^U(W) \tilde{\omega}_X^k(X) = \tilde{\omega}_k^U(W^\gamma) \bar{\omega}_X^k(X).$$

The Koszul formula then gives us that

$$|\tilde{\omega}_k^U(W^\gamma) - \bar{\omega}_k^U(W^\gamma)| \leq O(C^0)$$

and

$$|\tilde{\omega}_k^U(X) - \bar{\omega}_k^U(X)| \leq O(C^0).$$

Combining these displays, give us the second equation, and a similar argument gives us the third equation.

For $U = W$, the first proposition of this section gives us $d\bar{\omega}_X^{W^\alpha}(W^\gamma, X) = 0$, so

$$d\bar{\omega}_X^W(W^\gamma, X) = d\bar{\omega}_X^{W^\gamma}(W^\gamma, \zeta).$$

We also have to deal with

$$\sum \tilde{\omega}_k^W(W) \bar{\omega}_X^k(X) - \tilde{\omega}_k^W(X) \bar{\omega}_X^k(W^\gamma).$$

The previous proposition gives us $\tilde{\omega}_k^W(X) = \bar{\omega}_k^{W^\gamma}(X)$.

We also have $\bar{\omega}_X^k(X) = \langle \bar{\nabla}_X X, E_k \rangle$ and $\bar{\nabla}_X X \in \text{span}\{X, \text{grad}f\}$. The previous proposition gives us $\tilde{\omega}_X^W(W) = \bar{\omega}_X^W(W^\gamma)$. Since $\bar{\omega}_X^{W^\alpha}(W^\gamma) = 0$, we conclude that $\tilde{\omega}_X^W(W) = \bar{\omega}_X^{W^\gamma}(W^\gamma)$. The first proposition of the section gives us

$$\tilde{\omega}_{\frac{|\nabla f|}{|\bar{\nabla} f|}}^W(W) = \tilde{\omega}_{\frac{|\nabla f|}{|\bar{\nabla} f|}}^{W^\gamma}(W^\gamma)$$

and the second gives us

$$\tilde{\omega}_{\frac{|\nabla f|}{|\bar{\nabla} f|}}^{W^\gamma}(W^\gamma) = \bar{\omega}_{\frac{|\nabla f|}{|\bar{\nabla} f|}}^{W^\gamma}(W^\gamma).$$

So in any case we have,

$$\tilde{\omega}_k^W(W) \bar{\omega}_X^k(X) = \bar{\omega}_k^{W^\gamma}(W^\gamma) \bar{\omega}_X^k(X).$$

Combining displays we have

$$\begin{aligned} \tilde{R}(W, X, X, W) &= d\bar{\omega}_X^{W^\gamma}(W^\gamma, X) + \sum \bar{\omega}_k^{W^\gamma}(W^\gamma) \bar{\omega}_X^k(X) - \omega_k^{W^\gamma}(X) \bar{\omega}_X^k(W^\gamma) \\ &= \bar{R}(W^\gamma, X, X, W^\gamma). \end{aligned}$$

□

6. LONG TERM CHEEGER PRINCIPLE

In the presence of a group of isometries, G , a method for perturbing the metric on a manifold, M , of nonnegative sectional curvature is proposed in [Cheeg]. Various special cases of this method were first studied in [Berg3] and [BourDesSent]. An exposition can be found in [Muet]. Although this technique has been used repeatedly in the literature, our impression is that it is not widely understood.

To understand the effect of a Cheeger deformation on the curvature of a nonnegatively curved manifold, in our view, it is crucial to exploit the ‘‘Cheeger reparametrization’’ of the Grassmannian. We will review the definition of the Cheeger reparametrization below. For now we recall (see e.g. [PetWilh1])

Proposition 6.1. *Let (M, g_{Cheeg}) be a Cheeger deformation by G of the nonnegatively curved manifold (M, g) . Then modulo the Cheeger reparametrization,*

- 1:** *If a plane P is positively curved with respect to g , then it is positively curved with respect to g_{Cheeg} .*
- 2:** *If a plane P has a nondegenerate projection onto the orbits of G and ‘‘corresponds’’ to a positively curved plane in G , then P is positively curved with respect to g_{Cheeg} .*

The meaning of ‘‘corresponds’’ will be explained below.

In this section we will discuss a generalization of this result to manifolds that do not necessarily have nonnegative curvature. This result is used in [PetWilh2].

Proposition 6.2. *Let (M, g_{Cheeg}) be a Cheeger deformation by G of (M, g) . Then modulo the Cheeger reparametrization,*

- 1: *If a plane P is positively curved with respect to g , then it is positively curved with respect to g_{Cheeg} .*
- 2: *If a plane P has a nondegenerate projection onto the orbits of G and “corresponds” to a positively curved plane in G , then P is positively curved with respect to a Cheeger deformed metric, provided the Cheeger deformation is “run for a sufficiently long time”.*

The meaning of “run for a sufficiently long time” will also be explained below.

To explain these results we offer a review that is sufficient for our purposes. None of this review is original, and in fact some of it is copied verbatim from [PetWilh1], whose main contribution to the theory of Cheeger deformations is expository.

If G is a compact group of isometries of M , then we let G act on $G \times M$ by

$$(6.2) \quad g(p, m) = (pg^{-1}, gm).$$

If we endow G with a biinvariant metric and $G \times M$ with the product metric, then the quotient of (6.2) is a new metric on M . It was observed in [Cheeg], that in a certain sense we may expect the new metric to have more curvature and less symmetry than the original metric. The “sense” in which this is true is modulo the Cheeger reparametrization.

The quotient map for the action (6.2) is

$$q_{G \times M} : (g, m) \mapsto gm.$$

The vertical space for $q_{G \times M}$ at (g, m) is

$$\mathcal{V}_{q_{G \times M}} = \{(-k, k) \mid k \in \mathfrak{g}\}$$

where the $-k$ in the first factor stands for the value at g of the Killing field on G given by the circle action

$$(\exp(tk), g) \mapsto g \exp(-kt)$$

and the k in the second factor is the value of the Killing field

$$m \mapsto \frac{d}{dt} \exp(tk)m$$

on M at m .

We recall from [Cheeg], [PetWilh1] that there is a reparametrization of the tangent space, that we will call the Cheeger reparametrization. It is given by

$$v \mapsto Dq_{G \times M}(\hat{v})$$

where

$$\hat{v} \equiv (k_v, v)$$

is the vector tangent to $G \times M$ that is horizontal for $q_{G \times M} : G \times M \rightarrow M$, and projects to v under $\pi_2 : G \times M \rightarrow M$.

From now on we will assume that the metric on the G -factor in $G \times M$ is biinvariant. This means that we have only a one parameter family $(M, g_l)_{l \in \mathbb{R}}$ of Cheeger deformed metrics, where l denotes the scale of the biinvariant metric in $G \times M$. As $l \rightarrow \infty$, (M, g_l) converges to the metric on the M factor in $G \times M$, so we will often call the original metric g_∞ [Pet].

With an understanding of the Cheeger re-parameterization the proof of Proposition 6.1 is now clear. $Dq_{G \times M}(\hat{P})$ is positively curved if \hat{P} is positively curved, and \hat{P} is positively curved if its projection onto either M or G is positively curved. Since the projection onto M is P , we get the conclusion of Proposition 6.1.

The proof of Proposition 6.2 is only a little harder. If P happens to be positively curved, then so is \hat{P} and hence also $Dq_{G \times M}(\hat{P})$.

On the other hand, if

$$\begin{aligned} \hat{P} &= \text{span} \{ \hat{v}, \hat{w} \} \\ &= \text{span} \{ (k_v, v), (k_w, w) \} \end{aligned}$$

when $l = 1$, then for arbitrary l ,

$$\hat{P} = \text{span} \left\{ \left(\frac{k_v}{l^2}, v \right), \left(\frac{k_w}{l^2}, w \right) \right\}.$$

So

$$\begin{aligned} \text{curv}_{(M, g_l)} \left(Dq_{G \times M} \left(\frac{k_v}{l^2}, v \right), Dq_{G \times M} \left(\frac{k_w}{l^2}, w \right) \right) &\geq \text{curv}_{G, l} \left(\frac{k_v}{l^2}, \frac{k_w}{l^2} \right) + \text{curv}_M(v, w) \\ &= \frac{1}{l^6} \text{curv}_{G, 1}(k_v, k_w) + \text{curv}_M(v, w) \end{aligned}$$

where $\text{curv}_{G, l}$ stands for the curvature with respect to the biinvariant metric with scale l , and $\text{curv}_{G, 1}$ stands for the curvature with respect to the biinvariant metric with scale 1. Thus if $\text{curv}_{G, 1}(k_v, k_w)$ happens to be positive, then the term $\frac{1}{l^6} \text{curv}_{G, 1}(k_v, k_w)$ will dominate the term $\text{curv}_M(v, w)$, when l is sufficiently small, and we conclude that

$$\text{curv}_{(M, g_l)} \left(Dq_{G \times M} \left(\frac{k_v}{l^2}, v \right), Dq_{G \times M} \left(\frac{k_w}{l^2}, w \right) \right) > 0.$$

The utility of using the Cheeger reparametrization is undeniable. As we have seen, it provides a simple way to track changes of curvature. It also preserves horizontal spaces of Riemannian submersions, [PetWilh1].

Proposition 6.3. *Let $A_H : H \times M \rightarrow M$ be an action that is by isometries with respect to both g_∞ and g_l . Let \mathcal{H}_{A_H} denote the distribution of vectors that are perpendicular to the orbits of A_H .*

Then u is in \mathcal{H}_{A_H} with respect to g_∞ if and only if $Dq_{G \times M}(\hat{u})$ is in \mathcal{H}_{A_H} with respect to g_l . In fact,

$$g_\infty(u, w) = g_l(u, Dq_{G \times M}(\hat{w}))$$

for all $u, w \in TM$.

Proof. Starting with the left and side we take the horizontal lifts to $G \times M$

$$g_l(u, Dq_{G \times M}(\hat{w})) = g_{G \times M} \left((0, u) - (0, u)^\vee, \hat{w} \right)$$

Since \hat{w} is horizontal this becomes

$$\begin{aligned} g_l(u, Dq_{G \times M}(\hat{w})) &= g_{G \times M}((0, u), \hat{w}) \\ &= g_\infty(u, w). \end{aligned}$$

□

Notational Convention: Let

$$q_{G \times M} : G \times (M, g_\infty) \longrightarrow (M, g_l)$$

be a Cheeger submersion. Suppose that $\pi : M \longrightarrow B$ is a Riemannian submersion with respect to both g_∞ and g_l . It follows that z is horizontal for $\pi : M \longrightarrow B$ with respect to g_∞ if and only if $Dq_{G \times M}(\hat{z})$ is horizontal for π with respect to g_l . To keep the notation simpler, we can think of this correspondence as a parameterization of the horizontal space, \mathcal{H}_{π, g_l} , of π with respect to g_l by the horizontal space, $\mathcal{H}_{\pi, g_\infty}$ of π with respect to g_∞ . We can then denote vectors and planes in \mathcal{H}_{π, g_l} by the corresponding vectors and planes in $\mathcal{H}_{\pi, g_\infty}$. Unless otherwise indicated, we will do this throughout the remainder of this paper.

6.1. Quadratic Nondegeneracy and Cheeger Deformations. Cheeger deformations and the Cheeger reparametrization also play a role in verifying the Quadratic Nondegeneracy Condition.

Proposition 6.4. *Let (E, g_∞) be nonnegatively curved and*

$$q_{H \times E} : H \times (E, g_\infty) \longrightarrow (E, g_l)$$

a Cheeger submersion. Let M be as in Theorem 2.4 and be obtained as a Riemannian submersion

$$\pi : (E, g_l) \longrightarrow M.$$

Suppose that X is orthogonal to the orbits of H on (E, g_∞) , X, W, Z and V are π -horizontal with respect to g_∞ , and

$$\text{span} \left\{ D\pi \circ Dq_{H \times E}(\hat{X}), D\pi \circ Dq_{H \times E}(\hat{W}) \right\}$$

is one of the zero curvature planes of M . Then the nondegeneracy condition holds for

$$\text{span} \left\{ D\pi \circ Dq_{H \times E}(\hat{X}) + \sigma \cdot D\pi \circ Dq_{H \times E}(\hat{Z}), D\pi \circ Dq_{H \times E}(\hat{W}) + \tau \cdot D\pi \circ Dq_{H \times E}(\hat{V}) \right\}$$

if and only if any of the following hold

- *For the original nonnegatively curved metric g_∞*

$$\sigma^2 \text{curv}^{g_\infty}(Z, W) + 2\sigma\tau (R^{g_\infty}(X, W, V, Z) + R^{g_\infty}(X, V, W, Z)) + \tau^2 \text{curv}^{g_\infty}(X, V) > 0.$$

- *$\text{curv}^H(D\pi_H(\hat{Z}), D\pi_H(\hat{W})) > 0$, where $\pi_H : H \times G \longrightarrow H$ is projection to the H -factor.*
- $\left| \tau \cdot A_{\hat{X}}^{q_{H \times E}} \hat{V} + \sigma \cdot A_{\hat{Z}}^{q_{H \times E}} \hat{W} \right|^2 > 0.$
- $\left| \tau \cdot A_{Dq_{H \times E}(\hat{X})}^\pi Dq_{H \times E}(\hat{V}) + \sigma \cdot A_{Dq_{H \times E}(\hat{Z})}^\pi Dq_{H \times E}(\hat{W}) \right|^2 > 0.$

Proof. For g_∞ we have

$$P^{g_\infty}(\sigma, \tau) = \text{curv}^{g_\infty}(X + \sigma Z, W + \tau V) \geq 0.$$

The constant and linear terms are 0. So the total quadratic term is nonnegative, otherwise (E, g_∞) would have a negative curvature, near $\text{span}\{X, W\}$.

Since $\text{curv} \left(D\pi \circ Dq_{H \times E} \left(\hat{X} \right), D\pi \circ Dq_{H \times E} \left(\hat{W} \right) \right) = 0$, it follows that $A_{\hat{X}}^{q_{H \times E}} \hat{W} = 0$. Therefore, writing A for $A^{q_{H \times E}}$ and omitting the hats,

$$\begin{aligned}
 0 &\leq \langle A_{X+\sigma Z}(W + \tau V), A_{X+\sigma Z}(W + \tau V) \rangle \\
 &= \tau^2 \langle A_X V, A_X V \rangle + \sigma \tau \langle A_X V, A_Z W \rangle + \sigma \tau^2 \langle A_X V, A_Z V \rangle + \sigma \tau \langle A_Z W, A_X V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle \\
 &\quad + \sigma^2 \langle A_Z W, A_Z W \rangle + \sigma \tau^2 \langle A_Z V, A_X V \rangle + \sigma^2 \tau \langle A_Z V, A_Z W \rangle + \sigma^2 \tau^2 \langle A_Z V, A_Z V \rangle \\
 &= \tau^2 \langle A_X V, A_X V \rangle + 2\sigma \tau \langle A_X V, A_Z W \rangle + \sigma^2 \langle A_Z W, A_Z W \rangle \\
 &\quad + 2\sigma \tau^2 \langle A_X V, A_Z V \rangle + 2\sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau^2 \langle A_Z V, A_Z V \rangle \\
 &= |\tau A_X V + \sigma A_Z W|^2 + 2\sigma \tau^2 \langle A_X V, A_Z V \rangle + 2\sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau^2 \langle A_Z V, A_Z V \rangle
 \end{aligned}$$

In particular, the effect of the Cheeger A -tensor on the total quadratic term is nonnegative and given by $|\tau A_{\hat{X}} \hat{V} + \sigma A_{\hat{Z}} \hat{W}|^2$. The same argument gives that the effect of the A -tensor of π on the total quadratic term is nonnegative and given by $\left| \tau A_{Dq_{H \times E}(\hat{X})}^\pi Dq_{H \times E}(\hat{V}) + \sigma A_{Dq_{H \times E}(\hat{Z})}^\pi Dq_{H \times E}(\hat{W}) \right|^2$.

Because X is orthogonal to the orbits of H , the only quadratic term that can be nonzero in the H -factor is $\text{curv}^H \left(D\pi_H \left(\hat{Z} \right), D\pi_H \left(\hat{W} \right) \right)$.

Since this is also nonnegative, we have decomposed the total quadratic term as a sum of four nonnegative quantities. If any one of these quantities is positive, then the total quadratic term is positive. If on the other hand, all four quantities are 0, then the total quadratic term is 0. \square

Remark 6.5. *In light of the main result of [Tapp2], one might expect that the two A -tensor conditions could be omitted if (E, g_∞) is a biinvariant metric on a compact Lie group. However, because the total quadratic term is not the curvature of a plane, we are not for the moment aware of how to prove this.*

7. CURVATURE COMPRESSION PRINCIPLE

On the Gromoll–Meyer sphere Cheeger deformations can have a huge quantitative impact on the formula

$$e^{-2f} \text{curv}(X, W) = s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6)$$

for $\text{curv}(X, W)$ after a (Tangential Partial) conformal change (Proposition 4.12).

In fact, by running one of our Cheeger deformations for a long time, we shall see that the vast bulk of the first three of these terms is compressed into a small neighborhood of the poles of X .

What happens to these curvatures is a lot like what happens to \mathbb{R}^2 under the long term Cheeger deformation by the standard $SO(2)$ -action. The metric becomes a paraboloid that is very flat except near the fixed point, $(0, 0)$, where there is a lot of curvature. In this example, the radial field, ∂_r , plays the role of our field, X , and the lengths of the circles centered about the origin play the role of our function ψ .

Here we describe an abstraction of what happens on the Gromoll–Meyer sphere, whose starting point is the fiber scaling theorem, 4.1.

Let $\pi : (M, g_\infty) \rightarrow B$ be a Riemannian submersion. Let $G = G_1 \times G_2$ act isometrically on M and by symmetries of π . Let $g_{\nu, l}$ be the metric on M obtained by doing the Cheeger deformation with $G = G_1 \times G_2$ on M , where the scale on the

G_1 factor in $(G_1 \times G_2) \times M$ is ν and the scale on the G_2 factor in $(G_1 \times G_2) \times M$ is l .

Because of our curvature formula, we are interested in how the length of the π -horizontal part, $W^{\mathcal{H}}$, of a G_1 -Killing field W is affected by Cheeger deforming with $(G_1 \times G_2)$. In the Gromoll–Meyer sphere we will consider the case when ν is very small and

$$l = O\left(\nu^{1/3}\right).$$

So we adopt these hypotheses for our abstract framework here. We set

$$\begin{aligned}\psi_\infty &\equiv |W^{\mathcal{H},g_\infty}|_{g_\infty} \quad \text{and} \\ \psi_{\nu,l} &\equiv |W^{\mathcal{H},g_{\nu,l}}|_{g_{\nu,l}}.\end{aligned}$$

Our goal is to obtain a formula for $\psi_{\nu,l}$ in terms of ψ_∞ .

Lemma 7.1. *Let K_W^1 be the Killing field on G_1 that corresponds to W . Suppose W lies in the direction of the projection of $W^{\mathcal{H},g_\infty}$ onto the orbits of G_1 , and that*

$$\rho = \frac{1}{|K_W^1|_{\text{bi}}}.$$

Let $K_{W,M}^2$ be a vector in the direction of the projection of $W^{\mathcal{H},g_\infty}$ onto the orbit of G_2 . We normalize $K_{W,M}^2$ so that $|K_{W,M}^2|_{\text{bi}}^2 = 1$, where K_W^2 is the corresponding Killing field on G_2 . Then

$$\psi_{\nu,l}^2 = \frac{\psi_\infty^2}{\rho^2 \frac{\psi_\infty^2}{\nu^2} + \frac{\varphi_\infty^4}{l^2 \psi_\infty^2} + 1}$$

where

$$\varphi_\infty^4 \equiv \langle K_{W,M}^2, W^{\mathcal{H},g_\infty} \rangle^2.$$

Remark 7.2. *Note that the above definition of $K_{W,M}^2$ does not preclude the possibility of $K_{W,M}^2$ varying from point to point.*

Proof. We have

$$\begin{aligned}\psi_\infty &= |W^{\mathcal{H},g_\infty}|_{g_\infty} \\ &= \left| \frac{\langle W, W^{\mathcal{H},g_\infty} \rangle}{|W^{\mathcal{H},g_\infty}|} \right| \\ \langle W, W^{\mathcal{H},g_\infty} \rangle &= \psi_\infty |W^{\mathcal{H},g_\infty}| = \psi_\infty^2\end{aligned}$$

Because of the formula

$$g_\infty(u, w) = g_{\nu,l}(u, Dq_{G \times M}(\hat{w})),$$

$Dq_{G \times M}(\widehat{W^{\mathcal{H}, g_\infty}})$ is horizontal with respect to $g_{\nu, l}$. So setting $G_{\nu, l} \equiv (G_1, \nu \text{bi}) \times (G_2, l \text{bi})$

$$\begin{aligned} \psi_{\nu, l} &\equiv \left| W^{\mathcal{H}, g_{\nu, l}} \right|_{g_{\nu, l}} \\ &= \left| \left\langle W, Dq_{G \times M} \left(\frac{\widehat{W^{\mathcal{H}, g_\infty}}}{\left| \widehat{W^{\mathcal{H}, g_\infty}} \right|_{G_{\nu, l} \times M}} \right) \right\rangle_{g_{\nu, l}} \right| \\ &= \frac{1}{\left| \widehat{W^{\mathcal{H}, g_\infty}} \right|_{G_{\nu, l} \times M}} \langle W, W^{\mathcal{H}, g_\infty} \rangle_{g_\infty} \\ &= \frac{\psi_\infty^2}{\left| \widehat{W^{\mathcal{H}, g_\infty}} \right|_{G_{\nu, l} \times M}} \end{aligned}$$

where we use the general formula

$$g_\infty(u, w) = g_l(u, Dq_{G \times M}(\hat{w}))$$

for the next to last equation.

We have

$$\begin{aligned} \widehat{W^{\mathcal{H}, g_\infty}} &= \left(\frac{\langle W^{\mathcal{H}, g_\infty}, W \rangle}{|W|_{M, g_\infty}^2} \frac{K_W^1}{\nu^2} \frac{|W|_{M, g_\infty}^2}{|K_W^1|_{\text{bi}}^2}, \frac{\langle W^{\mathcal{H}, g_\infty}, K_{W, M}^2 \rangle}{|K_{W, M}^2|_{M, g_\infty}^2} \frac{K_W^2}{l^2} \frac{|K_{W, M}^2|_{M, g_\infty}^2}{|K_W^2|_{\text{bi}}^2}, W^{\mathcal{H}, g_\infty} \right) \\ &= \left(\frac{K_W^1}{\nu^2} \frac{\langle W^{\mathcal{H}, g_\infty}, W \rangle}{|K_W^1|_{\text{bi}}^2}, \frac{K_W^2}{l^2} \frac{\langle W^{\mathcal{H}, g_\infty}, K_{W, M}^2 \rangle}{|K_W^2|_{\text{bi}}^2}, W^{\mathcal{H}, g_\infty} \right) \\ &= \left(\rho^2 \left(\frac{K_W^1}{\nu^2} \langle W^{\mathcal{H}, g_\infty}, W \rangle \right), \frac{K_W^2}{l^2} \langle W^{\mathcal{H}, g_\infty}, K_{W, M}^2 \rangle, W^{\mathcal{H}, g_\infty} \right). \end{aligned}$$

This gives us

$$\begin{aligned} \left| \widehat{W^{\mathcal{H}, g_\infty}} \right|^2 &= \rho^2 \frac{\langle W, W^{\mathcal{H}, g_\infty} \rangle^2}{\nu^2} + \frac{\langle K_{W, M}^2, W^{\mathcal{H}, g_\infty} \rangle^2}{l^2} + |W^{\mathcal{H}, g_\infty}|^2 \\ &= \rho^2 \frac{\psi_\infty^4}{\nu^2} + \frac{\varphi_\infty^4}{l^2} + \psi_\infty^2, \end{aligned}$$

and hence

$$\begin{aligned} \psi_{\nu, l}^2 &= \frac{\psi_\infty^4}{\rho^2 \frac{\psi_\infty^4}{\nu^2} + \frac{\varphi_\infty^4}{l^2} + \psi_\infty^2} \\ &= \frac{\psi_\infty^2}{\rho^2 \frac{\psi_\infty^2}{\nu^2} + \frac{\varphi_\infty^4}{l^2 \psi_\infty^2} + 1} \end{aligned}$$

□

Straightforward calculation gives us formal derivatives of $\psi_{\nu, l}$ in some unspecified direction

Proposition 7.3.

$$\begin{aligned}\psi'_{\nu,l} &= \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right) \frac{\psi_{\nu,l}^3}{\psi_\infty^3} \\ \psi''_{\nu,l} &= \left(\psi''_\infty - \frac{6\varphi_\infty^2\varphi'_\infty}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)'' \right) \frac{\psi_{\nu,l}^3}{\psi_\infty^3} \\ &\quad - 3 \frac{\psi'_\infty}{\psi_\infty} \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right) \frac{\psi_{\nu,l}^3}{\psi_\infty^3} \\ &\quad + 3 \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \frac{\psi_{\nu,l}^5}{\psi_\infty^6}\end{aligned}$$

For the remainder of the section, we restrict our attention to a curve $\gamma : [0, \frac{\pi}{4}] \rightarrow M$ on which

$$\begin{aligned}\left| \frac{\varphi_\infty}{\psi_\infty} \right| &\text{ is bounded} \\ \psi''_\infty &\leq 0, \\ \psi_\infty(0) &= \psi''_\infty(0) = 0, \\ \left(\frac{\varphi_\infty}{\psi_\infty} \right)' &\text{ is bounded, and} \\ \psi'_\infty(0) &= \varphi'_\infty(0) = 1.\end{aligned}$$

The next result gives a quantitative description of how $(\psi'_{\nu,l})^2$ is compressed as $\nu \rightarrow 0$.

Proposition 7.4. *For ν sufficiently small and $l = O(\nu^{1/3})$, we have*

$$\begin{aligned}(\psi'_{\nu,l})^2 \Big|_{[0,\nu]} &\geq \frac{97}{100} \frac{1}{\rho^2 + 1}, \\ (\psi'_{\nu,l})^2 \Big|_{[\nu^\beta, \frac{\pi}{4}]} &\leq O\left(\nu^{\frac{14}{3} - 6\beta}\right),\end{aligned}$$

for any fixed $\beta < \frac{7}{9}$.

Remark 7.5. *Keeping in mind that ρ is a fixed “background constant”, $\rho = \frac{1}{|K_W^1|_{\text{bi}}}$, that is independent of ν , these formulas tell us that $(\psi'_{\nu,l})^2$ is large on the small interval $[0, \nu]$, and then rapidly becomes very small, its generic order being $\nu^{\frac{14}{3}}$.*

Proof. Setting $D^2 = \rho^2 \frac{\psi_\infty^2}{\nu^2} + \frac{\varphi_\infty^4}{l^2 \psi_\infty^2} + 1$ we have

$$\begin{aligned}\frac{\psi_{\nu,l}^2}{\psi_\infty^2} &= \frac{1}{D^2} \\ &= \frac{\nu^2}{\rho^2 \psi_\infty^2 + \frac{\nu^2 \varphi_\infty^4}{l^2 \psi_\infty^2} + \nu^2}\end{aligned}$$

So

$$(\psi'_{\nu,l})^2 = \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \frac{1}{D^6}$$

On $[0, \nu]$ our hypotheses imply that $\frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)'$ is much smaller than ψ'_∞ ; so on $[0, \nu]$ we have

$$\begin{aligned} (\psi'_{\nu,t})^2 &\geq \frac{99}{100} (\psi'_\infty)^2 \frac{1}{D^6} \\ &\geq \frac{98}{100} \frac{1}{\rho^2 \frac{\psi_\infty^2}{\nu^2} + 1} \\ &\geq \frac{97}{100} \frac{1}{\rho^2 \frac{t^2}{\nu^2} + 1} \\ &\geq \frac{97}{100} \frac{\nu^2}{\rho^2 t^2 + \nu^2} \\ &\geq \frac{97}{100} \frac{1}{\rho^2 + 1} \end{aligned}$$

We get the upper estimate on $[\nu^\beta, \frac{\pi}{4}]$ by using

$$\begin{aligned} \psi'_\infty &\leq 1, \\ \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)' &= O\left(\frac{\psi_\infty^3}{l^2}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{D^6} &= \left(\frac{\nu^2}{\rho^2 \psi_\infty^2 + \frac{\varphi_\infty^4}{\psi_\infty^2} \frac{\nu^2}{l^2} + \nu^2} \right)^3 \\ &\leq O\left(\frac{\nu^2}{\rho^2 \psi_\infty^2 + \psi_\infty^2 \frac{\nu^2}{l^2} + \nu^2} \right)^3 \\ &\leq O\left(\frac{\nu^2}{\rho^2 \nu^{2\beta} + \nu^2} \right)^3 \\ &\leq \left(\frac{\nu^2}{\rho^2 \nu^{2\beta}} \right)^3 \\ &\leq \frac{1}{\rho^2} \nu^{6(1-\beta)} \end{aligned}$$

So

$$\begin{aligned} (\psi'_{\nu,t})^2 \Big|_{[\nu^\beta, \frac{\pi}{4}]} &\leq O\left(\frac{1}{\rho^2} \frac{\nu^{6(1-\beta)}}{l^4} \right) \\ &= O\left(\frac{1}{\rho^2} \nu^{6(1-\beta) - \frac{4}{3}} \right) \\ &= O\left(\nu^{\frac{14}{3} - 6\beta} \right) \end{aligned}$$

□

The results in the remainder of this section will be used in [PetWilh2], but not in this paper.

Lemma 7.6. *In addition to the assumptions above suppose that*

$$\text{curv}^M(X, W^{\mathcal{H}, g_\infty}) \geq C_1 \psi_\infty^2$$

for some positive constant C_1 .

Let $\gamma : [0, \pi/4] \longrightarrow M$ be as above, then for any $\beta > 0$

$$- (D_X (\psi_{\nu,l} D_X \psi_{\nu,l})) > 0,$$

provided $t \geq \frac{\nu}{\sqrt{3\rho}} + \beta\nu$, and ν is sufficiently small.

Proof. Because the second derivative of $\psi_{\nu,l}$ is so complicated, we divide the proof of the first inequality into the case where $t \geq O(\nu^{1/2})$ and the case where $t \leq O(\nu^{1/2})$. Since $\psi_\infty \equiv |W^{\mathcal{H},g_\infty}|_{g_\infty}$ and $\psi_{\nu,l} \equiv |W^{\mathcal{H},g_{\nu,l}}|_{g_{\nu,l}}$ we have using Proposition 4.6

$$\begin{aligned} -\psi_{\nu,l} \psi''_{\nu,l} &= \text{curv}^B (X, W^{\mathcal{H},g_{\nu,l}}) \\ &\geq \text{curv}^M (X, W^{\mathcal{H},g_{\nu,l}}) \\ &= \psi_{\nu,l}^2 \text{curv}^M \left(X, \frac{W^{\mathcal{H},g_{\nu,l}}}{|W^{\mathcal{H},g_{\nu,l}}|^2} \right) \\ &= \psi_{\nu,l}^2 \text{curv}^M \left(X, Dq_{G \times M} \left(\frac{\widehat{W^{\mathcal{H},g_\infty}}}{|\widehat{W^{\mathcal{H},g_\infty}}|^2} \right) \right) \\ &= \frac{\psi_{\nu,l}^2}{|\widehat{W^{\mathcal{H},g_\infty}}|^2} \text{curv}^M \left(X, Dq_{G \times M} \left(\widehat{W^{\mathcal{H},g_\infty}} \right) \right) \\ &\geq \frac{\psi_{\nu,l}^2}{|\widehat{W^{\mathcal{H},g_\infty}}|_{G_{\nu,l} \times M}^2} C_1 \psi_\infty^2 \end{aligned}$$

Using $|\widehat{W^{\mathcal{H},g_\infty}}|^2 = \psi_\infty^2 D^2$ and $\psi_{\nu,l}^2 = \frac{\psi_\infty^2}{D^2}$ we have

$$\begin{aligned} -\psi_{\nu,l} \psi''_{\nu,l} &\geq \frac{\psi_\infty^2}{D^2} \frac{1}{\psi_\infty^2 D^2} C_1 \psi_\infty^2 \\ &= C_1 \frac{\psi_\infty^2}{D^4} \end{aligned}$$

and

$$(\psi'_{\nu,l})^2 = \frac{\left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2}{D^6}$$

So it would be enough to prove

$$\left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \leq C_1 \psi_\infty^2 D^2$$

For $t \geq O(\nu^{1/2})$

$$\left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \leq (\psi'_\infty)^2 + O\left(\frac{t^3}{l^2}\right) + O\left(\frac{t^6}{l^4}\right)$$

and since $D^2 = \rho^2 \frac{\psi_\infty^2}{\nu^2} + \frac{\varphi_\infty^4}{l^2 \psi_\infty^2} + 1$,

$$C_2 t^2 \left(1 + \rho^2 \frac{t^2}{\nu^2}\right) \leq C_1 \psi_\infty^2 D^2,$$

for another positive constant C_2 . So the desired inequality would follow from

$$(\psi'_\infty)^2 \leq C_2 t^2 \left(1 + \rho^2 \frac{t^2}{\nu^2}\right)$$

or

$$1 \leq O\left(\frac{t^4}{\nu^2}\right)$$

or

$$t \geq O\left(\nu^{1/2}\right).$$

For $t \leq O\left(\nu^{1/2}\right)$,

$$\begin{aligned} (\psi'_{\nu,l})^2 &= \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)'\right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \\ &\leq \left(\psi'_\infty - O\left(\frac{\nu^{3/2}}{l^2}\right)\right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6}, \end{aligned}$$

$$\begin{aligned} |\psi_{\nu,l} \psi''_{\nu,l}| &\geq \left| \left(\psi''_\infty - \frac{6\varphi_\infty^2 \varphi'_\infty}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)' - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)'' \right) \frac{\psi_{\nu,l}^4}{\psi_\infty^3} \right. \\ &\quad \left. - 3 \frac{\psi'_\infty}{\psi_\infty} \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)' \right) \frac{\psi_{\nu,l}^4}{\psi_\infty^3} \right. \\ &\quad \left. + 3 \left(\psi'_\infty - \frac{2\varphi_\infty^3}{l^2} \left(\frac{\varphi_\infty}{\psi_\infty}\right)' \right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right| \\ &\geq \left| \left(\psi''_\infty - O\left(\frac{t^2}{l^2}\right) - O\left(\frac{t^3}{l^2}\right) \right) \frac{\psi_{\nu,l}^4}{\psi_\infty^3} \right. \\ &\quad \left. - 3 \frac{\psi'_\infty}{\psi_\infty} \left(\psi'_\infty - O\left(\frac{t^3}{l^2}\right) \right) \frac{\psi_{\nu,l}^4}{\psi_\infty^3} \right. \\ &\quad \left. + 3 \left(\psi'_\infty - O\left(\frac{t^3}{l^2}\right) \right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right| \end{aligned}$$

Since $t \leq O\left(\nu^{1/2}\right)$ we conclude that

$$|\psi_{\nu,l} \psi''_{\nu,l}| \geq \left| (\psi''_\infty) \frac{\psi_{\nu,l}^4}{\psi_\infty^3} - 3 (\psi'_\infty)^2 \left(\frac{\psi_{\nu,l}^4}{\psi_\infty^3} - \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right) \right| + O,$$

where “ O ” stands for a quantity that is too small to matter.

Recalling that $\frac{\psi_\infty^2}{\psi_{\nu,l}^2} = D^2 = \rho^2 \frac{\psi_\infty^2}{\nu^2} + \frac{\varphi_\infty^4}{l^2 \psi_\infty^2} + 1$ we have

$$\begin{aligned} \left(\frac{\psi_{\nu,l}^4}{\psi_\infty^4} - \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right) &= \frac{1}{D^4} - \frac{1}{D^6} \\ &= \frac{D^2 - 1}{D^6} \\ &= \frac{\rho^2 \frac{\psi_\infty^2}{\nu^2}}{D^6} + O \\ &= \frac{\rho^2 \psi_\infty^2}{\nu^2 D^6} + O \end{aligned}$$

Since $\psi_\infty''(0) = 0$, it follows that for $t \leq O(\nu^{1/2})$,

$$|\psi_{\nu,l} \psi_{\nu,l}''| \geq 3 (\psi_\infty')^2 \frac{\rho^2 \psi_\infty^2}{\nu^2 D^6} + O.$$

Thus our total derivative is positive when

$$\begin{aligned} (\psi_\infty')^2 \left(\frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right) &\leq 3 (\psi_\infty')^2 \frac{\rho^2 \psi_\infty^2}{\nu^2 D^6} + O, \text{ or} \\ \left(\frac{1}{D^6} \right) &\leq 3 \rho^2 \frac{\psi_\infty^2}{\nu^2 D^6} + O, \text{ or} \\ \nu &\leq \sqrt{3} \rho \psi_\infty + O, \text{ or} \end{aligned}$$

Since $\psi_\infty'(0) = 1$, $\psi_\infty''(0) = 0$, this is equivalent to

$$t \geq \frac{\nu}{\sqrt{3} \rho} + O$$

□

Lemma 7.7.

$$(7.7) \quad \left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}''} [\psi_{\nu,l} \psi_{\nu,l}']' \right| \leq \max \left\{ \psi_{\nu,l}^2, \frac{\nu^2}{3\rho^2} \right\}.$$

Proof. From the previous result we have that for $t \geq \frac{\nu}{\sqrt{3}\rho} + O$, $|\psi_{\nu,l} \psi_{\nu,l}'| \leq |\psi_{\nu,l} \psi_{\nu,l}''|$, so

$$\left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}''} [\psi_{\nu,l} \psi_{\nu,l}']' \right| \leq \psi_{\nu,l}^2$$

For some $t \leq \frac{\nu}{\sqrt{3}\rho} + O$ the above inequality fails, but then we have

$$\left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}''} [\psi_{\nu,l} \psi_{\nu,l}']' \right| \leq \left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}''} [\psi_{\nu,l}']^2 \right|$$

Estimating as in Proposition 7.4 we have that for $t \leq \frac{\nu}{\sqrt{3}\rho} + O$

$$[\psi_{\nu,l}']^2 \leq \left[(\psi_\infty') \frac{\psi_{\nu,l}^3}{\psi_\infty^3} \right]^2 + O$$

and

$$\begin{aligned}\psi''_{\nu,l} &= \psi''_{\infty} \frac{\psi_{\nu,l}^3}{\psi_{\infty}^3} - 3(\psi'_{\infty})^2 \left(\frac{\psi_{\nu,l}^3}{\psi_{\infty}^4} - \frac{\psi_{\nu,l}^5}{\psi_{\infty}^6} \right) + O \\ &= -3(\psi'_{\infty})^2 \left(\frac{\psi_{\nu,l}^3}{\psi_{\infty}^4} - \frac{\psi_{\nu,l}^5}{\psi_{\infty}^6} \right) + O\end{aligned}$$

Using $D^2 = \rho^2 \frac{\psi_{\infty}^2}{\nu^2} + \frac{\varphi_{\infty}^4}{l^2 \psi_{\infty}^2} + 1$ we have

$$\begin{aligned}\left(\frac{\psi_{\nu,l}^3}{\psi_{\infty}^4} - \frac{\psi_{\nu,l}^5}{\psi_{\infty}^6} \right) &= \frac{1}{\psi_{\infty}} \left(\frac{1}{D^3} - \frac{1}{D^5} \right) \\ &= \frac{1}{\psi_{\infty}} \left(\frac{D^2 - 1}{D^5} \right) \\ &= \frac{\rho^2}{\nu^2} \left(\frac{\psi_{\infty}}{D^5} \right) + O\end{aligned}$$

So

$$\begin{aligned}\left| \frac{\psi_{\nu,l}}{\psi''_{\nu,l}} [\psi_{\nu,l} \psi'_{\nu,l}]' \right| &\leq \left| \frac{\psi_{\nu,l}}{\psi''_{\nu,l}} [\psi'_{\nu,l}]^2 \right| \\ &\leq \frac{1}{3(\psi'_{\infty})^2 \frac{\rho^2}{\nu^2} \left(\frac{\psi_{\infty}}{D^5} \right)} \left[(\psi'_{\infty})^2 \frac{\psi_{\nu,l}^7}{\psi_{\infty}^6} \right] \\ &= \frac{\nu^2 D^5}{3\rho^2 (\psi_{\infty})} \left[\frac{1}{D^6} \right] \psi_{\nu,l} \\ &= \frac{\nu^2 D^5}{3\rho^2 (\psi_{\infty})} \left[\frac{1}{D^6} \right] \frac{\psi_{\infty}}{D} \\ &= \frac{\nu^2}{3\rho^2} \left[\frac{1}{D^2} \right]\end{aligned}$$

Since $D^2 \geq 1$, we get the desired inequality. \square

8. SYNERGY

In Section 4, we detailed an abstract setting for which fiber scaling produces integrally positive curvature on initially flat totally geodesic tori. We also explained how, with a few extra hypotheses, this deformation can be combined with a conformal change to produce positive curvature on a single initially flat torus. In Section 5, we described an abstract framework that will allow to use a tangential partial conformal change to put positive curvature on all the initially flat tori in the Gromoll-Meyer sphere, simultaneously. However, we are not aware of any way to combine Cheeger deformations, fiber scaling and tangential partial conformal changes to put positive curvature on the Gromoll-Meyer sphere. The problem is that these deformations only produce positive curvature to higher order on the initially flat tori. In principle, such a deformation could produce positive curvature, but much more needs to be verified. As far as we can tell this verification must fail for the Gromoll-Meyer sphere.

We described in Sections 2 and 3 a method, called orthogonal partial conformal change, that will allow us to change the metric on the Gromoll–Meyer sphere to one that

- has nonnegative curvature,
- the same zero curvatures,

and to which we will be able to apply a combination of Cheeger deformations, fiber scaling, and partial conformal changes and get positive curvature.

In this section, we discuss in an abstract setting, how the orthogonal partial conformal change of Sections 2 and 3 will play a role in making our problem more solvable. This will involve a synergy between the curvature compression principle, fiber scaling, and the orthogonal partial conformal change.

To allow for a slightly less intertwined exposition we will explain this synergy as it applies to a single torus. This will allow us to use a conformal change in place of the tangential conformal change.

The addition of the orthogonal partial conformal change will aid us in verifying the positivity of the curvatures of planes of the form

$$\text{span}\{X, W + \tau V\},$$

where V is perpendicular to X , W , and $W^{\mathcal{H}}$, and $\tau \in \mathbb{R}$. It is necessary that such planes have positive curvature, but of course it is not sufficient.

The curvature of $\text{span}\{X, W + \tau V\}$ is a quadratic polynomial in τ

$$Q(\tau) = \text{curv}(X, W) + 2\tau R(W, X, X, V) + \tau^2 \text{curv}(X, V)$$

whose minimum value is

$$\text{curv}(X, W) - \frac{R(W, X, X, V)^2}{\text{curv}(X, V)}.$$

Proposition 8.1. *Let M be nonnegatively curved and let X, W satisfy the hypotheses of Section 4. After scaling the fibers of the Riemannian submersion π and performing the conformal change described in subsection 4.1,*

$$\begin{aligned} \text{curv}(X, W) - \frac{R(W, X, X, V)^2}{\text{curv}(X, V)} &= s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X(\psi D_X \psi) - s^4 I'' |W|^2 \\ &\quad - s^4 (D_X \psi)^2 \frac{\left\langle \frac{W^{\mathcal{H}}}{|W^{\mathcal{H}}|}, A_X V \right\rangle^2}{\text{curv}(X, V)} + O(s^6), \end{aligned}$$

provided V is perpendicular to X , W , and $W^{\mathcal{H}}$, and $\text{Hess}^{g_s}(f)(W, V) = 0$ where f is as in Section 4.

Remark 8.2. *In our application $\text{Hess}^{g_s}(f)(W, V)$ will not be 0, but it will be too small to matter.*

Proof. According to Proposition 4.12, the first four terms are just $\text{curv}(X, W)$.

Because X and W are initially tangent to a totally geodesic flat in a nonnegatively curved manifold our initial curvature, R^{old} , satisfies

$$R^{\text{old}}(W, X)X = 0.$$

In particular,

$$R^{\text{old}}(W, X, X, V) = 0$$

Our hypotheses on V combined with Lemma 4.8 give us that after fiber scaling

$$R^{g_s}(W, X, X, V)^2 = s^4 (D_X \psi)^2 \left\langle \frac{W^\mathcal{H}}{|W^\mathcal{H}|}, A_X V \right\rangle^2.$$

It remains to verify that this formula continues to hold after our conformal change. After the conformal change we have

$$\begin{aligned} e^{-2f} \langle R^{\text{new}}(W, X) X, V \rangle &= \langle R^{g_s}(W, X) X, V \rangle \\ &\quad - g_s(W, V) \text{Hess}^{g_s}(f)(X, X) - g_s(X, X) \text{Hess}^{g_s}(f)(W, V) \\ &\quad + g_s(X, V) \text{Hess}^{g_s}(f)(W, X) \\ &\quad + g_s(W, V) D_X f D_X f - g_s(X, X) g_s(W, V) |\text{grad} f|^2 \end{aligned}$$

Our hypotheses about V immediately simplifies this to

$$e^{-2f} \langle R^{\text{new}}(W, X) X, V \rangle = \langle R^{g_s}(W, X) X, V \rangle$$

□

In addition to the hypotheses of the previous Proposition we assume the following.

- We have the set up for the orthogonal partial conformal change of Section 3, with X and W tangent to one of the flats S and V tangent to the distribution \mathcal{O} of Section 3.
- There is a $G_1 \times G_2$ action on M as in Lemma 7.1, and the action of G_1 coincides with that of G from the fiber scaling of section 4.

We now carry out metric deformations in the following order.

- Cheeger deform with $G_1 = G$ and with the Cheeger parameter ν being small.
- Perform the *orthogonal partial conformal change* with φ as in Section 3.
- Scale the fibers of the Riemannian submersion $\pi : (M, g_0) \rightarrow B$, as in Section 4, and
- perform a conformal change with conformal factor f as in Subsection 4.1.

As usual we call the initial metric g and the final metric \tilde{g} . The metric obtained by omitting the orthogonal partial conformal change will be called \bar{g} .

Remark 8.3. *We have chosen to explain the synergy only for a single torus. Because of this our final deformation can be an actual conformal change rather than the tangential conformal change described in Section 5. The abstract framework of Section 5 will allow us to achieve the same results on all of the initially flat tori of the Gromoll–Meyer sphere using a tangential conformal change.*

We remind the reader that the notational convention about the Cheeger reparametrization described after Proposition 6.3 is in effect.

Lemma 8.4. *In addition to the hypotheses above assume that the ratio*

$$\frac{g\left(\frac{W^\mathcal{H}}{|W^\mathcal{H}|}, A_X V\right)^2}{\text{curv}^g(X, V)} \leq C$$

for all ν .

There is a function $\varkappa : (0, 1) \rightarrow \mathbb{R}_+$ with $\lim_{t \rightarrow 0} \varkappa(t) = 0$ so that for all $\tau \in \mathbb{R}$

$$(8.4) \quad Q(\tau) \equiv \text{curv}^{\tilde{g}}(X, W + \tau V) \geq (1 - \varkappa(\nu)) \text{curv}^{\tilde{g}}(X, W) > 0,$$

provided that the φ used in the orthogonal partial conformal change is chosen appropriately.

Remark 8.5. *The reader should note that this lemma not only shows that these curvatures are positive, but also shows that we can make them as close as we like to $\text{curv}^{\bar{g}}(X, W)$. This will be important for our computations on the Gromoll–Meyer sphere.*

Without the orthogonal partial conformal change we still get an estimate that roughly looks like

$$Q(\tau) \equiv \text{curv}^{\bar{g}}(X, W + \tau V) \geq \frac{1}{100} \text{curv}^{\bar{g}}(X, W) > 0$$

on the Gromoll–Meyer sphere. It turns out that this estimate is not good enough, but one like inequality 8.4 is.

Remark 8.6. *By carefully considering the exponents in Proposition 7.4 one can also make more precise statements about the behavior of allowable functions \varkappa near 0. We will not need this, and so have omitted it.*

Proof. For the moment assume that $\varphi \equiv 1$, (i.e., the orthogonal partial conformal change is not performed, and the resulting metric is called \bar{g} .)

Combining the previous lemma with our new hypothesis that there is a $C > 0$ so that

$$\frac{g\left(\frac{W^\varkappa}{|W^\varkappa|}, A_X V\right)^2}{\text{curv}^g(X, V)} \leq C$$

for all ν , we conclude that the minimum of $Q(\tau)$ satisfies

$$\begin{aligned} & \text{curv}^{\bar{g}}(X, W) - \frac{R^{\bar{g}}(W, X, X, V)^2}{\text{curv}^{\bar{g}}(X, V)} \\ & \geq s^4 (D_X \psi)^2 (1 - C) + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X(\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6) \end{aligned}$$

It follows from Proposition 4.14 that the sum of the first three terms on the right hand side has a negative integral over an integral curve γ of X that is parameterized as in Proposition 4.14. So the metric \bar{g} can not satisfy our conclusion. Depending on the precise value of C we may even get that the minimum of $Q(\tau)$ is negative somewhere along γ for all choices of I'' . In any event, our conclusion is false without the orthogonal partial conformal change.

It follows from Theorem 3.5 that the orthogonal partial conformal change does not affect $\text{curv}(X, W)$ and $R(W, X, X, V)$. Its effect on $\text{curv}(X, V)$ is given in Proposition 3.4 and is

$$(8.6) \quad \text{curv}^{\bar{g}}(X, V) = \text{curv}^g(X, V) - \varphi'' |V|_g^2 |X|_g^2 + O(C^1).$$

where we use φ'' for $D_X D_X(\varphi)$. The goal will now be to select φ'' appropriately so as to adjust our estimate for

$$(D_X \psi)^2 \frac{\tilde{g}\left(\frac{W^\varkappa}{|W^\varkappa|}, A_X V\right)^2}{\text{curv}^{\bar{g}}(X, V)}$$

Recall that

$$\varphi \equiv f \circ r$$

where r is a smooth distance function with gradient X and $f : \mathbb{R} \rightarrow \mathbb{R}$, and is constant outside of a compact interval, $[a, b]$. So in fact $\varphi'' = f'' \circ r$. Since f is constant outside of $[a, b]$

$$\int_{[a,b]} f'' = 0.$$

Equation 8.6 therefore gives us a way to redistribute $\text{curv}(X, V)$ along the integral curves of X .

Our curvature compression result, Proposition 7.4, and our estimate for the minimum of $Q(\tau)$ together suggest an appealing choice for f'' . Indeed Proposition 7.4 says, for example, that $(D_X \psi)^2 \leq \nu^{4.6}$ outside of an interval like $[0, \varkappa(\nu)]$. We choose f'' to be negative (and relatively large in absolute value) on an interval like $[0, O(\nu)]$ and “pay for this” by having f'' be positive (but relatively small) on $[\varkappa(\nu), \frac{\pi}{4}]$. With such a choice of φ we can make the integral over any integral curve of X satisfy

$$\varkappa(\nu) \int \text{curv}^{\bar{g}}(X, W) > \int \frac{R^{\bar{g}}(W, X, X, V)^2}{\text{curv}^{\bar{g}}(X, V)}$$

for the appropriately chosen function $\varkappa(\nu)$ with $\lim_{\nu \rightarrow 0} \varkappa(\nu) = 0$. Indeed we have made the denominator $\text{curv}^{\bar{g}}(X, V)$ larger on the region $[0, O(\nu)]$ where $R^{\bar{g}}(W, X, X, V)^2$ is relatively large. We have done this at the expense of making it very slightly smaller on $[\varkappa(\nu), \frac{\pi}{4}]$, but on this region $R^{\bar{g}}(W, X, X, V)^2$ is very small. So our redistribution of $\text{curv}(X, V)$ does in fact give us the desired inequality in an integral sense.

We obtain the point wise inequality by combining the integral inequality with a judicious choice of I'' . Namely that it be sufficiently negative on the the complement of $[0, O(\nu)]$. \square

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