Corrections and additions for 2nd Edition of Riemannian Geometry

I'd like to thank Victor Alvarez, Mayer Amitai Landau, Ciprian Manolescu, Jiayin Pan, Fred Wilhelm for finding several mistakes in my book.

12^{16} : Replace section 3.4. with the following more general exposition. I have now also added an elementary article about warped products on the webpage. It covers the issues addresses here in an even more elegant fashion.

In the rotationally symmetric examples we haven't discussed what happens when $\varphi(t) = 0$. In the revolution case, the curve clearly needs to have a vertical tangent in order to look smooth. To be specific, assume that we have $dt^2 + \varphi^2(t)d\theta^2$, $\varphi : [0, b) \to [0, \infty)$, where $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. All other situations can be translated or reflected into this position.

More generally we wish to consider metrics on $I \times S^{n-1}$ of the type $dt^2 + \varphi^2(t)ds_{n-1}^2$, where ds_{n-1}^2 is the canonical metric on $S^{n-1}(1) \subset \mathbb{R}^n$. These are the *rotationally symmetric metrics*. If we assume that $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$ we want to check that the metric extends smoothly near $t = 0$ to give a smooth metric on $\mathbb{R}^n$. The natural coordinate change to make is

$$x = tz$$

where $x \in \mathbb{R}^n$ and $t > 0$ and $z \in S^{n-1}(1) \subset \mathbb{R}^n$. Thus

$$ds_{n-1}^2 = \sum_{i=1}^{n} \left(dz^i\right)^2,$$

Keeping in mind that the constraint

$$\sum \left(z^i\right)^2 = 1$$

implies the relationship

$$\sum z^i dz^i = 0$$

between the restriction of the differentials to $S^{n-1}(1)$.

The standard metric on $\mathbb{R}^n$ can now be written as

$$\begin{aligned} \sum \left(dx^i\right)^2 &= \sum \left(z^i dt + t dz^i\right)^2 \\ &= \sum \left(z^i\right)^2 dt^2 + t^2 \left(dz^i\right)^2 + (tdt)\left(z^i dz^i\right) + \left(z^i dz^i\right)(tdt) \\ &= dt^2 + t^2 ds_{n-1}^2 \end{aligned}$$

when switching to polar coordinates.

In the general situation we have to do this calculation in reverse and check that the expression becomes smooth at the origin corresponding to $x^i = 0$. So we have to calculate dt and dz^i in terms of x^i. In fact we only need to observe that

$$\begin{aligned} 2tdt &= 2\sum x^i dx^i, \\ dt &= \frac{1}{t}\sum x^i dx^i \end{aligned}$$

and then from our knowledge of the Euclidean metric that

$$ds_{n-1}^2 = \frac{\sum \left(dx^i\right)^2 - dt^2}{t^2}$$

This gives us

$$\begin{aligned} dt^2 + \varphi^2(t) ds_{n-1}^2 &= dt^2 + \varphi^2(t) \frac{\sum \left(dx^i\right)^2 - dt^2}{t^2} \\ &= \left(1 - \frac{\varphi^2(t)}{t^2}\right) dt^2 + \frac{\varphi^2(t)}{t^2} \sum \left(dx^i\right)^2 \\ &= \left(\frac{1}{t^2} - \frac{\varphi^2(t)}{t^4}\right) \left(\sum x^i dx^i\right)^2 + \frac{\varphi^2(t)}{t^2} \sum \left(dx^i\right)^2 \end{aligned}$$

Thus we have to ensure that the functions

$$\frac{\varphi^2(t)}{t^2} \text{ and } \left(\frac{1}{t^2} - \frac{\varphi^2(t)}{t^4}\right)$$

are smooth, keeping in mind that $t = \sqrt{\sum (x^i)^2}$ is not differentiable at the origin. Assuming that $\varphi(0) = 0$ is necessary for the first function to be continuous at $t = 0$, while we have to additionally assume that $\dot{\varphi}(0) = 1$ for the second function to be continuous. Next we note that if

$$\varphi(t) = t + \sum_{k=1}^{\infty} a_k t^{2k+1}$$

then the two functions reduce to

$$\frac{\varphi^2(t)}{t^2} = \left(1 + \sum_{k=1}^{\infty} a_k t^{2k}\right)^2$$

and

$$\begin{aligned} \frac{1}{t^2} - \frac{\varphi^2(t)}{t^4} &= \frac{1}{t^2} - \left(\frac{1}{t} + \sum_{k=1}^{\infty} a_k t^{2k-1}\right)^2 \\ &= \frac{2}{t} \sum_{k=1}^{\infty} a_k t^{2k-1} + \left(\sum_{k=1}^{\infty} a_k t^{2k-1}\right)^2 \\ &= 2 \sum_{k=1}^{\infty} a_k t^{2k-2} + \left(\sum_{k=1}^{\infty} a_k t^{2k-1}\right)^2 \end{aligned}$$

Showing that they are smooth.

These conditions are all satisfied by the metrics $dt^2 + \mathrm{sn}_k^2(t) ds_{n-1}^2$, where $t \in [0, \infty)$ when $k \leq 0$ and $t \in [0, \frac{\pi}{\sqrt{k}}]$ for $k > 0$.

It is possible to show that in general the two functions are smooth iff $\varphi^{(\text{even})}(0)=0$ and $\dot{\varphi}(0)=1$, but will only need to use the real analytic case we established.

23^{14} : "extend" should be "extent"

23_9 : $(L_X\delta_{ij})$ should be $(L_X\delta_{ij})\,dx^i dx^j$

25_5: $[X,Y]$ should be $[Y,Z]$.

28_{15} : (Y,Z) should be (X,Y)

31^4: $\Gamma_{ij,k}$ should be $\Gamma_{ij,l}$

33^4 : Γ_{ij}^k should be Γ_{ij}^l

33^{13} : $\nabla_Y\nabla_X$ should be $\nabla_Y\nabla_X Z$

36^{14} : Z should be V.

40^2 : Replace the calculation of div (Ric) by

$$\begin{aligned}
2\operatorname{div}(\mathrm{Ric}) &= 2(n-1)\operatorname{div}(f\cdot I)\\
&= 2(n-1)\,df+2(n-1)\,f\operatorname{div}(I)\\
&= 2(n-1)\,df
\end{aligned}$$

40_{15} : Replace calculation by

$$\begin{aligned}
d\mathrm{scal}(W)|_p &= D_W\mathrm{scal}\\
&= \sum(\nabla_W R)(E_i,E_j,E_j,E_i)\\
&= \sum\left(\nabla_{E_j}R\right)(E_i,W,E_j,E_i)\\
&\quad -\sum(\nabla_{E_i}R)(E_j,W,E_j,E_i)\\
&= 2\sum\left(\nabla_{E_j}R\right)(E_i,W,E_j,E_i)\\
&= 2\sum\left(\nabla_{E_j}R\right)(E_j,E_i,E_i,W)\\
&= 2\sum\nabla_{E_j}\left(R\left(E_j,E_i,E_i,W\right)\right)\\
&= 2\sum\nabla_{E_j}g\left(\mathrm{Ric}\left(E_j\right),W\right)\\
&= 2\sum g\left(\left(\nabla_{E_j}\mathrm{Ric}\right)(W),E_j\right)\\
&= 2\operatorname{div}(\mathrm{Ric})(W)(p).
\end{aligned}$$

45^5 : In view of how the exterior derivative is defined in the appendix it is worth mentioning that Theorem 4 can be stated more succinctly as follows:

$$R(X,Y)\,\partial_r=\left(d^{\nabla}S\right)(X,Y).$$

92^{12} : Should be

$$\begin{aligned}
h\circ k(v_1,v_2,v_3,v_4) &= h(v_1,v_4)\,k(v_2,v_3)+h(v_2,v_3)\,k(v_1,v_4)\\
&\quad -h(v_1,v_3)\,k(v_2,v_4)+h(v_2,v_4)\,k(v_1,v_3)
\end{aligned}$$

92^{16} : Should be

$$R=\frac{\mathrm{scal}}{2n(n-1)}g\circ g+\frac{1}{n-2}\left(\mathrm{Ric}-\frac{\mathrm{scal}}{n}g\right)\circ g+W$$

118_8: Replace $\dot{\gamma}_v(0) = v$ with $\dot{\gamma}_v(a) = v$.

118_4 : Replace γ by γ_v and $\dot{\gamma}(0)$ with $\dot{\gamma}_v(0)$

124_{17} : Lemma 10 can also be proven by showing that $|dr| \leq 1$ (with equality only on vectors proportional to ∇r) and then noting

$$\begin{aligned} \ell(\gamma) &= \int_0^b |\dot{\gamma}| dt \\ &\geq \int_0^b |dr(\dot{\gamma})| \, dt \\ &\geq \left| \int_0^b d(r \circ \gamma) \, dt \right| \\ &= |r(q) - r(p)| \, . \end{aligned}$$

134_{14} : Instead of the sentence: "A careful....proved" Insert: Note that in coordinates this can be written as

$$g\left(x^i \partial_i, v^j \partial_j\right) = \delta_{ij} x^i v^j$$

Since

$$dr(v) = \delta_{ij} x^i v^j$$

we note that this can only be true if $\nabla r = \partial_r = x^i \partial_i$, which is our version of the Gauss lemma.

136_8 : The proof of Theorem 15 should be simplified as follows:

Proof. We use polar coordinates around $p \in M$ and the asymptotic behavior of g_r and Hess$r$ near p that was just established. We shall also use the fundamental equations

$$\begin{aligned} L_{\partial_r} g &= 2\text{Hess} r \\ \left(L_{\partial_r} \text{Hess} r\right)(X, Y) - \text{Hess}^2 r \, (X, Y) &= -R(X, \partial_r, \partial_r, Y) \, . \end{aligned}$$

that were introduced in chapter 2.

Our assumption that the curvature is constant implies that the second equation becomes

$$L_{\partial_r} \text{Hess} r - \text{Hess}^2 r = -k g_r .$$

We wish to show that

$$\begin{aligned} g &= dr^2 + g_r = dr^2 + \text{sn}_k^2(r) \, ds_{n-1}^2 , \\ \text{Hess} r &= \frac{\text{sn}_k'(r)}{\text{sn}_k(r)} g_r . \end{aligned}$$

This is almost automatic as the right hand sides both have the same initial conditions as g and Hess$r$ as we approach p and also simultaneously solve the two equations

$$\begin{aligned} L_{\partial_r} g &= 2\text{Hess} r, \\ L_{\partial_r} \text{Hess} r - \text{Hess}^2 r &= -k g_r . \end{aligned}$$

■

139^{19}: ...intergral curves for r... should be ...integral curves for ∂_r ...

147^{20}: The last $=$ in the display is incorrect. Use the law of cosines to find $\cos\angle(v,w)$.

149: Exercise 1: Assume that (M,g) has the property that all *unit* speed geodesics........

178^{13}: The display should read

$$\operatorname{inj}(M) \geq \frac{\pi}{\sqrt{K}} \text{ or } \operatorname{inj}(M) = \frac{1}{2}\,(\text{length of shortest closed geodesic})$$

187 : Much in Chapter 7 has been completely revised. Look at the link to Bochner-Lichnerowicz-Weitzenböck Formulas.

204: Index mistakes in Lemma 27. The claim is that

$$\delta^{k-1} = (-1)^{(k+1)(n-k)} * d^{n-k} *$$

And the proof is

If $\omega_1 \in \Omega^k(M)$, $\omega_2 \in \Omega^{k-1}(M)$, then

$$\begin{aligned}
(\delta^{k-1}\omega_1, \omega_2) &= (\omega_1, d^{k-1}\omega_2) \\
&= (-1)^{k(n-k)} (* * \omega_1, d^{k-1}\omega_2) \\
&= (-1)^{k(n-k)} \int_M *\omega_1 \wedge d^{k-1}\omega_2 \\
&= (-1)^{n-k+k(n-k)} \int_M d^{n-1}(*\omega_1 \wedge \omega_2) \\
&\quad -(-1)^{n-k+k(n-k)} \int_M (d^{n-k} * \omega_1) \wedge \omega_2 \\
&= (-1)^{n-k+k(n-k)} \int_M (d^{n-k} * \omega_1) \wedge \omega_2 \\
&= (-1)^{(k+1)(n-k)} (*d^{n-k} * \omega_1, \omega_2).
\end{aligned}$$

241^2: More precisely: $\operatorname{Fix}(\sigma)_0 \subset \operatorname{Iso}_p \subset \operatorname{Fix}(\sigma)$

256_7: The display should be

$$R(v,w) = \lim_{t\to 0} \frac{I - P_t}{t}$$

262^{20}: replace “closed” by “exact”.

275_{10} : Change the proof of Theorem 63 to be

Proof. First observe that for $k > 0$ there is nothing to prove, as we know that $b_1 = 0$ from Myers' theorem. Suppose we have chosen a covering $\bar{M}$ of M with torsion-free Abelian Galois group of deck transformations $\Gamma = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ such that for some $x \in \bar{M}$ we have

$$\begin{aligned}
d(x, \gamma_i(x)) &\leq 2\operatorname{diam}(M), \\
d(x, \gamma(x)) &> \operatorname{diam}(M), \gamma \neq 1.
\end{aligned}$$

Then we clearly have that all of the balls $B\left(\gamma(x), \frac{\mathrm{diam}(M)}{2}\right)$ are disjoint. Now set

$$I_r = \{\gamma \in \Gamma : \gamma = l_1 \cdot \gamma_1 + \cdots + l_{b_1} \cdot \gamma_{b_1}, |l_1|, ..., |l_{b_1}| \leq r\}.$$

Note that for $\gamma \in I_r$ we have

$$B\left(\gamma(x), \frac{\mathrm{diam}(M)}{2}\right) \subset B\left(x, r \cdot 2\mathrm{diam}(M) + \frac{\mathrm{diam}(M)}{2}\right).$$

All of these balls are disjoint and have the same volume, as γ acts isometrically. We can therefore use the relative volume comparison theorem to conclude that the cardinality of I_r is bounded from above by

$$\frac{\mathrm{vol}B\left(x, r \cdot 2\mathrm{diam}(M) + \frac{\mathrm{diam}(M)}{2}\right)}{\mathrm{vol}B\left(x, \frac{\mathrm{diam}(M)}{2}\right)} \leq \frac{v\left(n, k, r \cdot 2\mathrm{diam}(M) + \frac{\mathrm{diam}(M)}{2}\right)}{v\left(n, k, \frac{\mathrm{diam}(M)}{2}\right)}.$$

This shows that

$$\begin{aligned} b_1 &\leq |I_1| \\ &\leq \frac{v\left(n, k, 2\mathrm{diam}(M) + \frac{\mathrm{diam}(M)}{2}\right)}{v\left(n, k, \frac{\mathrm{diam}(M)}{2}\right)}, \end{aligned}$$

which gives us a general bound for b_1. To get a more refined bound we have to use I_r for larger r. If r is an integer, then

$$|I_r| = (2r+1)^{b_1}.$$

The upper bound for $|I_r|$ can be reduced to

$$\begin{aligned} \frac{v\left(n, k, b_1 \cdot r \cdot 2\mathrm{diam}(M) + \frac{\mathrm{diam}(M)}{2}\right)}{v\left(n, k, \frac{\mathrm{diam}(M)}{2}\right)} &\leq \frac{v\left(n, k, \left(b_1 \cdot r \cdot 2 + \frac{1}{2}\right) D\right)}{v\left(n, k, \frac{D}{2}\right)} \\ &= \frac{\int_0^{\left(b_1 \cdot r \cdot 2 + \frac{1}{2}\right)D} \left(\frac{\sinh\left(\sqrt{-k}t\right)}{\sqrt{-k}}\right)^{n-1} dt}{\int_0^{\frac{1}{2}D} \left(\frac{\sinh\left(\sqrt{-k}t\right)}{\sqrt{-k}}\right)^{n-1} dt} \\ &= \frac{\int_0^{\left(b_1 \cdot r \cdot 2 + \frac{1}{2}\right)D\sqrt{-k}} \sinh^{n-1}(t)\, dt}{\int_0^{\frac{1}{2}D\sqrt{-k}} \sinh^{n-1}(t)\, dt} \\ &= 2^n \left(b_1 \cdot r \cdot 2 + \frac{1}{2}\right)^n + \cdots \leq (b_1 \cdot 5 \cdot r)^n, \end{aligned}$$

where in the last step we assume that $D\sqrt{-k}$ is very small relative to r. By taking logarithms this is equivalent to

$$b_1 \log(2r+1) \leq n \log(b_1 \cdot 5 \cdot r), \text{ or}$$

or

$$
\begin{aligned}
\frac{b_1}{n} &\leq \frac{\log\left(b_1 \cdot 5 \cdot r\right)}{\log\left(2r+1\right)} \\
&\leq \frac{\log\left(5b_1\right)+\log r}{\log 2+\log r} \\
&= \frac{\frac{\log(5b_1)}{\log r}+1}{\frac{\log 2}{\log r}+1} \\
&\leq \frac{\log\left(5b_1\right)}{\log r}+1
\end{aligned}
$$

If $b_1 \geq n+1$, this is not possible when $r=\left(6b_1\right)^n$. Thus select $r=\left(6b_1\right)^n$ and the assume $D\sqrt{-k}$ is small enough that

$$
\frac{\int_0^{\left(b_1\cdot r\cdot 2+\frac{1}{2}\right)D\sqrt{-k}}\sinh^{n-1}\left(t\right)dt}{\int_0^{\frac{1}{2}D\sqrt{-k}}\sinh^{n-1}\left(t\right)dt} \leq \left(b_1\cdot 5\cdot r\right)^n
$$

in order to force $b_1 \leq n$. ■

363^{5-10} Replace with: ...To see this consider

$$
B\left(x,R/10\right) \subset B\left(p,R/5\right) \subset B\left(x.R/2\right) \subset B\left(p,R\right)
$$

Since $B\left(p,R\right)$ is assumed to be incompressible it follows that $B\left(x,R/2\right)$ does not deformation retract onto $B\left(x,R/10\right)$. Otherwise,

$$
\begin{aligned}
&\operatorname{rank}\left(H_*\left(B\left(x,R/10\right)\to B\left(x.R/2\right)\right)\right) \\
&\qquad\qquad = \operatorname{rank}\left(H_*\left(B\left(p,R/5\right)\to B\left(x.R/2\right)\right)\right) \\
&\qquad\qquad \geq \operatorname{rank}\left(H_*\left(B\left(p,R/5\right)\to B\left(p.R\right)\right)\right)
\end{aligned}
$$

This would imply that $\operatorname{cont}B\left(p,R\right) \leq \operatorname{cont}B\left(x,R/2\right)$ and thus contradict incompressibility of $B\left(p,R\right)$. We can now......

380^7 : Change X_n to X_p on right hand side.

The appendix on de Rham cohomology has been reworked. You can find a corrected version by going to my essay on manifold theory. The only mistakes where some confusion on the degrees of cohomology groups when proving Poincaré duality. Nevertheless I felt that many things should be reworked and hopefully clarified.