# 120A
# Final

**Instructor:** Peter Petersen, MS 6126.
**Office Hrs:** W10am-Noon, Th 1pm-3pm.
**Time:** F Dec 11th 8-11am.
**Place:** MS 5127
**Material:** Chapters 2,3,4,5.

**Exam Rules:** You are not allowed to used any electronic devises during the midterm, nor will the exam be open book. You are allowed to bring a regular piece of paper with whatever information on it that you feel could be helpful.

**Exam Problems:** Study all homework problems as well as other problems from the sections we have covered. Put special emphasis on:

2.4,6
3.5
4.3,4,5,7,8

Here is a suggested final:

1. Let $\alpha(s) = (x(s), y(s))$ be a planar unit speed curve. Show that the curvature can be computed by

$$\kappa(s) = y''x' - x''y'.$$

2. Let $\alpha(s)$ be a unit speed curve in $\mathbb{R}^3$. Prove that

$$(\alpha' \times \alpha'') \cdot \alpha''' = [\alpha', \alpha'', \alpha'''] = \kappa^2 \tau.$$

3. Prove that the concept of a vertex for a planar curve does not depend on the parametrization.

4. Let $\alpha : (a, b) \to \mathbb{R}^3$ be a unit speed curve with $\alpha''(s) \neq 0$ for all $s \in (a, b)$. Define

$$x(s, t) = \alpha(s) + t\alpha'(s).$$

Prove that $x$ defines a parametrized surface for $t \neq 0$. Compute its Gauss curvature.

5. For a surface of revolution $x(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ show that the metric coefficients are given by

$$\begin{aligned} [\mathrm{I}] &= \begin{bmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{bmatrix} \\ [\mathrm{II}] &= \frac{1}{\sqrt{\dot{r}^2 + \dot{z}^2}} \begin{bmatrix} \dot{r}\ddot{z} - \dot{z}\ddot{r} & 0 \\ 0 & r\dot{z} \end{bmatrix} \end{aligned}$$

6. Let $\gamma$ be a curve on $S^2(R)$. Prove that its normal curvature $\kappa_n$ is constant.

7. Assume that $(r(t), z(t))$ is unit speed and defines a surface of revolution by

$$x(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t)).$$

Show that the meridians are geodesics. Meridians are given by $t \to x(\theta, t)$, i.e., $\theta$ is fixed.

8. Let $\gamma(s) = x\left(\gamma^1(s), \gamma^2(s)\right)$ be a unit speed curve on a surface $M$. Prove that

$$\frac{dn}{ds} = -\mathrm{II}(T,T)\,T - \mathrm{II}(T,S)\,S,$$

where $T = \frac{d\gamma}{ds}$, $n$ is the normal to $M$, and $S = n \times T$.

9. Let $X, Y \in T_P M$ be an orthonormal basis for the tangent space at $P$ to the surface $M$. Prove that

$$H = \frac{1}{2}\left(\mathrm{II}(X,X) + \mathrm{II}(Y,Y)\right).$$

10. Let $\alpha : [0, L] \to \mathbb{R}^3$ be a closed curve with $0 \leq \kappa \leq R^{-1}$. Show that the length of $\alpha$ is $\geq 2\pi R$.