Markov Chains Expected time to Absorption

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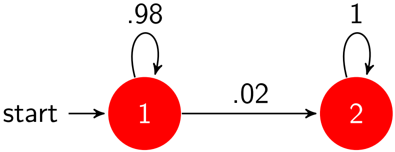
P_i, An Example

Let's for simplicity define

$$\mathrm{P}_i(X_n = j) = \mathrm{P}(X_n = j | X_0 = i)$$



$$P = \begin{bmatrix} .98 & .02 \\ 0 & 1 \end{bmatrix}$$

- There is one recurrent class $R = \{2\}$
- 1 is transient, $P^2 = \begin{bmatrix} 0.96 & 0.04 \\ 0 & 1 \end{bmatrix}, P^{32} = \begin{bmatrix} 0.52 & 0.48 \\ 0 & 1 \end{bmatrix}$,
  $P^{100} = \begin{bmatrix} 0.13 & 0.87 \\ 0 & 1 \end{bmatrix}, \cdots \Pi = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$
- Let T_i be the time (number of steps) needed to arrive at i

$$T_i = \text{Smallest } n \text{ such that } X_n = i$$

- $\mathrm{P}_1(T_2 = 1) = \mathrm{P}(X_1 = 2 | X_0 = 1) = .02$

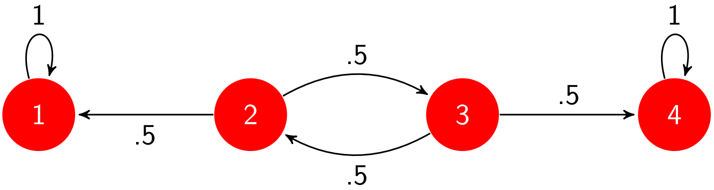
P_i, T_A, Example

- Let $S_1 \subset \mathbf{S}$

$$\mathrm{P}_i(A) = \mathrm{P}(A | X_0 = i)$$

$$T_{S_1} = \min\{n \geq 0 : X_n \in S_1\}$$

- Consider the MC



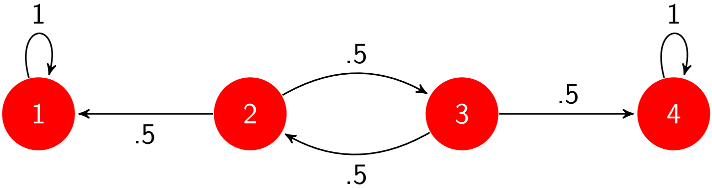
- $\mathrm{P}_1(T_4 < \infty) = 0, \mathrm{P}_4(T_4 < \infty) = 1$
- $\mathrm{E}_1(T_{1,4})$?, $\mathrm{E}_4(T_{1,4})$?
- $\mathrm{P}_2(T_4 < \infty)$?, $\mathrm{E}_2(T_{1,4})$?

P_i, T_A, Example Continued

Let $S_1 \subset \mathbf{S}$

$$\mathrm{P}_i(A) = \mathrm{P}(A|X_0 = i)$$

$$T_{S_1} = \min\{n \geq 0 : X_n \in S_1\}$$



- Start at 2 and consider the situation after making one step:

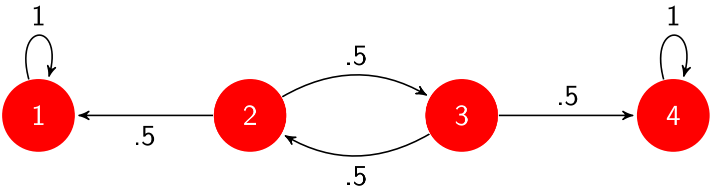
$$\begin{cases} \mathrm{P}_2(T_4 < \infty) = \frac{1}{2}(\mathrm{P}_1(T_4 < \infty) + \mathrm{P}_3(T_4 < \infty)) = \frac{1}{2}\mathrm{P}_3(T_4 < \infty) \\ \mathrm{E}_2\, T_{\{1,4\}} = 1 + \frac{1}{2}(\mathrm{E}_1\, T_{\{1,4\}} + \mathrm{E}_3\, T_{\{1,4\}}) = 1 + \frac{1}{2}\mathrm{E}_3\, T_{\{1,4\}} \end{cases}$$

P_i, T_A, Example

Let $S_1 \subset \mathbf{S}$

$$\mathrm{P}_i(A) = \mathrm{P}(A|X_0 = i)$$

$$T_{S_1} = \min\{n \geq 0 : X_n \in S_1\}$$



- Repeat the same technique fro 3 and E_3 to get:

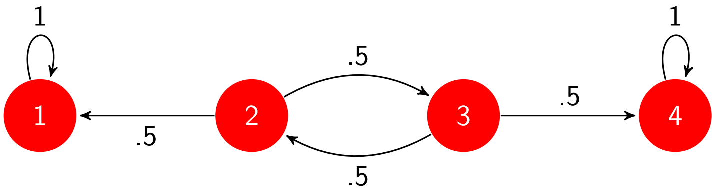
$$\begin{cases} \mathrm{P}_3(T_4 < \infty) = \frac{1}{2}(\mathrm{P}_2(T_4 < \infty) + \mathrm{P}_4(T_4 < \infty)) \\ \qquad = \frac{1}{2}\mathrm{P}_2(T_4 < \infty) + \frac{1}{2} \\ \mathrm{E}_3\, T_{\{1,4\}} = 1 + \frac{1}{2}(\mathrm{E}_2\, T_{\{1,4\}} + \mathrm{E}_4\, T_{\{1,4\}}) = 1 + \frac{1}{2}\mathrm{E}_2\, T_{\{1,4\}} \end{cases} \quad (1)$$

- Therefore,

$$\begin{cases} \mathrm{P}_3(T_4 < \infty) = \frac{1}{2}\mathrm{P}_2(T_4 < \infty) + \frac{1}{2} \\ \mathrm{P}_2(T_4 < \infty) = \frac{1}{2}\mathrm{P}_3(T_4 < \infty) \end{cases} \Rightarrow \begin{cases} \mathrm{P}_2(T_4 < \infty) = \frac{1}{3} \\ \mathrm{P}_3(T_4 < \infty) = \frac{2}{3} \end{cases}$$

- And

$$\begin{cases} \mathrm{E}_2 T_{\{1,4\}} = 1 + \frac{1}{2}\mathrm{E}_3 T_{\{1,4\}} \\ \mathrm{E}_3 T_{\{1,4\}} = 1 + \frac{1}{2}\mathrm{E}_2 T_{\{1,4\}} \end{cases} \Rightarrow \begin{cases} \mathrm{E}_2 T_{\{1,4\}} = \frac{1}{2} \\ \mathrm{E}_3 T_{\{1,4\}} = \frac{1}{2} \end{cases}$$



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$$T_j = \min\{n \geq 0 : X_n = j\}$$

- The index i of T_i is always an absorbing state.
- T_j is a minimum time we need to arrive at j: $\mathrm{P}_j(T_j = 0) = 1$.
- Assume i and j are two absorbing states, then

$$\mathrm{P}_i(X_n = j) = 0 = \mathrm{P}_j(X_n = i)$$

- We assume every recurrent class is a single absorbing state
- Therefore, the only interesting case is going from a transient to an absorbing state
- Let i be transient and j recurrent. It is possible that we never go fro i to j. In this case

$$\mathrm{P}_i(T_j < \infty) = 0, \quad \mathrm{P}_i(T = \infty) = 1$$

- By $\mathrm{P}_i(T_j < \infty) = p$ we mean

$$\mathrm{P}(X_n \text{ eventually becomes equal to the state } j | X_0 = j) = p$$

Absorption Probability Equations

The following theorem summarizes all our discoveries so far.

Theorem

Consider a MC such that each state is either transient or absorbing. Fix a particular absorbing state j. Then $\mathrm{P}_i(T_j < \infty)$ solve the following equations.

$$\begin{aligned} &\mathrm{P}_j(T_j < \infty) = 1 && \\ &\mathrm{P}_i(T_j < \infty) = 0 && \text{if } i \neq j \text{ are absorbing states} \\ &\mathrm{P}_i(T_j < \infty) = \sum_{k \in \mathbf{S}} \mathrm{P}_k(T_j < \infty) p_{i,k} && \text{if } i \text{ is transient} \end{aligned}$$

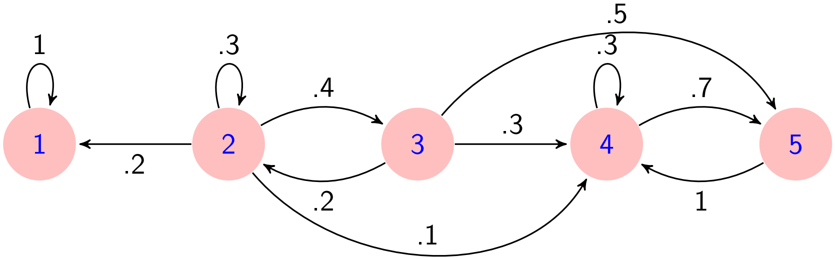
Proof.

The first and the second equations are trivial. The third one is the one step moving technique. □

Remark

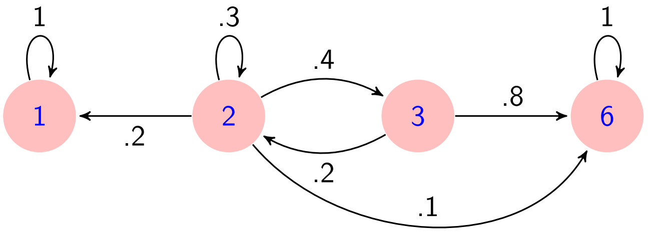
The solution to the Absorption Equations (AE) is unique (when **S** *is finite).*

Example 6.10



- Notice that $\{1\}$ and $\{4, 5\}$ are 2 recurrent classes.
- We would like to find $\mathrm{P}_3(T_{\{4,5\}} < \infty)$
- We lump the state 4 and 5 together to get the following MC:

Example 6.10 Continued



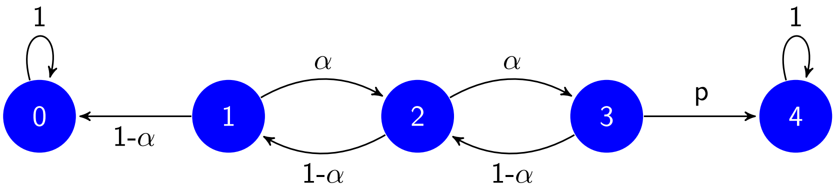
- In this figure we are looking for $\mathrm{P}_3(T_6 < \infty)$. By the AE

$$\begin{cases} p_3 = .2p_2 + .8p_6 \\ p_2 = .2p_1 + p_2 + .4p_3 + .1p_6 \end{cases} \tag{2}$$

- Since $p_1 := \mathrm{P}_1(T_6 < \infty) = 0$, and $p_6 := \mathrm{P}_6(T_6 < \infty) = 1$:

$$p_2 := \mathrm{P}_2(T_6 < \infty) = 21/31, \quad p_3 := \mathrm{P}_3(T_6 < \infty) = 29/31$$

Gambler's Ruin



- The gambler starts with \$2 in her pocket: $X_0 = 2$.
- At each step, idep. of the previous steps, she might win or lose \$1 with prob. α or $1 - \alpha$ res., where $\alpha \in (0, 1)$.
- The game ends either when she loses all she has, or accumulates \$4.
- The MC above models the game
- We are looking for $p_i := \mathrm{P}_i(T_4 < \infty)$, for $i = 0, , \cdots , 4$.

By AE we have

$$p_0 = 0,\ p_1 = \alpha p_2,\ p_2 = (1-\alpha)p_1 + \alpha p_3,$$
$$p_3 = (1-\alpha)p_2 + \alpha p_4,\ p_4 = 1$$

- p_0 and p_4 are called the boundary conditions
- These equations can uniquely be solved
- We now want to present an interesting method for solving this equation when $\mathbf{S} = \{0, \cdots, m\}$
- the equations are

$$p_0 = 0,\ \ p_m = 1,\ \ p_i = (1-\alpha)p_{i-1} + \alpha p_{i+1},\ \ i = 1, \cdots, m-1$$

- Since $(1-\alpha)(p_i - p_{i-1}) = \alpha(p_{i+1} - p_i)$, then $\delta_i = \rho\delta_{i-1}$ where

$$\delta_i = p_{i+1} - p_i, \qquad \rho = \frac{1-\alpha}{\alpha}$$

- Therefore, $\delta_i = \rho^i \delta_0.\ i = 1, \cdots, m-1.$

- So far we have

$$\delta_i = \rho^i \delta_0, \quad \delta_0 + \delta_1 + \cdots + \delta_{m-1} = p_m - p_0 = 1$$

The second equation gives us

$$\delta_0 + \rho\delta_0 + \cdots + \rho^{m-1}\delta_0 = 1$$

- Therefore if *rho* $\neq 1$, i.e., $\alpha \neq 1/2$:

$$\delta_0 = \frac{1}{1 + \rho + \cdots + \rho^{m-1}} = \frac{1-\rho}{1-\rho^m}$$

- And if $\rho = 1$, i.e if $\alpha = 1/2$, then $\delta_0 = \frac{1}{m}$
- Therefore

$$\begin{cases} \delta_i = \frac{1}{m} & \text{if } \alpha = \frac{1}{2} \\ \delta_i = \frac{1-\rho}{1-\rho^m}\rho^i & \text{if } \alpha \neq \frac{1}{2} \end{cases} \tag{3}$$

- Since $\delta_0 + \cdots + \delta_{i-1} = p_1 + (p_2 - p_1) + \cdots + p_i - p_{i-1} = p_i$, then

$$p_i = \begin{cases} \frac{i}{m} & \text{if } \alpha = \frac{1}{2} \\ \frac{1-\rho}{1-\rho^m}[1 + \rho + \cdots + \rho^{i-1}] & \text{if } \alpha \neq \frac{1}{2} \end{cases}$$

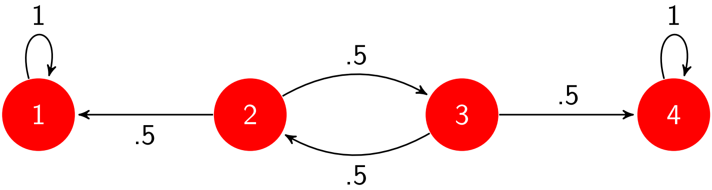
- Or

$$p_i = \begin{cases} \frac{i}{m} & \text{if } \alpha = \frac{1}{2} \\ \frac{1-\rho}{1-\rho^m}\frac{1-\rho^i}{1-\rho} & \text{if } \alpha \neq \frac{1}{2} \end{cases} \Rightarrow$$

- And finally the Gambler's Ruin probabilities are given by

$$\mathrm{P}_i(T_6 < \infty) = \begin{cases} \frac{i}{m} & \text{if } \alpha = \frac{1}{2} \\ \frac{1-\rho^i}{1-\rho^m} & \text{if } \alpha \neq \frac{1}{2} \end{cases}$$

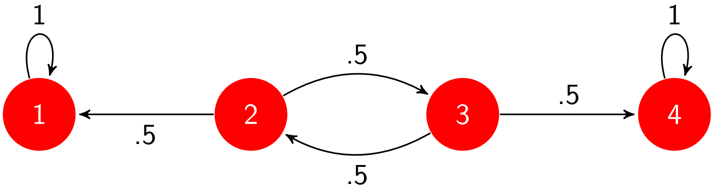
Expected Absorption time



- let $T_4 = \min\{n \geq 0 : X_n = 4\}$.
- Then T_4 is a r.v. We saw that $\mathrm{P}_1(T_4 < \infty) = 0$.
- $\mathrm{P}_1(T_4 < \infty) = 0$ is equivalent to $\mathrm{P}_1(T = \infty) = 1$.
- $T_4 = \infty$ means we will never reach 4 from 1 in finite time
- $T = \infty$, then average of T is infinite as well: $\mathrm{E}_1 T_4 = \infty$.
- $\mathrm{P}_2(T_4 < \infty) = \frac{1}{3}$, then $\mathrm{P}(T = \infty) = \frac{2}{3}$, then $\mathrm{E}_2 T_4 = \infty$.
- Therefore, for expectation we work with $T_{\{1,4\}}$, then $\mathrm{P}_i(T_{\{1,4\}} < \infty) = 1$.

Expected Absorption time

- The minimum time to reach to one of the absorbing states is called the "*absorption time*"
- The absorption time is always finite, and its mean can be calculated
- As we saw before for the MC below



- $\mathrm{E}_2 T_{\{1,4\}} = 1 + \frac{1}{2}(\mathrm{E}_3 T_{\{1,4\}} + \mathrm{E}_1 T_{\{1,4\}}) = 1 + \frac{1}{2}\mathrm{E}_3 T_{\{1,4\}}$.
- $\mathrm{E}_3 T_{\{1,4\}} = 1 + \frac{1}{2}(\mathrm{E}_2 T_{\{1,4\}} + \mathrm{E}_4 T_{\{1,4\}}) = 1 + \frac{1}{2}\mathrm{E}_2 T_{\{1,4\}}$.
- These two equations yield:

$$\mathrm{E}_2 T_{\{1,4\}} = \frac{1}{2} = \mathrm{E}_3 T_{\{1,4\}}$$

Mean Absorption Time

- Let i be a transient state and $S_1, \cdots, S_k$, be the recurrent states.

$$\mathrm{P}_i(T_{S_1 \cup \cdots \cup S_k} < \infty) = 1$$

$$\mathrm{P}_i(T_{S_1 \cup \cdots \cup S_k} < \infty) = \mathrm{P}_i(\text{Hitting time of a recurrent state} < \infty)$$

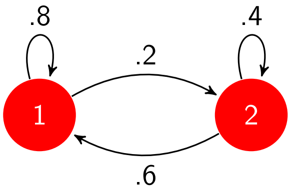
- Define $T_A = T_{S_1 \cup \cdots \cup S_k}$, then

- Theorem

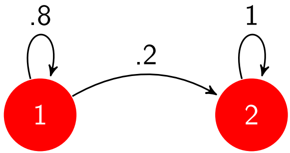
The expected times to absorption (Hit a recurrent state) are the unique solution to the equations

$$\begin{array}{ll} \mathrm{E}_i(T_A) = 0 & \textit{if } i \textit{ is recurrent} \\ \mathrm{E}_i(T_A) = 1 + \sum_{j \in \mathbf{S}} \mathrm{E}_j(T_A) p_{i,j} & \textit{if } i \textit{ is transient} \end{array} \quad (4)$$

Example, Mean First Passage time

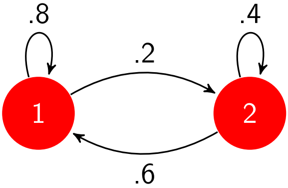


- Find $\mathrm{E}_1 T_2$. Notice that the sate 2 is not absorbing
- We change the MC such that 2 becomes an absorbing state, by setting $p_{2,2} = 1$, $p_{2,1} = 0$.
- This will not change T_2, bc T_2 only counts until we hit 2.



- $\mathrm{E}_1 T_2 = 1 + .8\mathrm{E}_1 T_2 + .2\mathrm{E}_2 T_2 = 1 + .8\mathrm{E}_1 T_2 \Rightarrow \mathrm{E}_1 T_2 = 5.$

Mean Recurrence time



- Find $\mathrm{E}_2 T_2^*$, where $T_2^* = \min\{n \geq 1 : X_n = 2\}$.
- If we are in 2, then $T_2 = 0$, and T_2^* is the recurrence time.
- We can calculate the recurrence time trough the passage time
- We first take one step, the calculate the passage time, $+ 1$
- $\mathrm{E}_2 T_2^* = 1 + .4\mathrm{E}_2 T_2^* + .6\mathrm{E}_1 T_2 \Rightarrow .6\mathrm{E}_2 T_2^* = 1 + .6(5)$.

$$\mathrm{E}_2 T_2^* = 20/3$$