Existence and Regularity of the Density for a Stochastic Heat Equation

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We study on torus $\mathbf{T}$, the stochastic partial differential equation (SPDE):

$$\begin{cases} u_t(t,x) = \mathbf{L}u(t,x) + \sigma(u(t,x))\dot{W}, & t \geq 0, x \in T \\ u(t,x) = u_0(x), & x \in T, \end{cases} \tag{1}$$

where $\mathbf{L}$ is the generator of a Lévy process $X := \{X_t\}$, $\sigma \in C_b^\infty(\mathbf{R})$, and $\dot{W}$ is the space-time white noise. Furthermore, let $\Phi : \mathbf{Z} \to \mathbf{C}$ denote the characteristic exponent of X. Define

A Family of Seminorms

$$\|f\|_{\beta,p} := \left\{ \sup_{t \geq 0, x \in \mathbf{T}} e^{-\beta p} \mathrm{E}|f(t,x)|^p \right\}^{1/p}, \quad \forall \beta \geq 0,\ p \geq 1.$$

Theorem 1 (The Mild Solution).

If:

$$\lim_{|n| \to \infty} \frac{\Re\Phi(n)}{\log |n|} \to \infty, \tag{2}$$

If $X' =^d X$ is independent of X, then $X - X'$ has local time, i.e

$$\Upsilon(\beta) := \sum_{-\infty}^{\infty} \frac{1}{\beta + 2\Re\Phi(n)} < \infty. \tag{3}$$

then the mild solution exists

Proof. Let $v_0(t,x) = \mathrm{E}^x(X_t)$, and for $n \geq 1$ define $v_n(t,x)$ through the Picard scheme

$$v_{n+1}(t,x) = v_0(t,x) + \int_0^t \int_{\mathbf{T}} q_{t-r}(x,z)\sigma(v_n(r,z))W(dr,dz).$$

There is $\beta > 0$ sufficiently large such that

$$\|u - v_n\|_{\beta,p} \to 0$$

$$\lim \sup_{t \to \infty} \frac{\ln \mathrm{E}|u(t,x)|^p}{t} < \infty.$$

□

Theorem 2 (Malliavin Derivatives).

If (2) and (3) hold then $u(t,x) \in D^\infty$ and furthermore

$$\lim \sup_{t \to \infty} \frac{\ln \mathrm{E}\|D^m u(t,x)\|^p_{H^{\otimes m}}}{t} < \infty. \tag{4}$$

Proof. If we let □

$$\Gamma^m_{t,x} v := \|D^m v(t,x)\|_{H^{\otimes m}},$$

then there is large β such that

$$\|\Gamma^m v_{n+1}\|^2_{\beta,p} \leq C_m \Upsilon(\frac{2\beta}{p})(1 + \|\Gamma^m v_n\|^2_{\beta,p}).$$

Since $\Upsilon(\beta) \to 0$ as $\beta \to \infty$, then by iteration

$$\sup_n \|\Gamma^1 v_{n+1}\|^2_{\beta,p} \leq \frac{C_m \Upsilon(\frac{2\beta}{p})}{1 - C_m \Upsilon(\frac{2\beta}{p})}.$$

Then $u(t,x) \in D^\infty$, and (4) holds

Theorem 3 (Existence and Smoothness of Density).

If in addition to conditions (2) and (3), σ is bounded below by some positive number κ

$$\sigma(x) \geq \kappa > 0, \quad \forall x \in \mathbf{T} \tag{5}$$

and there are $\alpha < \beta \in [1,2]$ such that $\alpha \leq \frac{2\beta}{\beta+1}$, and

$$cn^\alpha \leq \Phi(n) \leq Cn^\beta \tag{6}$$

then $u(t,x)$ has a smooth density at every $t \geq 0$ and $x \in \mathbf{T}$.