SOME VIEWS ON GLOBAL REGULARITY OF THE THIN FILM EQUATION

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Abstract. We introduce the thin film equation and the problem of proving positivity and global regularity on a periodic domain with a slip parameter \( n > 1 \). Several ideas in the direction of a proof are described. Some are recounted only briefly and qualitatively due to a lack of either an essential ingredient of the proof or enough time to carry it through. This document should be considered as both a record of the author’s effort up to this point and a source of inspiration for future work on the problem.

1. Introduction

We consider the equation for the evolution of the height of a thin film of viscous fluid with the simplifying assumption that the effect of surface tension is very strong compared to advection:

\[
\begin{align*}
  h_t + (f(h)h_{xxx})_x &= 0 \\
  h(x,0) &= h_0(x)
\end{align*}
\]  

(1)

We will derive this in the next section. Here \( f : \mathbb{R} \to \mathbb{R} \) is given, \( h : \mathbb{T}^1 \times [0,T) \to \mathbb{R} \) and \( h_0 : \mathbb{T}^1 \to (0,\infty) \), \( h_0 \in H^1(T) \) where \( \mathbb{T}^1 \) is the one-dimensional\(^1\) torus or, equivalently, a periodic arrangement on the whole real line. The choice of \( f \) is based on the most appropriate physical model for the interaction between the fluid and the substrate, see for example Greenspan [4]. For simplicity of the analysis it is standard to consider \( f(h) = |h|^n \) with \( n > 0 \) determining the extent of the slip. Starting from positive initial data, it is straightforward to observe that as long as a smooth solution exists and stays positive, \( f(h) \) is uniformly away from zero by compactness, so the equation is uniformly parabolic. Thus at least for short times, the solution is smooth and positive. One introduces a monotonic quantity called the “entropy” (to be discussed in greater detail later) which controls the thinness of the film and allows one to deduce that solutions, and in fact weak solutions, are nonnegative (see Bernis-Friedman [1]). Thus we will tend to omit the absolute value and consider \( f(h) = h^n \).

The essential conclusion of the above discussion is that to prove global well-posedness of (1), it suffices to prove that the equation preserves the strict positivity of the data. The purpose of this report is to recount several “good ideas” in this direction that are not successful in their current forms. We consider the problem only for \( n > 1 \) because the numerical results of Bertozzi et al. strongly indicate...

\[^1\text{The system could just as well be posed in several dimensions, but the one dimensional problem is already hard enough. The torus could also be replaced with an interval in } \mathbb{R} \text{ with the Neumann boundary data } h_x = h_{xxx} = 0.\]
breakdown of smoothness and positivity for smaller values \[3\]. On the other hand, it is quite straightforward to prove well-posedness for \(n \geq 4\). Indeed, approximating the equation with a uniformly parabolic nonlinearity (modifying \(f\)), the approximate solutions inherit the \(C^{1/2}_{\text{H"older}}\) property from the data, which is passed on to the exact solution upon taking a uniform limit. If \(h = 0\) at a point, it has to stay near zero in a H"older sense, which would force the entropy to be infinite unless \(n < 4\). The same idea can be refined to prove global regularity for \(n \geq 7/2\) (see \[2, 3\]). What follows are attempts to lower this bound further still.

2. Almost-transported quantities in the fluid interior

Since the dynamics of the free interface are ultimately determined by the dynamics of the fluid in the interior, it seems natural to understand the equations for that motion. A sensible place to start is by examining the derivation of (1) from the Navier-Stokes equations. What follows is known as the lubrication approximation and can be found in [4]. For \(t \in [0, T]\), let

\[
(u(t), v(t)) : \Omega(t) = \{(x, y) \in \mathbb{T}^1 \times (0, \infty) : y < h(x, t)\} \to \mathbb{R}^2
\]

be the velocity field of the fluid. First we have the kinetic equation to assert that the interface moves with the fluid:

\[
h_t + u|_{y=h}h_x = v|_{y=h}.
\]

Next, the surface tension \(\gamma\) which balances the fluid pressure \(p\):

\[
p = p_0 - \gamma h_{xx}.
\]

We assume the shear stress at the boundary vanishes,

\[
u_y|_{y=h} = 0,
\]

the fluid is incompressible,

\[
u_x + v_y = 0,
\]

and the fluid slips against the substrate at a velocity depending on the interface:

\[
u|_{y=0} = u_y|_{y=h}\kappa(h), \quad v|_{y=0} = 0.
\]

The choice of \(\kappa\) in (6), the so-called slip condition, is more a matter of physics than of analysis and there is a great deal of variability among authors. One finds that \(\kappa(h) = \frac{1}{3}(h^{n-2} - h)\) produces (1) with \(f(h) = h^n\) in the nonlinearity, which will be our choice in the interest of convenience. A significant note is that \(n = 3\) produces \(\kappa(h) = 0\) which is known as the “no-slip” condition.

Of course we are still yet to give the law of motion for the fluid. Here we invoke the assumption that pressure and viscosity are the only relevant forces in the fluid interior, and that the film is thin enough that the pressure gradient in the vertical direction is negligible. One arrives at the Stokes equation in the horizontal direction,

\[
p_x = \mu u_{yy}, \quad p_y = 0
\]
where $\mu$ is the viscosity. It is straightforward to integrate the equations (3) through (7) to find the velocity profiles
\begin{align*}
u &= -\frac{\gamma}{\mu} y(\kappa(h) + h\kappa'(h) + \frac{1}{2} y)h_{xxx} + \frac{\gamma}{\mu} y(-h\kappa(h) - \frac{1}{2} yh + \frac{1}{6} y^2)h_{xxxx}.
\end{align*}
Putting these into (2), we arrive at the equation
\begin{align*}
ht + \frac{\gamma}{\mu} (h^2(\frac{1}{3} h + \kappa(h)))h_{xxx} = 0
\end{align*}
and finally normalize $\gamma/\mu = 3$. Taking $f(h) = h^2(\frac{1}{3} h + \kappa(h))$, we recover the thin film equation (1) for an arbitrary slip condition.

Next let us rewrite the velocity field with the $\kappa$ that corresponds to $f(h) = h^n$:
\begin{align*}
u &= y(2h - (n-1)h^{n-2} - \frac{3}{2} yh)h_{xxx} + y(h^2 - h^{n-1} - \frac{3}{2} yh + \frac{1}{2} y^2)h_{xxxx}.
\end{align*}

One notices that defining the potential on $\Omega(t)$
\begin{align*}
\phi &= y(h^2 - h^{n-1} - \frac{3}{2} yh + \frac{1}{2} y^2)h_{xxx},
\end{align*}
the flow in the interior is given exactly by\(^2\)
\begin{align*}
(u, v) &= (\partial_y \phi, -\partial_x \phi) = \nabla^\perp \phi.
\end{align*}

One can go further and find several infinite families of potentials with different particle dynamics but induce the same boundary dynamics, that is to say equations (1) and (10)∪(2) are equivalent. For example, taking $n = 3$ in (9), one finds \(\phi = \frac{1}{2} y^2(y - 3h)h_{xxx}\) which suggests the potential
\begin{align*}
\phi &= \frac{1}{n-1} y^{n-1}(y - nh)h_{xxx}
\end{align*}
for a general $n$. It is trivial to compute that this too induces the correct boundary dynamics. It is this description of the internal fluid dynamics that leads to a very nice but incorrect proof that (1) preserves strict positivity of the data for a wider range of $n$.

**Conjecture.** If $n > 3/2$ and $u \in C^{1/8}_t C^{1/2}_x \cap L^2 T^1_H x_R^+$ is a weak solution\(^3\) of (1) with data $u_0 \in H^1(T^1)$ satisfying $u_0 > 0$, then $u > 0$ for all time.

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\(^2\)As a result, (1) can be rewritten as
\begin{align*}
\partial_t h = \nu \cdot \nabla^\perp \phi |_{y=h},
\end{align*}
where $\nu$ is the outward normal scaled by $\left(1 + h^2\right)^{1/2}$, the length element of the interface. This is in direct analogy to the Zakharov formulation of the water wave equations using the Dirichlet-to-Neumann operator [5].

\(^3\)There are several notions of weak solution, see eg. [2]. The ideas discussed here could be applied at a sequence of times approaching the blowup time to derive a contradiction. Thus one can imagine all the solutions to be in the classical sense as long as it is kept in mind that the high derivatives may not stay uniformly bounded as $t \to T^*$.
Sketch of an interesting argument which fails. Suppose the contrary, that is there exists a smallest $T^* > 0$ such that there exists $x_0 \in T^1$ with $u(x_0, T^*) = 0$. Recall from the discussion in the introduction that $u$ is $C^{1/2}$ in space, say for simplicity with norm 1. Let us identify $T^1$ with the real interval $[-1/2, 1/2]$ so that $u(x_0, T^*) = 0$ implies $|u(x, T^*)| \leq |x - x_0|^{1/2}$ for all $x \in T^1$.

The next observation is that according to (10), in a Lagrangian frame each particle moves along level sets of $\phi$ (as $\nabla \cdot \phi = 0$ for any potential) so $\phi$ is constant along “characteristics.” This suggests we might reach a contradiction by keeping track of the number of particles with very small $\phi$ and proving that there are more at the time of blowup than there were to begin with. (Note this is indeed a contradiction because of the incompressibility assumption.)

Consider the fluid region at time $T^*$ which lies within a horizontal distance of $0 < \delta \ll 1$ from $x_0$. Thanks to the 1/2-Hölder estimate, this region has volume on the order of $\delta^{3/2}$. Since the bulk of the particles in this region have height on the order of $\delta^{1/2}$, the potential defined in (11) is on the order $\delta^{n/2} |h_{xxx}|$. At time $t = 0$ when the interface is uniformly away from $h = 0$, $|\phi|$ is smallest for particles near the substrate. The same volume $\delta^{3/2}$ can be found in the region bounded above by height $\delta^{3/2}$, but now $\phi$ is on the order of $(\delta^{3/2})^{n-1}$. Since $\phi$ is only transported in a divergence-free manner, the volume of fluid with $|\phi|$ of a certain order is conserved, so it must be that $\delta^{n/2} |h_{xxx}| \sim (\delta^{3/2})^{n-1}$ as $\delta \to 0$, that is $|h_{xxx}| \sim \delta^{-3/2}$. It is well-known that $h_{xxx}$ is unbounded near a singularity (see eg. [3]) so it must be that $n \leq 3/2$. These heuristic estimates can be made more precise.

The range of $n$ can perhaps be widened even further by using estimates to the effect that $|h_{xxx}|$ grows at least with a power of $h^{-1}$ near a singularity.

Unfortunately, this method falls short in its current form. It is tempting to say that because particles follow level curves of the potential, it must be constant in Lagrangian coordinates. But this cannot be the case, as $\phi$ at a fixed Eulerian point changes with respect with time. In fact, as can be deduced from [2], $h_{xxx} \to 0$ (in, say, $L^2$) and $h \to \int h_0$ uniformly as $t \to \infty$ so $\phi \to 0$ everywhere. This argument might be completed by showing that over the short time scale on which blowup can occur (thanks to, for example, the uniform convergence of $h$ to its average), the change in $\phi$ along a characteristic is higher order than $\delta^{(3/2)(n-1)}$ so it can be neglected in the $\delta \to 0$ limit.\footnote{What follows is a strange idea far outside the scope of this paper: framing the thin film equation as the evolution of the height of a free interface above an incompressible vector field is physically natural because that is the precise scenario the equation was derived to model. But interestingly, there are infinitely-many qualitatively different choices of particle dynamics that yield the same equation (in addition to (9) which is derived from the physics). It is certainly the case that other evolution equations can be formulated this way—even ones that do not literally model a fluid interface. Plausibly, blowup can be globally excluded by exhibiting a geometric contradiction to the existence of a vector field satisfying the equation lying underneath a singularity.}

3. INCOMPRESSIBILITY NEAR A SINGULARITY

As before, suppose blowup first occurs at time $T^*$, located at $x_0$. One can imagine two ways a contradiction can be derived by examining the local movement of volume in the fluid region near the singularity. First, by Hölder continuity (again assuming the constant is 1), there is a region $U = \{(x, y) \in I \times (0, A) : y > |x - x_0|^{1/2}\}$ where $A > 0$ and $I$ is an interval containing $x_0$, such that $U$ was full of fluid at
some $t < T^*$ but is empty by $T^*$. Thus a particle in $U$ at a time $t < T^*$ would have $T^* - t$ time units to travel the distance to the complement of $U$.

Second, one could consider $V(t) = \{(x, y) \in I \times (0, \infty) : y < h(x, t)\}$. As $t \to T^*$, the volume of this region tends to decrease, so by incompressibility, the net fluid flow must be outward. One might derive a contradiction by showing the rate at which volume can exit the region is slower than the rate at which space is being taken away by the shrinking of $V(t)$. Or, that the only way to escape is to a thin layer above the substrate outside of $I \subset \mathbb{T}^1$, and bound how quickly volume can leave that layer (since the only way out is in the vertical direction).

How close this method gets to a contradiction depends heavily on the choice of $\phi$. For example, one could choose $\phi = h^{n-2}h_{xxx}$, so that $\partial_t h + \phi_x + \phi_y h_x |_{y=h} = 0$ trivially gives (1), but then particles freely move in the vertical direction exactly in parallel with the boundary so there is no hope of a contradiction. On the other hand, it is straightforward compute that the family of potentials

$$\phi = y^k h^{n-k} h_{xxx}, \quad k \in \mathbb{R}$$

produce the correct interface dynamics, as do convex combinations thereof (of course if one desires that the trajectories stay in the upper-half plane, $k > 0$ is necessary). Recalling (10), the $u$ and $v$ scale like $y^{k-1}$ and $y^k$ respectively for particles with small $y$ that are uniformly away from the interface. Therefore by taking $k$ very large, the velocities in thin strip above the substrate but away from a singularity can be made arbitrarily small. Moreover, writing $\phi = (y/h)^k h^{n} h_{xxx}$, it is clear that choosing $k$ large has little effect near a singularity where $y$ is close to $h$. It follows that as $t \to T^*$, $V(t)$ loses volume, leaving such a thin strip for the fluid to escape. But the fluid which started in that thin strip can barely leave by the choice of $k$ large. This hints of a potential obstruction to a singularity which should be explored.

4. Exploiting global monotonic quantities

The ideas discussed so far have not directly made use of the monotonic and conserved quantities enjoyed by (1). For classical solutions it is elementary to check that we have conservation of mass,

$$\frac{d}{dt} \int_{\mathbb{T}^1} h = 0,$$

conservation of energy,

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{T}^1} h_x^2 + \int h^2 h_{xxx}^2 = 0,$$

and a monotonic quantity called “entropy” that was originally introduced in [1],

$$\frac{d}{dt} \int_{\mathbb{T}^1} \frac{1}{h^{n-2}} + (n-2)(n-1) \int_{\mathbb{T}^1} h_x^2 = 0.$$

The last equality is only true and useful when $n > 2$. An analogous statement holds for $n = 2$ when $1/h^{n-2}$ is replaced by a logarithm.

These bounds offer quite a bit of control, and it is not absurd to suppose that they might be almost sufficient to prove global well-posedness for $n \geq 2$. Indeed, (15) was sufficient for $n \geq 4$ [1], and a refinement of it was sufficient for $n \geq 7/2$ [3]. An argument that additionally uses (13) may proceed as follows: in the event that the interface touches down, $\int 1/h^{n-2}$ is larger in a neighborhood of the singularity,
which forces $1/h^{n-2}$ to increase elsewhere. But too great an increase elsewhere would fail to conserve mass, violating (13). One can formulate increasingly careful estimates to this effect, but to date the results are only local. That is to say we can conclude the desired positivity if any of the following is true: the data is small (in $H^1$), the minimum of the data is not too small, the torus is small, or we consider only small times. In particular, it is not difficult to obtain global well-posedness under the assumption

$$L^{1/2} \| h_0 \|_{H^1} \leq c_n \min h_0$$

where $c_n$ is a constant depending on $n \in (2, 4)$ and $L = \| T \|$. One could hope to show that the small time when positivity is guaranteed contradicts upper bounds for $T^*$. For example, applying (15) along with the interpolation inequality $\| u \|_{L^4_{t}W^2_x} \geq \| u \|_{L^4_{t}W^1_x}^{4/3}/M$ where $M = \int h_0$ and Grönwall’s inequality, we find for a constant $d_n$

$$T^* \leq \left( \frac{1}{\min h_0^{n-2}} - \frac{d_n}{(\sqrt{L} \| h_0 \|_{H^1})^{n-2}} \right) \frac{L^6}{M^2}$$

which easily implies (16).

5. Shape of the interface at the singularity

It is straightforward to see for $n > 2$ that due to Hölder continuity of $h$ and the finiteness of the entropy (15), $h$ cannot have finite left or right derivatives at a singularity; otherwise $h$ will be too small over too much space for $1/h^{n-2}$ to be integrable. Thus we expect the liminf of the the left derivative to be $-\infty$ and the limsup of the right derivative to be $+\infty$ as $t \to T^*$. But by smoothness of $h$ before $T^*$, it has a global minimum where $h_x$ vanishes and $h_{xx} > 0$. From all these considerations, one may very informally expect that $h$ should have its third derivative large and positive approaching the minimum from the left, and large and negative from the right. This might be in direct conflict with the observation that at the local minimum from $h_x = 0$, (1) reduces to $h_t = h^n h_{xxxx}$, so $h_{xxxx} > 0$ on a set of positive measure.

Although this discussion hints at a substantial obstruction to a singularity for $n > 2$, there are many difficulties in making this reasoning rigorous. For example, we must contend with a severe lack of control over the high derivatives of $h$ near $T^*$. Furthermore we must settle for only knowing $h_{xxxx} > 0$ on a potentially small subset of times leading of to $T^*$.

6. Conclusion

This is just a sample of the author’s (so far) fruitless approaches to the thin film global regularity problem. More work is certainly warranted to push these ideas to their conclusions.

References


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