A paralinearization of the 2d and 3d gravity water wave system in infinite depth

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Abstract

We consider the 2d and 3d water waves system with gravity and no surface tension in infinite depth. Loss of derivatives from the Dirichlet-to-Neumann operator make studying solutions for long times difficult and until recently only local results were available. Several authors have since made use of paradifferential calculus to overcome these difficulties and prove global regularity in 2d and 3d with and without surface tension. The purpose of this thesis is to formulate the paralinearization of the system based on the Weyl quantization due to Deng-Ionescu-Pausader-Pusateri but with several key modifications. Namely, we work in lower regularity $L^2$-based Sobolev spaces and do not include surface tension. This makes the problem more difficult by reducing the regularity on the surface elevation. We flatten the interface to arrive at a paralinearization of the Dirichlet-to-Neumann operator. As a result we are able to paralinearize and symmetrize the entire system and derive a single equation for a single complex unknown. The result is suited for obtaining energy estimates that would be useful, for example, when proving rigorous modulation approximations to the water waves in various regimes.
I pledge my honor that this thesis represents my own work in accordance with University regulations.

-SPP
I would like to thank my advisor Fabio Pusateri for all his mentorship over the past two academic years. He has been essential not just in the success of this thesis but also in my developing interest in analysis and PDE. I am similarly indebted to Peter Constantin, Charles Fefferman, and Alexandru Ionescu who have continued to convince me of the beauty of this subject through their teaching and advising.

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Chapter 1

Introduction

1.1 Deriving the gravity water wave system

The purpose of this section is first to give Zakharov’s formulation of the gravity water wave system which will be used for the calculations that are to follow. The derivation is adapted from the one found in Chapter 11 of Sulem-Sulem [21]. We consider a region $\Omega(t) \subset \mathbb{R}^{d+1}$ depending on time of the form

$$\Omega(t) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < h(x, t)\}$$

where $h(x, t)$ is the elevation of a free surface at time $t$. We suppose the region $\Omega(t)$ is filled with a fluid which is evolving according to the incompressible, irrotational Euler equations with gravity of downward acceleration $g$, and that the interface moves with the fluid. Irrotationality implies that there exists a potential $\Phi(x, t)$ that is harmonic in $x \in \Omega$ with boundary conditions such that $\partial_y \Phi$ approaches zero as $y \to -\infty$ and at the interface the fluid velocity $u = \nabla \Phi$ satisfies the Euler equations. Indeed, we would like to have

$$0 = \partial_t u + u \cdot \nabla u + \nabla p + g = 0$$

$$= \partial_t \nabla \Phi + (\nabla \Phi \cdot \nabla) \nabla \Phi + \nabla p + (\vec{0}, g)$$

$$= \nabla (\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + p + gy).$$

(1.2)
At the interface, $y = h(x, t)$ and $p = 0$ so we require that $\Phi$ satisfies

$$\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + gh = 0. \quad (1.3)$$

We would also like a formula for $\partial_t h$. We have the system

$$\begin{cases}
\Delta x = -\Delta t \nabla_x \Phi(x, t) + o(\Delta t) \\
h(x, t + \Delta t) = h(x + \Delta x) + \Delta t \partial_y \Phi(x, t) + o(\Delta t)
\end{cases} \quad (1.4)$$

where $\nabla_x$ is the gradient taken in the $x$ direction or directions only. Rearranging and taking $\Delta t \to 0$, we get

$$\partial_t h = -\nabla_x \Phi \cdot \nabla h + \partial_y \Phi. \quad (1.5)$$

It was first observed by Zakharov [29] and later detailed by Craig and Sulem [10] that the resulting system

$$\begin{cases}
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + gh = 0 \\
\partial_t h + \nabla_x \Phi \cdot \nabla h - \partial_y \Phi = 0
\end{cases} \quad (1.6)$$

can be written in terms of the unknowns evaluated only at the free surface. In particular, define the trace of the velocity potential

$$\phi(x, t) = \Phi(x, h(x, t), t) \quad (1.7)$$

and the Dirichlet-to-Neumann map

$$N(h)\phi = \partial_n \Phi|_{z=h} \quad (1.8)$$

where $\partial_n$ is the derivative in the direction of the outward normal to the interface. For our purpose it will be more useful to consider the operator scaled by the length element of the interface,

$$G(h)\phi = \sqrt{1 + |\nabla h|^2} N(h)\phi. \quad (1.9)$$

Observe that we can explicitly write the outward unit normal vector from the interface as $(-\nabla h, 1)^t / \sqrt{1 + |\nabla h|^2}$ so, taking the directional derivative of $\Phi$ along it,

$$G(h)\phi = (-\nabla_x \Phi \cdot \nabla h + \partial_y \Phi)|_{y=h}. \quad (1.10)$$
Thus we can immediately rewrite the equation for $\partial_t h$ in terms of the potential only on the interface

$$\partial_t h = G(h)\phi. \quad (1.11)$$

To do the same for $\partial_t \Phi$, differentiating (1.7) in time and space with the chain rule respectively gives the formulas

$$\partial_t \Phi = \partial_t \phi - \partial_t h \partial_y \Phi, \quad (1.12)$$

$$\nabla_x \Phi = \nabla \phi - \nabla h \partial_y \Phi. \quad (1.13)$$

Combining (1.10) with (1.13), we get expressions for $\partial_y \Phi$ and $\nabla_x \Phi$ in terms of just $h$ and $\phi$ which we respectively name $B$ and $V$ to use later:

$$B = \partial_y \Phi = \frac{G(h)\phi + \nabla \phi \cdot \nabla h}{1 + |\nabla h|^2} \quad (1.14)$$

$$V = \nabla_x \Phi = \nabla \phi - B \nabla h. \quad (1.15)$$

Applying, in order, (1.12), (1.6), (1.11), (1.13),

$$\partial_t \phi = \partial_t \Phi + \partial_t h \partial_y \Phi \quad \begin{cases} 
\partial_t h = G(h)\phi \quad \text{(1.16)} \\
\partial_t \phi = -gh - \frac{1}{2} |\nabla \phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla|^2)}. 
\end{cases}$$

At last we can write down the desired system,

$$\begin{cases} 
\partial_t h = G(h)\phi \\
\partial_t \phi = -gh - \frac{1}{2} |\nabla \phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla|^2)}. 
\end{cases} \quad (1.17)$$
Another result of Zakharov is that these are canonical variables for the system with hamiltonian

\[ \mathcal{H}(h, \phi) = \frac{1}{2} \int_{\mathbb{R}^d} \phi G(h) \phi + g \frac{1}{2} \int_{\mathbb{R}^d} h^2 \]

\[ \sim \frac{1}{2} \langle \phi, |\nabla|^{1/2} \phi \rangle_{L^2} + g \frac{1}{2} \| \phi \|_{L^2}^2 \]

\[ = \frac{1}{2} \| |\nabla|^{1/2} \phi \|_{L^2}^2 + g \frac{1}{2} \| \phi \|_{L^2}^2 \]

which motivates considering our unknowns \( h \) and \( |\nabla|^{1/2} \phi \) to be in the \( L^2 \)-based Sobolev space \( H^N \) for some regularity \( N \).

### 1.2 Previous work on well-posedness and modulations

Even local well-posedness for this system (either as presented above with just gravity, just surface tension, or both), is nontrivial and the first results were proven only for small perturbations of a flat interface by Nalimov [19], Yosihara [28], and Craig [8]. Later the discovery of Wu that Taylor instability does not occur for topologically reasonable interfaces led to local well-posedness for arbitrary data in Sobolev spaces [24, 25]. Many more local-in-time results followed including, for example, Lindblad [18] which includes vorticity, Coutand-Shkoller [7] and Shatah-Zeng [20] with both vorticity and surface tension, Lannes [17] where the bottom is no longer flat, Christianson-Hur-Staffilani [6] which treats the dispersive nature of the system, and Alazard-Burq-Zuily [1, 2] which are of particular interest here because they rely on a paralinearization of the Dirichlet-to-Neumann operator. This successful paralinearization approach began with Alazard-Métivier [5].

Long-time existence for small data was first proved by Wu [26] for the 2d gravity waves using complex analysis and later in 3d by Germain-Masmoudi-Shatah [14] (using normal forms) and Wu [27]. Germain-Masmoudi-Shatah [15] also proved global regularity in 3d when only surface tension is present. Global regularity for the full water waves problem with both gravity and surface tension, which is harder due to weaker dispersion, was proved in 2d by Alazard-Delort [3, 4] and in 3d by Deng-Ionescu-Pausader-Pusateri [12] (which followed the work of Ionescu-Pusateri [16] on the 2d gravity waves). The details of this paradifferential
method will be discussed in great detail in the remaining chapters.

A related line of research concerns modulation approximations to the water waves systems in various regimes. It was first observed by Zakharov [29] that on length scales of $1/\epsilon$ and time scales up to $1/\epsilon^2$, modulations of the 2d gravity water waves in the form of a wave packet $\epsilon B(\epsilon \alpha)e^{ikx}$ behave (at least formally) like solutions of the nonlinear Schrödinger equation (or in 2d, the hyperbolic NLS). Craig-Sulem-Sulem [11] and Craig-Schanz-Sulem [9] were some of the first results proving rigorous bounds, although the difficulty remained of proving convergence to solutions as $\epsilon \to 0$ because there was not yet an existence result on time scales of $1/\epsilon^2$. Since then, the approximation for infinite depth gravity waves has been proven in 2d with complex analysis by Totz-Wu [23] and in 3d with Clifford analysis by Totz [22]. An essential idea there is to extend the time of existence by using a Riemann mapping to eliminate the quadratic terms from the system. More recently, Düll-Schneider-Wayne have been successful in the finite depth case using a normal form transformation, although this method assumes analyticity to avoid loss of derivatives [13].

1.3 Overview and future directions

Much of the content of the remaining sections is closely based on calculations that can be found in the appendices and first few chapters of Deng-Ionescu-Pausader-Pusateri [12]. While the main difficulty there is to use weighted spaces to track the global growth of the norms, here we neglect those issues and work simply in $L^2$-based Sobolev spaces and assume a fixed time of existence. A priority of the calculations in this work is to not assume more derivatives than necessary on the Hamiltonian variable $\sqrt{gh}+i|\nabla|^{1/2}\phi$, whereas in [12] much regularity is deliberately wasted. This effort is made more difficult by the absence of surface tension, which when nonzero puts the Hamiltonian variable in the form $(g-\sigma \Delta)^{1/2}h+i|\nabla|^{1/2}\phi$. Thus by considering only the gravity waves we are allowed one fewer derivative on $h$ compared to the full or capillary water waves. We leave the dimension of the interface $d = 1, 2$ arbitrary so the result can be applied in either case, and indeed less regularity is required when $d = 1$. 

We begin Chapter 2 by recalling many basic ideas from paradifferential calculus, in particular for the Weyl quantization as used in [12]. There we prove many of the estimates that will be needed for various compositions and commutators of paraproducts with symbols in suitable spaces. In Chapter 3 we consider the Dirichlet-to-Neumann map for a region bounded above by a \(d\)-dimensional interface in \(\mathbb{R}^{d+1}\). The main idea is to flatten the interface by considering a vertical coordinate \(y \leq 0\) which is 0 at the interface and solving Laplace’s equation in the interior of the fluid region. This allows us to explicitly expand the Dirichlet-to-Neumann map in terms of the paraproduct operators. Finally, in Chapter 4 we use the main theorem from Chapter 3 to paralinearize the entire water waves system and put it into a symmetric form. From there the problem can be easily written as an equation in a single complex unknown from which it is suitable to carry out energy estimates. The final form of the equation we reach is motivated by the NLS modulation problem. With potentially tighter estimates on the error terms and a suitable normal form to extend the time of existence, we expect to be able to obtain the energy estimates needed to prove the infinite depth NLS approximation to the gravity water waves in 2d and 3d. This would provide an alternative to the Riemann mapping method of Totz-Wu [23, 22], and one might hope it could be applied to a wider variety of situations (e.g. finite depth). We remark that there are many places where the estimates in the paralinearization presented here might be improved in the future. For example, there is likely further cancellation and other favorable structures to be found within the quadratic terms in (4.27). One may choose to exploit these differently depending on the intended application.
Chapter 2

Paradifferential calculus

As mentioned in the last chapter, one objective is to not assume more regularity on the Hamiltonian coordinate than is necessary. As a result, we will assume throughout that $h$ and $|\nabla|\phi$ are in the $L^2$-based Sobolev space $H^N$ where $N > 5d/2 + 3/2$.

2.1 Basic properties of the paraproduct

As usual in Littlewood-Paley theory, define a partition of unity for $\mathbb{R}^d \setminus \{0\}$ as follows: let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a radial function supported in the ball of unit radius 1 and identically 1 in the ball of radius 1/2. Take $\varphi(\xi) = \psi(\xi/2) - \psi(\xi)$ and, for $k \in \mathbb{Z}$, $\varphi_k(\xi) = \varphi(\xi/2^k)$. Clearly for $\xi \in \mathbb{R}^d \setminus \{0\}$, $1 = \sum_{k \in \mathbb{Z}} \varphi_k(\xi)$. Notation like $\varphi_{\leq k}$, $\varphi_{[-k_1,k_2]}$, etc. has the obvious meaning. Then we can define the Littlewood-Paley operators as Fourier multipliers with these symbols:

$$\mathcal{F} P_k f(\xi) = \varphi_k(\xi) \hat{f}(\xi).$$ (2.1)

Decompositions of this form allow us to formulate the paradifferential calculus which is the basis for many of our nonlinear estimates. As in [12], we choose to use the paradifferential calculus based on the Weyl quantization:

$$\hat{T}_a f(\xi) = \int_{\mathbb{R}^d} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \widehat{u}(\xi - \eta, \frac{\xi + \eta}{2}) \hat{f}(\eta) d\eta$$ (2.2)
where $\tilde{a}$ is partial Fourier transform in the first coordinate and $\chi = \varphi_{-20}$. This has the advantage over the usual operator that it is self-adjoint on $L^2$ (when it is well-defined there) which makes it easier to obtain energy estimates.

While working with paraproducts we will often have to estimate multilinear operators in the following multiplier form:

$$F_m[f_1, \ldots, f_\ell](\xi) = \int_{(\mathbb{R}^d)^{\ell-1}} m(\xi, \eta_2, \ldots, \eta_\ell) \hat{f}_1(\xi - \eta_2) \cdots \hat{f}_{\ell-1}(\eta_{\ell-1} - \eta_\ell) \hat{f}_\ell(\eta_\ell) d\eta_2 \cdots d\eta_\ell$$

(2.3)

**Lemma 2.1.** A multilinear operator $L_m$ defined by 2.3 satisfies, for exponents $p, q_1, \ldots, q_\ell \in [1, \infty]$,

$$\|L_m[f_1, \ldots, f_\ell]\|_{L^p} \lesssim \|m\|_{S^\infty} \|f_1\|_{L^{q_1}} \cdots \|f_\ell\|_{L^{q_\ell}}$$

(2.4)

when $\frac{1}{q_1} + \cdots + \frac{1}{q_\ell} = \frac{1}{p}$, where we have defined the symbol norm

$$\|f\|_{S^\infty} = \|F^{-1}f\|_{L^1}.$$  

(2.5)

**Proof.** Letting $K = F^{-1}(m)$, we can rewrite the operator as

$$L_m[f_1, \ldots, f_\ell](x) = \int_{(\mathbb{R}^d)^{\ell+1}} e^{ix \cdot \xi} e^{-iy_1 \cdot (\xi - \eta_1)} \cdots e^{-iy_{\ell-1} \cdot (\eta_{\ell-1} - \eta_\ell)} e^{-iy_\ell \cdot \eta_\ell} m(\xi, \eta_2, \ldots, \eta_\ell)$$

$$\times f_1(y_1) \cdots f_\ell(y_\ell) dy_1 \cdots dy_\ell d\eta_2 \cdots d\eta_\ell d\xi$$

$$= \int_{(\mathbb{R}^d)^{\ell}} K(z_1, \ldots, z_\ell) f_1(x - z_1) \cdots f_\ell(x - z_1 - \cdots - z_\ell) dz_1 \cdots dz_\ell$$

(2.6)

and the theorem follows directly from Young’s inequality. One can justify the formal calculations here and elsewhere by first assuming $m$ is a Schwartz function and arguing by density.

When decomposing bilinear operators according to the frequency of the result and the two inputs, frequency localized symbols in the form

$$m^{k,k_1,k_2}(\xi, \eta) = \varphi_k(\xi) \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta) m(\xi, \eta)$$

(2.7)

will often appear. Uniform control of these interactions is sufficient to control the entire operator.
Lemma 2.2. Suppose \( \|m^{k_1,k_2}\|_{S^\infty} \leq 1 \). Then for \( s > d/2 \),

\[
\|M[f,g]\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}
\]

where

\[
\mathcal{F}M[f,g](\xi) = \int m(\xi,\eta)\hat{f}(\xi - \eta)\hat{g}(\eta)d\eta.
\]

Proof. Expanding the frequency localized operator gives

\[
P_kM[P_{k_1}f,P_{k_2}g](x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \varphi_k(\xi) \int_{\mathbb{R}^d} m(\xi,\eta)\varphi_{k_1}(\xi - \eta)\hat{f}(\xi - \eta)\varphi_{k_2}(\eta)\hat{g}(\eta)d\eta d\xi
\]

\[
= \int_{(\mathbb{R}^d)^2} e^{ix\cdot\xi} \hat{f}(\xi - \eta)\hat{g}(\eta)m^{k_1,k_2}(\xi,\eta)d\eta d\xi
\]

and as a result, using Lemma 2.1 and the Bernstein inequalities to move between \( L^p \) spaces,

\[
2^{ks}\|P_kM[P_{k_1}f,P_{k_2}g]\|_{L^2} \sim 2^{ks}\|\mathcal{F}_\xi^{-1} \int_{\mathbb{R}^d} \hat{P}_{k_1}f(\xi - \eta)\hat{P}_{k_2}g(\eta)m^{k_1,k_2}(\xi,\eta)d\eta\|_{L^2}
\]

\[
\lesssim 2^{ks}\min\{\|P_{k_1}f\|_{L^\infty}\|P_{k_2}g\|_{L^2},\|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^\infty}\}\|m^{k_1,k_2}\|_{S^\infty}
\]

\[
\lesssim 2^{ks}\min\{2^{k_1d/2},2^{k_2d/2}\}\|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^2}.
\]

Note that \( |\xi| \sim 2^k, |\xi - \eta| \sim 2^{k_1} \), and \( |\eta| \sim 2^{k_2} \) so \( 2^k \sim |\xi| \leq |\xi - \eta| + |\eta| \sim 2^{k_1} + 2^{k_2} \). Thus
we need only consider \( k \leq \log_2(2^{k_3} + 2^{k_2}) + 2 \). We compute

\[
\sum_{k \geq 0} 2^{2ks} \| P_k M [P_{\geq 0} f, P_{\geq 0} g] \|_{L^2}^2 \leq \sum_{0 \leq k \leq \log_2(2^{k_3} + 2^{k_2}) + 2} 2^{2ks} \| P_k M [P_{k_1} f, P_{k_2} g] \|_{L^2}^2 
\]

\[
\lesssim \sum_{0 \leq k \leq \log_2(2^{k_3} + 2^{k_2}) + 2} 2^{ks} \| P_k M [P_{k_3} f, P_{k_2} g] \|_{L^2} 
\]

\[
\lesssim \sum_{0 \leq k \leq \log_2(2^{k_3} + 2^{k_2}) + 2} 2^{ks} 2^{\min(k_1, k_2)/2} \| P_{k_1} f \|_{L^2} \| P_{k_2} g \|_{L^2} 
\]

\[
\lesssim \sum_{k_1, k_2 \geq 0} (2^{k_1} + 2^{k_2}) 2^{\min(k_1, k_2)/2} 2^{-s(k_1 + k_2)} 
\]

\[
\times (2^{k_1} \| P_{k_1} f \|_{L^2}) (2^{k_2} \| P_{k_2} g \|_{L^2}) 
\]

\[
\lesssim \sup_{k_1, k_2 \geq 0} \left( (2^{k_1} + 2^{k_2}) 2^{\min(k_1, k_2)/2} 2^{-s(k_1 + k_2)} \right) 
\]

\[
\times \left( \sum_{k_1 \geq 0} 2^{2k_1} \| P_{k_1} f \|_{L^2}^2 \right)^{1/2} \left( \sum_{k_2 \geq 0} 2^{2k_2} \| P_{k_2} g \|_{L^2}^2 \right)^{1/2} 
\]

\[
\lesssim \| f \|_{H^s} \| g \|_{H^s}. 
\]

(2.12)

\[\square\]

**Lemma 2.3.** Paraproduct operators preserve frequency localization:

\[ P_k T_a f = P_k T_a P_{\geq k-4} f = P_k T_{a(x, \xi ) \varphi_{\leq k-4}} f = P_k T_{a(x, \xi)} f. \]  

(2.13)

**Proof.** We have

\[ (P_k T_a)^{\wedge} f (\xi) = \int_{\mathbb{R}^d} \psi_k (\xi) \chi\left( \left| \frac{\xi - \eta}{| \xi + \eta |} \right| \right) \tilde{a} (\xi - \eta, \frac{\xi + \eta}{2}) \hat{f} (\eta) d \eta. \]

(2.14)

The integrand is supported where \( 2^{k-1} \leq | \xi | \leq 2^{k+1} \) and \( | \xi - \eta | \leq 2^{-20} | \xi + \eta | \) which imply \( | \eta | \leq | \xi | + | \xi - \eta | \leq 2^{k+1} + 2^{-20} (2^{k+1} + | \eta |) \) and as a result \( | \eta | \leq 4 \cdot 2^k \). Similarly \( | \eta | \geq | \xi | - | \xi - \eta | \geq | \xi | - 2^{-20} | \xi + \eta | \geq (1 - 2^{-20}) | \xi | - 2^{-20} | \eta | \) implies \( | \eta | \geq | \xi | / 4 \). Thus multiplication by \( \varphi_{k-4} (\eta) - \varphi_{k-4} (\eta) \) has no effect. A similar argument proves the other equalities.  

\[\square\]
Definition 2.1. For $q \in \{2, \infty\}$ and $r \in \mathbb{Z}_+$, define the symbol norms

$$\|a\|_{M_{r,q}} = \sup_{\zeta} \|a_r(\cdot, \zeta)\|_{L^q_x} \text{ where } a_r(x, \zeta) = \sum_{|\alpha|+|\beta| \leq r} |\zeta|^{|\beta|} \partial_{\zeta}^\beta \partial_x^\alpha a(x, \zeta).$$ (2.15)

Lemma 2.4. As straightforward computations from the definitions, we have the inequalities

$$\|ab\|_{M_{r,q}} + \|\zeta\{|a, b\}\|_{M_{r-2,q}} \lesssim \|a\|_{M_{r,q}} \|b\|_{M_{r,\infty}}$$ (2.16)

where

$$\{a, b\} = \nabla_x a \cdot \nabla_\zeta b - \nabla_\zeta a \cdot \nabla_x b$$ (2.17)

is the Poisson bracket and, for $q \in \{2, \infty\}$, $k \in \mathbb{Z}$, and $s \in \mathbb{Z}_+$,

$$\|P_k a\|_{M_{r,q}} \lesssim 2^{-sk} \|P_k a\|_{M_{r+s,q}}.$$ (2.18)

Lemma 2.5. For $1 \leq q \leq \infty$ and $r > 2d$, we have

$$\|P_k T_a f\|_{L^q} \lesssim \|a\|_{M_{r,\infty}} \|P_{[k-2, k+2]} f\|_{L^q}$$ (2.19)

and

$$\|P_k T_a f\|_{L^\infty} \lesssim \|a\|_{M_{r,2}} \|P_{[k-2, k+2]} f\|_{L^\infty}.$$ (2.20)

Proof. By rescaling $f$, it is sufficient to prove the result for $k = 0$. By expanding out the Fourier transforms in the definition of $T_a f$ and assuming for the moment that $a$ is rapidly decreasing, we can express the $L^2$ inner product $\langle P_0 T_a h, g \rangle$ as

$$\langle P_0 T_a h, g \rangle = \int_{(\mathbb{R}^d)^2} \frac{\tilde{g}(x) h(y)}{2} I(x, y) dx dy$$ (2.21)

where

$$I(x, y) = \int_{(\mathbb{R}^d)^3} a(z, \frac{\xi + \eta}{2}) e^{i\xi \cdot (x - z)} e^{i\eta \cdot (y - z)} \chi\left(\frac{\xi - \eta}{\xi + \eta}\right) \varphi_0(\xi) d\eta d\xi dz$$

$$= \int_{(\mathbb{R}^d)^3} a(z, \xi + \theta/2) e^{i\theta \cdot (z-y)} e^{i\xi \cdot (x-y)} \chi\left(\frac{|\theta|}{|2\xi + \theta|}\right) \varphi_0(\xi) d\xi d\theta dz$$ (2.22)
and we have taken $\theta = \eta - \xi$. Operators in the form $(1 - \Delta)^{\alpha/2}$ are defined on the Fourier spectrum so they act on an exponential as $(1 - \Delta_x)^{\alpha/2} e^{ix \cdot \xi} = (1 + |\xi|^2)^{\alpha/2} e^{ix \cdot \xi}$. Thus one easily computes

$$
(1 + |x - y|^2)^{r/4} I(x, y) = \int_{\mathbb{R}^d} \frac{a(z, \xi + \theta/2)}{(1 + |z - y|^2)^{r/4}} \chi\left(\frac{|\theta|}{2|\xi + \theta|}\right) \varphi_0(\xi) \times (1 - \Delta \theta)^{r/4} (1 - \Delta \xi)^{r/4} (e^{i\theta (z - y)} e^{i\xi (x - y)}) d\xi d\theta dz
$$

from which integration by parts gives

$$
(1 + |x - y|^2)^{r/4} |I(x, y)| = \left| \int_{\mathbb{R}^d} (1 + |z - y|^2)^{-r/4} e^{i\theta (z - y)} e^{i\xi (x - y)} \times (1 - \Delta \theta)^{r/4} (1 - \Delta \xi)^{r/4} \left[a(z, \xi + \theta/2) \chi\left(\frac{|\theta|}{2|\xi + \theta|}\right) \varphi_0(\xi)\right] d\xi d\theta dz \right|
\lesssim \int_{\mathbb{R}^d} \|a\|_r (z, \xi + \theta/2) \chi\left(\frac{|\theta|}{2|\xi + \theta|}\right) \varphi_0(\xi) \lesssim 10 (\theta) d\xi d\theta dz
\lesssim \|a\|_{M_{r, \infty}},
$$

As a result, using Holder’s inequality with the measure $(1 + |x - y|^2)^{-2} dxdy$,

$$
|\langle P_0 T_a g, h \rangle| \lesssim \|a\|_{M_{8, \infty}} \int_{\mathbb{R}^d} |g(x)| |h(y)|(1 + |x - y|^2)^{-2} dxdy
\leq \|a\|_{M_{8, \infty}} \left( \int_{\mathbb{R}^d} |g(x)|^q (1 + |x - y|^2)^{-2} dxdy \right)^{1/q}
\times \left( \int_{\mathbb{R}^d} |h(y)|^{q'} (1 + |x - y|^2)^{-2} dxdy \right)^{1/q'}
= C\|a\|_{M_{8, \infty}} \|g\|_{L^q} \|h\|_{L^{q'}}
$$

where $q$ and $q'$ are dual exponents. Thus

$$
\|P_0 T_a f\|_{L^q} = \sup_{\|h\|_{L^{q'}} \leq 1} |\langle P_0 T_a f, h \rangle| \lesssim \|a\|_{M_{8, \infty}} \|f\|_{L^q}
$$

and we conclude the first inequality with Lemma 2.3. The argument for the second is analogous.
2.2 Commutator estimates

Lemma 2.6. The remainder

\[ \mathcal{H}(f, g) = fg - T_f g - T_g f. \]  

(2.27)

satisfies

\[ \|P_k \mathcal{H}(f, g)\|_{L^q} \lesssim \sum_{k', k'' \geq k - 40, |k' - k''| \leq 40} \min\{\|P_{k'} f\|_{L^q} \|P_{k''} g\|_{L^\infty}, \|P_{k'} f\|_{L^\infty} \|P_{k''} g\|_{L^q}\} \]  

(2.28)

which implies

\[ \|\mathcal{H}(f, g)\|_{H^s} \lesssim \|f\|_{H^{s + d/4}} \|g\|_{H^{s + d/4}} \]  

(2.29)

Thus \( \mathcal{H} \) is smoothing as long as \( s > d/2 \).

Proof. Observe that

\[ P_k \mathcal{H}(P_{k'} f, P_{k''} g) = 0 \]  

unless \( k', k'' \geq k - 40, |k' - k''| \leq 40 \).

(2.30)

Indeed, we can write

\[ \mathcal{F}(P_k \mathcal{H}(P_{k'} f, P_{k''} g))(\xi) = \int_{\mathbb{R}^d} \varphi_k(\xi) \varphi_{k'}(\xi - \eta) \varphi_{k''}(\eta) \times \left[ 1 - \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \chi\left(\frac{|\eta|}{2|\xi - \eta|}\right) \right] \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \]  

(2.31)

If \( k' < k - 40 \) or \( k' - k'' > 40 \), then in the support of \( \psi_k(\xi) \psi_{k'}(\xi - \eta) \psi_{k''}(\eta) \) it’s easy to compute that \( |\xi - \eta| \leq 2^{-21} |\xi + \eta| \) and \( |\eta| > 2^{-19} |2\xi - \eta| \) which makes the integrand vanish. Furthermore \( k' \) and \( k'' \) are symmetric so we need only consider the terms with \( k', k'' \geq k - 40 \) and \( |k' - k''| \leq 40 \). Thus

\[ \|P_k \mathcal{H}(f, g)\|_{L^q} \leq \sum_{k', k'' \geq k - 40, |k' - k''| \leq 40} \|P_k \mathcal{H}(P_{k'} f, P_{k''} g)\|_{L^q}. \]  

(2.32)

The first inequality then follows from Holder’s inequality and Lemma 2.1 since the symbol of the multilinear operator \( P_k T_{h_1} h_2 \) is smooth and compactly supported in \( \xi \) and \( \eta \) and a simple scaling computation shows that the symbol norm doesn’t depend on \( k \). For the second, we
have, by using the first inequality and the Bernstein inequality \( \|P_j f\|_{L^\infty} \lesssim 2^{jd/2} \|P_j f\|_{L^2} \),

\[
\|\mathcal{H}(f, g)\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} (1 + 2^{2ks}) \|P_k \mathcal{H}(f, g)\|_{L^2}^2
\leq \sum_{k \in \mathbb{Z}} (1 + 2^{2ks}) \left( \sum_{k', k'' \geq k-40, |k' - k''| \leq 40} 2^{\min\{k', k''\}d/2} \|P_{k'} f\|_{L^2} \|P_{k''} g\|_{L^2} \right)^2
\leq \left( \sum_{k', k'' \geq k-40, |k' - k''| \leq 40} (1 + 2^{\min\{k', k''\}d/2 + ks}) \|P_{k'} f\|_{L^2}^2 \right) \times \left( \sum_{k', k'' \geq k-40, |k' - k''| \leq 40} (1 + 2^{\min\{k', k''\}d/2 + ks}) \|P_{k''} g\|_{L^2}^2 \right).
\]

By explicitly summing over the permissible \( k \) and \( k'' \) (in the first factor) and \( k \) and \( k' \) (in the second factor), we get

\[
\|\mathcal{H}(f, g)\|_{H^{s+5}}^2 \lesssim \left( \sum_{k_1 \in \mathbb{Z}} (1 + 2^{k_1(s+d/2)}) \|P_{k_1} f\|_{L^2}^2 \right) \left( \sum_{k_2 \in \mathbb{Z}} (1 + 2^{k_2(s+d/2)}) \|P_{k_2} g\|_{L^2}^2 \right)
= \|f\|_{H^{s+2d/4}} \|g\|_{H^{s+2d/4}}.
\]

\[
(2.34)
\]

**Lemma 2.7.** The remainder \( E \) defined by

\[
T_a T_b = T_{ab} + \frac{i}{2} T_{(a,b)} + E(a, b)
\]

satisfies the following for \( 1 \leq q \leq \infty, k \geq -100, \) and \( r > 2d: \)

\[
\|P_k E(a, b) f\|_{L^q} \lesssim 2^{-2k} \|a\|_{\mathcal{M}_{3r/2+d+2, \infty}} \|b\|_{\mathcal{M}_{3r/2+d+2, \infty}} \|P_{[k-5,k+5]} f\|_{L^q}, \quad q \in \{2, \infty\}
\]

\[
\|P_k E(a, b) f\|_{L^2} \lesssim 2^{-2k} \|a\|_{\mathcal{M}_{3r/2+d+2, \infty}} \|b\|_{\mathcal{M}_{3r/2+d+2, \infty}} \|P_{[k-5,k+5]} f\|_{L^\infty},
\]

\[
(2.36)
\]

\[
\|P_k E(a, b) f\|_{L^2} \lesssim 2^{-2k} \|a\|_{\mathcal{M}_{3r/2+d+2, \infty}} \|b\|_{\mathcal{M}_{3r/2+d+2, \infty}} \|P_{[k-5,k+5]} f\|_{L^\infty}.
\]

**Proof.** Consider first the components of the symbols \( a \) and \( b \) with low frequencies below \( k - 100. \) For instance, with \( q \in \{2, \infty\} \) and \( a = P_{>k-100} a, \) repeated application of Lemmas 2.4
and 2.5 gives

\[
\|P_k E(a, b) f\|_{L^q} \leq \|P_k T_{P_{k-100}^a} P_{b} f\|_{L^q} + \|P_k T_{(P_{k-100}^a) b} f\|_{L^q} + \|P_k T_{(P_{k-100}^a b)} f\|_{L^q}
\]

\[
\lesssim \|P_{> k-100} a\|_{M_{r, \infty}} \|P_{k-2, k+2} b f\|_{L^q} + \|(P_{> k-100} a) b\|_{M_{r, \infty}} \|P_{k-2, k+2} f\|_{L^q}
\]

\[
+ \|P_{> k-100} a, b\|_{M_{r, \infty}} \|P_{k-2, k+2} f\|_{L^q}
\]

\[
\lesssim 2^{-2k} \|P_{> k-100} a\|_{M_{r+2, \infty}} \|b\|_{M_{r, \infty}} \|P_{k-5, k+5} f\|_{L^q}
\]

\[
+ 2^{-2k} \|P_{> k-100} a\|_{M_{r+2, \infty}} \|b\|_{M_{r, \infty}} \|P_{k-2, k+2} f\|_{L^q}
\]

\[
\lesssim 2^{-2k} \|a\|_{M_{r+2, \infty}} \|b\|_{M_{r, \infty}} \|P_{k-5, k+5} f\|_{L^q}
\]

(2.37)

where we have extracted the extra decay in \(k\) by estimating \(a\) in the stronger symbol space and using the assumption \(k \geq -100\). The argument is analogous for the high frequencies of \(b\) and for the other inequalities in the lemma. Thus we’re left only to consider the low frequencies, so assume \(a = P_{\leq k-100} a\) and \(b = P_{\leq k-100} b\). From the definitions,

\[
\mathcal{F}\{P_k (T_a T_b - T_{ab}) f\} (\xi) = \varphi_k(\xi) \int_{\mathbb{R}^d} \hat{f}(\eta) \varphi_{\leq k-100}(\xi - \eta) \varphi_{\leq k-100}(\theta - \eta) \times \left[ \bar{a}(\xi - \theta, \frac{\xi + \eta}{2}) \bar{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \bar{a}(\xi - \theta, \frac{\xi + \eta}{2}) \bar{b}(\theta - \eta, \frac{\xi + \eta}{2}) \right] d\eta d\theta
\]

(2.38)

and

\[
\mathcal{F}\{P_{\frac{i}{2}} T_{(a, b)} f\} (\xi) = \varphi_k(\xi) \int_{\mathbb{R}^d} \hat{f}(\eta) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) \times \left[ \frac{\theta - \eta}{2} \nabla \bar{a}(\xi - \theta, \frac{\xi + \eta}{2}) \bar{b}(\theta - \eta, \frac{\xi + \eta}{2}) - \bar{a}(\xi - \theta, \frac{\xi + \eta}{2}) \frac{\xi - \theta}{2} \nabla \bar{b}(\theta - \eta, \frac{\xi + \eta}{2}) \right] d\eta d\theta.
\]

(2.39)

Thus we can write \(P_k E(a, b)\) as \(U_1 + U_2 + U_3\) where \(U_3\) is given by the Fourier multiplier

\[
\mathcal{F}(U^3 f)(\xi) = \varphi_k(\xi) \int_{\mathbb{R}^d} \hat{f}(\eta) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) m^3(\xi, \eta, \theta) d\eta d\theta
\]

(2.40)
on the exponentials, we obtain the bound

\[ \int \frac{2^{-2k}}{(2^{-2k} + |x-y|^2)^{r/4}} \frac{2^{-2k}}{(2^{-2k} + |z-y|^2)^{r/4}} \frac{2^{-2k}}{(2^{-2k} + |w-y|^2)^{r/4}} F_{a,b}(z,w) dz dw \]

where

\[
\begin{align*}
m^1(\xi, \eta, \theta) &= \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \tilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \\
&\quad - \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \theta - \xi \nabla_{\xi} \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}), \\
m^2(\xi, \eta, \theta) &= \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \tilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \\
&\quad - \theta - \eta \nabla_{\xi} \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}), \\
m^3(\xi, \eta, \theta) &= \theta - \eta \nabla_{\xi} \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \\
&\quad \left[ \tilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \right].
\end{align*}
\]

We can rearrange \( U^1 \) to

\[
U^1 f(x) = \int_{\mathbb{R}^d} f(y) K^1(x,y) dy
\]

where the kernel is

\[
K^1(x,y) = \int_{\mathbb{R}^d} e^{-iy \cdot \eta} e^{ix \cdot \xi} \varphi_k(\xi) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) m^1(\xi, \eta, \theta) d\eta d\theta d\xi. \quad (2.43)
\]

From the fundamental theorem of calculus, one easily computes

\[
m^1(\xi, \eta, \theta) = \sum_{k,j} \int_0^1 \tilde{a}(\xi - \theta, \frac{\xi + \eta}{2})(\theta - \eta) \nabla_{\xi} \tilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) (1-s) ds
\]

(2.44)

so by changing variables \( \mu = \theta - \xi \) and \( \nu = \eta - \theta \) and expanding the Fourier transforms,

\[
K^1(x,y) = \int_0^1 \int_{\mathbb{R}^d} e^{-iy \cdot (\xi + \mu + \nu)} e^{ix \cdot \xi} e^{iz \cdot \mu} e^{iw \cdot \nu} \varphi_k(\xi) \varphi_{\leq k-100}(\mu) \varphi_{\leq k-100}(\nu) \\
\times \nabla_{\xi} \tilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) (1-s) ds.
\]

(2.45)

Integrating by parts in the frequency variables to put operators in the form \((2^{-2k} - \Delta_\xi)^{r/4}\) on the exponentials, we obtain the bound

\[
|K^1(x,y)| \lesssim \int \frac{2^{-2k}}{(2^{-2k} + |x-y|^2)^{r/4}} \frac{2^{-2k}}{(2^{-2k} + |z-y|^2)^{r/4}} \frac{2^{-2k}}{(2^{-2k} + |w-y|^2)^{r/4}} F_{a,b}(z,w) dz dw
\]

(2.46)
where

\[
F_{a,b}(z,w) = 2^{6k} \int_{(\mathbb{R}^d)^3} \left| (2^{-2k} - \Delta_\xi)^{r/4} (2^{-2k} - \Delta_\mu)^{r/4} (2^{-2k} - \Delta_\nu)^{r/4} \right|
\times \left\{ \partial_{x^i} a(z,\xi + \mu/2 + \nu/2) \partial_{x^j} b(w,\xi + \mu/2 + \nu/2 + s\mu/2) \right\} \mid d\xi d\mu d\nu
\]

(2.47)

Observe that if we estimate the symbols using \( |\cdot|_{3r/2+d+2} \), we need at most two derivatives to fall on the spacial variable, so the remaining \( 3r/2 + d \) may fall on the frequency. Thus, by definition of this norm, we gain a factor of \( |\xi + \mu/2 + \nu/2|^{-(3r/2+d)} \) or \( |\xi + \mu/2 + \nu/2 + s\mu/2|^{-(3r/2+d)} \) which, by the localization of \( \xi, \mu, \) and \( \nu \), are majorized by a constant on the order of \( 2^{-k(3r/2+d)} \). Thus

\[
|F_{a,b}(z,w)| \lesssim 2^{-(6+d)k} \int_{(\mathbb{R}^d)^3} \varphi(-4,4) (2^{-k}\xi) \varphi_{\leq k-100} (2^{-k}\mu) \varphi_{\leq k-100} (2^{-k}\nu) |a|_{\mathcal{M}_{3r/2+d+2,\infty}} |b|_{\mathcal{M}_{3r/2+d+2,\infty}} d\xi d\mu d\nu
\]

(2.48)

which implies

\[
|K^1(x,y)| \lesssim 2^{-(6-2d)k} \||a|_{\mathcal{M}_{3r/2+d+2,\infty}} ||b|_{\mathcal{M}_{3r/2+d+2,\infty}} |z|^2 |w|^2 |z|^r/4 |w|^r/4 dz dw
\]

(2.49)

where we have used scaling of the integral to extract the dependence on \( k \). The first desired inequality for the \( U_1 \) piece is now immediate. The other pieces and inequalities follow by analogous arguments.

**Definition 2.2.** Define the space of symbols \( \mathcal{M}_{\ell}^{r,p} \subset C(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}) \) with norm

\[
\|a\|_{\mathcal{M}_{\ell}^{r,p}} = \sup_{|\alpha|+|\beta| \leq r} \sup_{\zeta \in \mathbb{R}^d} \|\zeta\|^{\ell} \||\zeta\|^{|\beta|} \partial^\beta_\zeta \partial^\alpha_x a\|_{L^p_x}
\]

(2.50)

and let \( \mathcal{M}_{\ell}^r = \mathcal{M}_{\ell,\infty} \cap \mathcal{M}_{\ell,2}^r \).
Lemma 2.8. For a symbol \( a \in \mathcal{M}_r^\ell \) where \( r > 2d \), we have the embedding \( T_a H^s \subset H^{s-\ell} \). This holds in particular with \( \ell = 0 \) if \( a \in H^{r+d/2} \).

Proof. From the definitions, one easily has
\[
\|a(x, \zeta)\varphi \|_{M_{r,q}} \lesssim 2^{k_\ell} \|a\|_{\mathcal{M}_{r,q}^\ell}.
\] (2.51)

Using this with Lemmas 2.3 and 2.5, we get for \( k \geq 0 \)
\[
2^{k(s-\ell)}\|P_k T_a f\|_{L^2} = 2^{k(s-\ell)}\|P_k T_a(x,\zeta)\varphi \|_{L^2} \lesssim 2^{k(s-\ell)}\|a(x, \zeta)\varphi \|_{M_{r,\infty}} \|P[k-2, k+2] f\|_{L^2}
\] (2.52)

and summing in \( \ell^2 \) completes the proof of the first statement. Then we must argue that \( H^{r+d/2} \subset \mathcal{M}_r^\ell \). Obviously a symbol in \( H^{r+d/2} \) must not depend on \( \zeta \), in which case the \( \mathcal{M}_r^\ell \) norm easily reduces to \( \|a\|_{\dot{H}^r} + \|a\|_{\dot{W}^{r, \infty}} \). The first inequality is obvious and the second is a standard Sobolev embedding.

Lemma 2.9. Assume \( a \in \mathcal{M}_{r_1}^{\ell_1} \) and \( b \in \mathcal{M}_{r_2}^{\ell_2} \) where \( r_1 + r_2 > \ell_1 + \ell_2 + d + 2 \). Then we have the inclusions
\[
P_{\geq -100} E(a, b) H^{N+q} \subset H^{N+q-\ell_1-\ell_2+2}
\] (2.53)
\[
P_{\geq -100} (T_a T_b + T_b T_a - 2T_{ab}) H^{N+q} \subset H^{N+q-\ell_1-\ell_2+2},
\] (2.54)
\[
[T_a, T_b] H^{N+q} \subset H^{N+q-\ell_1-\ell_2+1},
\] (2.55)
\[
(T_a T_b - T_{ab}) H^{N+q} \subset H^{N+q-\ell_1-\ell_2+1}.
\] (2.56)

Proof. From the definitions, we have
\[
\mathcal{F} P_{k_1} a P_{k_2} b f(\xi) = \int_{\mathbb{R}^d} \chi\left(\frac{\xi}{\zeta + \eta}\right) \varphi_{k_1}(\xi - \eta)\tilde{a}(\xi - \eta, \frac{\xi + \eta}{2}) T_{P_{k_2}} \tilde{b}(\eta) d\eta
\]
\[
= \int_{\mathbb{R}^d} \chi\left(\frac{\xi}{\zeta + \eta}\right) \chi\left(\frac{\eta - \theta}{\eta + \theta}\right) \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta - \theta) \tilde{a}(\xi - \eta, \frac{\xi + \eta + \theta}{2}) \tilde{b}(\eta - \theta, \frac{\eta + \theta}{2}) f(\theta) d\theta d\eta
\] (2.57)
and

\[
\mathcal{F}T_{\theta_1}^aP_{\kappa_2}b\hat{f}(\xi) = \int_{\mathbb{R}^d} \chi\left(\frac{\xi - \eta}{\xi + \eta}\right)\hat{P}_{\kappa_1}^a a \ast \hat{P}_{\kappa_2} b(\xi - \eta, \frac{\xi + \eta}{2})\hat{f}(\eta) d\eta
\]

\[
= \int_{(\mathbb{R}^d)^2} \chi\left(\frac{\xi - \eta}{\xi + \eta}\right)\varphi_{\kappa_1}(\theta)\varphi_{\kappa_2}(\xi - \eta - \theta)\tilde{a}(\theta, \frac{\xi + \eta}{2})\tilde{b}(\xi - \eta - \theta, \frac{\xi + \eta}{2})\hat{f}(\eta) d\theta d\eta.
\]

(2.58)

Suppose that \(2^{k-1} \leq |\xi| \leq 2^{k+1},\ 2^{k_1-1} |\theta| \leq 2^{k_1+1},\ 2^{k_2-1} \leq |\xi - \eta - \theta| \leq 2^{k_2+1},\) and \(k_1, k_2 \leq k - 40.\) Then \(|2\xi - \theta| \geq 2^{k_1} - 2^{k_1+1} \geq 2^{k_1+40} - 2^{k_1+1} \geq 2^{k_1+39} \geq 2^{38}|\theta|\) so \(\chi(|\theta|/|2\xi - \theta|) = 1.\) Similarly, \(|\xi + \eta - \theta| = |2\xi - 2\theta - (\xi - \eta - \theta)| \geq 2 \cdot 2^{k_1} - 2^{k_1+1} \geq 2^{k_1+1} \geq 2^{k}\) and \(|\xi - \eta - \theta| \leq 2^{k-39}\) so \(\chi(|\xi - \eta - \theta|/|\xi + \eta - \theta|) = 1.\) The same calculation proves that \(\chi(|\xi - \eta|/|\xi + \eta|) = 1.\) Thus when \(k_1, k_2 < k - 40,\) we can rearrange the first expression by switching \(\theta\) and \(\eta,\) replacing \(\theta\) with \(\xi - \theta,\) and \(\eta\) with \(\xi - \eta - \theta\) to find

\[
\mathcal{F}(P_{\theta_1}T_{\theta_1}^aT_{\theta_2}b - T_{\theta_1}^aP_{\kappa_2}b)f(\xi) = \int_{(\mathbb{R}^d)^2} \varphi_k(\xi) \varphi_{\kappa_1}(\theta)\varphi_{\kappa_2}(\eta)\hat{f}(\xi - \eta - \theta)
\]

\[
\times \left[\tilde{a}(\theta, \frac{2\xi - \theta}{2})\tilde{b}(\eta, \frac{2\xi - \eta - 2\theta}{2}) - \tilde{a}(\theta, \frac{2\xi - \eta - \theta}{2})\tilde{b}(\eta, \frac{2\xi - \eta - \theta}{2})\right] d\theta d\eta.
\]

(2.59)

Note that, using the facts that the distance between \((2\xi - \eta - 2\theta)/2\) and \((2\xi - \eta - \theta)/2\) is localized around \(2^{k_1}\) while between \((2\xi - \eta - 2\theta)/2\) and \((2\xi - \eta - \theta)/2\) is around \(2^{k_2},\) we can estimate

\[
\|\tilde{a}(\theta, \frac{2\xi - \theta}{2})\tilde{b}(\eta, \frac{2\xi - \eta - 2\theta}{2}) - \tilde{a}(\theta, \frac{2\xi - \eta - \theta}{2})\tilde{b}(\eta, \frac{2\xi - \eta - \theta}{2})\|_{L^2_{\theta,\eta}}
\]

\[
\lesssim \|\tilde{a}(\theta, \frac{2\xi - \theta}{2}) - \tilde{a}(\theta, \frac{2\xi - \eta - \theta}{2})\|_{L^\infty_{\theta}L^2_\eta} \|\tilde{b}(\eta, \frac{2\xi - \eta - \theta}{2})\|_{L^\infty_{\eta}L^2_\theta}
\]

\[
+ \|\tilde{b}(\eta, \frac{2\xi - \eta - 2\theta}{2}) - \tilde{b}(\eta, \frac{2\xi - \eta - \theta}{2})\|_{L^\infty_{\eta}L^2_\theta} \|\tilde{a}(\theta, \frac{2\xi - \theta}{2})\|_{L^\infty_{\theta}L^2_\eta}
\]

\[
\lesssim 2^{k_2}\|\partial_\xi a\|_{L^\infty_{\xi}L^2_{\xi}(|\xi| < 2^k)}\|b\|_{L^\infty_{\xi}L^2_{\xi}(|\xi| < 2^k)} + 2^{k_1}\|\partial_\xi b\|_{L^\infty_{\xi}L^2_{\xi}(|\xi| < 2^k)}\|a\|_{L^\infty_{\xi}L^2_{\xi}(|\xi| < 2^k)}.
\]

(2.60)

In fact in the support of the integrand we know precisely the order of magnitude of \(|\eta|\) and \(|\theta|\). Exploiting this and the growth in \(\xi\) included in the symbol norm (assuming \(k \geq 0\), we
get

$$
\| \mathcal{F}(P_k T_{P_{k_1}} a T_{P_{k_2}} b - T_{P_{k_1}} a T_{P_{k_2}} b) f \|_{L^2} \lesssim \| \varphi_k(\xi) \varphi_{k_1}(\theta) \varphi_{k_2}(\eta) \hat{f}(\xi - \eta - \theta) \|_{L^2_{\xi,\eta,\theta}} \\
\times (2^{k_2} \| \partial_\zeta a \|_{L^\infty_{\zeta} L^2} \| b \|_{L^\infty_{\zeta} L^2} + 2^{k_1} \| \partial_\zeta b \|_{L^\infty_{\zeta} L^2} \| a \|_{L^\infty_{\zeta} L^2}) \\
\lesssim 2^{(k_1 + k_2) d/2} \| P_k f \|_{L^2} \| a \|_{\mathcal{M}_1^d} \| b \|_{\mathcal{M}_2^d} \\
\times (2^{k_2} 2^{k(\ell_1 - r_1 + 1)} 2^{k(\ell_2 - r_2)} + 2^{k_1} 2^{k(\ell_2 - r_2 + 1)} 2^{k(\ell_1 - r_1)}).
$$

(2.61)

Explicitly summing the result over just these $k_1$ and $k_2$,

$$
\sum_{k_1, k_2 < k - 40} \| P_k T_{P_{k_1}} a T_{P_{k_2}} b - T_{P_{k_1}} a T_{P_{k_2}} b) f \|_{L^2} \lesssim 2^{k(\ell_1 + \ell_2 - r_1 - r_2 + d + 2)} \| P_k f \|_{L^2}
$$

(2.62)

which completes the proof of this inequality for the frequency region $\{k_1, k_2 < k - 40\}$. The proof for the remaining frequencies is analogous to the computation for $2^k \ll 2^{k_1} + 2^{k_2}$ in Lemma 2.2.

□
Chapter 3

Paralinearizing the Dirichlet-to-Neumann map

3.1 Flattening the interface

Let $\Omega = \{(x, z) \in \mathbb{R}^{d+1} : z \leq h(x)\}$ and $\Phi$ the harmonic extension of $\phi$ into $\Omega$. Flatten the surface by writing $u(x, y) = \Phi(x, h(x) + y)$ for $y \leq 0$. $u|_{y=0} = \phi$ and $\partial_y u|_{y=0} = B$ imply

$$G(h)\phi = (1 + |\nabla h|^2)\partial_y u|_{y=0} - \nabla h \cdot \nabla u|_{y=0}. \quad (3.1)$$

Also, the condition $\Delta x,z \Phi = 0$ with the definition $\Phi(x, z) = u(x, z - h(x))$ requires

$$0 = \Delta x u = \partial^i(u_i(x, z - h(x)) - h_i(x)u_g(x, z - h(x))) + \partial_z(u_g(x, z - h(x))) \quad (3.2)$$

We will also consider this equation with some error terms $\epsilon_a$ and $\epsilon_b$ satisfying

$$\partial_y \epsilon_a + |\nabla| \epsilon_b = \Delta x u - 2\nabla x h \cdot \nabla z \partial_y u - \Delta_x h \partial_y u + (1 + |\nabla x h|^2)\partial_y^2 u. \quad (3.3)$$

Collecting the terms $\partial_y^2 u$ and $\Delta x u$, this rearranges to give the system

$$(\partial_y^2 - |\nabla|^2)u = \partial_y(\nabla u \cdot \nabla h - |\nabla h|^2 \partial_y u + \epsilon_a) + \nabla h \cdot \nabla \partial_y u + \Delta h \partial_y u + |\nabla| \epsilon_b$$

$$= \partial_y(\nabla u \cdot \nabla h - |\nabla h|^2 \partial_y u + \epsilon_a) + |\nabla| \left(\frac{\nabla}{|\nabla|} \cdot (\partial_y u \nabla h) + \epsilon_b\right) \quad (3.4)$$

$$= \partial_y Q_a + |\nabla| Q_b$$
where we have set
\[ Q_a = \nabla u \cdot \nabla h - \alpha \partial_y u + e_a, \quad Q_b = R(\partial_y u \nabla h) + e_b \]  \tag{3.5}
where \( R \) is the vector of Riesz transforms and for simplicity, \( \alpha = |\nabla h|^2 \). We would like a Duhamel formula for this system to run a fixed point argument. Taking the Fourier transform in \( x \) we get an ODE
\[ \hat{\partial}_t^2 u(\xi) - |\xi|^2 \hat{u}(\xi) = \partial_y \hat{Q}_a(\xi) + |\xi| \hat{Q}_b(\xi) \] \tag{3.6}
which has homogeneous solutions \( e^{f'(y)|\xi|} \) where \( f'(y)^2 = 1 \). Thus using a suitable integrating factor,
\[ \hat{u}(y)(\xi) = e^{y|\xi|} \left( \psi + \frac{1}{2} \int_{-\infty}^{0} e^{s|\xi|} (\partial_y \hat{Q}_a(s) + |\xi| \hat{Q}_b(s)) ds \right) \\
+ \frac{1}{2} \int_{-\infty}^{0} e^{y-s|\xi|} (\partial_y \hat{Q}_a(s) + |\xi| \hat{Q}_b(s)) ds \]
\[ = e^{y|\xi|} \left( \psi + \frac{1}{2} \int_{-\infty}^{0} e^{s|\xi|} (-|\xi| \hat{Q}_a(s) + |\xi| \hat{Q}_b(s)) ds \right) \\
+ \frac{1}{2} \int_{-\infty}^{0} e^{y-s|\xi|} (-|\xi| \hat{Q}_a(s) + |\xi| \hat{Q}_b(s)) ds \] \tag{3.7}
where in both terms we have integrated by parts the \( Q_a \) component, and \( \psi \) depends on the potential at the interface \( y = 0 \).

**Lemma 3.1.** (i) Assume \( t \in [0, T] \) is fixed, \( ||h||_{H^N} \lesssim \epsilon \), and
\[ ||| \nabla^{1/2} \psi |||_{H^{N+p}} \leq A < \infty, \quad ||e_a||_{L^2_y H^{N+p}} + ||e_b||_{L^2_y H^{N+p}} \leq A \epsilon \] \tag{3.8}
for some \( p \in (d/2 - N, -1] \). Defining the Banach space \( L_p \) of \( C((-\infty, 0] : \dot{H}^{1/2}) \) functions with the norm
\[ ||g||_{L_p} = ||\nabla g||_{L^2_y H^{N+p}} + ||\partial_y g||_{L^2_y H^{N+p}} + ||\nabla^{1/2} g||_{L^2_y H^{N+p}}, \] \tag{3.9}
there is a unique solution \( u \in L_p \) of the equation
\[ u(y) = e^{y|\xi|} \left( \psi - \frac{1}{2} \int_{-\infty}^{0} e^{s|\xi|} (Q_a(s) - Q_b(s)) ds \right) \\
+ \frac{1}{2} \int_{-\infty}^{0} e^{y-s|\xi|} (\text{sgn}(y - s)Q_a - Q_b(s)) ds. \] \tag{3.10}
This solution \( u \) solves the equation \( (\partial_y^2 - |\nabla|^2)u = \partial_y Q_a + |\nabla| Q_b \) with \( \|u\|_{L_p} \lesssim A \).
(ii) Suppose instead that
\[
\|\nabla|^{1/2}\psi\|_{H^{N+p}} \leq A, \quad \|\partial_y^j \xi\|_{L_2^2 H^{N+p-j}} + \|\partial_y^j \xi\|_{L_2^2 H^{N+p-1/2-j}} \leq A\epsilon
\]  
for \( \epsilon \in \{\epsilon_a, \epsilon_b\} \) and \( j \in \{0, 1, 2\} \). Then
\[
\|\partial_y^j (\partial_y u - |\nabla|u)\|_{L_2^2 H^{N+p-j}} + \|\partial_y^j (\partial_y u - |\nabla|u)\|_{L_2^2 H^{N+p-1/2-j}} \lesssim A
\]

Proof. This follows from a fixed point argument in the ball of radius \( A \) in \( L_p \) for the functional
\[
\Phi(u) = e|\nabla| \left( \psi - \frac{1}{2} \int_{-\infty}^{\infty} e|\nabla|((Q_a(s) - Q_b(s))ds) \right)
\]
\[
+ \frac{1}{2} \int_{-\infty}^{0} e^{-|y-s||\nabla|} (sgn(y-s)Q_a(s) - Q_b(s))ds.
\]

Observe that if \( \|u\|_{L_p} \lesssim A \), then (since \( N + p > d/2 \))
\[
\|Q_a\|_{L_2^2 H^{N+p}} = \|\nabla u \cdot \nabla h - |\nabla h|^2 \partial_y u + \epsilon_a\|_{L_2^2 H^{N+p}}
\]
\[
\lesssim \|u\|_{L_p}\|\nabla h\|_{H^{N+p}} + \|\nabla h\|_{H^{N+p}}^2 \|u\|_{L_p} + \|\epsilon_a\|_{L_2^2 H^{N+p}}
\]
\[
\lesssim A\epsilon
\]

and similarly for \( Q_b \). For two points \( u \) and \( u' \) in the ball, we would like to estimate
\[
\Phi_u - \Phi_{u'} = -\frac{1}{2} e|\nabla| \int_{-\infty}^{0} e|\nabla| \left( [Q_a(s) - Q_a'(s)] - [Q_b(s) - Q_b'(s)] \right) ds
\]
\[
+ \frac{1}{2} \int_{-\infty}^{0} e^{-|y-s||\nabla|} (sgn(y-s)[Q_a(s) - Q'_a(s)] - [Q_b(s) - Q'_b(s)])\) ds
\]
in the \( L_p \) norm. First observe that
\[
\|e|\nabla|\psi\|_{L_p} = 2\|((1 + \xi^2)^{(N+p)/2})|\xi||e^{|\xi|\nabla|}\hat{\psi}(\xi)|_{L_2^2} + \|((1 + |\xi|^2(N+p)/2)|\xi|^{1/2}|e^{|\xi|\nabla|}\hat{\psi}(\xi)|_{L_2^2}
\]
\[
= 2e\|((1 + \xi^2)^{(N+p)/2})|\xi|-1/2|e^{|\xi|\nabla|}\hat{\psi}(\xi)|_{L_2^2} + \|((1 + |\xi|^2(N+p)/2)|\xi|^{1/2}\hat{\psi}(\xi)|_{L_2^2}
\]
\[
\sim \||\nabla|^{1/2}\psi\|_{H^{N+p}}
\]

By letting \( \delta Q(s) = Q(s) - Q'(s) \) for \( Q \in \{Q_a, Q_b\} \), we can write \( \Phi_u - \Phi_{u'} \) as a sum of terms in any of the forms
\[
\int_{-\infty}^{0} e^{(y-s)|\nabla|} \delta Q(s)ds, \quad \int_{-\infty}^{0} e^{-|y-s||\nabla|} \int_{-\infty}^{0} \delta Q(s)ds, \quad \int_{-\infty}^{0} e^{-|y-s||\nabla|} \delta Q(s)ds
\]  
(3.17)
which can be readily estimated in any of the norms composing the $L_p$ spaces with the help of (3.14) and (3.16). For example,

$$
\|\nabla |1/2 \int_{-\infty}^{0} e^{-|y-s|\|\nabla|1 (y-s) \delta Q(s) ds \|_{L_2 y H^{N+p}}
= \|(1 + |\xi|^2(N+p)/2 |\xi|^{1/2} \int_{-\infty}^{0} e^{-|y-s|\|\xi|1 (y-s) \delta Q(s) ds \|_{L_2 y L_2^2}
\lesssim \|(1 + |\xi|^2(N+p)/2 |\xi|^{1/2} \int_{-\infty}^{0} e^{-2|y-s|\|\xi|1 (y-s) ds \|_{L_2 y L_2^2}^{1/2} \int_{-\infty}^{0} \delta Q(s) ds \|_{L_2 y L_2^2}^{1/2}
= \|(1 + |\xi|^2(N+p)/2 |\xi|^{1/2} |\xi|^{-1/2} \int_{-\infty}^{0} \delta Q(s) ds \|_{L_2 y L_2^2}^{1/2}
= \|\delta Q\|_{L_2 y H^{N+p}}
\quad \text{(3.18)}
$$

and the rest may be computed similarly. Thus the fixed point argument closes and (i) immediately follows.

For (ii), let us directly differentiate (3.10). Both $\partial_y u$ and $\|\nabla|$ clearly act the same on the first term. For the second, since $\partial_y (e^{-|y-s|\|\xi|sgn(y-s)) = e^{2|y-s|\|\xi|(-|\xi| + \delta(y-s))$, we have

$$
\partial_y u(y) - |\nabla| u(y) = Q_a(y) + \int_{-\infty}^{0} |\nabla| e^{-|s-y|\|\nabla|1_+ (y-s) (Q_b(s) - Q_a(s)) ds. \quad \text{(3.19)}
$$

By a similar argument as the last part, we get

$$
\|Q\|_{L_2 y H^{N+p}} + \|Q\|_{L_2 y H^{N+p-1/2}} \lesssim A\epsilon \quad \text{(3.20)}
$$

for $Q \in \{Q_a, Q_b\}$ and

$$
\|\partial_y u - |\nabla| u\|_{L_2 y H^{N+p}} \lesssim \|Q_a\|_{L_2 y H^{N+p}} + \int_{-\infty}^{y} |\nabla| e^{-|s-y|\|\nabla| (Q_b(s) - Q_a(s)) ds \|_{L_2 y H^{N+p}}
\lesssim A\epsilon + \|(1 + |\xi|^2(N+p)/2 |\xi| \int_{-\infty}^{y} e^{-|s-y|\|\xi|} (Q_b(s) - Q_a(s)) ds \|_{L_2 y L_2^2}
\lesssim A\epsilon + \|(1 + |\xi|^2(N+p)/2 |\xi|^{-1} \|Q_b(y) - Q_a(y)\|_{L_2 y L_2^2}
\lesssim A\epsilon.
\quad \text{(3.21)}
$$
A similar calculation proves the same estimate in the $L^\infty_y H^{N+p-1/2}$ norm,

$$
\| \partial_y u - |\nabla u| \|_{L^\infty_y H^{N+p-1/2}} \lesssim \epsilon \quad (3.22)
$$

which is the desired result for $j = 0$. To prove the result for $j = 1$, we rearrange (3.3) to get

$$
\partial_y^2 u - |\nabla|^2 u = (1 + |\nabla_y h|^2)^{-1}(\Delta u |\nabla_x h|^2 + 2\partial_y \nabla_x u \cdot \nabla_x h + \partial_y u \Delta_x h + \partial_y \epsilon_a + |\nabla| \epsilon_b). \quad (3.23)
$$

which appears in the bound

$$
\| \partial_y (\partial_y u - |\nabla| u) \|_{L^p_y H^{N+s}} = \| (\partial_y^2 u - |\nabla|^2 u) - |\nabla| (\partial_y u - |\nabla| u) \|_{L^p_y H^{N+s}} \quad (3.24)
$$

$$
\lesssim \| \partial_y^2 u - |\nabla|^2 u \|_{L^p_y H^{N+s}} + \| \partial_y u - |\nabla| u \|_{L^p_y H^{N+s-1}}.
$$

The expression in (3.23) is easily estimated term by term for the desired values of $p$ and $s$ using the assumed bounds for $u$ and $h$, provided $N + p - 3/2 > d/2$. Similarly for $j = 2$, we write

$$
\partial_y^2 (\partial_y u - |\nabla| u) = \partial_y (\partial_y^2 u - |\nabla|^2 u) + \partial_y |\nabla| (|\nabla| u - \partial_y u) \quad (3.25)
$$

and the same argument applies.

\[\square\]

**Lemma 3.2.** Supposing $\| h \|_{H^N} + \| |\nabla|^{1/2} \psi \|_{H^N} \lesssim \epsilon$ and $\epsilon_a = \epsilon_b = 0$ with $u$ as in Lemma 3.1, we have

$$
\partial_y u = |\nabla| u + \nabla h \cdot \nabla u + N_2[h, u] + E^{(3)} \quad (3.26)
$$

for an error $E^{(3)}$ satisfying

$$
\| E^{(3)} \|_{L^2_y H^{N-1} \cap L^\infty_y H^{N-3/2}} \lesssim \epsilon \quad (3.27)
$$

where $N_2$ is defined as the multiplier

$$
\mathcal{F} N_2[h, \phi] (\xi) = \int_{(\mathbb{R}^d)^2} n_2(\xi, \eta) \hat{h}(\xi - \eta) \hat{\phi}(\eta) d\eta, \quad n_2(\xi, \eta) = \xi \cdot \eta - |\xi| |\eta| \quad (3.28)
$$

which satisfies the symbol bounds

$$
\| n_2^{k_1, k_2} \|_{S^\infty} \lesssim 2^{|\min\{k_1, k_2\}|} 2^{k_2}. \quad (3.29)
$$
Proof. Define $u^{(1)} = e^{\theta |\nabla|} \psi$, $Q_a^{(1)} = \nabla u^{(1)} \cdot \nabla h$, and $Q_b^{(1)} = \mathcal{R}(\partial_{\eta} u^{(1)} \nabla h)$. As in the proof of Lemma 3.1 in the case $p = -1$, the bounds of the type (3.18), followed by (3.20), directly imply

\[
\| |\nabla|^{1/2}(u - u^{(1)})\|_{L^\infty_{\mathcal{N}} H^{N-1}} + \| |\nabla|(u - u^{(1)})\|_{L^2_{\mathcal{N}} H^{N-1}} \\
+ \| \partial_{\eta}(u - u^{(1)})\|_{L^\infty_{\mathcal{N}} H^{N-3/2}} + \| \partial_{\eta}(u - u^{(1)})\|_{L^2_{\mathcal{N}} H^{N-1}} \\
\lesssim \| Q_a - Q_b \|_{L^2_{\mathcal{N}} H^{N-1}} \\
\lesssim \varepsilon^2
\]

(3.30)

since what we had called $A$ was $\| |\nabla|^{1/2}\psi\|_{H^{N-1}}$ which we are now taking to be on the order of $\varepsilon$. As a direct result of this and the assumed bound on $h$, since $Q_a - Q_a^{(1)} = \nabla(u - u^{(1)}) \cdot \nabla h - |\nabla h|^2 \partial_{\eta} u$ and $Q_b - Q_b^{(1)} = \mathcal{R}(\partial_{\eta}(u - u^{(1)}) \nabla h)$, we have

\[
\| Q - Q^{(1)} \|_{L^\infty_{\mathcal{N}} H^{N-3/2}} + \| Q - Q^{(1)} \|_{L^2_{\mathcal{N}} H^{N-1}} \lesssim \varepsilon^3.
\]

(3.31)

It follows from this, (3.19), and once again the integral estimates of the type (3.18) and (3.21) that

\[
\| \partial_{\eta} u - |\nabla| u - |\nabla h| \cdot \nabla u - \int_{-\infty}^{\gamma} |\nabla| e^{-|s-y|} |\nabla|(Q_b^{(1)}(s) - Q_a^{(1)}(s)) ds \|_{L^2_{\mathcal{N}} H^{N-1} \cap L^\infty_{\mathcal{N}} H^{N-3/2}} \\
= \| \int_{-\infty}^{\gamma} |\nabla| e^{-|s-y|} |\nabla|((Q_b(s) - Q_b^{(1)}(s)) - (Q_a(s) - Q_a^{(1)}(s))) ds - |\nabla h|^2 \partial_{\eta} u \|_{L^2_{\mathcal{N}} H^{N-1} \cap L^\infty_{\mathcal{N}} H^{N-3/2}} \\
\lesssim \| Q_a - Q_a^{(1)} \|_{L^2_{\mathcal{N}} H^{N-1} \cap L^\infty_{\mathcal{N}} H^{N-3/2}} + \| Q_b - Q_b^{(1)} \|_{L^2_{\mathcal{N}} H^{N-1} \cap L^\infty_{\mathcal{N}} H^{N-3/2}} + \varepsilon^3 \\
\lesssim \varepsilon^3.
\]

(3.32)

Thus we are left with showing that the integral operator above differs from $N_2[h, u]$ by an acceptable error. Indeed,

\[
\mathcal{F}\{Q_b^{(1)}(s) - Q_a^{(1)}(s)\}(\xi) = \mathcal{F}\{\mathcal{R}(\partial_{\eta} u^{(1)} \nabla h) - \nabla u^{(1)} \cdot \nabla h\}(\xi) \\
= \frac{i\xi}{|\xi|} \partial_{\eta} u^{(1)} * (i\xi \hat{h}) - (i\xi u^{(1)}) * (i\xi \hat{h}) \\
= \int_{\mathbb{R}^d} \left[ \partial_{\eta}(e^{i\eta|\eta|} \hat{\psi}(\eta)) \frac{i\xi}{|\xi|} - i\eta e^{i\eta|\eta|} \hat{\psi}(\eta) \right] \cdot i(\xi - \eta) \hat{h}(\xi - \eta) d\eta \\
= \int_{\mathbb{R}^d} \left[ \eta \cdot (\xi - \eta) - |\eta| \frac{\xi \cdot (\xi - \eta)}{|\xi|} \right] \hat{h}(\xi - \eta) e^{i\eta|\eta|} \hat{\psi}(\eta) d\eta
\]

(3.33)
which implies, using $\int_{-\infty}^{y} e^{\xi - y} e^{s|\eta|} ds = e^{y|\eta|}/(\xi + |\eta|)$,

$$
\mathcal{F}\{\int_{-\infty}^{y} |\nabla| e^{\xi - y} |\nabla| (Q_{b}^{(1)}(s) - Q_{a}^{(1)}(s)) ds\}(\xi)
= \int_{\mathbb{R}^d} \left[ \eta \cdot (\xi - \eta) - \frac{\xi \cdot (\xi - \eta)}{|\xi|} |\eta| \right] \frac{|\xi|}{|\xi| + |\eta|} \hat{h}(\xi - \eta) e^{y|\eta|} \hat{\psi}(\eta) d\eta
= \mathcal{F}N_2[h, u^{(1)}](\xi).$$

(3.34)

To conclude, we observe that this equals $N_2[h, u]$ up to a cubic error,

$$
\|N_2[h, u - u^{(1)}]\|_{L^2_y H^{N-1} \cap L^\infty_y H^{N-3/2}} \lesssim \epsilon^3,
= \mathcal{F}N_2[h, u^{(1)}](\xi).$$

(3.35)

which follows from (3.30).

Let us now introduce a new variable

$$
\omega = \phi - T_B h
$$

(3.36)

which will replace the fluid potential $\phi$ in our subsequent results. This is known as Alinhac’s good unknown and makes it possible to carry out the paralinearization in a way that does not lose derivatives.

**Lemma 3.3.** Define $a = a^{(1)} + a^{(0)}$ and $b = b^{(1)} + b^{(0)}$ where

$$
a^{(1)} = \frac{\zeta \cdot \nabla h}{\sqrt{1 + \alpha}}, \quad a^{(0)} = -\frac{1}{2\sqrt{1 + \alpha}} \{ \sqrt{1 + \alpha}, b^{(1)} \} \varphi \geq 0(\zeta),
$$

$$
b^{(1)} = \sqrt{|\zeta|^2 - (a^{(1)})^2}, \quad b^{(0)} = \frac{1}{2b^{(1)}} (-2a^{(1)} a^{(0)} - \{ a^{(1)}, b^{(1)} \} \varphi \geq 0(\zeta)).
$$

Then

$$
(T_{\sqrt{1+\alpha}} \partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}} \partial_y - iT_a - T_b)\omega = Q_0 + \tilde{C}
$$

(3.38)

where

$$
\tilde{C} \in \epsilon^3 L^\infty_y H^{N-1/2} \cap L^2_y H^N, \quad Q_0 \in \epsilon^2 L^\infty_y H^{N+1/2} \cap L^2_y H^{N+1},
$$

$$
\tilde{Q}_0(\xi, y) = \int q_0(\xi, \eta) \hat{h}(\xi - \eta) \hat{u}(\eta, y) d\eta,
$$

(3.39)
Combining this with the identities satisfies the symbol bound where the error is and have

Proof. Substituting into (3.2) the definition of $\omega$ and applying the paralinearization, we have

\[
T_{1+\alpha} \partial_{y}^{2} \omega + \Delta \omega - 2T_{\nabla h} \nabla \partial_{y} \omega - T_{\Delta h} \partial_{y} \omega = Q + C
\]

where the errors are

\[
\begin{align*}
Q &= -2\mathcal{H}(\nabla h, \nabla \partial_{y} u) - \mathcal{H}(\Delta h, \partial_{y} u) \\
C &= \partial_{y}(T_{1+\alpha} T_{\nabla h}^{2} h) + T_{\Delta u} - 2T_{\nabla h} T_{\nabla \partial_{y} u} - T_{\Delta h} T_{\partial_{y} u} h \\
&+ 2(T_{\nabla h} T_{\nabla h} - T_{\nabla h} T_{\nabla \partial_{y} u}) \nabla h + T_{\nabla h} H(\nabla h, \nabla h) + H(\alpha, \partial_{y}^{2} u).
\end{align*}
\]

Then one can directly compute

\[
(T_{\sqrt{1+\alpha}} \partial_{y} - iT_{1+\alpha} + T_{b}) (T_{\sqrt{1+\alpha}} \partial_{y} - iT_{1+\alpha} - T_{b})
\]

\[
= T_{1+\alpha} \partial_{y}^{2} - (2T_{a} \sqrt{1+\alpha} + T_{T_{a} \sqrt{1+\alpha}}) i \partial_{y} - T_{1+\alpha}^{2} - T_{b}^{2} - T_{1+\alpha} T_{b}^{0} + \mathcal{E}
\]

where the error is

\[
\mathcal{E} = (T_{\sqrt{1+\alpha}} T_{\sqrt{1+\alpha}} - T_{1+\alpha}) \partial_{y}^{2} - (T_{a} T_{\sqrt{1+\alpha}} + T_{T_{a} \sqrt{1+\alpha}}) i \partial_{y} - T_{1+\alpha} T_{b}^{0} \partial_{y}
\]

\[
- ([T_{\sqrt{1+\alpha}}, T_{b}^{(1)}] - iT_{(\sqrt{1+\alpha}, b^{(1)})}) \partial_{y} + (T_{a}^{2} - T_{a}^{2}) + (T_{b}^{2} - T_{b}^{2}) + i[T_{a}, T_{b}^{0} + T_{1+\alpha} T_{b}^{0} + T_{a} T_{b}^{0} + T_{(\sqrt{1+\alpha}, b^{(1)})} \varphi_{0}(\xi), (\xi, \eta)]
\]

Combining this with the identities

\[
2a \sqrt{1+\alpha} + \{\sqrt{1+\alpha}, b^{(1)}\} = 2\xi \cdot \nabla h + \{\sqrt{1+\alpha}, b^{(1)}\} \varphi_{-1}(\xi)
\]

\[
a^2 + b^2 + \{a^{(1)}, b^{(1)}\} \varphi_{0}(\xi) = |\xi|^2 + (a^{(0)})^2 + (b^{(0)})^2,
\]
and (3.42), we obtain

\[
\begin{align*}
(T\sqrt{1+\alpha}\partial_y \omega) 
= & (-\Delta \omega + 2T\nabla h \partial_y \omega + T\Delta h \partial_y \omega + Q + \mathcal{C}) - (T_2\nabla h i\partial_y \omega + T\{\sqrt{1+\alpha,b(1)}\} \varphi_{\leq -1}(\zeta) i\partial_y \omega) \\
& - (T_{|\zeta|^2} \omega + T(a^{(0)})^2 + (b^{(1)})^2 \omega) + \mathcal{E} \omega \\
= & Q + C + \mathcal{E} \omega - T(a^{(0)})^2 - (b^{(1)})^2 \omega - T\{\sqrt{1+\alpha,b(1)}\} \varphi_{\leq -1}(\zeta) i\partial_y \omega - (\Delta \omega + T_{|\zeta|^2} \omega) \\
& + (2T\nabla h \nabla + T\Delta h - i T_{2\zeta \nabla h}) \partial_y \omega.
\end{align*}
\]

(3.48)

Clearly, since \(T_a(\zeta)\) is the Fourier multiplier \(a(D)\), the term \(\Delta \omega + T_{|\zeta|^2} \omega\) vanishes. Furthermore,

\[
\mathcal{F}\{iT_2\nabla h f(\xi)\} = i \int_{\mathbb{R}^d} \chi(\frac{|\xi - \eta|}{|\xi + \eta|}) \frac{2\xi + \eta}{2} \cdot \nabla h(\xi - \eta) \tilde{f}(\eta) d\eta
\]

(3.49)

so, using the Leibniz rule,

\[
2T\nabla h \nabla + T\Delta h - iT_{2\zeta \nabla h} = 2T\nabla h \nabla + T\Delta h - \nabla \cdot T\nabla h - T\nabla h \nabla = 0.
\]

(3.50)

Thus

\[
(T\sqrt{1+\alpha}\partial_y - iT_a + T_b)(T\sqrt{1+\alpha}\partial_y - iT_a - T_b) \omega
\]

(3.51)

\[= Q + C + \mathcal{E} \omega - T(a^{(0)})^2 + (b^{(0)})^2 \omega - T\{\sqrt{1+\alpha,b(1)}\} \varphi_{\leq -1}(\zeta) i\partial_y \omega.
\]

This can clearly be absorbed into the third order error. Defining the symbol

\[
b_k^{(0)} = -\varphi_{\geq 0}(\zeta) \frac{\zeta_j \zeta_k}{2|\zeta|^2} \partial_j \partial_k h,
\]

(3.52)

one can compute that

\[
\mathcal{E} \omega - (T_{2|\zeta|b^{(0)}} - T_{|\zeta|T_{b^{(0)}}} - T_{b^{(0)}T_{|\zeta|}}) \omega - (iT_{\zeta \nabla h, T_{|\zeta|}} + T_{(\zeta \nabla h, |\zeta|) \varphi_{\geq 0}(\zeta)}) \omega
\]

(3.53)

also falls within the cubic error. For example, using the fact that symbols depending only on \(\zeta\) are Fourier multipliers and therefore composition is just multiplication of the symbols,
then up to cubic errors we have
\[
(T_2 - T_b^2) \omega = (T_{(\xi+b_1(0)+\epsilon^2M_0)^2} - T_{(\xi+b_1(0)+\epsilon^2M_0)^2}) \omega
\]
(3.54)
which cancels with the first term subtracted off above. Analogous arguments using the
expansions in Lemma 4.2 prove that the last two terms of \(E\) cancel at the top order with
\[
(T_{2(\xi| \zeta|b_1(0)\epsilon^2M_0r)} - T_{b_1(0)\epsilon^2T(\xi| \zeta|b_1(0)\epsilon^2M_0r)}) \omega
\]
and that the rest of \(E\) is already an acceptable cubic error.

**Lemma 3.4.** Let
\[
U = (T_{\sqrt{1+\alpha}} - iT_a - T_b) \omega
\]
(3.55)
and
\[
\hat{M}_0[f,g](\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^d} m_0(\xi,\eta) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta, \quad m_0(\xi,\eta) = \frac{q_0(\xi,\eta)}{|\xi| + |\eta|}. \tag{3.56}
\]
Then
\[
P \geq -10 (U_0 - \hat{M}_0[h, \phi]) \in \epsilon^3 H^{N+1/2}
\]
(3.57)
where \(U_0\) is the restriction of \(U\) to the interface.

**Proof.** Take
\[
\hat{U} = T_{(1+\alpha)^{1/4}} U, \quad \sigma = \frac{b - ia}{\sqrt{1+\alpha}}, \quad f = (1 + \alpha)^{1/4} - 1.
\]
(3.58)
Then a direct computation gives
\[
T_{(1+\alpha)^{1/4}} (\partial_y + T_{\sigma}) \hat{U} = T_{(f+1)^2}(\partial_y + T_{\sigma}) T_{f+1} U
\]
(3.59)
\[
= (T_{(f+1)^2} - T_{(f+1)^2}) U + \partial_y (T_{(f+1)^2} U)
\]
\[
+ [T_{f+1}, T_{\sigma}] T_{f+1} U + \sigma T_{(f+1)^2} U.
\]
All but the first term can be easily estimated with Lemma 2.9 to give, after applying the
definitions of \(f\) and \(\sigma\) and the result of Lemma 3.3,
\[
T_{(1+\alpha)^{1/4}} (\partial_y + T_{\sigma}) \hat{U} = (\partial_y T_{(f+1)^2} + T_{(f+1)^2} \sigma U) + \epsilon^3 (L_\infty^\infty H^{N-1/2} \cap L_\infty^2 H^N)
\]
(3.60)
\[
= (T_{\sqrt{1+\alpha}} \partial_y + T_b - ia) (T_{\sqrt{1+\alpha}} \partial_y - iT_a - T_b) \omega
\]
\[
= Q_0 + \epsilon^3 (L_\infty^\infty H^{N-1/2} \cap L_\infty^2 H^N).}
Then since Lemma 3.3 gives that $Q_0$ is a quadratic error, we have $T_{(1+\alpha)^{-1/2}-1}Q_0$ is a cubic error, and thus, with Lemma 2.8,

\[ (\partial_y + T_\sigma)\tilde{U} = Q_0 + T_{(1+\alpha)^{-1/4}}Q_0 + \epsilon^3 T_{(1+\alpha)^{-1/4}}(L_y^\infty H^{N-1/2} \cap L_y^2 H^N) \]

\[ = Q_0 + \epsilon^3 (L_y^\infty H^{N-1/2} \cap L_y^2 H^N) \]

(3.61)

and we define $C_2$ to be this cubic error. Next consider the quantities $M_0[h,u]$ and $M_0[h,\partial_y u]$. We can easily find that these are in $\epsilon^2 L_y^\infty H^{N+3/2} \cap L_y^2 H^{N+2}$ and $\epsilon^2 L_y^\infty H^{N+1/2} \cap L_y^2 H^{N+1}$.

For example, applying Lemma 2.2 and the fact that $S^\infty$ is an algebra with the bound (3.41),

\[ \|P_kM_0|P_k| h, P_k u\|_{L_y^\infty H^{N+3/2}} \lesssim \left\| \left( \frac{q_0(\xi, \eta)}{\left| \xi \right| + \left| \eta \right|} \right)^{k,k_1,k_2} \|S\| \|P_k h\|_{L_y^\infty H^{N+3/2}} \|P_k u\|_{L_y^\infty H^{N+3/2}} \]

\[ \lesssim (2^{-k} + 2^{-k_2}) 2^{4k_1-k_2} 1_{[-40,\infty)}(k_2 - k_1) + 2^{3k_1+k_2} 1_{(-\infty,4]}(k_2)) \]

\[ \times 2^{3/2}k_1 \|P_k h\|_{L_y^\infty H^{N}} 2^{2k_2} \|P_k |\nabla| u\|_{L_y^\infty H^{N-1}}. \]

(3.62)

Thanks to the gain in the symbol $q_0/(\left| \xi \right| + \left| \eta \right|)$ and the restriction of the relevant frequencies to the region $|2^{k_1} - 2^{k_2}| \leq 4 \cdot 2^k$, this is summable in $\ell_k^2 \ell_{k_1} \ell_{k_2}$ as required. As a result of this with Lemma 2.8, we have

\[ \tilde{U} - M_0[h,u] = T_{(1+\alpha)^{1/4}}(U - M_0[h,u]) + T_{(1+\alpha)^{1/4}-1}M_0[h,u] \]

(3.63)

Thus by frequency localization (Lemma 2.3) and Lemma 2.8 with the symbol $T_{(1+\alpha)^{-1/4}}$, to prove $P_{\geq -10}(U - M_0[h,u]) \in \epsilon^3 H^{1/2}$ at the boundary, it suffices to prove for all $y \leq 0$ that

\[ P_{\geq -20}(\tilde{U} - M_0[h,u]) \in \epsilon^3 H^{N+1/2}. \]

(3.64)

We compute from the definitions

\[ \mathcal{F}(Q - (\partial_y + |\nabla|)M_0[h,u] - M_0[h,|\nabla| u - \partial_y u])(\xi) \]

\[ = \int_{\mathbb{R}^d} \left( 1 - \frac{|\xi|}{|\xi| + |\eta|} - \frac{|\eta|}{|\xi| + |\eta|} \right) q_0(\xi, \eta) \tilde{h}(\xi - \eta) \tilde{\eta}(\eta, y) d\eta = 0 \]

(3.65)

which implies

\[ (\partial_y + T_\sigma)(\tilde{U} - M_0[h,u]) = (\partial_y + T_\sigma)\tilde{U} - (\partial_y + |\nabla|)M_0[h,u] - T_{\sigma - |\xi|}M_0[h,u] \]

\[ = C_2 + M_0[h,|\nabla| u - \partial_y u] - T_{\sigma - |\xi|}M_0[h,u] \]

(3.66)
Using part (ii) of Lemma 3.1 and the fact that $\sigma$ is $|\zeta|$ plus a first order error, we find that this quantity is a cubic error $C_3 \in e^3(L_y^\infty \cap L_y^2H^N)$. Projecting this onto the $2^k$ frequency and rearranging,

$$(\partial_y + |\nabla|)P_k V = P_k C_3 - P_k T_{\sigma - |\zeta|} V$$  \hspace{1cm} (3.67)

where we have set $V = \tilde{U} - M_0[h,u]$. Multiplying by $e^{y|\nabla|}$ and integrating, we obtain the integral equation

$$P_k V(y) = \int_{-\infty}^{y} e^{(s-y)|\nabla|} (P_k C_3(s) - P_k T_{\sigma - |\zeta|} V(s)) ds.$$  \hspace{1cm} (3.68)

Define the frequency localized parts of the desired Sobolev norm,

$$X_k = \sup_{y \leq 0} 2^{k(N+1/2)} \|P_k V(y)\|_{L^2}.$$  \hspace{1cm} (3.69)

To see how this acts on the second term in the integral equation,

$$2^{k(N+1/2)} \int_{-\infty}^{y} \|e^{(s-y)|\nabla|} P_k T_{\sigma'} V(s)\|_{L^2} ds = 2^{k(N+1/2)} \int_{-\infty}^{y} \|e^{(s-y)|\xi|} \mathcal{F}(P_k T_{\zeta} V(s))\|_{L^2} ds$$

$$\sim 2^{k(N+1/2)} \int_{-\infty}^{y} e^{(s-y)2^{k-4}} 2^k \|P_k T_{\zeta} V(s)\|_{L^2} ds$$

$$\lesssim 2^{k(N+1/2)} \int_{-\infty}^{y} e^{(s-y)2^{k-4}} 2^k ds \sup_{y \leq 0} \sum_{|k'-k| \leq 4} \|P_{k'} V(y)\|_{L^2}$$

$$\lesssim e^{2^{k(N+1/2)}} \sum_{|k'-k| \leq 4} \sup_{y \leq 0} \|P_{k'} V(y)\|_{L^2}$$

$$= \epsilon \sum_{|k'-k| \leq 4} X_{k'}.$$  \hspace{1cm} (3.70)

Thus, using Cauchy-Schwarz,

$$X_k \lesssim \sup_{y \leq 0} 2^{k(N+1/2)} \int_{-\infty}^{y} e^{(s-y)2^{k-4}} \|P_k C_3(s)\|_{L^2} ds + \epsilon \sum_{|k'-k| \leq 4} X_{k'}$$

$$\lesssim 2^{kN} \left( \int_{-\infty}^{0} \|P_k C_3(s)\|_{L^2} ds \right)^{1/2} + \epsilon \sum_{|k'-k| \leq 4} X_{k'}$$  \hspace{1cm} (3.71)

which we sum in $\ell^2(\mathbb{Z})$ to find

$$\|X_k\|_{\ell^2(\mathbb{Z})} \lesssim \epsilon \|X_k\|_{\ell^2(\mathbb{Z})} + \sum_{k \in \mathbb{Z}} 2^{2kN} \int_{-\infty}^{0} \|P_k C_3(s)\|_{L^2}^2 ds \right)^{1/2}.$$  \hspace{1cm} (3.72)
Because $C_3$ is known to be in $\ell^2 L^2_y H^N$, we conclude that
\[ \|X_k\|_{\ell^2(\mathbb{Z})} \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{2kN} \|P_k C_3\|_{L^2_y L^2_x}^2 \right)^{1/2} \lesssim \|C_3\|_{L^2_y H^N} \lesssim \varepsilon^3. \] (3.73)

This proves (3.64) as claimed.

\[ \square \]

### 3.2 The main theorem for the expansion

**Theorem 3.1.** Assume $N > 5d/2 + 3/2$ and
\[ \|h\|_{H^N} + \|\nabla|^{1/2}\phi\|_{H^N} \lesssim \varepsilon. \] (3.74)

Define
\[ B = \frac{G(h)\phi + \nabla_x h \cdot \nabla_x \phi}{1 + |\nabla h|^2}, \quad V = \nabla_x \phi - B\nabla_x h, \quad \omega = \phi - TBh. \] (3.75)

Then
\[ G(h)\phi = T_{\lambda_{DN}}\omega - \text{div}(TVh) + G_2 + \varepsilon^3 H^{N+1/2} \] (3.76)

where
\[ \lambda_{DN} = \lambda^{(1)} + \lambda^{(0)}, \]
\[ \lambda^{(1)}(x, \zeta) = \sqrt{(1 + |\nabla h|^2)|\zeta|^2 - (\zeta \cdot \nabla h)^2}, \] (3.77)
\[ \lambda^{(0)}(x, \zeta) = \left( \frac{(1 + |\nabla h|^2)^2}{2\lambda^{(1)}} \left\{ \frac{\lambda^{(1)}}{1 + |\nabla h|^2} \frac{\zeta \cdot \nabla h}{1 + |\nabla h|^2} + \frac{1}{2} \Delta h \right\} \right) \varphi_{\geq 0}(\zeta) \]

and
\[ G_2 = G_2(h, |\nabla|^{1/2}\omega) \in \mathcal{C}^2 H^N, \quad \hat{G}_2(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^d} g_2(\xi, \eta)\hat{h}(\xi - \eta)|\eta|^{1/2}\hat{\omega}(\eta)d\eta \] (3.78)

where
\[ \|g_2^{k,k_1,k_2}\|_{S^{\infty}} \lesssim 2^{k_2 \min(k_1,k_2)/2} \left( \frac{1 + 2^{\min(k_1,k_2)}}{1 + 2^{\max(k_1,k_2)}} \right)^{7/2}. \] (3.79)
Note that when \( d = 1 \), clearly \( \lambda^{(1)} = |\zeta| \) and as a result \( \lambda^{(0)} \) takes the simpler form

\[
\lambda^{(0)}(x, \zeta) = \left( \frac{(1 + |\partial_x h|^2)^2}{2|\zeta|} \left( \frac{|\zeta|}{1 + |\partial_x h|^2}, \frac{\zeta \partial_x h}{1 + |\partial_x h|^2} \right) + \frac{1}{2} \partial^2_{xx} h \right) \varphi \geq 0(\zeta) \tag{3.80}
\]

Thus

\[
\lambda_{DN} = |\zeta| - \frac{1}{2} \partial^2_{xx} h \varphi \geq 0(\zeta). \tag{3.81}
\]

**Proof.** Fix \( y = 0 \) so we’re at the interface. Then

\[
G(h)\phi = (1 + \alpha)\partial_y u - \nabla h \cdot \nabla u
\]

\[
= T_{1+\alpha}\partial_y u + T_{\partial_y u} + H(\alpha, \partial_y u) - T_{\nabla h} \nabla u - T_{\nabla u} \nabla h - H(\nabla h, \nabla u) \tag{3.82}
\]

(since \( T_1 \) is obviously the identity). To talk about derivatives of \( \omega \) in the vertical direction, we can extend it naturally into the fluid domain as \( \omega = u - T_{\partial_y u} h \). As a result \( T_{1+\alpha}\partial_y u = T_{1+\alpha}\partial_y \omega + T_{1+\alpha} T_{\partial_y u} h \) and \( T_{\nabla h} \nabla u = T_{\nabla h} \nabla \omega + T_{\nabla h} T_{\partial_y u} h + T_{\nabla h} T_{\partial_y u} \nabla h \). Thus we can rewrite \( G(h)\phi \) in terms of the good unknown:

\[
G(h)\phi = T_{1+\alpha}\partial_y \omega - T_{\nabla h} \nabla \omega - T_{\nabla u} \nabla h + T_{\nabla h} T_{\partial_y u} \nabla h + (T_{\partial_y u} - 2T_{\nabla h} T_{\partial_y u} \nabla h)
\]

\[
+ T_{1+\alpha} T_{\partial_y u} h - T_{\nabla h} T_{\partial_y u} h + H(\alpha, \partial_y u) - H(\nabla h, \nabla u). \tag{3.83}
\]

Rearranging the first term,

\[
T_{1+\alpha}\partial_y \omega = T_{\sqrt{1+\alpha}}(T_b + iT_a)\omega + T_{\sqrt{1+\alpha}} M_0[h, \phi] + T_{\sqrt{1+\alpha}} (U_0 - M_0[h, \phi]) + (T_{1+\alpha} - T_{\sqrt{1+\alpha}}^2) \partial_y \omega. \tag{3.84}
\]

Note that we can write \( T_{\sqrt{1+\alpha}} = T_{\sqrt{1+\alpha} - 1} + T_1 \) so immediately, Lemma 3.4 gives that the high frequencies of \( T_1(U_0 - M_0[h, \phi]) \) are in \( c^3 H^{N+3/2} \). To make use of Lemma 2.8, since the derivatives of \( x - (\sqrt{1 + x^2} - 1) \) are each bounded for positive \( x \), we can bound the symbol \( ||\sqrt{1 + |\nabla h|^2} - 1||_{H^{3/2+d/2}} \lesssim ||\nabla h||_{H^{3/2+d/2}} \lesssim ||h||_{H^N} \) so by Lemma 2.3 followed by Lemma 3.4,

\[
P_{2-6} T_{\sqrt{1+\alpha} - 1}(U_0 - M_0[h, \phi]) = P_{2-6} T_{\sqrt{1+\alpha} - 1} P_{2-15}(U_0 - M_0[h, \phi])
\]

\[
= P_{2-6}(c^3 H^{N+1/2}) \in c^3 H^{N+1/2}. \tag{3.85}
\]
Finally from Lemma 2.9, we have

$$P_{2-6}(T_{1+α} - T^2_{\sqrt{1+α}})∂_y ω = P_{2-6}(T_{(\sqrt{1+α})^2} - T^2_{\sqrt{1+α}-1})∂_y ω \in ε^3 H^{N+1/2}. \tag{3.86}$$

Thus, letting $C$ be an error satisfying $P_{2-6}C \in ε^3 H^{N+1/2}$, we have

$$T_{1+α}∂_y ω = T_{\sqrt{1+α}}(T_b + iT_a)ω + M_0[h, φ] + C \tag{3.87}$$

and, with a rearrangement can express the Dirichlet-to-Neumann map as

$$G(h)φ = T_{\sqrt{1+α}}(T_b + iT_a)ω + M_0[h, φ] + C - T\nabla h\nabla ω - \text{div}(T_V h) + C_1 + C_2 - \mathcal{H}(\nabla h, \nabla u) \tag{3.88}$$

where $V$ is naturally extended into the fluid region as $V = ∇_x u - ∂_y u ∇_x h$ and the cubic terms are given by

$$C_1 = (T_{∂_y u}(∇ h \cdot ∇ h) - 2T\nabla h T_{∂_y u}∇ h) + \mathcal{H}(α, ∂_y u), \tag{3.89}$$

$$C_2 = (T_{\text{div} V} + T_{1+α} T_{∂_y^2 u} - T\nabla h T_{∂_y u})h + (T\nabla h T_{∂_y u} - T_{∂_y u}∇ h)∇ h.$$

We can easily rearrange $C_1$ to get

$$C_1 = T_{∂_y u}\mathcal{H}(∇ x, ∇ x h) + 2[T_{∂_y u}, T\nabla x h]\nabla x h + \mathcal{H}(α, ∂_y u) \tag{3.90}$$

From the definition $\text{div} V = ∆ x u - ∇_x ∂_y u \cdot ∇ x h - ∂_y u ∆ x h$ which, combined with (3.2), gives

$$\text{div} V = ∇_x h \cdot ∇_x ∂_y u - (1 + α)∂_y^2 u. \tag{3.91}$$

Thus we can rewrite $C_2$ as

$$C_2 = (T\nabla x h \cdot ∇_x ∂_y u - T\nabla h T_{∂_y u})h + (T_{1+α} T_{∂_y^2 u} - T_{(1+α)∂_y^2 u})h + (T\nabla h T_{∂_y u} - T_{∂_y u}∇ h)∇ h \tag{3.92}$$

so each of the terms can be estimated with Lemma 2.9. Simply applying definitions and noting that $b^{(1)}\sqrt{1+α} = λ^{(1)}$,

$$T_{\sqrt{1+α}} T_b ω = (T_b^{\sqrt{1+α}} + i T_{\sqrt{1+α}b} + E(\sqrt{1+α} - 1, b))ω = (T_{λ^{(1)}} + T_b^{(0)}\sqrt{1+α} + i T_{\sqrt{1+α}b^{(1)}})ω + i T_{\sqrt{1+α}b^{(0)}}ω + E(\sqrt{1+α} - 1, b)ω \tag{3.93}$$
The final two terms fall in $c^3 \mathcal{H}^{N+1/2}$. Similarly,

$$i T_{\sqrt{1+\alpha}} T_a \omega - T_{\nabla h} \nabla \omega = (iT_a \sqrt{1+\alpha} - \frac{1}{2} T_{\sqrt{1+\alpha},a}) + i E(\sqrt{1+\alpha} - 1, a) \omega - T_{\nabla h} \nabla \omega. \quad (3.94)$$

Observe that

$$\mathcal{F} T \zeta \cdot \nabla \omega(\xi) = \int_{\mathbb{R}^d} \chi(\frac{1}{2} \sqrt{1+\alpha}) + \frac{1}{2} T_{\nabla h} \omega(\eta) d\eta = \frac{1}{i} \mathcal{F} (T_{\Delta h} \omega + T_{\nabla h} \nabla \omega)(\xi). \quad (3.95)$$

Thus, using $a = a^{(1)} + a^{(0)}$ and $\zeta \cdot \nabla h = a^{(1)} \sqrt{1+\alpha}$,

$$iT_{\sqrt{1+\alpha}} T_a \omega - T_{\nabla h} \nabla \omega = (iT_a^{(0)} \sqrt{1+\alpha} + \frac{1}{2} T_{\Delta h} - \frac{1}{2} T_{\sqrt{1+\alpha},a^{(1)}}) \omega + c^3 \mathcal{H}^{N+1/2} \quad (3.96)$$

where $-\frac{1}{2} T_{\sqrt{1+\alpha},a^{(0)}} + E(\sqrt{1+\alpha} - 1, b) \in c^3 \mathcal{H}^{N+1/2}$ has been absorbed into the error.

Adding these,

$$T_{\sqrt{1+\alpha}} (iT_a + T_b) \omega - T_{\nabla h} \nabla \omega = T_{\lambda(1)} \omega + T_m \omega + T_{\frac{1}{2} \sqrt{1+\alpha},a^{(1)}} + ia^{(0)} \sqrt{1+\alpha} \omega + c^3 \mathcal{H}^{N+1/2} \quad (3.97)$$

where we’ve defined

$$m = b^{(0)} \sqrt{1+\alpha} - \frac{1}{2} \{\sqrt{1+\alpha}, a^{(1)}\} + \frac{1}{2} \Delta h. \quad (3.98)$$

Then since

$$\frac{i}{2} \{\sqrt{1+\alpha}, b^{(1)}\} + ia^{(0)} \sqrt{1+\alpha} = \frac{i}{2} \{\sqrt{1+\alpha}, b^{(1)}\} (1 - \varphi \geq 0(\zeta)), \quad (3.99)$$

the final term is in $c^3 \mathcal{H}^{N+1/2}$ to yield

$$T_{\sqrt{1+\alpha}} (iT_a + T_b) \omega - T_{\nabla h} \nabla \omega = T_{\lambda(1)} \omega + T_m \omega + c^3 \mathcal{H}^{N+1/2}. \quad (3.100)$$

As a result,

$$G(h) \phi = T_{\lambda(1)} \omega + T_m \omega + M_0 [h, \phi] - \mathcal{H}(\nabla h, \nabla u) - \text{div}(T_{\nabla h}) + C + c^3 \mathcal{H}^{N+1/2}. \quad (3.101)$$
Then we can compute from the definitions

\[ m = \frac{(1 + \alpha)^{3/2}}{2\lambda^{(1)}} \left\{ \sqrt{1 + \alpha} \left( \frac{\lambda^{(1)}}{1 + \alpha} \cdot \nabla h + \nabla \frac{\lambda^{(1)}}{1 + \alpha} \cdot \nabla h \right) \right\} \varphi \geq 0(\zeta) - \frac{1}{2} \left\{ \sqrt{1 + \alpha} \cdot \nabla h \right\} + \frac{1}{2} \Delta h \]

\[ = \frac{(1 + \alpha)^{3/2}}{2\lambda^{(1)}} \left[ \left\{ \frac{\lambda^{(1)}}{1 + \alpha} \cdot \nabla h \right\} - \frac{\lambda^{(1)}}{1 + \alpha} \cdot \nabla h \right] \varphi \geq 0(\zeta) - \frac{1}{2} \left\{ \sqrt{1 + \alpha} \cdot \nabla h \right\} + \frac{1}{2} \Delta h \]

\[ = \frac{1}{2} (1 + \alpha)^{1/2} \left\{ \sqrt{1 + \alpha} \cdot \nabla h \right\} \varphi \geq 0(\zeta) + \left( \frac{1 + \alpha}{2\lambda^{(1)}} \right) \left\{ \frac{\lambda^{(1)}}{1 + \alpha} \cdot \nabla h \right\} + \frac{1}{2} \Delta h \]

\[ = \lambda^{(0)} - \frac{1}{2} \left\{ \sqrt{1 + \alpha} \cdot \nabla h \right\} \varphi \leq 1(\zeta) + \frac{1}{2} \Delta h \varphi \leq 1(\zeta). \]

(3.102)

Thus

\[ P \geq 7G(h)\phi = P \geq 7(T_{DN\omega} - \text{div}(T_{Vh}) + M_0[h, u] - \mathcal{H}(\nabla h, \nabla u)) + \epsilon^3 H^{N+1/2} \] (3.103)

where \( M_0[h, u] - \mathcal{H}(\nabla h, \nabla u) \) has symbol

\[ |\eta|^{1/2} g_2(x, \eta) = \frac{g_0(\xi, \eta)}{|\xi| + |\eta|} + \left[ 1 - \chi\left( \frac{|\xi - \eta|}{|\xi + \eta|} \right) - \chi\left( \frac{|\eta|}{2|\xi - \eta|} \right) \right] (\xi - \eta) \cdot \eta \]

\[ = \chi\left( \frac{|\xi - \eta|}{|\xi + \eta|} \right) (|\xi| - |\eta|)^2 \left[ \frac{\xi \cdot \eta - |\xi||\eta|}{|\xi + \eta|^2} \varphi \geq 0(\frac{\xi + \eta}{2}) + \frac{1}{2} \varphi \leq 1(\frac{\xi + \eta}{2}) \right] \]

\[ + \left[ 1 - \chi\left( \frac{|\xi - \eta|}{|\xi + \eta|} \right) - \chi\left( \frac{|\eta|}{2|\xi - \eta|} \right) \right] (\xi \cdot \eta - |\xi||\eta|) \]

(3.104)

and we can conclude that \( g_2 \) admits the desired bound (3.79), completing the proof of the theorem for large frequencies. At last we can consider the low frequencies:

\[ P \leq 6G(h)\phi = P \leq 6(\partial_y u - \nabla h \cdot \nabla u) + P \leq 6(|\nabla h|^2 \partial_y u) \] (3.105)

but observe that the second term is third order and falls in the error \( H^{N+1/2} \). Then using
the fact that \( \partial_y u = |\nabla|u + \nabla h \cdot \nabla u + N_2[h, u] + \mathcal{E}^{(3)} \),

\[
P \leq 6 G(h) \phi = P \leq 6(|\nabla|\omega + |\nabla|T_{\partial_y h} + N_2[h, u] + \mathcal{E}^{(3)})
\]

\[
= P \leq 6(|\nabla|\omega - \text{div}(T_V h)) + P \leq 6(\text{div}(T_V h) + |\nabla|T|\omega h + N_2[h, u]) + P \leq 6|\nabla|T_{\partial_y h} - |\nabla|\omega h
\]

\[
= P \leq 6(T_{\lambda_D N}\omega - \text{div}(T_V h)) + P \leq 6 G_2 + P \leq 6 |\nabla|T_{\nabla h \nabla u + N_2[h, u] + \mathcal{E}^{(3)} + |\nabla|T_{\partial_y h}^h h}
\]

(3.106)

where we have taken

\[
P \leq 6 G_2 = P \leq 6(\text{div}(T_V h) + |\nabla|T|\omega h + N_2[h, u])
\]

(3.107)

Since \( |\nabla| = T_{\lambda_D N} \) up to a third order error and the third term is entirely in the error, we are only left with showing the second order term \( P \leq 6 G_2 \) is given by a symbol with the claimed bounds.  \( \square \)
Chapter 4

Symmetrization of the equations

The objective of this chapter is to prove the following theorem to reduce the water waves system to an equation in a single complex unknown:

**Theorem 4.1.** Assume as before that a solution \((h, \phi)\) of the water waves system has \(\|h\|_{HN} + \|\nabla|^{1/2}\phi\|_{HN} \lesssim \epsilon\) and \(\lambda_{DN}\) is as in Theorem 3.1. Then letting

\[
\ell(x) = -g|\nabla|h, \quad \Sigma = \sqrt{\lambda_{DN}}(g + \ell),
\]

the variable defined by

\[
U = T_{\sqrt{g+\ell}}h + iT_{\Sigma}T_{1/\sqrt{g+\ell}}\omega
\]

is given by

\[
U = \sqrt{gh} + i|\nabla|^{1/2}\omega + \epsilon^2 H^N
\]

and satisfies the equation

\[
(\partial_t + iT_\Sigma + iT_{V-\zeta})U = Q + \epsilon^3 H^N
\]

where the quadratic term \(Q\) is a sum of bilinear operators in the forms \(A[U, U]\), \(A[U, \bar{U}]\), and \(A[\bar{U}, \bar{U}]\) where \(A\) is an operator as in (2.9) which does not lose derivatives.

To prove the theorem we must combine the paralinearization of the Dirichlet-to-Neumann map obtained in the last chapter with a paralinearization of the rest of the system.
Lemma 4.1. The water wave system takes the form

\[
\begin{aligned}
\partial_t h &= T_{\lambda D N} \omega - \text{div}(T_V h) + G_2 + \epsilon^3 H^{N+1/2} \\
\partial_t \omega &= -gh - T_I h - T_V \nabla \omega + \Omega_2 + \epsilon^3 H^N
\end{aligned}
\]  

(4.5)

where \( G_2 \) is as in Theorem 3.1 and

\[
\Omega_2 = \frac{1}{2} \mathcal{H}(|\nabla|, |\nabla|) - \frac{1}{2} \mathcal{H}(\nabla \omega, \nabla \omega) \in \epsilon^2 H^{N+2}.
\]

Proof. The equation for \( \partial_t h \) comes immediately from Theorem 3.1. For the second equation, observe from the definitions that \( V + B \nabla h = \nabla \phi \) and as a result

\[
-\frac{1}{2} |\nabla \phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla h|^2)} = -\frac{(V + B \nabla h)^2}{2} + \frac{(1 + |\nabla h|^2)B^2}{2}
\]

\[
= \frac{B^2 - 2BV \cdot \nabla h - |V|^2}{2}.
\]

Then using the definition \( \omega = \phi - T_B h \) and the first equation \( \partial_t h = G(h)\phi = B - V \cdot \nabla h \),

\[
\partial_t \omega = \partial_t \phi - T_{\partial h} h - T_B \partial_t h
\]

\[
= -gh - \frac{1}{2} |\nabla \phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla h|^2)} - T_{\partial h} h - T_B (B - V \cdot \nabla h)
\]

\[
= -gh + \frac{B^2 - 2BV \cdot \nabla h - |V|^2}{2} - T_{\partial h} h - T_B B + T_B (V \cdot \nabla h).
\]

Applying the definition of \( \omega \) then of \( V \),

\[
T_V \nabla \omega = T_V \nabla \omega - T_V \nabla T_B h = T_V V + T_V (B \nabla h) - T_V \nabla T_B h
\]

(4.9)

so we may add \( T_V V + T_V (B \nabla h) - T_V \nabla T_B h \) to (4.8):

\[
\partial_t \omega = -gh - T_V \nabla \omega + \frac{B^2 - 2BV \cdot \nabla h - |V|^2}{2} - T_{\partial h} h - T_B B + T_B (V \cdot \nabla h)
\]

\[
+ T_V V + T_V (B \nabla h) - T_V \nabla T_B h.
\]

(4.10)

Applying the definition of \( \mathcal{H} \) to the products \( V \cdot V \), \( B^2 \), and \( BV \cdot h \), this rearranges to

\[
\partial_t \omega = -gh - T_{\partial h} h - T_V \nabla \omega + I + II
\]

(4.11)

where

\[
I = \frac{1}{2} B^2 - T_B B - \frac{1}{2} |V|^2 + T_V V = \frac{1}{2} \mathcal{H}(B, B) - \frac{1}{2} \mathcal{H}(V, V) = \Omega_2 + \epsilon^3 H^{N+2},
\]

\[
II = -BV \cdot \nabla h + T_B (V \cdot \nabla h) + T_V (B \nabla h) - T_V \nabla T_B h.
\]

(4.12)
Then
\[ II = -T_V \nabla B + \mathcal{H}(B, V \cdot \nabla h) + T_V (B \nabla h) - T_V T_B \nabla h - T_V T_{\nabla B} h \]
\[ = \mathcal{H}(B, V \cdot \nabla h) + (T_V T_{\nabla B} B - T_V \nabla h B) + T_V \mathcal{H}(B, \nabla h) - T_V T_{\nabla B} h. \]  
(4.13)

By Lemmas 2.6 and 2.9 this error is in $\epsilon^3 H^N$. To rewrite the $T_\partial B h$ term, we compute
\[ \partial_t B = \partial_t |\nabla| \omega + \epsilon^2 H^{N-1/2} = \partial_t |\nabla|(\phi - T_B h) + \epsilon^2 H^{N-1/2} \]
\[ = \partial_t |\nabla| \phi + \epsilon^2 H^{N-1/2} = \partial_t G(h) \phi + \epsilon^2 H^{N-2} \]  
and by the second part of Lemma 3.2, \( \partial_t G(h) \phi = |\nabla| \partial_t \phi + \epsilon^2 H^{N-2} \). Using this and also the system (1.17),
\[ \partial_t B = |\nabla| \partial_t \phi + \epsilon^2 H^{N-2} \]
\[ = |\nabla| \left( -gh - \frac{1}{2} \nabla \phi^2 + \frac{(G(h) \phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla|^2)} \right) \]
\[ = \ell + \epsilon^2 H^{N-2}. \]  
(4.15)

To properly estimate the error terms in the next step we will need some Taylor-like expansions of the symbols.

**Lemma 4.2.** We have
\[ \lambda^{(1)} = |\zeta|(1 + \epsilon^2 M^0_r), \]
\[ \lambda_{DN} = |\zeta| + \lambda^{(0)}_1 + \epsilon^2 M^1_r, \]  
(4.16)

and
\[ (g + \ell)^p = g^p(1 - p|\nabla|h + \epsilon^2 M^0_r), \]
\[ \lambda^p_{DN} = |\zeta|^p(1 + p\lambda^{(0)}_1(x, \zeta)/|\zeta| + \epsilon^2 M^0_p), \]
\[ \Sigma = (g|\zeta|)^{1/2} + \Sigma_1 + \epsilon^2 M^0_r \]
where \( r = N - d/2 - 1 \) and
\[ \lambda^{(0)}_1(x, \zeta) = \frac{1}{2} \left( \Delta h - \frac{\zeta_i \zeta_k}{|\zeta|^2} \partial_j \partial_k h \right) \phi_{\geq 0}(\zeta), \]
\[ \Sigma_1(x, \zeta) = \frac{g^{1/2}}{4|\zeta|^{1/2}} \left( \Delta h - \frac{\zeta_i \zeta_j}{|\zeta|^2} \partial_i \partial_j h \right) \phi_{\geq 0}(\zeta) - \frac{(g|\zeta|)^{1/2}}{2}|\nabla|h = \epsilon M^1_r. \]  
(4.18)
Similarly for the time derivatives, we have

\[ \partial_t \sqrt{g + \ell} = \frac{1}{2} g^{1/2} \Delta \omega + \epsilon^2 \mathcal{M}^0_t, \]
\[ \partial_t \sqrt{\lambda_{DN}} \frac{1}{2 \sqrt{|\zeta|}} \partial_t \lambda_1^{(0)} + \epsilon^2 \mathcal{M}^{1/2}_t. \]  

(4.19)

**Proof.** For the first equality in (4.16), observe that we can write

\[ \lambda_1^{(0)} = |\zeta|(1 + \sqrt{1 + |\nabla h|^2 \sin^2 \angle(\nabla h, \zeta) - 1}) \]

(4.20)

where \( \angle(\xi, \eta) \) represents the angle between vectors in \( \mathbb{R}^d \). The error part \( (\nabla h, \zeta) \mapsto \sqrt{1 + |\nabla h|^2 \sin^2 \angle(\nabla h, \zeta) - 1} \) is clearly smooth with bounded derivatives in \( \nabla h \). Moreover, it is elementary to argue that

\[ |\partial_\beta \zeta \sin^2 \angle(\nabla h, \zeta)| = |\zeta| - |\beta| \text{ as } |\zeta| \to \infty. \]

Thus

\[ \| \lambda_1^{(0)} - 1 \|_{\mathcal{M}^0} = \sup_{\alpha + |\beta| \leq p} \sup_{\zeta \in \mathbb{R}^d} \| |\zeta|^{1/2} \partial_\beta \omega (\sqrt{1 + |\nabla h|^2 \sin^2 \angle(\nabla h, \zeta) - 1}) \|_{L^2 L^\infty} \leq \| h \|_{H^{r+1} \cap W^{r+1, \infty}} \lesssim \epsilon^2. \]  

(4.21)

For the second expansion in (4.16), observe that if we decompose \( \lambda^{(0)} = \lambda_1^{(0)} + \lambda_2^{(0)} \) where \( \lambda_1^{(0)} \) is defined above, then a straightforward calculation from the definitions shows that \( \lambda_2^{(0)} = \epsilon^2 \mathcal{M}^0_t \). It follows from this and the first expansion of (4.16) that

\[ \lambda_{DN} - |\zeta| - \lambda_1^{(0)} = \lambda_1^{(1)} + (\lambda^{(0)} - \lambda_1^{(0)}) - |\zeta| \]
\[ = |\zeta|(1 + \epsilon^2 \mathcal{M}^0_t) + \epsilon^3 \mathcal{M}^0_t - |\zeta| \]
\[ = \epsilon^2 \mathcal{M}_t \]  

(4.22)

as desired.

To prove the first equation in (4.17), observe

\[ \|(g + \ell)^p - g^p(1 - p|\nabla h|)\|_{\mathcal{M}^0_{N-d/2-1}} = g^p \|(1 - |\nabla|)h - (1 - p|\nabla|)\|_{\mathcal{M}^0_{N-d/2-1}} \]
\[ = \sup_{|\alpha| \leq N-d/2-1} \|pg^p \partial_\alpha \omega \|_{L^2 \cap L^\infty} \lesssim \| h \|_{H^{r+1} \cap W^{r+1, \infty}}^2 \lesssim \epsilon^2. \]  

(4.23)
The corresponding expansion for $\lambda_{DN}^0$ follows analogously from the expansion for $\lambda_{DN}$. The third follows from the first two:

$$
\Sigma = \lambda_{DN}^{1/2}(g + \ell)^{1/2}
= |\zeta|^{1/2}(1 + \frac{\lambda_1}{2|\zeta|} + \epsilon^2 \mathcal{M}_r)g^{1/2}(1 - \frac{1}{2} |\nabla|h + \epsilon^2 \mathcal{M}_r)
= (g|\zeta|)^{1/2} + \frac{g^{1/2}}{2|\zeta|^{1/2}}\lambda_1 - \frac{1}{2}(g|\zeta|)^{1/2} |\nabla|h + \epsilon^2 \mathcal{M}_r
= (g|\zeta|)^{1/2} + \Sigma + \epsilon^2 \mathcal{M}_r.
$$

(4.24)

To prove (4.19), observe that

$$
\partial_t h = \nabla |\omega| + \mathcal{H}^{N-1/2} + \frac{1}{2}g^{-1/2}(\nabla |\omega| + \epsilon^2 \mathcal{M}_r)\nabla |(\nabla |\omega| + \epsilon^2 \mathcal{H}^{N-1/2})
= \frac{1}{2}g^{1/2} \Delta \omega + \epsilon^2 \mathcal{M}_r.
$$

(4.25)

The argument is analogous for the second equation in (4.19).

\[\square\]

**Lemma 4.3.** Let

$$
H = T_{\sqrt{g + \ell}}h, \quad \Psi = T_{\Sigma} T_{1/\sqrt{g + \ell}}\omega.
$$

(4.26)

Then for $(h, \phi)$ satisfying the system (1.17),

$$
\begin{aligned}
\partial_t H - T_{\Sigma} \Psi + iT_{V, \zeta} H &= -\frac{1}{2}T_{\sqrt{g + \ell} \text{div} V} h + A_1[h, \omega] + \epsilon^3 \mathcal{H}^N \\
\partial_t \Psi + T_{\Sigma} + iT_{V, \zeta} \Psi &= \frac{1}{2}T_{\lambda_{DN}^{1/2} \text{div} V} \omega + A_2[h, \omega] + \epsilon^3 \mathcal{H}^N
\end{aligned}
$$

(4.27)

where $A_1, A_2$ are given by bilinear symbols of the form (2.9) that do not lose derivatives.

**Proof.** We have

$$
\frac{1}{\sqrt{g + \ell}} \{V \cdot \zeta, \sqrt{g + \ell}\} = \frac{1}{2} \frac{g}{(g + \ell)} V \cdot \nabla |\nabla|h.
$$

(4.28)
Since \( V \in \epsilon H^{N-3/2} \) and \( \nabla |\nabla| h \in \epsilon H^{N-2} \), this symbol is in \( \epsilon^2 H^{N-2} \subset \mathcal{M}_N^{0-d/2-2} \). Thus by Lemma 2.8, for \( F \in \{ H, \Psi \} \),

\[
i T_{(V,\xi,\sqrt{g+\ell})/\sqrt{g+\ell}} F \in \epsilon^2 H^N. \tag{4.29}
\]

We then compute using the first equation in the paralinearized system (4.5)

\[
\partial_t H - T_{\Sigma} \Psi + iT_{V,\xi} H + \frac{1}{2} T_{\sqrt{g+\ell}} \text{div} V h = T_{\partial_t \sqrt{g+\ell}} h + T_{\sqrt{g+\ell} dV} h - T_{\Sigma}^2 T_{1/\sqrt{g+\ell}} \omega \\
+ iT_{V,\xi} H + \frac{1}{2} T_{\sqrt{g+\ell} \text{div} V h} \\
= T_{\partial_t \sqrt{g+\ell}} h + T_{\sqrt{g+\ell} dV} h \lambda_{DN} \omega - T_{\sqrt{g+\ell} \text{div} (V h)} \\
+ T_{\sqrt{g+\ell} G_2} + \epsilon^3 T_{\sqrt{g+\ell} H^N} - T_{\Sigma}^2 T_{1/\sqrt{g+\ell}} \omega \\
+ iT_{V,\xi} H + \frac{1}{2} T_{\sqrt{g+\ell} \text{div} V h}. \tag{4.30}
\]

Observe that, since \( \xi = (\xi - \eta) + \eta \), we have the Leibnitz rule

\[
\text{div} (T \sqrt{g+\ell} h(\xi)) = F^{-1}(i\xi \cdot \int_{\mathbb{R}^d} \chi(\frac{|\xi - \eta|}{|\xi + \eta|}) \hat{V}(\xi - \eta) \hat{h}(\eta) d\eta) \\
= F^{-1}(i\xi \cdot \int_{\mathbb{R}^d} \chi(\frac{|\xi - \eta|}{|\xi + \eta|}) \left\{ i(\xi - \eta) \cdot \hat{V}(\xi - \eta) \hat{h}(\eta) + \hat{V}(\xi - \eta) \cdot i\eta \hat{h}(\eta) \right\} d\eta) \tag{4.31}
\]

and by a similar calculation

\[
i T_{V,\xi} h = \frac{1}{2} T_{\text{div} V h} + T_V \nabla h. \tag{4.32}
\]

With this we can rewrite

\[
T_{\sqrt{g+\ell} \text{div} (T \sqrt{g+\ell} h)} = \frac{1}{2} T_{\sqrt{g+\ell} \text{div} V h} + iT_{\sqrt{g+\ell} T_{V,\xi} h} \tag{4.33}
\]

and as a result

\[
\partial_t H - T_{\Sigma} \Psi + iT_{V,\xi} H + \frac{1}{2} T_{\sqrt{g+\ell} \text{div} V h} = I + II + III \tag{4.34}
\]

where

\[
I = (T_{\sqrt{g+\ell} T_{\lambda_{DN}}} - T_{\Sigma}^2 T_{1/\sqrt{g+\ell}}) \omega \\
II = iT_{V,\xi} H - iT_{\sqrt{g+\ell} T_{V,\xi} h} \tag{4.35}
\]

\[
III = T_{\partial_t \sqrt{g+\ell} h} - \frac{1}{2} (T_{\sqrt{g+\ell} \text{div} V} - T_{\sqrt{g+\ell} \text{div} V}) h + T_{\sqrt{g+\ell} G_2} + \epsilon^3 T_{\sqrt{g+\ell} H^N+1/2}. 
\]
Repeatedly applying the definition of $E(a, b)$ first on $T_{\Sigma}^2$ and then on $T_{\Sigma^2 T_1/\sqrt{\gamma + \ell}}$, we can write $I$ as

$$I = (T_{\sqrt{g + \ell} \lambda_{DN}} + \frac{i}{2} T_{\{\sqrt{g + \ell} \lambda_{DN}\}} + E(\sqrt{g + \ell}, \lambda_{DN})) \omega$$

$$- (T_{\Sigma^2/\sqrt{\gamma + \ell}} + \frac{i}{2} T_{\{\Sigma^2, 1/\sqrt{\gamma + \ell}\}} + E(\Sigma^2, 1/\sqrt{g + \ell}) + E(\Sigma, \Sigma) T_1/\sqrt{\gamma + \ell}) \omega.$$

Right from the definitions we also have

$$\sqrt{g + \ell} \lambda_{DN} = \Sigma^2/\sqrt{g + \ell}$$

and

$$\{\Sigma^2, 1/\sqrt{g + \ell}\} = -\nabla_\zeta \lambda_{DN} (g + \ell) \nabla_x \frac{1}{\sqrt{g + \ell}}$$

$$= \nabla_\zeta \lambda_{DN} (g + \ell) \frac{\nabla_x \sqrt{g + \ell}}{g + \ell} = \{\sqrt{g + \ell}, \lambda_{DN}\}.$$

which cause the first two terms in both of the lines to cancel out, leaving

$$I = E(\sqrt{g + \ell}, \lambda_{DN}) \omega - E(\Sigma^2, 1/\sqrt{g + \ell}) \omega - E(\Sigma, \Sigma) T_1/\sqrt{\gamma + \ell} \omega.$$  

Using Lemma 4.2, we can write the first term as

$$E(\sqrt{g + \ell}, \lambda_{DN}) \omega = E(g^{1/2} - \frac{1}{2} g^{1/2} |\nabla| h + e^2 \mathcal{M}^0_1, |\zeta| + \lambda_1^{(0)} + e^2 \mathcal{M}^1_1) \omega$$

$$= -\frac{1}{2} g^{1/2} E(|\nabla| h, |\zeta|) \omega + e^3 H^N.$$

the second term as

$$E(\Sigma^2, 1/\sqrt{g + \ell}) = E(g \zeta) + \Sigma^2 + 2 \sqrt{g} |\Sigma| + e^2 \mathcal{M}^0_1, \frac{1}{2} g^{-1/2} |\nabla| h + e^2 \mathcal{M}^0_1$$

$$= \frac{1}{2} g^{1/2} E(|\zeta|, |\nabla| h) \omega + e^3 H^N,$$

and the third as

$$E(\Sigma, \Sigma) T_1/\sqrt{\gamma + \ell} \omega = g E(|\zeta|^{1/2}, |\zeta|^{1/2}) T_1/\sqrt{\gamma + \ell} \omega + E(\Sigma_1, \Sigma_1) T_1/\sqrt{\gamma + \ell} \omega$$

$$+ (E(|\zeta|^{1/2}, \Sigma_1) + E(\Sigma_1, |\zeta|^{1/2})) T_1 + \frac{1}{2} |\nabla| h + e^2 \mathcal{M}^0_0 \omega + e^3 H^N$$

$$= E(|\zeta|^{1/2}, \Sigma_1) \omega + E(\Sigma_1, |\zeta|^{1/2}) \omega + e^3 H^N$$

where we have used the straightforward facts that $E(a(\zeta), a(\zeta)) = 0$ and $T_1$ is the identity.

Collecting these results, we get

$$I = -\frac{1}{2} g^{1/2} E(|\nabla| h, |\zeta|) \omega + \frac{1}{2} g^{1/2} E(|\zeta|, |\nabla| h) \omega + E(|\zeta|^{1/2}, \Sigma_1) \omega + E(\Sigma_1, |\zeta|^{1/2}) \omega + e^3 H^N.$$  

(4.43)
To understand the first two terms, we compute from the definition

\[ E(|\nabla|_h, |\nabla|_h)\omega - E(|\nabla|_h, |\nabla|_h)\omega = T_{|\nabla|_h}T_{|\nabla|_h}\omega - T_{|\nabla|_h}T_{|\nabla|_h}\omega - iT_{|\nabla|_h, |\nabla|_h}\omega \]

\[ = T_{|\nabla|_h}(|\nabla|_h - |\nabla|_h)\omega + T_{|\nabla|_h}(|\nabla|_h - |\nabla|_h)\omega - iT_{|\nabla|_h, |\nabla|_h}\omega \]

\[ = \mathcal{F}^{-1} \int_{\mathbb{R}^d} b(\xi, \eta) \hat{h}(\xi - \eta) \hat{\omega}(\eta) d\eta \]

where we have defined the symbol

\[ b(\xi, \eta) = \chi(\frac{|\xi - \eta|}{|\xi + \eta|})(|\xi| - |\eta|) \left( \frac{|\xi| + |\eta|}{|\xi + \eta|} - 1 \right) \]

which gains a derivative. For the second two terms, one easily computes

\[ E(|\nabla|^{1/2}_h, \Sigma_1)\omega + E(\Sigma_1, |\nabla|^{1/2}_h)\omega = (T_{|\nabla|^{1/2}_h}T_{\Sigma_1} + T_{\Sigma_1}T_{|\nabla|^{1/2}_h} - 2T_{|\nabla|^{1/2}_h, \Sigma_1})\omega \]

We know from Lemma 4.2 that this loses two half derivatives from the symbols, but by Lemma 2.9 it gains two from the commutator structure. To see that it has the claimed bilinear form with a suitable symbol, we easily calculate from the definition of \( T \) that

\[ \mathcal{F}(\sqrt{|\nabla|^{1/2}_h}T_{\Sigma_1} + T_{\Sigma_1}\sqrt{|\nabla|^{1/2}_h} - 2T_{|\nabla|^{1/2}_h, \Sigma_1})\omega(\xi) \]

\[ = \int_{\mathbb{R}^d} \chi(\frac{|\xi - \eta|}{|\xi + \eta|}) \left( |\xi|^{1/2} + |\eta|^{1/2} - |\xi + \eta|^{1/2} \right) \hat{\Sigma}_1(\xi - \eta, \frac{\xi + \eta}{2}) \hat{\omega}(\eta) d\eta \]

and

\[ \hat{\Sigma}_1(\xi, \zeta) = \left[ \frac{g^{1/2}}{4|\xi|^{1/2}} \right] \left( -|\xi|^2 + \frac{\zeta_i \zeta_j}{|\xi|^2} \zeta_i \zeta_j \right) \varphi_{\geq 0}(\zeta) - \frac{(g|\zeta|^{1/2})^2}{2|\xi|} \hat{h}(\xi). \]

As a result, the commutator term is given as a bilinear operator as usual, with the symbol

\[ b(\xi, \eta) = -\frac{g^{1/2}}{2} \chi(\frac{|\xi - \eta|}{|\xi + \eta|}) \left( |\xi|^{1/2} + |\eta|^{1/2} - |\xi + \eta|^{1/2} \right) \]

\[ \times \left( \frac{|\xi|^2 \sin^2 \angle(\xi, \xi + \eta)}{\sqrt{2}|\xi + \eta|^{1/2}} \right) \varphi_{\geq 0}(\frac{\xi + \eta}{2}) + \frac{|\xi + \eta|^{1/2}}{2} \right) \]

For \( II \), we have

\[ II = T_{V \cdot \zeta} H - T_{\sqrt{g + \ell}}T_{V \cdot \zeta} T_{V \cdot \zeta}^{-1} H \]

\[ = [T_{V \cdot \zeta}, T_{\sqrt{g + \ell}}]T_{\sqrt{g + \ell}}^{-1} H \]

\[ = [T_{V \cdot \zeta}, T_{\sqrt{g + \ell}}](T_1 - T_1 T_{\sqrt{g + \ell}}) T_{\sqrt{g + \ell}}^{-1} H. \]
For the first term we have, using Lemma 4.2,
\[
[T_{V\cdot\zeta}, T_{\sqrt{g+\ell}T}]_{1/\sqrt{g+\ell}T} = [T_{V\cdot\zeta}, T_{g^{1/2}(1 - \frac{1}{2}h)\nabla|\omega| + e^2\mathcal{M}_0}]T_{g^{-1/2}(1 + \frac{1}{2}h)\nabla|\omega| + e^2\mathcal{M}_0}H
\]
\[
= [T_{V\cdot\zeta}, -\frac{1}{2}T_{|\omega|}](\text{Id} + \frac{1}{2}T_{|\omega|})H + e^3H^N. \tag{4.51}
\]
Here the commutator with Lemma 2.9 gains back the derivative lost in the \(V\cdot\zeta\) symbol.

The argument that the second term is in \(e^3H^N\) is analogous; one expands the symbol \(\sqrt{g+\ell}\) in the left-most operator and then applies Lemma 2.9.

For the first term of \(III\), (4.19) gives us
\[
T_{\partial_t T_{\sqrt{g+\ell}}} = \frac{1}{2}g^{1/2}\Delta\omega + e^3H^N.
\]
which can clearly be written with a symbol in the desired form. For the second group of terms, since \(V\) is defined as \(\nabla\times\omega\) of the potential for the fluid velocity and \(|\nabla|^{1/2}\phi\) is assumed to be in \(H^N\), we clearly have \(V = \nabla\omega + e^3H^{N-1/2}\). Along with (4.17), this implies the second group of symbols can be written as
\[
(T_{\sqrt{g+\ell}T_{\text{div}}} - T_{\sqrt{g+\ell}T_{\text{div}}}V)h = (T_{g^{1/2}(1 - \frac{1}{2}h)\nabla|\omega| + e^2\mathcal{M}_0})T_{\nabla|\omega| + e^3H^{N-1/2}}
\]
\[
= \frac{1}{2}g^{1/2}(T_{|\omega|}T_{\nabla|\omega|} - T_{|\omega|}h\nabla|\omega|)h + e^3H^N. \tag{4.53}
\]
This completes the derivation of the first equation in (4.27), and the second follows by an analogous calculation.

This lemma all but completes the proof of the main theorem of this chapter.

**Proof of Theorem 4.1.** The expression \(U = \sqrt{gh + i|\nabla|^{1/2}\omega + e^2H^N}\) follows from the definition due to the expansions in Lemma 4.2. To prove the main equation (4.3), observe that \(U = H + i\Psi\). Then by Lemma 4.3 and a straightforward computation,
\[
(\partial_t + iT_{\Sigma} + iT_{V\cdot\zeta})U = (\partial_t H - T_{\Sigma}\Psi + iT_{V\cdot\zeta}H) + i(\partial_t\Psi + T_{\Sigma} + iT_{V\cdot\zeta}\Psi)
\]
\[
= -\frac{1}{2}T_{\sqrt{g+\ell}\text{div}}Vh + \frac{i}{2}T_{\sqrt{\mathcal{D}N}\text{div}}V\omega + A_1[h,\omega] + A_2[h,\omega] + e^3H^N. \tag{4.54}
\]
Let us argue that the operators $A_i[h, \omega]$ can be written as combinations of $A[U, U]$, $A[U, \bar{U}]$, and $A[\bar{U}, \bar{U}]$ as claimed. We already argued using Lemma 4.2 that up to $\epsilon M_0^r$ errors (which will fall into the cubic term $\epsilon^3 H^N$), $h \sim H$ and $\omega \sim \Psi$. Thus

$$A_i[H, \Psi] = A_i[U + \bar{U}, \frac{U - \bar{U}}{2i}]$$

$$= \frac{1}{4i}(A_i[U, U] - A_i[U, \bar{U}] - A_i[U, \bar{U}] + A_i[U, U]).$$

(4.55)

Of course $A_i[\bar{U}, \bar{U}]$ becomes $A'_i[U, \bar{U}]$ after the change of variables $\eta \mapsto \xi - \eta$.

The proof is complete if $T\sqrt{g + \ell} \text{div} V h$ and $T\sqrt{\lambda N} \text{div} V \omega$ can be absorbed into the quadratic and cubic errors, and indeed this follows from Lemma 4.2 in exactly the manner as the estimates in the proof of Lemma 4.3.
References


