TILING SIMPLY CONNECTED REGIONS WITH RECTANGLES

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Abstract. In [BNRR], it was shown that tiling of general regions with two rectangles is NP-complete, except for few trivial special cases. In a different direction, Rémila [Rém2] showed that for simply connected regions and two rectangles, the tileability can be solved in quadratic time (in the area). We prove that there is a finite set of at most 10^6 rectangles for which the tileability problem of simply connected regions is NP-complete, closing the gap between positive and negative results in the field. We also prove that counting such rectangular tilings is #P-complete, a first result of this kind.

1. Introduction

The study of *finite tilings* is a classical subject of interest in both theoretical and recreational literature [Gol, GS]. In the *tileability problem*, a finite set of tiles **T** is fixed, and a region is an input. This problem is known to be polynomial in some cases, and NP-complete in others (see [Pak]). Over the years, the hardness results were successively simplified (in statement, not in proof), with both sets of tiles and the regions becoming more restrictive. This paper is a new step in this direction.

In [BNRR], it was shown that tiling of general regions with two bars is NP-complete, except for the case of dominoes. In a different direction, Rémila [Rém2] (building on the ideas in [KK, Thu]), showed that for *simply connected regions* and two rectangles, the tileability can be solved in quadratic time (in the area). The following theorem closes the gap between these polynomial and NP-complete results.

Theorem 1.1 (Main Theorem) *There exists a finite set* **R** *of at most* 10^6 *rectangular tiles, such that the tileability problem of simply connected regions with* **R** *is* NP*-complete.*

Our proof of the Main Theorem is split into two parts. In the first part, we use the language of *Wang tiles* to reduce the Cubic Monotone 1-in-3 SAT problem, known to be NP-complete, to the **T**-tileability of simply connected regions with Wang tiles. In the second part, we reduce Wang tileability to tileability with rectangular tiles. Both our reductions are *parsimonious* and are used to prove that counting the number of tilings of simply connected regions is also hard, via reduction from 2SAT.

Theorem 1.2 *There exists a finite set* **R** *of at most* 10^6 *rectangular tiles, such that counting the number of tilings of simply connected regions with* **R** *is* #P*-complete.*

Although #P-completeness is known for tilings of general regions with right tromino and square tetromino [MR], nothing is known for tilings with rectangles, or for tilings of simply connected regions. We refer to Section 7 for the history of the problem, references, and further remarks.

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2. Definitions and basic results

2.1. **Ordinary tiles.** Consider the integer lattice $\mathbb{Z}^2$ as a union of closed unit squares with pairwise disjoint interiors. A *region* is a finite union of such unit squares such that the interior is connected. An (*ordinary*) *tile* is a finite simply connected region.

A *tileset* **T** is a collection of tiles. Given a region Γ and a tileset **T**, a **T**-*tiling* of Γ is a union of translated copies of tiles from **T** with pairwise disjoint interiors. If a region admits a **T**-tiling then it is **T**-*tileable*. We may simply say *tiling* and *tileable* when **T** is understood. Consider the following decision problems regarding tileability:

Simply Connected Tileability
Instance: Simply connected region Γ, finite tileset **T**.
Decide: Whether Γ is **T**-tileable?

Simply Connected **T**-Tileability
Instance: Simply connected region Γ.
Decide: Whether Γ is **T**-tileable?

An input region can be given by the (finite) union of the squares it contain.

Theorem 2.1 *If both region* Γ *and tileset* **T** *are part of the input,* Simply Connected Tileability *is* NP*-complete even in the* 1*-dimensional case.*

Although elementary, we include a short proof of this theorem in Section 6.3. By the theorem, if the tileset **T** is part of the input, we easily obtain that Simply Connected Tileability is NP-complete. For the rest of the paper, we will focus on finding a fixed **T** such that Simply Connected **T**-Tileability is NP-complete.

2.2. **Wang tiles.** The *edges* of an ordinary tile are the unit-length edges on the boundary. Given a set of colors and an ordinary tile τ, a *generalized Wang tile* is an assignment of colors to the edges of τ. Note that an (*ordinary*) *Wang tile* is a generalized Wang tile of a unit square. The region Γ we are trying to tile will also have specified colors on its boundary. A region is (*Wang*) *tileable* if there is a tiling where incident edges have the same color, including on the boundary of the region (see Figure 1). If a tileset consists of (generalized) Wang tiles, tileability always mean Wang tileability.

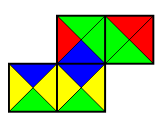


Figure 1: A Wang tiling.

The following result is of independent interest, and its proof is a key step towards the proof of the Main Theorem.

Theorem 2.2 *There exists a set* **T** *of* 23 *ordinary tiles, such that* Simply Connected **T**-Tileability *is* NP*-complete.*

2.3. **Relational Wang tiles.** Let us consider a more general setting. A set of *relational Wang tiles* is a collection of squares $\mathbf{W}$ and the following data. The vertical (respectively horizontal) *Wang relation* $V(\tau, \tau')$ (respectively $H(\tau, \tau')$) specify that $\tau' \in \mathbf{W}$ is allowed to be placed immediately below (respectively to the right of) $\tau \in \mathbf{W}$. The *boundary tiles* of a region Γ is a map from the exterior edges of Γ to the tiles $\mathbf{W}$. By abuse of language, we define the notion of tiling in this context: a $\mathbf{W}$*-tiling* of a region Γ is a map $\pi : \Gamma \to W$ such that tiles placed next to each other satisfy the Wang relations. Whenever a tile is adjacent to an exterior edge, we check the Wang relations as if the boundary tile corresponding to the edge is on the other side of the edge.

2.4. **Complexity.** A decision problem Q is a question with output either yes or no, depending on the input. Formally, it is a function $Q : \{0,1\}^* \to \{0,1\}$, where $\{0,1\}^*$ denotes finite binary sequences where instances of the problem are encoded. Suppose Q_1 and Q_2 are decision problems. We say Q_1 is *reducible* to Q_2 if there is a function $f : \{0,1\}^* \to \{0,1\}^*$ such that f is computable by a polynomial time algorithm and $Q_1(x) = Q_2(f(x))$ for all x. Such a function is called a *decision problem reduction* and *transforms* an instance of the first problem to one of the second. We will consider many problems that are NP-complete. Since the problems are finite, they will all be trivially in NP. To prove NP-hardness, we reduce a known NP-complete problem to the problem in question.

We will embed Cubic Monotone 1-in-3 SAT as a tiling problem. Let $X = \{x_1, \ldots, x_n\}$ be a set of boolean variables. A (*monotone* 1*-in-*3) *clause* C is a set of three variables. A (*cubic monotone* 1*-in-*3) *expression* E is a finite collection $\mathcal{C}$ of monotone 1-in-3 clauses, where each variable $x_i \in X$ occurs three times. We say such E is (1*-in-*3) *satisfiable* if there is an assignment of boolean values $\{0,1\}$ to the variables $x_i \in X$ such that each clause in E contains precisely one variable receiving 1 (and thus two variables receiving 0).

Cubic Monotone 1-in-3 SAT
Instance: Set X of variables, cubic monotone expression E.
Decide: Whether E is 1-in-3 satisfiable?

The following result was shown by Gonzalez in the language of exact covers:

Theorem 2.3 ([Gon, MR]) Cubic Monotone 1-in-3 SAT *is* NP*-complete.*

Given an expression E, we can associate a bipartite graph G with vertex set $X \sqcup \mathcal{C}$, where a variable $x \in X$ is adjacent to a clause $C \in \mathcal{C}$ if $x \in C$. Moore and Robson showed something stronger, that this problem is NP-complete even if we require the associated graph to be planar. They did this by reducing from Planar 1-in-3 SAT, which is NP-complete [Lar, MuR].

2.5. **Counting problems.** Given a decision problem Q, a *counting problem* associated with Q is a function $c_Q : \{0,1\}^* \to \mathbb{N}$ such that $c_Q(x) > 0$ if and only if $Q(x) = 1$. Each decision problem we will consider has a *natural* associated counting problem by counting the number of witnesses. Indeed, for a satisfiability problem, a witness to satisfiability is a satisfying assignment. So instead of asking whether satisfying assignments exist, one asks *how many* satisfying assignments there are. Similarly, for tileability, one counts the number of tilings. From now on, we only consider the natural associated counting problems.

Suppose $f : \{0,1\}^* \to \{0,1\}^*$ is a decision problem reduction from Q_1 to Q_2, that is, $Q_1 = Q_2 \circ f$. This reduction is *parsimonious* if $c_{Q_1} = c_{Q_2} \circ f$. That is, the witnesses of an instance in Q_1 is in bijective correspondence with witnesses of the transformed instance in Q_2. Moreover, this bijection will usually be fairly obvious.

Parsimonious reductions have the additional benefit of proving counting results using the same reduction. The class #P consists of the counting problems associated with decisions problems in NP. A counting problem is #P-complete if it is in #P and every #P question can be reduced to it. Thus if there is a parsimonious reduction from Q_1 to Q_2, then if Q_1 is #P-complete, then so is Q_2.

One main goal is to reduce Cubic Monotone 1-in-3 SAT to a tiling problem Simply Connected $\mathbf{T}$-Tileability for some fixed $\mathbf{T}$. This reduction will turn out to be parsimonious, hence the number of satisfying assignments of a given instance of the satisfiability problem can be calculated by counting the number of tilings of the transformed instance.

3. Reduction lemmas

3.1. **Basic reductions.** In this section we consider five classes of Tileability problems. Let $\mathcal{T}$ be a collection of tiles and $\mathcal{R}$ be a collection of regions. A decision problem in $(\mathcal{T}, \mathcal{R})$-Tileability consists of a *fixed* tileset $\mathbf{T} \subset \mathcal{T}$, receives some $\Gamma \in \mathcal{R}$ as input, and outputs whether Γ is $\mathbf{T}$-tileable.

We say $(\mathcal{T}, \mathcal{R})$-Tileability is *linear time reducible* to $(\mathcal{T}', \mathcal{R}')$-Tileability if for any finite tileset $\mathbf{T} \subset \mathcal{T}$, there exists a finite tileset $\mathbf{T}' \subset \mathcal{T}'$ and a *reduction map* $f : \mathcal{R} \to \mathcal{R}'$ that is computable in linear time (in the complexity of $\Gamma \in \mathcal{R}$), such that $\Gamma \in \mathcal{R}$ is $\mathbf{T}$-tileable if and only if $f(\Gamma)$ is $\mathbf{T}'$-tileable. If, moreover, that $(\mathcal{T}', \mathcal{R}')$-Tileability is linear time reducible to $(\mathcal{T}, \mathcal{R})$-Tileability, then they are *linear time equivalent*. Note that the transformation of the tilesets need not be efficient nor bijective.

For instance, if $\mathcal{T}$ is the collection of all rectangular tiles and $\mathcal{R}$ consists of simply connected regions, then $(\mathcal{T}, \mathcal{R})$-Tileability is a class of problems regarding tiling simply connected regions with rectangular tiles. To simplify the notation, we drop the prefix in $(\mathcal{T}, \mathcal{R})$-Tileability, when the sets $\mathcal{T}$ and $\mathcal{R}$ are clear.

Lemma 3.1 (Tileability Equivalence Lemma) *The following classes of* Simply Connected Tileability *problems are linear time equivalent:*

- (i) Tileability *with a fixed set of rectangular tiles.*
- (ii) Tileability *with a fixed set of n ordinary tiles.*
- (iii) Tileability *with a fixed set of n generalized Wang tiles.*
- (iv) Tileability *with a fixed set of ordinary Wang tiles.*
- (v) Tileability *with a fixed set of relational Wang tiles.*

Moreover, the size of the tileset can be preserved in the reductions between (ii) *and* (iii).

Proof. Proof of the implications (i)⇒(ii)⇔(iii)⇒(iv)⇒(v) are elementary and given below. The implication (v)⇒(i) is stated separately as Lemma 3.3 and proved in the next section.

We may consider a rectangular tile as an ordinary tile, which in turn is a monochromatic generalized Wang tile. Therefore the implications (i)⇒(ii)⇒(iii) are immediate, where the reductions are by identity maps.

(iii)⇒(iv). Given a set of generalized Wang tiles, color each interior edge with a new color not used anywhere else, and consider each square as a separate ordinary Wang tile (see Figure 2). These tiles are forced to reassemble themselves as the original generalized Wang tiles. The reduction map is again the identity.

(iv)⇒(v). It is obvious how to define the Wang relations to mimic the colored Wang tiles without increasing the number of tiles. However, to encode the boundary conditions, we may need to introduce less than 4χ tiles, where χ is the number of colors permitted on the boundary. Indeed, to specify a color c on the top boundary, we need to choose an (arbitrary) tile whose bottom color is c. If no such tile exists, we must add a new tile to do so. If we do not involve the new tile in any Wang relations in the other directions, then it will never be used in the actual tiling, and thus will not affect tileability. We do the same for the other three directions.

The final implication (v)⇒(i) is more difficult and is the content of Lemma 3.3 and proved in a later section.

To preserve the number of tiles in (iii)⇒(ii), scale the generalized Wang tile and replace each colored edge by an appropriate rectilinear zig-zag curve to encode the matching rules (see Figure 3). □

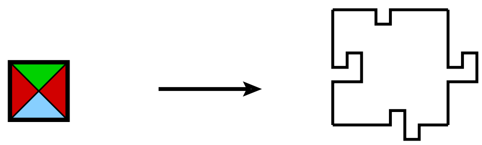


Figure 2: Replacing each colored edge by a zig-zag curve to get ordinary tiles.

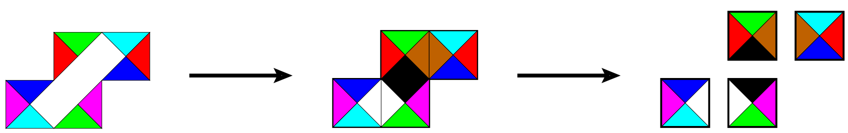


Figure 3: From generalized Wang tiles to ordinary ones.

3.2. **Two main reductions.**

Lemma 3.2 (First Reduction Lemma) *There exists a set* $\mathbf{T}$ *of at most* 23 *generalized Wang tiles such that* Simply Connected $\mathbf{T}$-Tileability *is* NP-*complete. Moreover, this will be achieved by a parsimonious reduction from* Cubic Monotone 1-in-3 SAT.

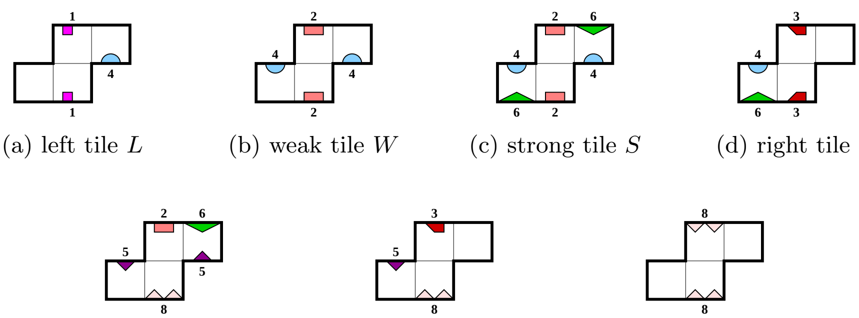
Lemma 3.3 (Second Reduction Lemma) *For a set* $\mathbf{W}$ *of at most* k *(ordinary) Wang tiles with* c *(boundary) colors, there exists a set* $\mathbf{R}$ *of at most* $32(k+c)^2$ *rectangular tiles with the following property. Given a simply connected colored region* Γ*, there is a simply connected region* Γ' *such that* Γ *is* $\mathbf{W}$*-tileable if and only if* Γ' *is* $\mathbf{R}$*-tileable. Moreover, this reduction is parsimonious and can be computed in linear time.*

We may transform the set of 23 generalized Wang tiles afforded by Lemma 3.2, according to the procedure outlined in (iii)⇒(ii) of Lemma 3.1, in order to obtain Theorem 2.2 using 23 ordinary tiles. Similarly, using the transformation of Lemma 3.3, we conclude the result for rectangular tiles in Theorem 1.1 (see Subsection 6.1). Theorem 1.2 can be shown by modifying the proof of Lemma 3.2 to achieve a parsimonious reduction from, say, 2SAT, whose associated counting problem is #P-complete (see Subsection 6.2).

4. Proof of the First Reduction Lemma (Lemma 3.2)

4.1. **General setup.** The goal of this section is to construct a set of generalized Wang tiles that could be used to solve Cubic Monotone 1-in-3 SAT. Each expression will be encoded as a colored rectangular boundary. Tiles corresponding to variables and clauses will appear on the left and right sides of the region, respectively. The variable tiles will "transmit" its state (0 or 1) through "wires" to the clause tiles; each clause tile will "check" if precisely one out of three signals it receives is 1. The path of the transmissions will be regulated by placing "crossover tiles" that allow signals to crossover at specific locations. The positioning of such tiles will be enforced by using a combination of "control tiles" that follow instructions encoded on the boundary. Empty spaces will be filled by "filler tiles."

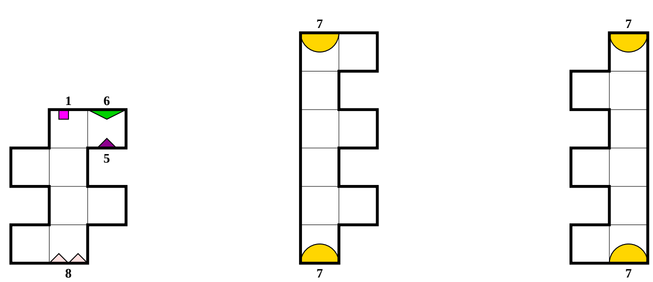
4.2. **Tileset T.** Let **T** be a tileset with the 7 small tiles shown in Figure 4 and the 3 big tiles in Figure 5. Some horizontal edges are colored by their labels; all unlabeled edges are colored by 0, which is omitted in the figures for clarity, but acts as any other ordinary color.



(a) left tile L (b) weak tile W (c) strong tile S (d) right tile R

(e) bottom tile B (f) corner tile K (g) filler tile F

Figure 4: Tiles in tileset **T**.



(a) crossover tile X (b) variable tile V (c) clause tile C

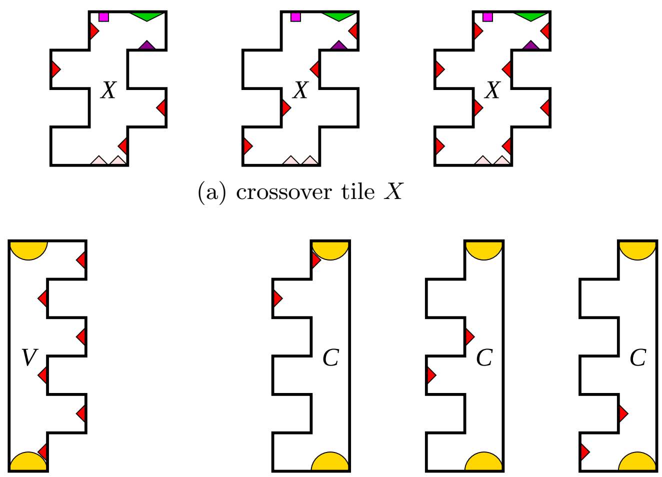
Figure 5: More tiles in **T**.

4.3. **Tileset T′.** Recall that the vertical edges of our tiles in **T** are all colored with 0. Form **T′** by recoloring the vertical edges of tiles in **T** as follows. Given each small tile $\tau \in \mathbf{T}$ in

Figure 4, we introduce a variant by coloring all its vertical edges with 1. The color of the vertical edges is called the *parity* of τ. Include both this variant and the original in $\mathbf{T}'$.

Given a rectangular array of these tiles, the parities are consistent across each row and are independent across the columns. Intuitively, these tiles act as wires that can transmit data (parity of the tile) horizontally across the region.

We continue defining $\mathbf{T}'$. We add three new versions of the crossover tile X as in Figure 6a. Intuitively, this allows the data transmissions to *crossover*. We also add a variant of the variable tile V, as in Figure 6b, where all the right vertical edges are colored with 1. The use of this tile indicates an assignment of 1 to the variable. Finally, we *replace* the clause tile C by the three shown in Figure 6c, where each tile has one out of three pairs of left vertical edges colored with 1. Thus $\mathbf{T}'$ consists of 23 tiles.



(a) crossover tile X

(b) variable tile

(c) clause tile

Figure 6: Variations of tiles in $\mathbf{T}$.

We will place the variable tiles on the left and the clause tiles on the right. It remains to send the data from the variables to the correct clauses. We achieve this by specifying boundary colors to force crossover tiles to appear at the desired locations.

4.4. **Reduction construction.** Our goal is to embed the decision problem Cubic Monotone 1-in-3 SAT as a tiling problem. Given a cubic monotone 1-in-3 SAT expression E with variables $X = \{x_1, \ldots, x_n\}$ and clauses $\mathcal{C} = \{C_1, \ldots, C_n\}$, consider it as a permutation $\sigma = \sigma_E \in S_{3n}$ in the symmetric group on $3n$ letters as follows. Think of σ as a bijection from the ordered multiset $X' = \{x_1, x_1, x_1, x_2, \ldots, x_n\}$ to the ordered multiset $\mathcal{C}' = \{C_1, C_1, C_1, C_2, \ldots, C_n\}$, where each variable and each clause is listed three times. For each $x_i \in C_j$, we have $\sigma(x_i) = C_j$ once. Now identify each multiset with $[3n] = \{1, 2, \ldots, 3n\}$ to get σ as a permutation in S_{3n}. Let $s_i = (i, i+1)$ be an adjacent transposition for $i \in [3n-1]$. Write $\sigma = s_{i_1} s_{i_2} \ldots s_{i_d}$ as a product of adjacent transpositions, with $d = O(n^2)$.[1]

[1]For illustration purposes, it is often convenient to encode the product of adjacent transpositions using *wiring diagrams*, as shown in Figures 7a and 8a.

Let c_k be the color sequence $01(02)^{k-1}63$. Define a rectangular region $\Gamma = \Gamma_E$ as follows. The height of Γ is $6n$ and the vertical edges are colored with 0. The width is the length of the color sequence $7c_{i_1}c_{i_2}\dots c_{i_d}07$, which is used as the top boundary. The bottom boundary is $700\dots07$. The following result demonstrates the ability to place the crossover tile X at arbitrary depth of a large rectangular region.

Sublemma 4.1 *The region Γ admits a unique $\mathbf{T}$-tiling.*

Proof. The left and right sides are forced to be filled with variable and clause tiles, respectively. Now consider the section in between.

For $k \geq 1$ and $\ell \geq 0$, consider a row of tiles $LW^kS^\ell R$ (meaning an L tile followed by a W tiles k times, an S tile ℓ times, and ending with an R tile). The bottom color sequence is $01(02)^k(62)^\ell 63$. One easily checks that the unique way to tile the next row is with $LW^{k-1}S^{\ell+1}R$.

If $k = 0$, we get the case where we have a row $LS^\ell R$ with bottom color sequence $01(62)^\ell 63$. The unique way to tile the next row is with $XB^\ell K$.

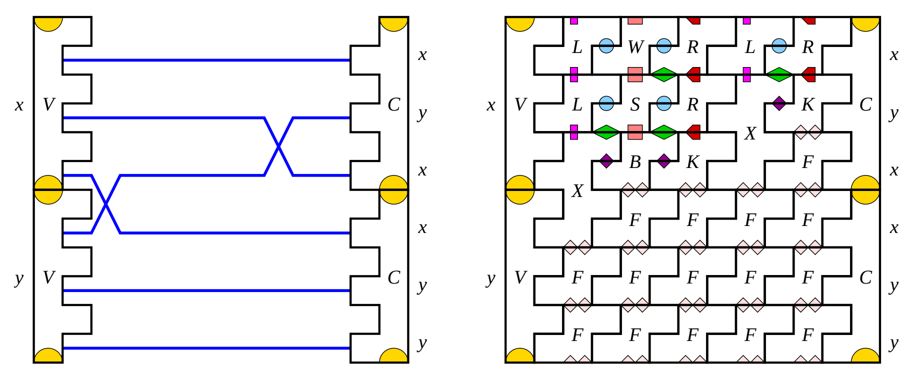
The section below will be filled by null tiles N. Thus every section below c_i is filled uniquely, with the crossover tile X occupying rows i and $i+1$ in the first column. □

The above proof is illustrated with two examples in the next subsection.

Corollary 4.2 *The expression E is satisfiable if and only if Γ_E is $\mathbf{T}'$-tileable. Moreover, the reduction is parsimonious, that is, the number of tilings of Γ_E is the number of satisfying assignments for E.*

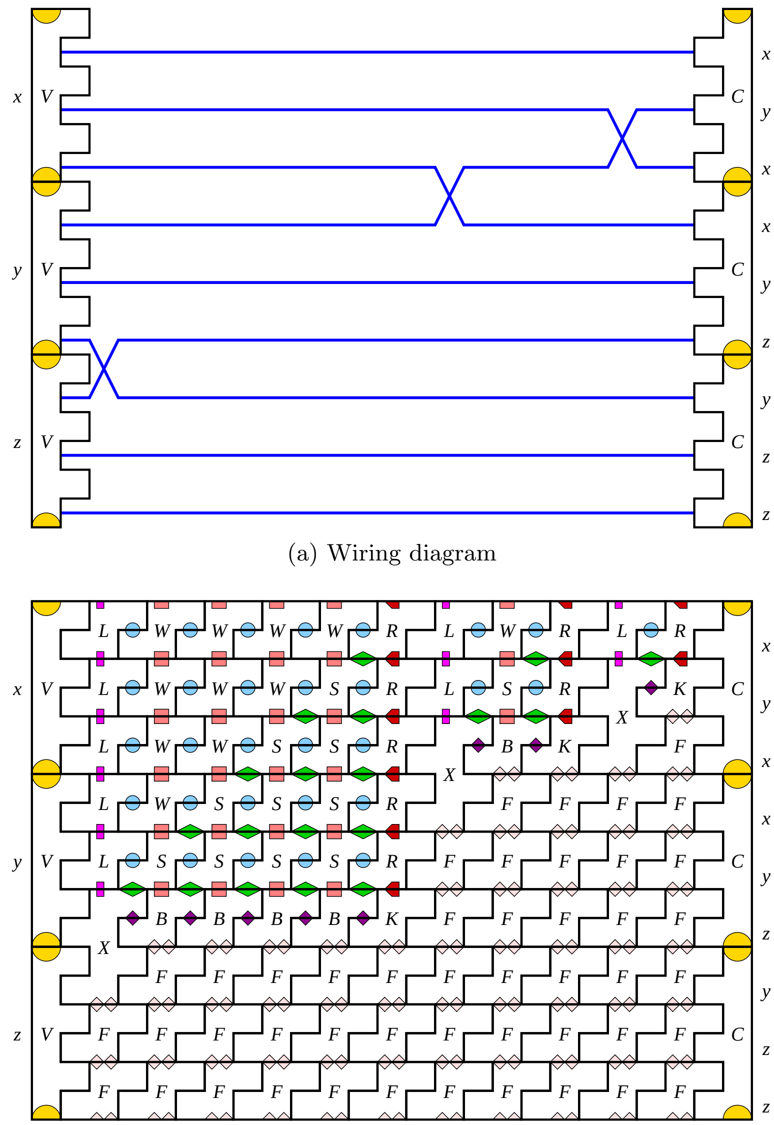
The corollary follows immediately from the construction given above, and concludes the proof of Lemma 3.2.

4.5. **Examples of the tiling construction.** In Figure 7 we show how to place a crossover tile in a special case, corresponding to expression $\{(x,y,x),(x,y,y)\}$. We illustrate the crossings with a *wiring diagram* and then give a complete Wang tiling. In Figure 8 below we give a bigger example of the wiring diagram and the unique Wang tiling, corresponding to expression $\{(x,y,x),(x,y,z),(y,z,z)\}$.



(a) Wiring diagram (b) Unique tiling

Figure 7: A small example.



(a) Wiring diagram

(b) Unique tiling

Figure 8: A bigger example.

5. Proof of the Second Reduction Lemma (Lemma 3.3)

5.1. **Basics.** In this section, we provide a further connection between Wang tiles and ordinary rectangular tiles (by making a reduction from the latter to the former). Recall that by Lemma 3.1, we can replace generalized Wang tiles with relational Wang tiles.

Without loss of generality, we may assume that the Wang relations are irreflexive, that is, there is no tile τ such that $H(\tau, \tau)$ or $V(\tau, \tau)$. Indeed, suppose $H(\tau, \tau)$. First remove this relation, then make a copy τ' of τ, satisfying all the same relations. Now change the relation so $H(t, \tau')$ if and only if $t = \tau$, and add $H(\tau', \tau)$. This replaces each long sequence of τ by an alternating sequence of τ and τ', starting with τ. Clearly this does not affect tileability nor the number of such tilings. We do this sequentially over all $H(\tau,\tau)$ and $V(\tau,\tau)$, making the Wang relations irreflexive.

From now on, assume we are given a fixed set $\mathbf{W}$ of relational Wang tiles whose relations H and V are irreflexive. Our goal is to produce a fixed set $\mathbf{R}$ of rectangular tiles with the following property: Given any simply connected region Γ with specified boundary tiles, we can produce (in linear time) a simply connected region Γ' such that Γ is $\mathbf{W}$-tileable if and only if Γ' is $\mathbf{R}$-tileable. Moreover, the number of $\mathbf{W}$-tilings of Γ will be the same as the number of $\mathbf{R}$-tilings of Γ'.

For simplicity, we first consider the case where we are given an $r \times c$ rectangular region Γ with specified boundary tiles.

5.2. **Expansion.** From this point on, we only consider tiling using rectangular tiles. Fix M and ε to be positive integers. Given a region Γ_0, we obtain an (M,ε)-*expansion* Γ by scaling Γ_0 by a factor of M and then *perturb* it by moving each corner vertex of the boundary curve of the region Γ, at most ε in each direction, such that Γ is still a region (with rectilinear edges). Recall that a (rectangular) tile is just a simply connected region, thus the notion of (M,ε)-expansion of a tile is defined. A tileset $\mathbf{T}$ is an (M,ε)-*expansion* of a tileset $\mathbf{T}_0$ if each $\tau \in \mathbf{T}$ is an (M,ε)-expansion of some $\tau_0 \in \mathbf{T}_0$.

A tiling π of a region Γ is an (M,ε)-*expansion* of a tiling π_0 of some region Γ_0 if it can be obtained by dilating by a factor of M, and then perturbing the tiles and the region by at most ε as above. Notice that after scaling, each tile may grow or shrink in each dimension by at most 2ε, and can *shift* around from its starting point by at most ε.

Given a tileset $\mathbf{T}_0$ and an (M,ε)-expansion $\mathbf{T}$, a region Γ *respects the expansion* if there is a unique region Γ_0 such that any $\mathbf{T}$-tiling of Γ is an (M,ε)-expansion of a $\mathbf{T}_0$-tiling of Γ_0.

Intuitively, we will choose $M > 100\varepsilon$, say, and carefully perturb only a few tiles, so that when consider tilings of regions respecting the expansion, we can essentially predict what the new tiling can be based on the original tiling.

5.3. **Rectangular tiles $\mathbf{R}_0$ and the region $\Gamma_0(r,c)$.** Consider the following tileset:

$$\mathbf{R}_0 = \{f = R(34,11),\ w = R(31,14),\ s = R(10,10),\ h = R(11,31),\ v = R(14,34)\},$$

where $R(a,b)$ denotes a rectangle of height a and width b (see Figure 9). For a rectangle t, write $\mathbf{ht}\,t$ and $\mathbf{wd}\,t$ for its height and width, respectively.

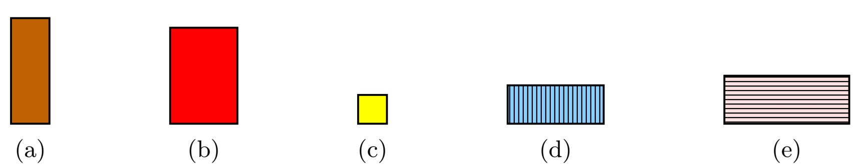


Figure 9: Rectangular tiles: (a) fixed rectangle f, (b) fixed rectangle w, (c) flexible square s, (d) flexible rectangle h, and (e) flexible rectangle v.

Now consider the region $\Gamma_0(r,c)$ defined as follows (see Figure 10). On each vertical side, there are r *protrusions* of height $\mathbf{ht}\,h$ and width $\mathbf{wd}\,s$, separated by height $\mathbf{ht}\,f$. On each horizontal side, there are c *cavities* of width $\mathbf{wd}\,v$ and height $\mathbf{ht}\,s$, separated by width $\mathbf{wd}\,f$.

Sublemma 5.1 *The unique* $\mathbf{R}_0$*-tiling of* $\Gamma_0(r,c)$ *consists of* r *rows and* c *columns of the* w *tile.*

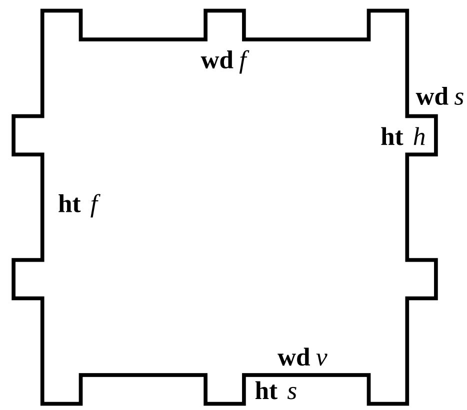


Figure 10: Boundary region $\Gamma_0(2,2)$.

Proof. Let z and ε be natural numbers, where z is sufficiently larger than ε. Consider the set of tiles $f = R(3z+4\varepsilon, z+\varepsilon)$, $w = R(3z+\varepsilon, z+4\varepsilon)$, $s = R(z,z)$, $h = R(z+\varepsilon, 3z+\varepsilon)$, and $v = R(z+4\varepsilon, 3z+4\varepsilon)$.

First we establish a few definitions. A horizontal (vertical) segment of a region is *bounded* if the region extends downward (to the right) on both sides of the segment. For $t \in \{v,h\}$, a *pair* (t,s) is the configuration of placing the tile s above or below t, aligned on the left. The *orientation* of the pair is positive (negative) if s is placed below (above). Similarly, for $t \in \{w,f\}$, a pair (t,s) is obtained by placing s to the left or right of t, aligned on top. The orientation is positive (negative) if s is placed to the right (left). A bounded segment is *tiled by a tile* (*pair*) if in all tilings, the tile (pair) is adjacent to the segment.

We will tile the region $\Gamma_0(r,c)$ in steps, as indicated by the numbers labeled on Figure 11. Notice that each bounded horizontal segment of width $\mathbf{wd}\,f$ on the top border must be tiled by f tiles, labeled 1. Similarly on the left, the bounded vertical segments of height $\mathbf{ht}\,h$ must be tiled by h tiles, labeled 2. This creates a bounded vertical segment of height $\mathbf{ht}\,v + \mathbf{ht}\,s$ on the top left corner, which is tiled by the pair (v,s), labeled 3. It is obvious that it needs to be positively oriented, lest there be a hole of width $\mathbf{wd}\,v - \mathbf{wd}\,s$ and height $\mathbf{ht}\,s$, which cannot be filled.

This creates a new bounded horizontal segment of width $\mathbf{wd}\,w + \mathbf{wd}\,s$, which is tiled by the pair (w,s), labeled 4. If w is on the left, it will create a bounded horizontal segment of width $\mathbf{wd}\,f + \mathbf{wd}\,s$ to its left. Otherwise, if w is on the right, several s will be forced to appear on the left and still create the same bounded segment. Therefore, the (w,s) pair creates the bounded segment, regardless of how it is oriented.

This bounded horizontal segment of width $\mathbf{wd}\,f + \mathbf{wd}\,s$ is again tiled by an (f,s) pair, labeled 5. Like the (v,s) pair above, this needs to be positively oriented. This creates the bounded vertical segment of height $\mathbf{ht}\,v + \mathbf{ht}\,s$, tiled by a pair (v,s), labeled 6. In either orientation, it bounds the vertical segment of width $\mathbf{wd}\,w$ above, concluding that the (w,s) pair (labeled 4) we placed above needs to be positively oriented. Furthermore, this bounds the vertical segment of height $\mathbf{ht}\,h + \mathbf{ht}\,s$, again tiled by the pair (h,s), labeled 7. As before, in either orientation, we have a bounded vertical segment of height $\mathbf{ht}\,v + \mathbf{ht}\,s$, which necessarily needs to be tiled by the positively oriented pair (v,s), labeled 8. This creates a bounded horizontal segment of width $\mathbf{wd}\,w + \mathbf{wd}\,s$.

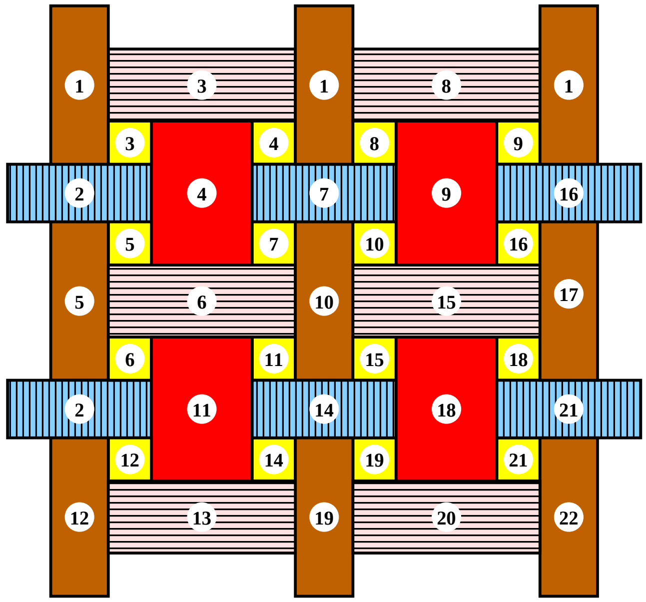


Figure 11: Unique base tiling labeled by order.

We continue in like manner, working our way on the antidiagonal from top right to bottom left. Each time we place the pair (w, s), forcing the adjacent pair (h, s) placed in the previous stage to be positively oriented. Then we place (f, s), forcing the adjacent (v, s) to be positively oriented as well. This procedure repeats with (w, s) and (f, s) in an alternating fashion. The last (f, s) will be placed in positive orientation and creates a bounded vertical segment of height $\mathbf{ht}\, v + \mathbf{ht}\, s$.

Similarly, we work from bottom left to top right on the next antidiagonal. We alternate between placing (v, s) and (h, s) pairs, positively orienting the (w, s) and (f, s) pairs in the previous stage, respectively. This continues until the entire region is filled.

It remains to pick z and ε so all the bounded segments encountered are tiled uniquely by a tile or a pair (in either orientation). One can check that $z = 10$ and $\varepsilon = 1$ works. $\square$

5.4. **Expansion $\mathbf{R}$ of $\mathbf{R}_0$.** We will now define a clever set of perturbed expansion tiles that will correspond to the relational Wang tiles. Only the tiles s, h, and v will have perturbations. Let $\mathbf{W} = \{\tau_1, \ldots, \tau_n\}$ be the fixed set of relational Wang tiles with irreflexive horizontal and vertical Wang relations H and V, respectively. Fix $\varepsilon = 5^n$ and $M = 100\varepsilon$ for the remainder of the section. Let $\mathbf{R}$ be an (M, ε)-expansion of $\mathbf{R}_0$ as follows:

For $t \in \{s, h, v\}$, let $t(a, b)$ be the scaled version of t with height and width increased by a and b, respectively. Imagine that the h and v tiles can stretch horizontally and vertically, respectively, and the s tiles can stretch in both directions. Then the w tiles, having no perturbations, will only shift around a little (by at most ε). See Figure 12. A w tile will be shifted to the right and down by 5^i to *represent* the Wang tile τ_i. To restrict the shifts to only those sizes, we replace s with the appropriate perturbed versions. Namely, for each i, introduce four tiles with perturbations $s(\pm 5^i, \pm 5^i)$, where all four combinations of signs are included. To enforce the Wang relations, for each $\tau_i, \tau_j \in \mathbf{W}$ such that $V(\tau_i, \tau_j)$ or $H(\tau_i, \tau_j)$, we introduce the perturbation $v(5^j - 5^i, 0)$ or $h(0, 5^j - 5^i)$, respectively. This is the set $\mathbf{R}$ we will use.

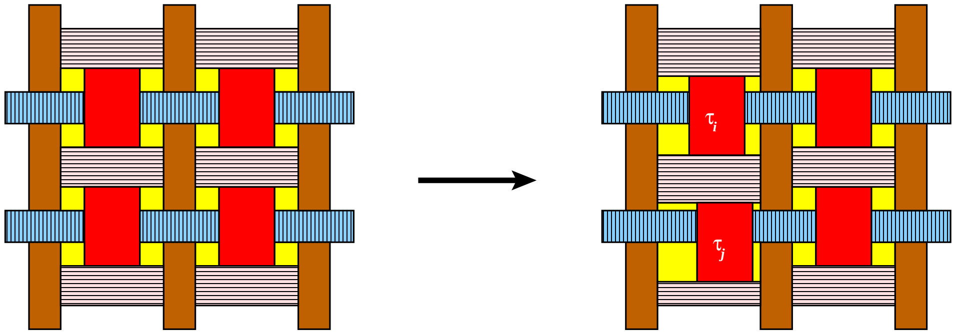


Figure 12: Shifting an expansion of the unique tiling to represent Wang tiles.

5.5. **Rectangular tiling.** Obtain an (M, ε)-expansion $\Gamma(r, c)$ of $\Gamma_0(r, c)$ by scaling with a factor of M and then perturbing it as follows. Recall that there are r protrusions on each vertical side and c cavities on each horizontal side. Each protrusion or cavity corresponds to a boundary tile of Γ in a natural way. Perturb the protrusion or cavity to the right or down, respectively, by 5^i units if it corresponds to τ_i.

Sublemma 5.2 *The (M, ε)-expansion $\Gamma(r, c)$ of $\Gamma_0(r, c)$ respects the expansion $\mathbf{R}$ of $\mathbf{R}_0$.*

Proof. The f tiles are fixed and force the perturbations to stay local. The w tiles have two degrees of freedom. They can move $\pm 5^i$ in each direction, as regulated by the s tiles. □

We now return to the proof of Lemma 3.3. It is clear that given a Wang $\mathbf{W}$-tiling of the rectangle Γ with boundary, we will get an $\mathbf{R}$-tiling of $\Gamma(r, c)$. Indeed, simply take the unique tiling of $\Gamma_0(r, c)$ as afforded by Sublemma 5.1, scale by a factor of M, and then shift each w tile to the right and down by 5^i if it represents τ_i, and adjust the other tiles in the obvious way.

Conversely, if we are given an $\mathbf{R}$-tiling of $\Gamma(r, c)$, we wish to recover the $\mathbf{W}$-tiling of Γ. This is achieved using the following two sublemmas, both of which can be proved by dividing by $5^{\min(i,j,k,\ell)}$ and then considering the equation modulo 5; we omit the (easy) details.

Sublemma 5.3 *The equation $5^i - 5^j = 5^k + 5^\ell$ does not admit a solution in $\mathbb{N}$.*

Therefore each w tile will shift to the right and down (as opposed to shifting left or up), and hence indeed represents a Wang tile τ_i for some i.

Sublemma 5.4 *The equation $5^i - 5^j = 5^k - 5^\ell$ does not admit non-trivial solutions in $\mathbb{N}$, except if $i = j$ or $i = k$.*

If a w tile representing τ_j is to the right of a w tile representing τ_i, then $h(0, 5^j - 5^i)$ must be in $\mathbf{R}$. By the sublemma above, the differences $5^j - 5^i$ are all distinct (recall that the Wang relations are irreflexive, so $i = j$ does not happen), therefore we must have had $H(\tau_i, \tau_j)$ as part of the Wang relation. Similarly for the vertical Wang relation V. So by reading off the associated tile τ_i from the shifts of each w tile, we get a Wang $\mathbf{W}$-tiling of Γ.

This completes the construction of $\Gamma_0(r, c)$ for the case when Γ is a rectangle. For the general case, when Γ is a simply connected region, the proof follows verbatim after replacing $\Gamma(r, c)$ and $\Gamma_0(r, c)$ by appropriate regions.

It remains to get the upper bound estimates on the number of rectangles involved in the construction. Suppose we are given a set of k ordinary Wang tiles using c colors (on the boundary). By Lemma 3.1 we can equivalently consider a set of less than $k + 4c$ relational Wang tiles. To satisfy irreflexivity, we might have to double each of the k tiles twice (once for H and once for V), resulting in $n = |\mathbf{W}| < 4k + 4c$ tiles. When making $\mathbf{R}$, we will have one each of f and w tiles. There will be $4n$ perturbed s tiles and at most n^2 perturbed h and v tiles each. In total,

$$|\mathbf{R}| \leq 2n^2 + 4n + 2 = 2(n+1)^2 \leq 32(k+c)^2.$$

This concludes the proof of Lemma 3.3.

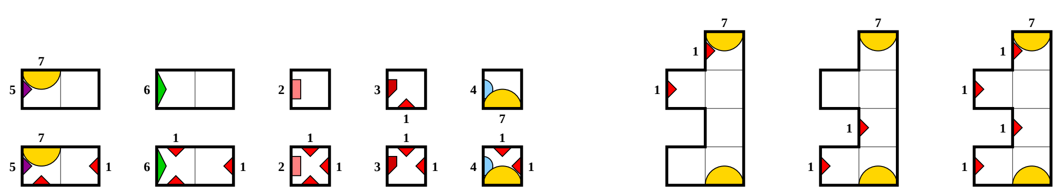
6. Proof of theorems

6.1. **Proof of Theorem 1.1.** In the proof of Lemma 3.2 in Section 4, we constructed the set $\mathbf{W}$ of 23 generalized Wang tiles using 9 colors, such that Simply Connected **W**-Tileability is NP-complete. It remains to count the total number of rectangles we obtain from the series of reduction constructions.

First, we compute the number of ordinary Wang tiles given by the transformation in Lemma 3.1. Observe that the total area of tiles in $\mathbf{W}$ is $9 \cdot 5 + 8 \cdot 4 + 4 \cdot 14 = 133$. Therefore we can break them into 133 ordinary Wang tiles by adding $133 - 23$ more colors. But as these colors do not appear on the boundary, they need not be counted. Hence, in Lemma 3.3, we can take $k = 133$ and $c = 9$, thus giving us at most 10^6 rectangles. □

6.2. **Proof of Theorem 1.2.** First, note that the reduction in the proof of Theorem 1.1 is parsimonious. However, there seems to be no #P-completeness result for the #Cubic Monotone 1-in-3 SAT problem. This is easy to fix by making a similar reduction from the 2SAT problem, whose associated counting problem is #P-complete.

An instance of 2SAT is a set of variables and a collection of clauses. Each clause is a disjunction of two literals, where each literal is either a variable or a negated variable. The problem is to decide whether there is a satisfying assignment such that each clause has at least one true literal. We modify the proof of Lemma 3.2 to obtain a parsimonious reduction from 2SAT. By replacing the two variations of the variable tile by the ones shown in Figure 13a, we may set up unnegated and negated copies of a single variable. Indeed, with a sequence of $5(26)^{r-1}36(26)^{s-1}4$ as colors on the left vertical edge, we create a list of $r + s$ variables, where the last s are negated. By replacing the three variants of the clause tile by the three obvious candidates in Figure 13b, we force each clause to be satisfied.



(a) tiles for variables

(b) clause tiles

Figure 13: Tiles in tileset **T**.

Notice that the modified tileset has a smaller total area, and has the same number of colors used on the boundary. Therefore as in the proof of Theorem 1.1, we apply Lemma 3.3 to conclude that 10^6 rectangles suffice. □

6.3. **Proof of Theorem 2.1.** We consider the following two decision problems:

SUBSET-SUM
Instance: Set $S = \{s_1, \ldots, s_n\} \subset \mathbb{N}$, target $t \in \mathbb{N}$.
Decide: Does there exists $S' \subseteq S$ such that $\sum_{s \in S'} s = t$?

UNBOUNDED SUBSET-SUM
Instance: Set $S = \{s_1, \ldots, s_n\} \subset \mathbb{N}$, target $t \in \mathbb{N}$.
Decide: Does there exists $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $\sum_{i=1}^{n} x_i s_i = t$?

Lemma 6.1 SUBSET-SUM *reduces to* UNBOUNDED SUBSET-SUM.

Proof. Suppose we are given an instance (S, t) of SUBSET-SUM. Pick large m such that $2^m > t2^n + 2^n - 1$; $m > n + \log_2 t$ suffices. Let $u_i = 2^m + 2^{i-1}$, $v_i = 2^m + 2^{i-1} + s_i 2^n$ for each $i \in [n]$. Let $w = n2^m + t2^n + 2^n - 1$. Do UNBOUNDED SUBSET-SUM on set $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and target w. That is, choose some numbers (with repetitions) to sum to w. As $u_i, v_i > 2^m$ for all i, and $w < (n+1)2^m$, we can choose at most n numbers. To satisfy the n lowest bits of w, we must choose at least n numbers. Hence we must choose precisely one of u_i or v_i for each i. The set of v_i chosen corresponds to the original s_i. □

As SUBSET-SUM is NP-complete, so is UNBOUNDED SUBSET-SUM, which is precisely TILEABILITY in dimension one, proving Theorem 2.1.

7. Final remarks and open problems

7.1. In the tiling literature, the original theoretical emphasis was on tileability of the plane, the decidability and aperiodicity. The problem was often stated in the equivalent language of Wang tiles [Ber, Rob2, Wang]. Unfortunately, there does not seem to be any standard treatment of the *finite Wang tiling* problems. Although some equivalences in the Lemma 3.1 are routine, such as the reduction in Figure 2, others seem to be new. We present full proofs for completeness.

7.2. Historically, finite tilings were a backwater of the tiling theory, with coloring arguments being the only real tool [Gol]. On a negative (complexity) side, originally, the tileability problem was studied for general regions, where the tiles were part of the input. The NP-completeness of this most general problem is given in [GJ, §GP13]. When the set of tiles is fixed, NP-completeness was shown for general regions and various fixed small sets of tiles (see [MR] and [BNRR] building on the earlier unpublished work by Robson).

On the positive side, papers of Thurston [Thu] and Conway & Lagarias [CL] introduced the *height function* and the *tiling group* interrelated approaches. The key underlying idea is the use of combinatorial group theory applied to the boundary word of the *simply connected regions*, so the tilings become *Van Kampen diagrams* of the corresponding tiling group. This approach allowed numerous applications to perfect matchings [Cha], tile invariants [Korn, MP, Reid2], tileability [She], various local move connectivity results [KP, Rém1], classical geometric problems [Ken1], applications to colorings and mixing time [LRS], etc. More relevant to this paper, the breakthrough result by C. and R. Kenyon [KK] proved that tileability with bars of simply connected regions can be decided in polynomial time.

This result was further extended to all pairs of rectangles by Rémila in [Rém2], and by Korn [Korn] to an infinite family of *generalized dominoes*. Our Main Theorem puts an end to the hopes that these results can be extended to larger sets of rectangles.

7.3. We conjecture that in the Main Theorem (Theorem 1.1), the number of rectangles can be reduced down to 3, thus matching the lower bound (Rémila's tileability algorithm for the case of two rectangles). As a minor supporting evidence in favor of this conjecture, let us mention that the proofs in [KK, Rém2] are crucially based on *local move connectivity*, which fails for three general rectangles. In the absence of algebraic methods, there seem to be no other (positive) approach to tileability.

7.4. This result of Main Theorem can be contrasted with a large body of positive results on tiling rectangular regions with a fixed set of rectangles.

Theorem 7.1 ("Tiling rectangles with rectangles" Theorem [LMP]) *For every finite set* $\mathbf{R}$ *of rectangular tiles, the tileability problem of an* $[M \times N]$ *rectangle can be decided in* $O(\log M + \log N)$ *time.*

Note that the Theorem 7.1 has linear time complexity for the rectangular regions written *in binary.* This result is based on the pioneer results by Barnes [Bar1, Bar2] applying commutative algebra, the *finite basis theorem* [DK] (see also [Reid1]), and the *transfer matrix method* (see e.g. [Sta, Ch. 4]).

It seems, tilings of rectangles have additional structure, which general regions do not have. See e.g. [BSST, C+, Rob1] for assorted results on the subject. On the other hand, as shown in Theorem 2.1, when the tiles are part of the input, deciding tileability can be NP-hard, and the proof can be used to show that counting tilings is #P-hard. Note that the results in [LMP] only discuss tileability, not counting. It would be interesting to obtain general results on the local move connectivity and hardness of counting results for tilings of rectangular regions with rectangles.

Let us also mention a large body of hardness results in the 1-dimensional case, which are somewhat related to Theorem 2.1 (see e.g. [GJ, Ram]).

7.5. Although counting perfect matchings in general graphs is #P-complete, for the grid graphs a *Pfaffian formula* gives a count for the number of domino tilings for any (not necessarily simply connected) region; this formula can be applied in polynomial time [LP] (see also [Ken2]). In a different direction, Moore and Robson [MR] conjecture that already for two bars, the problem is #P-complete for general regions. They note that the corresponding reductions in [BNRR, MR] are not parsimonious. Thus, until now, the #P-completeness was open for any finite set of rectangular tiles, even for general regions.

We make a stronger conjecture that for every tileset $\mathbf{T}$ of two bars $[1 \times k]$ and $[\ell \times 1]$, where $k, \ell \geq 2$, $(k, \ell) \neq (2, 2)$, the counting of tilings by $\mathbf{T}$ of simply connected regions is #P-complete. In particular, the number 10^6 in Theorem 1.2 can be decreased to 2. There is no direct evidence in favor of this, except that the general combinatorial counting problems tend to be #P-complete unless there is a special algebraic formula counting them. Furthermore, when it comes to tile counting, there seem to be no direct benefit of simple connectivity of the regions, so such result is likely to be equally hard as for general regions. We refer to [Jer] for the introduction and references.

7.6. By a simple modification of the Wang tiles, we can also get a parsimonious reduction from SAT. For that, first, we can introduce wire splitters and the NOT gate. By doing so, we remove the "cubic" and "monotone" constraints, respectively. These would play the same role as crossover tiles, and require a separate color on the boundary for each. This would also increase the set of tiles by introducing new variants for the V and L tiles as well. We omit the details.

We can then introduce the AND gate in a similar fashion, again with a new control color on the top and new versions of the V, C and L tiles. This gives the embedding of SAT. This reduction is parsimonious in the same way as the reduction in Theorem 1.2, which implies that the associated counting problem is also #P-complete.

Let us compute the total number of rectangles necessary for this construction. First, this would increase the number of Wang tiles from 23 to no more than $23 \cdot 8$. Then, the same argument as above gives the 10^8 bound in the number of rectangular tiles. We omit the (easy) calculation and details.

7.7. The reductions in this paper can be used to prove *uniqueness* results on tileability with rectangles, i.e. whether there exists a unique tiling of a region with a given set of rectangular tiles. In [BNRR], the problem was completely resolved in the case of two bars. An even simpler solution follows from [KK] in this case. Since all tilings are local move connected, taking the "minimal tiling" constructed by the algorithm in [KK] and trying all potential moves gives an easy polynomial time test. More generally, Rémila [Rém2] showed that for two general rectangles one can go from one to another with certain non-local moves which are easy to describe. Again, since he produces the "minimal tiling," his algorithm can be used to decide unique tileability with two rectangles.

Now, our approach, via reduction from the general SAT problem (see above) shows that for a certain finite set of rectangles, uniqueness of tilings of a simply connected region is as hard as UNIQUE SAT, which is co-NP-hard and has been extensively studied [BG, VV]. This seems to be the first result of this type.

7.8. A version of Theorem 2.2 was first announced in Levin's original short note on NP-completeness [Lev] but, to the best of our knowledge, the proof has never been published. Curiously, the NP-completeness of tileability of general regions with tiles as part of the input was also announced in [GJ], referencing an unpublished preprint (yet?). Of course, now we have much stronger results.

7.9. The domino tileability is a special case of perfect matching, solvable in quadratic time on all planar bipartite graphs [LP]. However, Thurston's classical domino tileability algorithm [Thu] (see also [Cha]) is linear time (in the area) on all simply connected regions.

7.10. Although Theorem 7.1 extends directly to bricks in higher dimensions [LMP], this is an exception rather than the rule. In fact, almost no other positive tileability results extend to higher dimensions, even Thurston's algorithm mentioned above (see, however, [RY]). We refer to [Korn] for more on this.

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