A SHORT PROOF OF RIGIDITY OF CONVEX POLYTOPES

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ABSTRACT. We present a much simplified proof of Dehn's theorem on the infinitesimal rigidity of convex polytopes. Our approach is based on the ideas of Trushkina [11] and Schramm [10].

Introduction

Let $P \subset \mathbb{R}^3$ be a simplicial convex polytope. Define a *continuous deformation* $\{P_t : t \in [0,1]\}$ of $P = P_0$ to be a family of convex polytopes with the same combinatorial structure, the same corresponding edge lengths, and continuity on vertices. We say that P is *continuously rigid* if every such deformation is a rigid motion in $\mathbb{R}^3$. It is a classical corollary from the Cauchy theorem that every simplicial polytope P is continuously rigid (see below). In this article we present a simple proof of Dehn's theorem, which also implies the continuous rigidity.

Let $\boldsymbol{v}_i(t) = \overrightarrow{Ov_i}$ be a vector from the origin O into the vertex v_i in P_t. Think of vectors $\boldsymbol{v}'_i(t)$ as of *velocities* of vertices v_i. For an edge length $|v_iv_j|$ to be constant under the deformation we need $\|\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\|' = 0$, where $\|\boldsymbol{w}\| = (\boldsymbol{w}, \boldsymbol{w}) = |\boldsymbol{w}|^2$. Thus, in particular, at $t = 0$ we have:

$$0 = \frac{d}{dt} \left\| \boldsymbol{v}_i(t) - \boldsymbol{v}_j(t) \right\|_{t=0} = \frac{d}{dt} \left\| \left(\boldsymbol{v}_i(0) - \boldsymbol{v}_j(0) \right) + t \left(\boldsymbol{v}'_i(0) - \boldsymbol{v}'_j(0) \right) \right\|_{t=0}$$
$$= 2 \left(\boldsymbol{v}_i(0) - \boldsymbol{v}_j(0), \boldsymbol{v}'_i(0) - \boldsymbol{v}'_j(0) \right).$$

This leads to the following definition of infinitesimal rigidity.

Let $V = \{v_1, \ldots, v_n\}$ and E be the set of vertices and edges of P. We will assume that vertices v_1, v_2 and v_3 form a triangular face of P; they are called *base vertices*, and face $(v_1v_2v_3)$ is called *base triangle*. Suppose we are given a vector $\boldsymbol{a}_i \in \mathbb{R}^3$, for every $i \in \{1, \ldots, n\}$. We say that the set of vectors $\{\boldsymbol{a}_i, 1 \le i \le n\}$ defines an *infinitesimal rigid motion* if

$$(*) \qquad (\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i - \boldsymbol{a}_j) = 0, \quad \text{for every } (v_i, v_j) \in E.$$

An infinitesimal rigid motion is called *planted* if the base velocities vectors are equal to zero: $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \boldsymbol{a}_3 = \boldsymbol{0}$. Finally, we say that a simplicial polytope $P \subset \mathbb{R}^3$ is *infinitesimally rigid* if every planted infinitesimal rigid motion is trivial: $\boldsymbol{a}_i = \boldsymbol{0}$, for all $i \in \{1, \ldots, n\}$. Here is the main result of this paper:

Dehn's Theorem. *Every simplicial convex polytope in $\mathbb{R}^3$ is infinitesimally rigid.*

Of course, the restriction to planted infinitesimal rigid motions is necessary, as the usual rigid motions of P in $\mathbb{R}^3$ can define nontrivial infinitesimal rigid motions. By the argument above, the infinitesimal rigidity implies the continuous rigidity:

Corollary (Cauchy). *Every simplicial convex polytope in $\mathbb{R}^3$ is continuously rigid.*

In this paper we present a new proof of Dehn's theorem, based on the approach by Trushkina [11] (see also a followup [12]). Unfortunately, the technical details in Trushkina's paper are somewhat complicated. We substitute Trushkina's definition of the *inversion* with the one given by Schramm [10] (for different purposes). We should mention an important survey [4] (see §4.6), which helped us to translate the ideas in [10] into the language of infinitesimal rigidity.

Before we conclude, let us repeat that the continuous rigidity of convex polytopes follows from the classical Cauchy theorem [1, 3, 4]. Dehn's theorem was established by Dehn in [5] in an equivalent language of the *static rigidity*. This result became fundamental in the modern study of rigidity of frameworks and non-convex polyhedra [4, 14], and a number of proofs have been found. We refer to [2, 8] for the exposition of Dehn's original proof, to [6, 7, 9, 13] for applications and modern treatment, and to [2, 4] for further references.

1. Proof of infinitesimal rigidity

First, note that equations $(*)$ above say that the difference in velocities along an edge is orthogonal to this edge of the polytope. Think of velocity vectors as vector functions on vertices of P which are equal to $\mathbf{0}$ on base vertices v_1, v_2, v_3. The idea of the proof is to enlarge the set of such functions and prove a stronger result.

As before, let $V = \{v_1, \ldots, v_n\}$ and E be the set of vertices and edges of a simplicial convex polytope $P \subset \mathbb{R}^3$. Consider the set of all vector sequences $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$, $\boldsymbol{a}_i \in \mathbb{R}^3$, such that for every edge $(v_i, v_j) \in E$ we have one of the following three possibilities:

1. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) = (\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) = 0,$
2. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) < 0$ and $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) < 0,$
3. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) > 0$ and $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) > 0.$

In other words, we require that projections of velocity vectors $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ onto edge (v_i, v_j) have the same signs. We say that a vertex v_i is *dead* if $\boldsymbol{a}_i = \boldsymbol{0}$; it is *live* otherwise. We need to prove that for every vector sequence $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ as above, if the base vertices are dead, then all vertices $v_i \in V$ are dead. By definition of the infinitesimal rigidity, this would immediately imply the theorem.

Denote by $\Gamma = (V, E)$ the graph of P. Since P is simplicial, Γ is a plane triangulation. Consider an orientation of edges of Γ in the direction of projections of the velocity vectors. More precisely, we orient edges $v_i \to v_j$ in case **2**, orient them $v_i \leftarrow v_j$ in case **3**, and leave them unoriented in case **1**. Clearly, the edges adjacent to dead vertices are unoriented.

Consider two edges $e = (v_i, v_j)$ and $e' = (v_i, v_r)$, $e, e' \in E$, with a common vertex v_i, such that $(v_i v_j v_r)$ is a face in P. We say that edges e and e'

  ○ have *one inversion* if one of them is oriented into v_i, and the other out of v_i,
  ○ have *zero inversions* if both of them are oriented into v_i or out of v_i,
  ○ have a *half-inversion* if one of the edges is oriented and the other is unoriented,
  ○ have *one inversion* if both of them are unoriented and v_i is a live vertex,
  ○ have *zero inversions* if both of them are unoriented and v_i is a dead vertex.

When we talk about the number of inversions in a subgraph, in a triangle, or around a vertex, we mean the total sum of inversions between pairs of edges involved. For example, we say that a graph *has at least q inversions* if this sum is $\geq q$.

A triangle is called *active* if at least one of its vertices is live; it is called *inactive* otherwise. Now consider different orientations of an active triangle $(v_i v_j v_r)$ where vertex v_i is live (see some of them in Figure 1). A simple enumeration of all possible cases gives the following result:

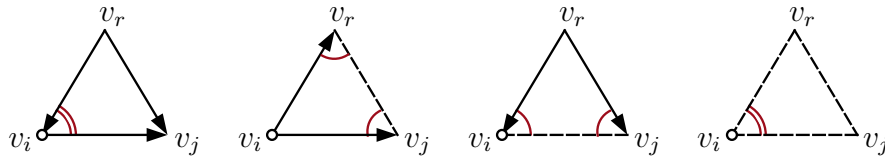**Lemma 1.** *Every active triangle has at least one inversion.*

PSfrag replacements

FIGURE 1. Different orientations of $(v_i v_j v_r)$, where vertex v_i is live.

This lemma gives a lower bound on the number of inversions in Γ. To get an upper bound, we count inversions around vertices:

Lemma 2. *There are at most two inversions around every live vertex.*

We postpone the proof of the lemma until we finish the proof of the theorem. Consider what this gives us when the only unoriented edges in Γ are the base edges (v_1, v_2), (v_1, v_3), and (v_2, v_3). In this case we have $(n-3)$ live vertices. Recall also that there are no inversions around dead vertices. Thus, by Lemma 2, there are at most $2(n-3) = 2n-6$ inversions in Γ. On the other hand, recall that a triangulation with n vertices always has $(2n-4)$ triangles (see e.g. [3]), and, by assumption, only one of them is inactive. Thus, by Lemma 1, there are at least $(2n-5)$ inversions in Γ, a contradiction.

We use the same strategy in general case. Remove from Γ all inactive triangles together with all edges and vertices which belong only to inactive triangles. Denote by $H = (V', E')$ a connected component of the remaining graph. Since Γ is planar, the induced subgraph H of Γ has a well defined boundary ∂H. Denote by k the number of vertices in ∂H (all of them dead), and by ℓ the number of connected components of ∂H. Finally, denote by m the number of vertices in $H \smallsetminus \partial H$ (some of them live and some possibly dead). By Lemma 2, there are at most $2m$ inversions in H.

Let us now estimate the number of inversions via the number t of triangles in H. Observe that the total number of vertices and edges in H is given by

$$|V'| = m + k, \quad \text{and} \quad 2|E'| = k + 3t.$$

On the other hand, there are $f = (t + \ell)$ faces in H taken together with the boundary components. Substituting these values into Euler's formula $|V'| - |E'| + f = 2$, we conclude that graph H has exactly $t = 2m + k + 2\ell - 4$ triangles. Since there is at least one inactive triangle $(v_1 v_2 v_3)$, we have $k \geq 3$ and $\ell \geq 1$. Therefore, by Lemma 1, there are at least

$$t = 2m + k + 2\ell - 4 \geq 2m + 3 + 2 - 4 = 2m + 1$$

inversions in H, a contradiction. $\square$

2. Proof of Lemma 2.

Let us consider all possibilities one by one, and check the claim in each case. Suppose a vertex v_i is adjacent to three or more unoriented edges. This means that $\boldsymbol{a}_i$ is orthogonal to at least three vectors spanning $\mathbb{R}^3$. Therefore, $\boldsymbol{a}_i = \boldsymbol{0}$ and v_i is a dead vertex with zero inversions.

Suppose now that v_i is adjacent to exactly two unoriented edges e, e' in P. This means that $\boldsymbol{a}_i \neq \boldsymbol{0}$ is orthogonal to a plane spanned by these edges. Observe that e, e' separate the edges oriented into v_i from those oriented out of v_i. Thus, there are either two half-inversions and one inversion if the edges e, e' are adjacent, or four half-inversions if e, e' are not adjacent (see Figure 2).
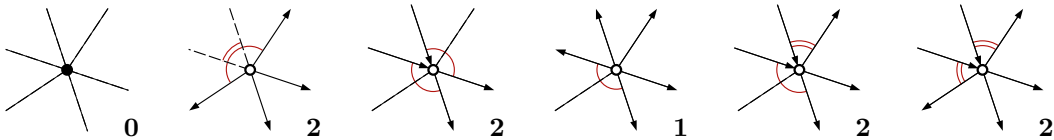
PSfrag replacements

Figure 2. The number of inversions around a vertex in different cases.

Next, suppose that v_i is adjacent to exactly one unoriented edge e. Since $\boldsymbol{a}_i \neq \boldsymbol{0}$ in this case, consider a plane containing v_i and orthogonal to $\boldsymbol{a}_i$. This plane contains e and separates the edges oriented into v_i from those oriented out of v_i. Therefore, there are either two half-inversions if all other edges are oriented into v_i, or out of v_i, or two half-inversions and one inversion otherwise.

Finally, if v_i is a live vertex that is not adjacent to any unoriented edges, then the plane orthogonal to $\boldsymbol{a}_i$ separates the edges into two parts: those oriented into v_i from those oriented out of v_i. Thus, there are exactly two inversions in this case. $\square$

Final remarks. Let us note that Dehn's theorem and Cauchy's corollary hold in the generality of all convex polytopes. The reduction to the simplicial case is straightforward: triangulate the surface of P by adding diagonals of the faces. We skip the details. We should mention also that Dehn's original proof is also based on a graph-theoretic argument, very much different from the one presented here.

The proof we presented above easily splits into two parts: global and local, not unlike the original proof of the Cauchy theorem (see e.g. [3, 4]). The local part (Lemma 2), while different, has the same flavor as the the sign changes lemma in

Cauchy's proof. It is even more similar to the local part of other proofs of Dehn's theorem (see [4, 8, 13]). The global part is a graph-theoretic argument somewhat similar in style and complexity to the double counting argument used in the proof of the Cauchy theorem, with both arguments based on Euler's formula.

Finally, let us mention that the result of Trushkina is a bit stronger as she allows the base vertices not to be on the same face. When applied to the infinitesimal rigidity of polytopes, this gives an equivalent result, but the graph-theoretic claim becomes harder to prove. Similarly, Schramm proves a much stronger result about certain delicate graph labeling [10]. Interestingly, at some point he employs a counting angles argument, essentially reproving Euler's formula.

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