SPECIAL CASE OF NHLF

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This is a quick followup on our paper [1]. Let $\lambda = (k^\ell)$, $\mu = \big((k-1)^{\ell-1}\big)$. Then, clearly,

$$f^{\lambda/\mu} \, = \, \big|\mathrm{SYT}(k, 1^{\ell-1})\big| \, = \, \binom{k+\ell-2}{k-1}.$$

The excited diagrams $D \in \mathcal{E}(\lambda/\mu)$ are the complements to paths $\gamma \in \Upsilon(k,\ell)$ from $(1,k)$ to $(\ell,1)$ in $[\lambda]$. Thus, we have:

$$\big|\mathcal{E}(\lambda/\mu)\big| \, = \, \big|\Upsilon(k,\ell)\big| \, = \, \binom{k+\ell-2}{k-1}.$$

Using the symmetry of λ, the NHLF gives the following summation:

$$(\diamond) \qquad (k+\ell-1)! \sum_{\gamma\in\Upsilon(k,\ell)} \prod_{(i,j)\in\gamma} \frac{1}{i+j-1} \, = \, \binom{k+\ell-2}{k-1}.$$

Denote the product above by X_γ:

$$X_\gamma \, := \prod_{(i,j)\in\gamma} \frac{1}{i+j-1}\,.$$

The equation $(\diamond)$ then implies following curious result.

Corollary: *Let $\gamma \in \Upsilon(k,\ell)$ be a grid path chosen uniformly at random. Then:*

$$\mathbb{E}[X_\gamma] \, = \, \frac{1}{(k+\ell-1)!}\,.$$

The formula has a strange probabilistic feel. Here is another way to understand it. Take a path γ from $(1,1)$ to (ℓ,k). Then $X_\gamma = 1/(k+\ell-1)!$, for all such γ. In other words, the r.v. X_γ averages out to the same value no matter whether they are $(1,1) \to (\ell,k)$ or $(1,k) \to (\ell,1)$.

The corollary can be proved directly by induction, see here:
`http://math.stackexchange.com/a/1591493/17176`

The induction argument generalizes the statement above: $\mathbb{E}[X_\gamma]$ is equal to the same value for random paths $\gamma : (a,b) \to (c,d)$, as for random paths $\gamma : (a,d) \to (c,b)$, for all $a,b,c,d \geq 1$. It would be very interesting to find a probabilistic explanation of this equality.

[1] A. Morales, I. Pak and G. Panova, Hook formulas for skew shapes, preprint (2015), 40 pp.

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