# RIGIDITY AND POLYNOMIAL INVARIANTS OF CONVEX POLYTOPES

MAKSYM FEDORCHUK⋆ AND IGOR PAK⋆

July 22, 2004

Abstract. We present an algebraic approach to the classical problem of constructing a simplicial convex polytope given its planar triangulation and lengths of its edges. We introduce *polynomial invariants* of a polytope and show that they satisfy polynomial relations in terms of squares of edge lengths. We obtain sharp upper and lower bounds on the degree of these polynomial relations. In a special case of *regular bipyramid* we obtain explicit formulae for some of these relations. We conclude with a proof of Robbins Conjecture [R2] on the degree of generalized Heron polynomials.

## 1. Introduction

Rigidity of convex polytopes is a classical subject initiated almost two hundred years ago with *The Cauchy Rigidity Theorem.* The work on rigidity and metric geometry of 3-dimensional convex polytopes was later continued by A. D. Aleksandrov and his school, and by now has become a subject on its own right, with a number of techniques and applications (see e.g. [C4, CW, GSS]). In an important development [S1, S2, S3, S4], Sabitov introduced *polynomial relations* for the volumes and "small diagonals" of nonconvex polyhedra[1]. In this paper we introduce an algebraic approach to problems of rigidity by establishing the existence of such relations for convex polytopes in great generality. One can view these polynomial relations as a quantitative version of the classical rigidity theorems. We conclude with a number of examples and special cases. The proofs involve ideas from algebraic geometry and some delicate calculations.

Fix a planar triangulation G and a function $\ell$ on its edges: $\ell(e_i) = \ell_i$. The *Cauchy Rigidity Theorem* says that every continuous deformation of $P$ which preserves the edge lengths is in fact a rigid (Euclidean) motion. Cauchy further extended his result to show that $P$ is in fact the only polytope, up to translations and rotations in $\mathbb{R}^3$, with the same lengths and combinatorics of edges (see Section 2.5 for a precise statement). In a different direction, existence of polytopes with a given and edge length is a difficult problem only partly resolved by the celebrated *Aleksandrov's Existence Theorem* (see Section 2.6). We are interested in the following problem:

- *Construct a polytope $P$ from its graph* G *and edge lengths $\ell_1, \ldots, \ell_m$,*

where by *construct* we mean compute its metric invariants, such as lengths of the diagonals, i.e. compute the coordinates of vertices $v_1, \ldots, v_n$ of (any rigid motion of) $P$. This classical problem has been the source of inspiration for decades, and remains largely open. In recent years it reappeared in a computer science context [E] and in mathematical literature [S3, S4]. Our algebraic approach to the problem gives a theoretical solution and at the same time limits the possibility of an effective "practical" solution.

The main objects studied in this paper are *realization spaces* of polytopes (see Section 2.3) and *polynomial invariants* (see Section 6). Broadly speaking, realization spaces are moduli

⋆Department of Mathematics, MIT, Cambridge, MA, 02139. Email: maksym@mit.edu, pak@math.mit.edu.

[1] Throughout the paper we always use the word *polytope* for convex bodies and the word *polyhedron* for nonconvex bodies.

spaces of embeddings of vertices of a graph G into $\mathbb{R}^3$, which correspond to polytopes combinatorially equivalent to a given simplicial polytope $P$ with graph G. Similarly, polynomial invariants are, roughly, polynomials in coordinates of the polytope vertices, invariant under Euclidean transformations. We use the geometry of realization spaces to prove general results on polynomial invariants. The following theorem is the main result of the paper.

Denote by $f_i$ the squared length of the edge $e_i$ written as a polynomial in the coordinates.

**Theorem 1.1.** *Let* G *be a plane triangulation with $m$ edges, and let $\mathfrak{I} = \mathfrak{I}_{\mathrm{G}}(\cdot)$ be a polynomial invariant of convex polytopes with graph* G. *Then $\mathfrak{I}$ satisfies the following nontrivial* polynomial relation:

$$(\circ) \qquad C_N\, \mathfrak{I}^N \,+\, C_{N-1}\, \mathfrak{I}^{N-1} \,+\, \ldots \,+\, C_1\, \mathfrak{I} \,+\, C_0 \,=\, 0,$$

*where coefficients $C_r \in \mathbb{C}[f_1, \ldots, f_m]$ depend only on the graph* G *and the invariant $\mathfrak{I}$. Moreover, for the degree $N$ of* polynomial relation $(\circ)$ *we have: $N \le 2^m$.*

The upper bound in the theorem will be further strengthened in Section 6, where it is stated in terms of realizations of polytopes, and will be coupled with the lower bound.

The theorem is an extension of Sabitov's work [S1, S2, S3, S4], where he showed that one can construct a polytope $P$ by finding explicit *polynomial relations* for the length $d_{ij}$ of the diagonal joining vertices $v_i$ and $v_j$, for certain pairs $(i,j)$. We have in this case:

**Corollary 1.2.** *Let* G *be a plane triangulation with $m$ edges, and let $P$ be a convex polytope with graph* G. *Then for every a pair $(i,j)$ of vertices in* G, *the length of the diagonal $d_{ij}$ is a root of a nonzero polynomial*

$$(*) \qquad c_N\, x^N \,+\, c_{N-1}\, x^{N-1} \,+\, \ldots \,+\, c_1\, x \,+\, c_0\,,$$

*where coefficients $c_r \in \mathbb{C}[\ell_1, \ldots, \ell_m]$ depend only on the graph* G *and the pair $(i,j)$. Moreover, the degree $N$ is at most $4^m$.*

Let us emphasize here that $f_i$ are polynomials in the coordinates, while the edge lengths $\ell_i$ and diagonal lengths $d_{ij}$ are real numbers. This distinction will be important here and in the future.

Observe that once we have all $d_{ij}$, we can immediately construct the polytope $P$. Indeed, without loss of generality, we can assume that vertices $v_1$, $v_2$ and $v_3$ form a triangular face with given coordinates. Now use the diagonal lengths $d_{1i}$, $d_{2i}$ and $d_{3i}$ to compute the coordinates of $v_i$. See Section 10 for further remarks on the history and references.

To see the connection between Corollary 1.2 and the Cauchy Rigidity Theorem simply observe that nonzero equations $(*)$ imply that the diagonal lengths take a discrete set of values. Since a simplicial polytope $P$ can be constructed given its diagonal lengths, this implies that there can be at most a finite number of *realizations* of $P$. Thus, the polytope cannot be continuously deformed with its edge lengths being preserved.

Now, the rigidity argument fails for edge lengths $\{\ell_i\}$ when all polynomials $c_r$ are zero. This is related to existence of nonrigid polyhedra, called *flexors*. Such polytopes must have edge lengths so that all polynomial coefficients $c_r$ vanish for (usually) several polynomial invariants, corresponding to a certain subset of the diagonals. In fact, Sabitov shows that relations $(*)$ are nonzero on "small diagonals" of convex polytopes.

Interestingly, Sabitov's approach is motivated by the *Bellows Conjecture* for flexors. Existence of flexors was a long standing open problem until their celebrated discovery by Connelly [C1, C3] (see also [D, S5]). The Bellows Conjecture states that the volume of flexors remains invariant under continuous transformation. Sabitov showed [S3, S4] that the volume of flexors satisfies the polynomial relation $(\circ)$, with the polynomial $C_N(\ell_1, \ldots, \ell_m) = \pm 1$. This implies Bellows Conjecture by the reasoning as above. Similarly, for convex polytopes Theorem 1.1 gives the following result.

**Corollary 1.3.** *Let* G *be a plane triangulation with $m$ edges, and let $P$ be a convex polytope with graph* G. *Then the volume* $\mathrm{vol}(P)$ *is a root of a nonzero polynomial*

$$(**) \qquad c_N\, x^N \,+\, c_{N-1}\, x^{N-1} \,+\, \ldots \,+\, c_1\, x \,+\, c_0\,,$$

*where coefficients $c_r \in \mathbb{C}[\ell_1, \ldots, \ell_m]$ depend only on the graph* G. *Moreover, the degree $N$ is at most $2^m$.*

We introduce polynomial invariants of $P$ in Section 6 and prove for them polynomial relations ($\circ$) and the degree condition. The proof naturally splits into two parts. First, we look into the proof of the Cauchy Rigidity Theorem and extract an algebraic ingredient needed in the algebraic part. We then prove a key algebraic lemma (Generic Freeness Lemma 4.1) and deduce the results from there.

Our interest in the degrees of polynomial relations is rooted in our belief that the problem of constructing polytopes is computationally intractable. We elaborate further on this problem in Section 10.

Our second reason to study the degrees of polynomial relations lies in the following unexpected application. Let $\mathtt{S}(a_1, \ldots, a_n)$ denote the area of a $n$-gon inscribed in a circle with the side lengths $a_1, \ldots, a_n$. Note that $\mathtt{S}(\cdot)$ is a symmetric function of $n$ variables [R2]. Define the sequence $\{\Delta_i\}$ as follows:

$$(\clubsuit) \qquad \Delta_k \,:=\, \frac{2k+1}{2}\binom{2k}{k} \,-\, 2^{2k-1} \,=\, \sum_{i=0}^{k-1}(k-i)\binom{2k+1}{i},$$

which has a combinatorial interpretation as the total number of (both convex and nonconvex) $(2k+1)$-gons inscribed in a circle. The initial terms of the sequence are: $\Delta_1 = 1$, $\Delta_2 = 7$, $\Delta_3 = 38$, $\Delta_4 = 187$, $\Delta_5 = 874$, etc.

**Theorem 1.4.** (Robbins Conjecture) *For every $n$ there exists a nonzero polynomial*

$$(\mathfrak{s}) \qquad c_\nu\, x^\nu \,+\, c_{\nu-1}\, x^{\nu-1} \,+\, \ldots \,+\, c_1\, x \,+\, c_0\,,$$

*with a root $\mathtt{S}^2(a_1, \ldots, a_n)$ and coefficients $c_i \in \mathbb{C}[a_1^2, \ldots, a_n^2]$. Moreover, the smallest possible degree $\nu = \nu(n)$ of such polynomial is equal to $\Delta_k$ if $n = 2k+1$, and $2\Delta_k$ if $n = 2k+2$.*

One can view ($\mathfrak{s}$) as generalizations of the Heron formula for the area of a triangle (see Section 2.8 below). In this language, Robbins computed *generalized Heron polynomials* for $n = 4, 5, 6$, conjectured Theorem 1.4, and made additional conjectures based on these observations [R1, R2].

From the above combinatorial interpretation of $\Delta_k$ Robbins derived the lower bound as in the theorem (ibid.), but the upper bound remained elusive. The effort to obtain explicit formulas for generalized Heron polynomials has recently received much attention [L], when Robbins and Roskies partially resolved[2] the case $n = 7$.

Until now no connection between the Robbins Conjecture and the rigidity of polytopes has been established. In Section 7.6 we compute polynomial relations ($\circ$) for the 'main diagonal' of a regular bipyramid. Then, in Section 8 we use these results to prove Robbins Conjecture. Finally, in Section 10 we outline additional applications and directions for future research.

*On exposition.* The nature of this paper is rather unusual as we use elements of Rigidity Theory, Algebraic Geometry, and classical Geometric Combinatorics. Rather than make the paper a patchwork between the fields, we made a decision to include some background material and several known results. We hope this helps to illustrate and illuminate our approach, and allows the reader unfamiliar with the subject to follow the presentation. The paper is now nearly self-contained, with possible exception of basic results in Algebraic Geometry. The history, connections to known results, and pointers to references are postponed till Section 10.

[2] This work continues after Robbins' death and these calculations were recently reported in [MRR].

## 2. Definitions and basic results

2.1. **Graph of a polytope.** Throughout the paper we consider only simplicial polytopes, unless explicitly stated otherwise. As in the introduction, let $P$ be a convex simplicial polytope in $\mathbb{R}^3$. Vertices of $P$ are denoted by $v_1, \ldots, v_n$, edges by $e_1, \ldots, e_m$, and edge lengths by $\ell_1, \ldots, \ell_m$. Let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Since every face is a triangle, Euler's formula gives $m = 3n - 6$.

Denote by G the *graph* (1-skeleton) of $P$. The cyclic order on edges adjacent to the same vertex gives it a structure of a *plane graph* [Bo]. In our case, graph G is, in fact, 3-connected and has a unique (up to isomorphism) embedding into a sphere. By a slight abuse of notation, we use $v_i$ and $e_j$ to refer to vertices and edges of G as well.

We say that G is a *triangulation* if each face of G (including the outside face) is a triangle. Clearly, a graph of a simplicial polytope is a triangulation.

A pair $\mathcal{L} = (\mathrm{G}, L)$ is called *length diagram* of $P$, where $L\colon E \to \mathbb{R}_+$ given by $L(e_j) = (\ell_j)^2$ is a *length function* on edges of G. As we mentioned earlier, The Cauchy Rigidity Theorem says that two convex polytopes with the same length diagram are congruent.

Let $W \subset \mathbb{R}\langle E\rangle$ be any subspace of length functions $L$ as above. The pair $\mathcal{W} = (\mathrm{G}, W)$ is called *length family*, and corresponds to a *family of polytopes* containing $P$. The dimension $d = \dim(W)$ is called *degree of freedom* of the length family $\mathcal{W}$.

Many natural "polytopes" are in fact families of polytopes as opposed to length diagrams of individual polytopes. For example, *regular tetrahedron* and *regular octahedron* correspond to a complete graph $K_4$ and a complete tripartite graph $K_{2,2,2}$ with all edges of the same length. The dimension of the subspace $W$ is one in this case.

2.2. **Examples of families of polytopes.** Here is a natural way to obtain the length family as above. Let $E = \sqcup_{i=1}^d E_i$ be a *partition* of $E$ into disjoint sets of edges, and let $W$ be a vector space of all length functions such that edges from the same set $E_i$ have the same length: $L(e_j) = L(e_r)$ for all $e_j$ and $e_r$ in the same $E_i$. Clearly, the number $d$ of subsets $E_i$ is equal to the degree of freedom of a corresponding length family $\mathcal{W}$.

For example, an *equilateral bipyramid* corresponds to a length family $\mathcal{F} = (\mathrm{G}_n, W)$, where $\mathrm{G}_n$ is a graph on $(n+2)$ vertices $V = \{v_1, \ldots, v_n, u_1, u_2\}$ with edges $E = E_0 \sqcup E_1$, where $E_0 = \{(u_1, v_1), \ldots, (u_1, v_n), (u_2, v_1), \ldots, (u_2, v_n)\}$ and $E_1 = \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. The degree of freedom $d(\mathcal{W})$ is equal to 2.

Observe that equal lengths of edges in $E_0$ imply that points $v_1, \ldots, v_n$ lie of the intersection of two spheres (of the same radius). Thus they lie on the same plane $H$. Now equal lengths of edges in $E_1$ imply that $v_1, \ldots, v_n \in H$ form a regular $n$-gon, and thus the polytope $P$ is a union of two regular pyramids.

A different example is a *regular bipyramid* which corresponds to a length family $\mathcal{F} = (\mathrm{G}, W)$, where $\mathrm{G}_n$ is a graph defined above with edges $E = E_0 \sqcup_{i=1}^n E_i$. Here $E_0$ is as above, $E_i = \{(v_i, v_{i+1})\}$, for $1 \le i < n$, and $E_n = \{(v_1, v_n)\}$. In this case points $v_1, \ldots, v_n$ again lie on the same plane $H$, and the corresponding polytope $P$ is a union of a $n$-pyramid and its reflection with respect to $H$. In this case the degree of freedom $d(\mathcal{W})$ is equal to $n+1$.

2.3. **Realization space.** Define a *realization* of a graph G to be a map $\varphi\colon V \to \mathbb{R}^3$, and let $\mathfrak{M}_\circ(\mathrm{G})$ be the set of all realization of G. We say that a realization is *convex* if the points $\varphi(v_1), \ldots, \varphi(v_n) \in \mathbb{R}^3$ are in a strictly convex position. In other words, we cannot have

$$\varphi(v_i) = \sum_{j \neq i} \alpha_j \varphi(v_j), \quad \text{where } \alpha_i \text{ are positive numbers such that } \ \alpha_1 + \ldots + \alpha_n = 1.$$

In addition, we require that the edges of the convex polytope spanned by $v_i$ coincide with the edges of G. Define a natural action of the group of Euclidian motions $\mathbb{G} = \mathrm{SO}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ on all realizations of G:

$$[g\varphi](v_i) := g \cdot \varphi(v_i)\,, \quad \text{where } g \in \mathbb{G} \text{ and } v_i \in V.$$

Without loss of generality, we shall assume that vertices $v_1$, $v_2$ and $v_3$ form a triangle in G. We say that a realization $\varphi$ is *planted* if $\varphi(v_1) = (0,0,0)$, $\varphi(v_2) = (a,0,0)$, and $\varphi(v_3) = (b,c,0)$ for some $a,b,c \in \mathbb{R}$. Observe that for every realization $\varphi$ of G there is a unique $g \in \mathbb{G}$ such that $g\varphi$ is planted. Unless stated otherwise, from this point on we consider only planted realizations.

Define *realization space* of the graph G to be the set $\mathfrak{M}(\mathrm{G})$ of convex planted realizations of G. Similarly, define realization space of the length diagram $\mathcal{L} = (\mathrm{G}, L)$ to be the set $\mathfrak{M}(\mathrm{G}, L) \subset \mathfrak{M}(\mathrm{G})$ of realizations of G which preserve the distances $|\varphi(v_i) - \varphi(v_j)|^2 = L(v_i, v_j)$. Finally, realization space of the length family $\mathcal{W} = (\mathrm{G}, W)$ is defined to be the set $\mathfrak{M}(\mathrm{G}, W) \subset \mathfrak{M}(\mathrm{G})$ of realizations of G such that the edge length function satisfies $L \in W$, where the lengths function $L\colon E \to \mathbb{R}$ is defined by $L(v_i, v_j) := |\varphi(v_i) - \varphi(v_j)|^2$.

Note that the condition that vertices form a convex polytope is an open condition on the coordinates and, therefore, the realization space $\mathfrak{M}(\mathrm{G})$ is an open subset of $\mathbb{R}^m$. The original motivation behind this paper was the study of realization spaces $\mathfrak{M}(\mathrm{G}, W)$ of general families of polytopes. We will return to this problem in Section 10.

2.4. **Complex realizations.** One can modify the notion of a realization of the graph G to the field $\mathbb{C}$: a *complex realization* is a map $\varphi\colon V \to \mathbb{C}^3$ and the corresponding 'squared length function' is a map $L\colon E \to \mathbb{C}$. Formally, define $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ to be a complexification of $W$. Now let $\mathfrak{M}_{\mathbb{C}}(\mathrm{G}, W)$ be the set of complex realizations $\varphi$ with $L \in W_{\mathbb{C}}$.

While the notion of convexity does not translate directly, we still have definitions of 'planted polytopes', of realization space of 'polytope families', etc. As we will see in the next section, there are certain advantages of working with an algebraically closed field.

Let us also mention here that our 'complex realizations' of G can contain 'degenerate configurations' such as those where two or more vertices are mapped into the same point. As we mention in Remark 5.2, it is often useful to avoid such realizations.

2.5. **Cauchy and Aleksandrov rigidity theorems.** Let G be a 3-connected plane graph. The Steinitz Theorem says that there always exists a convex polytope $P$ with graph G. In other words, the realization space $\mathfrak{M}(\mathrm{G})$ is nonempty.

Suppose $\mathrm{G} = (V, E)$ is a graph of a (planted) simplicial polytope $P$, and let $L\colon E \to \mathbb{R}_+$ be the corresponding length function. The Cauchy Rigidity Theorem basically says that the realization space $\mathfrak{M}(\mathrm{G}, L)$ of the length diagram $\mathcal{L} = (\mathrm{G}, L)$ consists of isolated points[3]. Cauchy further showed that the realization space $\mathfrak{M}(\mathrm{G}, L)$ of the length diagram $\mathcal{L} = (\mathrm{G}, L)$ contains only one point corresponding to $P$. Aleksandrov Rigidity Theorem shows that polytope $P$ is uniquely determined by the metric space of the boundary $\partial P$; in other words the "diagonals" defining triangulation of the unfolding, are also uniquely determined.

2.6. **Aleksandrov existence theorem.** Consider the boundary $S = \partial P$ as a metric space. Of course, $S$ is determined by the edge lengths $\ell_i$ and the way graph G is embedded into a sphere. Clearly, the metric space $S$ is *locally convex*, i.e. the sum of the angles in the triangles adjacent to every vertex is at most $2\pi$. This property can be turned into a definition: metric space $S$ is an *Aleksandrov space* if $S$ is locally convex and homeomorphic to a sphere (see [BGP]). A classical *Aleksandrov's Existence Theorem*[4] says that for every Aleksandrov space $S$, there exists a convex polytope $Q$ whose boundary $\partial Q$ is isometric to $S$. Unfortunately, the proof uses the Inverse Function Theorem and cannot be used to construct polytope $P$.

---

[3]Recall that by definition the realization space $\mathfrak{M}(\mathrm{G}, L)$ contains only *convex* realizations.

[4]Until recently this work was somewhat overlooked in the West. Even now there is still no published exposition of the proof in English. The interested reader should consult [A2] or its German translation.

2.7. **The volume of a polytope.** Let us show that the volume $\mathrm{vol}(P)$ of a convex simplicial polytope $P$ is a polynomial in the coordinates of the vertices $v_1, \ldots, v_n \in \mathbb{R}^3$. Indeed, fix an orientation of the boundary $S = \partial P$ of the polytope $P$, i.e. an orientation of all triangles $(v_i, v_j, v_k) \in S$. We claim that

$$(\diamondsuit) \qquad \mathrm{vol}(P) \, = \, \frac{1}{6} \sum_{(v_i,v_j,v_k)\in S} \det\bigl(v_i, v_j, v_k\bigr).$$

Indeed, let $S = S_+ \cup S_-$, where $S_+$ is the union of triangles which have $\det\bigl(v_i, v_j, v_k\bigr) \geq 0$, and let $S_-$ be a union of triangles which have $\det(v_i, v_j, v_k) < 0$. Let $P_+$ ($P_-$ resp.) be a convex hull of the origin 0 and $S_+$ ($S_-$ resp.). Now observe that $\mathrm{vol}(P) = \mathrm{vol}(P_+) - \mathrm{vol}(P_-)$, and $P_+$ ($P_-$) is a union of simplices $(0, v_i, v_j, v_k)$ whose volume is $\frac{1}{6}\bigl|\det\bigl(v_i, v_j, v_k\bigr)\bigr|$. Since each determinant is a polynomial in coordinates of $v_i$, so is the volume of $P$.

The argument above extends to all (nonconvex) simplicial polyhedra and easily generalizes to higher dimension.

2.8. **The volume of a simplex.** Let $v_1, \ldots, v_n \in \mathbb{R}^{n-1}$ be vertices of a $(n-1)$-dimensional simplex $P$. As before, denote by $f_{ij}$ the square of the lengths of the diagonal $(v_i, v_j)$, $1 \leq i < j \leq n$. The volume of $P$ can be computed by the *Cayley-Menger determinant*:

$$(\heartsuit) \qquad \mathrm{vol}^2(P) \, = \, \frac{(-1)^n}{2^{n-1}(n-1)!^2} \, \det \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & f_{12} & f_{13} & \ldots & f_{1n} \\ 1 & f_{12} & 0 & f_{23} & \ldots & f_{2n} \\ 1 & f_{13} & f_{23} & 0 & \ldots & f_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_{1n} & f_{2n} & f_{3n} & \ldots & 0 \end{pmatrix}.$$

Thus, the volume of a simplex is a root of a quadratic equation in the squared edge lengths $f_{ij}$, in any dimension. We refer to [GK, § 3.6.1] for the references (see also [Be, CSW]).

One can think of ($\heartsuit$) as a generalization of the *Heron formula* for the area of a triangle $S$ with side length $a, b$ and $c$:

$$(\triangle) \qquad \mathrm{area}^2(S) \, = \, \rho(\rho-a)(\rho-b)(\rho-c) \, = \, \frac{1}{16}\bigl(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\bigr),$$

where $\rho = \frac{1}{2}(a+b+c)$ is half the perimeter of $S$.

2.9. **The area of an inscribed quadrilateral.** Recall the classical *Brahmagupta formula* for the area of a convex quadrilateral $T$ with side lengths $a, b, c, d$, and which is inscribed into a circle:

$$(\boxplus) \qquad \mathrm{area}^2(T) \, = \, (\rho-a)(\rho-b)(\rho-c)(\rho-d),$$

where $\rho = \frac{1}{2}(a+b+c+d)$ is half the perimeter of $T$ (see e.g. [Had, § 255]). As we show in Section 8, formulas by Heron and Brahmagupta give polynomial relations ($\mathfrak{s}$) as in Theorem 1.4.

## 3. Characteristic map

Let $P$ be a planted convex simplicial polytope in $\mathbb{R}^3$ with the graph $\mathrm{G} = (V, E)$. As in the introduction, denote by $E = \{e_1, \ldots, e_m\}$ the set of edges of $P$, and let $\ell_r$ denote the lengths of $e_r$, $1 \leq r \leq m$. Formally, the squared length of edge $e_r = (v_i, v_j)$ between vertices $v_i = (x_i, y_i, z_i)$ and $v_j = (x_j, y_j, z_j)$ is given by a quadratic polynomial

$$f_r = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$$

.

Recall that the number of 'free coordinates' in a planted simplicial polytope is equal to $3n-6$ as six coordinates are set to be zero. Since $m = 3n-6$ in every triangulation, we can define the following *characteristic map* of the graph G:

$$\mathbf{F} = \mathbf{F}_{\mathrm{G}} \colon \mathbb{R}^m \to \mathbb{R}^m \text{ defined as } \mathbf{F} = (f_1, \ldots, f_m)$$

**Proposition 3.1.** *The characteristic map* $\mathbf{F}$ *is regular on* $\mathfrak{M}(G)$.

The proof will follow one of the standard proofs of the Cauchy Rigidity Theorem. We include it here for completeness and so that we can use the notation and technique in the future.

We start with the following two simple lemmas.

**Lemma 3.2.** *Let* $e_1 = (w, v_1)$, ..., $e_k = (w, v_k)$ *be the edge vectors of a simplicial convex polytope* $P$ *in* $\mathbb{R}^3$, *given the same cyclic order, and assume that vertex* $w = 0$ *is in the origin. Suppose* $\alpha_1 e_1 + \ldots + \alpha_k e_k = 0$ *is a nontrivial linear combination. Then there are at least four sign changes (in the cyclic order) in a sequence* $\alpha_1, \ldots, \alpha_k$.

*Proof.* By convexity, all $v_i$ lie on the same side of some hyperplane $H$ going through 0. Thus, if all $\alpha_i \geq 0$, (there are no sign changes), the linear combination $u = \alpha_1 e_1 + \ldots + \alpha_k e_k$ also lies on the same side of $H$, which is impossible.

Suppose now there are only two sign changes. Without loss of generality we can assume that $\alpha_1, \ldots, \alpha_m \geq 0$, and $\alpha_{m+1}, \ldots, \alpha_k \leq 0$. Again, by convexity there exists a hyperplane $H$ which goes through the origin and separates $e_1, \ldots, e_m$ from $e_{m+1}, \ldots, e_k$ (see Figure 1). But then all vectors $e_1, \ldots, e_m$, $-e_{m+1}, \ldots, -e_k$ lies again on the same side of $H$, and thus so is the linear combination $u = \alpha_1 e_1 + \ldots + \alpha_k e_k$. Since $u = 0$, this is impossible. □

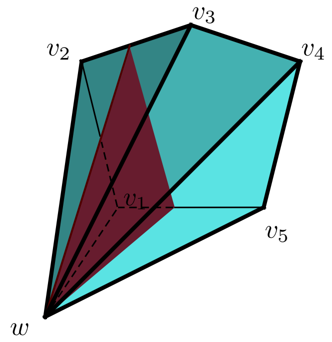


FIGURE 1. Hyperplane separating edges $(w, v_1)$, $(w, v_2)$ and $(w, v_3)$, $(w, v_4)$, $(w, v_5)$ of a polytope $P$.

**Lemma 3.3.** *Let* $\mathrm{G} = (V, E)$ *be a plane triangulation, and suppose vertices* $(v_1, v_2, v_3)$ *form a triangle. Assume that the edges of* G *are labeled by* $\{+, -, 0\}$ *so that each vertex except for* $v_1, v_2, v_3$, *either has all edges labeled* 0, *or has at least four sign changes in the cyclic order. Then all edges are labeled* 0. *Here a* sign change *is a pair of consecutive edges (in the cyclic order) adjacent to the same vertex and labeled with different* $+$ *or* $-$ *signs.*

*Proof.* We refer to edges and vertices of the *base* triangle $(v_1, v_2, v_3)$ as *base edges* and *base vertices*. We say that edges labeled $+$ or $-$ are *marked*, and those with 0 are *unmarked*. We call a vertex $v$ *clean* if all edges adjacent to $v$ are unmarked.

We first consider a case when there are exactly four sign changes at each nonbase vertex.

Let $n$ denote the number of nonbase vertices in G. Use induction to show that the number of nonbase edges in G is $m = 3n$. Thus, the total number of edges is $3n + 3$, and the number of faces in the plane triangulation G is $r = 2(3n + 3)/3 = 2n + 2$.

Let us compute the total number of sign changes. At each nonbase vertex we have at least four sign changes, hence in total we must have at least $4n$ sign changes. On the other hand, there are at most two sign changes in each triangular face. Moreover at each face adjacent to the base at most two edges are marked, so we can have only one sign change there. Finally, in the base face we have no sign changes. Hence the total number of sign changes is at most $2(r - 4) + 3 = 4n - 1$, a contradiction.

Now consider a graph G with clean vertices. Given such a graph, we delete all the clean vertices along with their adjacent faces and edges to obtain a graph G′. The case when G′ is a triangulation is considered above. Hence assume that G′ is not simplicial. Let $d$ denote the number of triangles needed to triangulate all non-triangular faces of G′. If $n$ is a number of vertices of G′, then the number of its faces is $F = 2n - 4 - d$. Since we deleted at least one vertex with adjacent faces and G′ is not simplicial, $d$ is at least 2. Assume first that the base face of G is present in G′. We have at most two sign changes in all faces except the base face, where there are none. Therefore, there are at most

$$2(F - 1) \,=\, 2(2n - 5 - d) = 4n - 10 - 2d \le 4n - 14$$

sign changes. In a different direction, each nonbase vertex has at least four sign changes. Therefore there are at least $4(n - 3) = 4n - 12$ sign changes, a contradiction.

Similarly, if the base face is absent in G′, then there are at most 2 base vertices left in G′, and thus by counting sign changes in faces we conclude that there are at least $4(n - 2)$ sign changes. On the other hand, by counting sign changes around vertices we conclude that there are at most $2(2n - 4 - d)$ sign changes, a contradiction. □

*Proof of Proposition 3.1* The Jacobian of the characteristic map $\mathbf{F}$ is a $m \times m$ matrix of differentials $\big(df_1, \ldots, df_m\big)^{\mathrm{T}}$. Assume that rows of Jacobian are linearly dependent at a point $s \in \mathfrak{M}(\mathrm{G})$ for a plane graph G of a convex polytope $P_s \subset \mathbb{R}^3$ defined as a convex hull of the image points $\mathbf{F}(s)$. Then there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that

$$\alpha_1 \cdot df_1(s) \,+\, \ldots \,+\, \alpha_m \cdot df_m(s) \,=\, 0.$$

Using the explicit formula for $df_r$ corresponding to edges $e_r = (i, j)$, and restricting to three columns corresponding to a nonbase vertex $v_i$ we have

$$\sum_j \alpha_r \cdot (x_i - x_j, y_i - y_j, z_i - z_j) \,=\, 0,$$

where summation is taken over all edges $e_r = (v_i, v_j) \in E$ incident to the vertex $v_i$. The linear dependence of the rows implies that there is an assignment of weights $\alpha_r$ on respective nonbase edges $e_r$, such that the weighted sum of edge vectors at each of the nonbase vertices is 0.

Now, label each edge $e_r$ of the plane graph G (of the convex polytope $P_s$) with the sign of the coefficient $\alpha_r$. If $\alpha_i = 0$, label it with 0. By Lemma 3.2 and 3.3 we conclude that all $\alpha_r = 0$, which completes the proof. □

## 4. Generic Freeness Lemma

Recall that by Proposition 3.1 the Jacobian of the characteristic map $\mathbf{F} = (f_1, \ldots, f_m)$ does not vanish on $\mathfrak{M}(\mathrm{G})$, an open subset of $\mathbb{R}^m$. Therefore, the Jacobian is a nontrivial polynomial and polynomials $f_i$ are algebraically independent.

Recall, that a morphism $f\colon X \to Y$ is *étale* at a point $p$ if $df_p\colon T_pX \to T_{f(p)}Y$ is an isomorphism. Note that nonvanishing of the Jacobian is an equivalent condition for $\mathbf{F}$ to be étale at the point.

Let $Z$ be a hypersurface on which the Jacobian vanishes. Then the image $\mathbf{F}(Z)$ is a Zariski-locally closed subset of $\mathbb{C}^m$ of dimension at most $m-1$. Therefore, $\mathbf{F}(Z)$ is not dense in $\mathbb{C}^m$. Denote $V = \mathbb{C}^m - \overline{\mathbf{F}(Z)}$ and observe that $\mathbf{F}\colon \mathbf{F}^{-1}(V) \to V$ is étale by definition. Denote by $U$ the open subset of the preimage $\mathbf{F}^{-1}(V)$. Since $\mathbf{F}$ is étale, it is open on $U$ and hence dominant, i.e. its image is dense. This property of characteristic map is crucial in the proof of the lemma below.

We are interested in polynomial relations ($\circ$) as in Theorem 1.1 for all polynomial invariants. The fact that the characteristic map is étale on a nontrivial open subset of $\mathbb{C}^m$ allows us to establish the existence of a polynomial relation ($\circ$) for every polynomial invariant and to estimate the degree $N$ of this polynomial relation. The proof of Theorem 1.1, the main result of this paper, uses the following technical lemma.

**Lemma 4.1** (Generic freeness)**.** *Let $\Bbbk$ be an algebraically closed field and $X$ be an irreducible algebraic variety of dimension $m$ over $\Bbbk$. Denote by $A$ the ring of rational functions on $X$, and let $f_1, \ldots, f_m \in A$. Suppose $\mathbf{F} = (f_1, \ldots, f_m) : X \to \Bbbk^m$ is a dominant morphism. Then the algebraic closure of $B = \Bbbk[f_1, \ldots, f_m]$ inside $A$ is the whole $A$. Moreover, for any $g$ from $A$, the degree of $g$ over $B$ is equal to the number of different values that $g$ takes on the preimages of a general point in $\Bbbk^m$.*

*Proof.* First, let us show that every element $g$ in $A$ is algebraic over $B$. Indeed, the ring $A$ has transcendence degree $m$. Elements $f_1, \ldots, f_m$ and $g$ lie in this ring and cannot be algebraically independent. However, elements $f_i$ are algebraically independent since $\mathbf{F}$ is dominant. We conclude that $g$ is algebraic over $B$. Furthermore, since $B \subset A$ are integral domains and $g \in A$ is algebraic over $B$, there exists a minimal irreducible polynomial $F_g \in B[t]$, such that $F_g(g) = 0$.

Note that the coefficients of $F_g$ are polynomial in $f_1, \ldots, f_m$. Since values of $\mathbf{F} = (f_1, \ldots, f_m)$ lie in $\Bbbk^m$, we, by abuse of notation, also denote by $F_g$ the minimal polynomial in $m+1$ variables such that $F_g(g(a)) = F_g\big(f_1(a), \ldots, f_m(a), g(a)\big)$ for all $a$ in $X$.

We can now estimate the degree $d$ of $g$ over $B$. Consider a morphism $\widehat{\mathbf{F}} = (\mathbf{F}, g)\colon X \to \Bbbk^{m+1}$. Clearly, $\mathbf{F}$ is a composition of $\widehat{\mathbf{F}}$ and a projection $\pi\colon \Bbbk^{m+1} \to \Bbbk^m$ on the first $m$ coordinates. The image of $\widehat{\mathbf{F}}$ lies in the hyperplane $H$ defined by the polynomial $F_g$. Clearly, a generic point in $\Bbbk^m$ has $d$ preimages in $H$ under $\pi$. On the other hand, since $\mathbf{F}$ is dominant, $\widehat{\mathbf{F}}$ is also dominant. Therefore, for a generic point all its preimages under $\pi$ lie in the image of $\widehat{\mathbf{F}}$. We conclude that $d$ is equal to the number of preimages of a generic point from $\Bbbk^m$ under projection $\pi$ which lie in the image of $\widehat{\mathbf{F}}$. Finally, this number is exactly the number of values that $g$ takes on the preimages of a generic point from $\Bbbk^m$ under $\mathbf{F}$. $\square$

## 5. Geometry of realization space of polytopes

The following theorem describes the realization space of a given length family $\mathcal{W} = (\mathrm{G}, W)$ of polytopes with a graph G and an edge function given by $L \in \mathcal{W}$. We also show that this result implies The Cauchy Rigidity Theorem.

**Theorem 5.1.** *Let $\mathcal{W} = (\mathrm{G}, W)$ be the length family with degree of freedom $\dim(W) = d$. Then the realization space $\mathfrak{M}(\mathrm{G}, W) \subset \mathbb{R}^m$ is a smooth real manifold of real dimension $d$. This manifold is an open subset of real points of an ambient Zariski-locally closed irreducible set $X(\mathrm{G}, W)$ of complex dimension $d$ in $\mathbb{C}^m$.*

*Proof.* Let $\mathbf{F}\colon \mathbb{C}^m \to \mathbb{C}^m$ be a complexification of the characteristic map of the graph G. Recall that polynomial map $\mathbf{F}$ is étale on Zariski open subset $U \subset \mathbb{C}^m$ where the Jacobian of $\mathbf{F}$ is nonvanishing. From Proposition 3.1 we conclude that $U$ contains $\mathfrak{M}(\mathrm{G}, W)$.

The subspace $W_{\mathbb{C}} \subset \mathbb{C}^m$ is an irreducible algebraic set, so $W_{\mathbb{C}} \cap \mathbf{F}(U)$ is also irreducible. Since $\mathbf{F}\colon U \to \mathbf{F}(U)$ is étale and surjective, the set $X(\mathrm{G}, W) := \mathbf{F}^{-1}(W_{\mathbb{C}}) \cap U$ is closed and irreducible in $U$. Since $\mathbf{F}$ is étale, we have

$$\dim_{\mathbb{C}}(X(\mathrm{G}, W)) = \dim_{\mathbb{C}}(W_{\mathbb{C}}) = \dim_{\mathbb{R}}(W) = d.$$

On the other hand, from the analytic point of view, the map $\mathbf{F}\colon \mathfrak{M}(\mathrm{G}) \to \mathbb{R}^m$ is regular by Proposition 3.1. Therefore, the preimage of $W$

$$S_{real} = \mathbf{F}^{-1}(W) \cap \mathbb{R}^m$$

is a real smooth manifold of the same dimension $d$. This implies the result. □

**Remark 5.2.** The set $X(\mathrm{G}, W)$ will be used in the future discussion in place of the whole $\mathfrak{M}_{\mathbb{C}}(\mathrm{G}, W)$. In many ways the space $X(\mathrm{G}, W)$ behaves better than $\mathfrak{M}_{\mathbb{C}}(\mathrm{G}, W)$. In particular, $X(\mathrm{G}, W)$ is irreducible and has the correct dimension $d = \dim(W)$.

We say that a continuous deformations of a convex polytope $P$ *preserves faces* if it defines an isometry on each of the facets. We are now ready to deduce the Cauchy Rigidity Theorem from our results.

**Corollary 5.3** (The Cauchy Rigidity Theorem)**.** *There are no continuous deformations of a convex polytope $P \subset \mathbb{R}^3$ which preserve the faces of $P$.*

*Proof.* A triangulation with diagonals of faces of a convex polytope $P$ satisfies the conditions of Lemma 3.3. Hence it suffices to prove the result when $P$ is simplicial. Then every continuous deformation as in the statement of the corollary must preserve the graph G of $P$ (a plane triangulation) and the edge lengths. As in the introduction, the result follows from the fact that there exists only a finite number of convex simplicial polytopes with the same graph and edge lengths.

First, we can always assume that the convex polytope and all deformations are planted; use a Euclidian motion to translate these into a planted position otherwise. Now fix a length diagram $\mathcal{L} = (\mathrm{G}, L)$ and consider a one-dimensional family of polytopes $\mathcal{W} = (\mathrm{G}, W)$, where $W = \mathbb{R}\langle L\rangle$. By Theorem 5.1 the (real) realization space $\mathfrak{M}(\mathrm{G}, W)$ is also one-dimensional. Therefore, the realization space $\mathfrak{M}(\mathrm{G}, L)$ is finite since each point in $\mathfrak{M}(\mathrm{G}, L)$ corresponds to a line in $\mathfrak{M}(\mathrm{G}, W)$. This completes the proof. □

## 6. Polynomial invariants of a polytope

Let $v_1, \ldots, v_n \in \mathbb{R}^3$ denote the vertices of a convex polytope in the realization space of the graph G. Let $\mathfrak{I}\colon \mathbb{R}^{3n} \to \mathbb{R}$ be a polynomial in the coordinates of $v_i$. Denote by $\mathbb{G} = \mathrm{SO}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ the group of all Euclidean motions of $\mathbb{R}^3$. We say that $\mathfrak{I}(\cdot)$ is a *polynomial invariant* of G if $\mathfrak{I}(\cdot)$ is invariant under the natural action of the group $\mathbb{G}$ on all convex polytopes $P$ with graph G. The squares of diagonal lengths $(d_{ij})^2$ are the basic examples of polynomial invariants. Another example in the volume $\mathrm{vol}(P)$ which is clearly invariant under $\mathbb{G}$, and is a polynomial as shown in Section 2.7.

We write $\mathfrak{I}(P)$ for the value of $\mathfrak{I}(\cdot)$ on vertices of $P$. Since $\mathfrak{I}(P)$ is an invariant under the action of $\mathbb{G}$ on $P$, everywhere below we restrict our attention only to values of polynomial invariants of planted polytopes. We are now ready to prove the main result of this paper.

**Proof of Theorem 1.1.** As we showed in Proposition 5.1, the complex realization space $\mathfrak{M}_{\mathbb{C}}(\mathrm{G}, W)$ contains a locally closed irreducible subset $X(\mathrm{G}, W)$ of dimension $d = \dim(W)$. We define a polynomial map $H = (h_1, \ldots, h_d)\colon X(\mathrm{G}, W) \to \mathbb{C}^d$, where the length functions $h_i$ form a basis in the vector space $W$, i.e. $\langle h_1, \ldots, h_d\rangle = W$. Map $H$ is a composition of the characteristic map $\mathbf{F}$ corresponding to G and a projection onto $d$ coordinates. Since $\mathbf{F}$ is dominant on $X(\mathrm{G}, W)$ and the projection is also dominant, so is $H$. Moreover, the invariant $\mathfrak{I}$ is a rational function on $X(\mathrm{G}, W)$, as in Lemma 4.1. From Lemma 4.1, the polynomial invariant $\mathfrak{I}$ is algebraic over $\mathbb{C}[h_1, \ldots, h_d]$ and the degree of $\mathfrak{I}$ is exactly the number $N$ of values the polynomial invariant $\mathfrak{I}$ takes on preimages under $H$ of the general point from $W$.

Recall that $h_i = \sum_j c_{ij} f_j$ are linear combinations of quadratic polynomials $f_j$ defining squared lengths of edges. The number $N$ of preimages of $(b_1, \ldots, b_d) \in \mathbb{C}^d$ is the number of solutions of the linear system of $m$ quadratic equations:

$$f_j \,=\, a_j, \quad \text{where } \; 1 \le j \le m, \; \text{ which satisfy}$$
$$\sum_j c_{ij}\, a_j \,=\, b_i, \quad \text{where } \; 1 \le i \le d,$$

and $(a_1, \ldots, a_m) \in \mathbf{F}(X) \subset \mathbb{C}^m$. For a general point in $\mathbb{C}^m$, the number of solutions of this system is finite. By Bezout theorem, we conclude $N$ is at most $2^m$. This completes the proof of Theorem 1.1. $\square$

Let us note that Corollary 1.3 follows immediately from Theorem 1.1 since volume is a polynomial invariant. Similarly, Theorem 1.1 implies that squares of diagonal lengths $(d_{ij})^2$ are roots of polynomial relations of degree at most $2^m$. Substituting $(d_{ij})^2$ in place of $\mathfrak{I}$ into the polynomial relation ($\circ$) we obtain the polynomial relation ($*$) for $d_{ij}$ of degree at most $4^m$, which proves Corollary 1.2.

Let $\mathcal{W} = (\mathrm{G}, W)$ be the length family, and let $\mathfrak{M}(\mathrm{G}, W)$ be the realization space of $\mathcal{W}$. Denote by $F_{\mathcal{W},\mathfrak{I}}$ the minimal polynomial relation for $\mathfrak{I}$ restricted to the realization space $\mathfrak{M}(\mathrm{G}, W)$, and let

$$\eta(\mathcal{W}, \mathfrak{I}) \,:=\, \deg F_{\mathcal{W},\mathfrak{I}}.$$

It is intuitively clear that the smaller the degree of freedom $d = \dim(W)$, the smaller is the degree of the minimal polynomial relation for $\mathfrak{I}$. The following result makes this intuition precise.

**Proposition 6.1.** *Let $\mathcal{W}_1 = (\mathrm{G}, W_1)$ and $\mathcal{W}_2 = (\mathrm{G}, W_2)$ be two length families with $W_2 \subset W_1$. Let $\mathfrak{I}$ be a polynomial invariant of the graph* G. *Then $\eta(\mathcal{W}_1, \mathfrak{I}) \ge \eta(\mathcal{W}_2, \mathfrak{I})$.*

*Proof.* It suffices to prove the result for $\dim(W_2) = \dim(W_1) - 1$. Denote by $F_1 = F_{\mathcal{W}_1,\mathfrak{I}}$ and $F_2 = F_{\mathcal{W}_2,\mathfrak{I}}$ the minimal polynomial relations for $\mathfrak{I}$ restricted to $X_1 = X(\mathrm{G}, W_1)$ and $X_2 = X(\mathrm{G}, W_2)$, respectively. Now, the set $X_2$ is contained in the set $X_1$ of dimension $d$, where $d = \dim(W_1)$ is the degree of freedom of $\mathcal{W}_1$. The characteristic map of $\mathcal{W}_1$ is

$$H = (f_1, \ldots, f_d) \colon \; X_1 \,\to\, \mathbb{C}^d.$$

Without loss of generality assume that $X_2 \subset H^{-1}(x_1, x_1, x_2, \ldots, x_{d-1}) \subset X_1$. Let

$$F_1(x) \,=\, C_r x^r + \ldots + C_1 x + C_0\,,$$

where $C_i \in \mathbb{C}[f_1, \ldots, f_d]$. Since $F_1$ is minimal, we have $(C_d, \ldots, C_0) = 1$. Therefore there exists $i$, such that $(f_1 - f_2)$ does not divide $C_i$. Since $C_i$ vanishes on the whole $X_2$ only if $(f_1 - f_2) \mid C_i$, we conclude that all $C_i$ do not vanish simultaneously on $X_2$. This implies that the restriction of $F_1$ to $X_2$ is nontrivial and has degree less or equal than $\deg F_1$. Therefore, the degree of the minimal polynomial $F_2$ is less or equal than degree of $F_1$. $\square$

We conclude with the upper and lower bounds on the degree $\eta(\mathcal{W}, \mathfrak{I})$.

**Corollary 6.2** (The Upper Bound)**.** *In the notation above, let $m$ be the number of edges of* G. *Then the degree $\eta(\mathcal{W}, \mathfrak{I}) \le 2^m$. Moreover, $\eta(\mathcal{W}, \mathfrak{I})$ is at most the number of complex realizations $|\mathfrak{M}_{\mathbb{C}}(\mathrm{G}, L)|$, for a general $L \in W$, where $\mathcal{W} = (\mathrm{G}, W)$.*

*Proof.* Theorem 1.1 gives an upper bound $\eta(\mathrm{G}, \mathfrak{I}) \le 2^m$ on the degree of the polynomial relation $(\circ)$ for the case when the degree of freedom $\deg(W) = m$. On the other hand, Proposition 6.1 gives $\eta(\mathcal{W}, \mathfrak{I}) \le \eta(\mathrm{G}, \mathfrak{I})$.

For the second claim, recall the use of Bezout theorem in the last step of the proof of Theorem 1.1. Instead, observe that the number of complex realizations $|\mathfrak{M}_{\mathbb{C}}(\mathrm{G}, L)|$ is equal to the number of preimages of $\mathbf{F}$. Following the proof of Theorem 1.1, we obtain the result. □

**Corollary 6.3** (The Lower Bound)**.** *In the notation above, let $\mathfrak{I}_{\mathbb{C}}(\mathcal{L})$ denote the set of values of a polynomial invariant $\mathfrak{I}$ on complex realizations in $X(\mathrm{G}, L)$. Then the degree $\eta(\mathcal{W}, \mathfrak{I}) \ge \big|\mathfrak{I}_{\mathbb{C}}(\mathcal{L})\big|$ for every length diagram $\mathcal{L} \in \mathcal{W}$. Moreover, the inequality becomes an equality for almost all length diagrams.*

*Proof.* Indeed, different values of polynomial invariant $\mathfrak{I}$ correspond to different roots of polynomial relation $(\circ)$ as in Theorem 1.1. This implies that the degree of every nontrivial relation is at least $\big|\mathfrak{I}_{\mathbb{C}}(\mathcal{L})\big|$. The second claim follows from Lemma 4.1. □

**Remark 6.4.** The upper and lower bounds are tight (see the next section), but in general may not be close to each other. In fact, the number of complex (and even real) realizations of the length diagram can be infinite, as shown by certain diagonals in flexors. On the other hand, in several special cases, such as regular bipyramid, these bounds determine the degree precisely.

## 7. Examples

Let us summarize the results we obtained so far from a combinatorial point of view. Consider a planar 3-connected triangulation G, let $E = \sqcup_i E_i$ be a partition of edges, and let $W$ be the corresponding subspace of (squared) length functions constant on each $E_i$. We consider the realization space $\mathfrak{M}(\mathrm{G}, W)$ of convex planted polytopes with graph G and (squared) lengths of edges in $W$.

Given a "combinatorial" polynomial invariant $\mathfrak{I}$ (such as a squared length of the diagonal $d_{ij}^2$ or the volume) of G, restricted to realization space $\mathfrak{M}(\mathrm{G}, W)$, there exists a minimal polynomial relation $F_{\mathcal{W},\mathfrak{I}}$ for $\mathfrak{I}$, whose coefficients are polynomials in the squared edge lengths $f_i$ (Theorem 1.1). The degree $\eta(\mathcal{W}, \mathfrak{I}) = \deg F_{\mathcal{W},\mathfrak{I}}$ is an important combinatorial invariant, perhaps the most interesting when the partition is maximal, in which case it is denoted $\eta(\mathrm{G}, \mathfrak{I})$.

It seems, the problem of computing $\eta(\mathrm{G}, \mathfrak{I})$ and $\eta(\mathcal{W}, \mathfrak{I})$ in general, is difficult even in very simple examples. Below we present several special cases to illustrate the results.

7.1. **Tetrahedron.** For $\mathrm{G} = K_4$ and a squared length function $L(i, j) > 0$, $1 \le i < j \le 4$, there is a unique tetrahedron $P$ up to reflections and Euclidean motions. In contrast, there are eight planted realizations of $P$, all real and obtained one from another by reflections with respect to axis hyperplanes: $|\mathfrak{M}(\mathrm{G}, L)| = 8$. There are no diagonals to be determined, but the volume is a polynomial invariant. On these eight realizations the volume takes two values: $\pm\mathrm{vol}(P)$. Thus, by the lower bound lemma 6.2 the volume is a root of a polynomial of degree at least two. In fact, as shown in $(\heartsuit)$, Section 2.8, the volume is a root of a quadratic polynomial. In the notation above $\eta(\mathrm{G}, \mathrm{vol}) = 2$. It is instructive to compare this value with the (very weak) first part of the upper bound: $\eta(\mathrm{G}, \mathrm{vol}) \le 2^6 = 64$. The second part gives a better bound $\eta(\mathrm{G}, \mathrm{vol}) \le 8$ in this case.

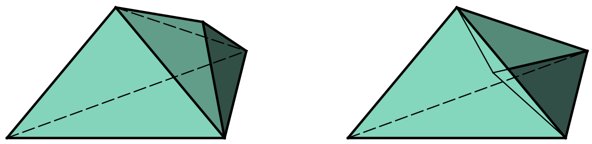

FIGURE 2. Two realizations of a triangular bipyramid.

7.2. **Triangular bipyramid.** Let $\mathrm{G} = K_{3,2}$ and fix a general squared length family $\mathcal{W}$. As in Section 2.2, denote the vertices $v_1, v_2, v_3, u_1, u_2$. Observe that up to reflections and Euclidean motions there are two complex realizations of $(\mathrm{G}, \mathcal{W})$: one convex and one nonconvex (see Figure 7.2).

There are four different values of the volume: $\mathrm{vol}(P) \in \{A{+}B, A{-}B, -A{-}B, -A{+}B\}$, where $A = \mathrm{vol}(v_1, v_2, v_3, u_1)$ and $B = \mathrm{vol}(v_1, v_2, v_3, u_4)$. Thus we have the lower bound $\eta(\mathrm{G}, \mathrm{vol}) \geq 4$ for the minimal degree of polynomial relation in terms of squared edge length functions $f_i$. The following polynomial relation achieves the bound and shows that $\eta(\mathrm{G}, \mathrm{vol}) = 4$:

$$(\dagger) \qquad F(x) \,:=\, (x - A - B)(x + A - B)(x - A + B)(x + A + B).$$

Indeed, observe that

$$(\dagger\dagger) \qquad F(x) \,=\, x^2 \,-\, 2(A^2 + B^2)x^2 \,+\, (A^2 - B^2)^2.$$

Since $A^2$ and $B^2$ are quadratic polynomials in the squared edge lengths, the polynomial relation $F(x)$ is a degree four polynomial relation.

Similarly, there is only one diagonal whose squared length is a polynomial invariant which we denote by $\mathfrak{I}$. Again, the lower and first part of the upper bound give:

$$2 \,\leq\, \eta(\mathrm{G}, \mathfrak{I}) \,\leq\, 2^9 \,=\, 512.$$

The second part in the upper bound can be shown to give $\eta(\mathrm{G}, \mathfrak{I}) \leq 16$. Already in this simple case we do not know the precise value of $\eta(\mathrm{G}, \mathfrak{I})$. It would be interesting to compute explicitly the minimal polynomial relation $F_{\mathrm{G},\mathfrak{I}}$.

7.3. **Snake polytopes.** Consider a polytope $P_k$ obtained by gluing together $k$ tetrahedra as follows. Start with the base $(v_1, v_2, v_3)$ and consider the tetrahedra $\tau_1 = (v_1, v_2, v_3, u_1)$, $\tau_2 = (u_1, v_2, v_3, u_2)$, $\tau_3 = (u_2, v_2, v_3, u_3)$, $\ldots$, and $\tau_k = (u_{k-1}, v_2, v_3, u_k)$ (see Figure 3). Let $\mathrm{G}_k$ denote the graph of $P_k$.

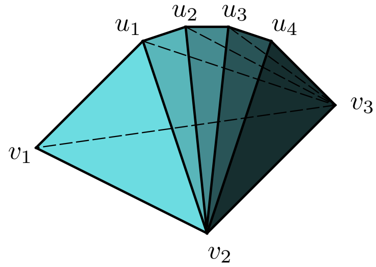


FIGURE 3. Snake polytope $P_4$.

Observe that every two successive tetrahedra can be glued in two ways. Taking orientation into account, this gives $2^k$ possible values of the volume. Thus $\eta(\mathrm{G}_k, \mathrm{vol}) \geq 2^k$. The following product is the minimal polynomial relation for the volume:

$$(\natural) \qquad F_k(x) := \prod_{\varepsilon_1\in\{\pm 1\}} \ldots \prod_{\varepsilon_k\in\{\pm 1\}} \bigl(x+\varepsilon_1 A_1+\ldots+\varepsilon_k A_k\bigr),$$

where $A_i = \mathrm{vol}(\tau_i)$ is the volume of $i$-th tetrahedron $\tau_i$. Note that

$$F_k(\ldots,-A_i,\ldots) = F_k(\ldots,A_i,\ldots),$$

which implies that $F_k$ is a polynomial in squared volumes $\mathrm{vol}^2(\tau_i)$. Since $\mathrm{vol}^2(\tau_i)$ is a polynomial in squared edge lengths polynomials $f_i$, we conclude that ($\natural$) gives the minimal polynomial relation for the volume of $P_k$.

Note that the number of edges in $P_k$ is $m = 3(k+1)$, so the first part of the upper bound gives $\eta(\mathrm{G}_k,\mathrm{vol}) \le 2^m = 8^{k+1}$. Similarly, the second part of the upper bound can be shown to give $\eta(\mathrm{G}_k,\mathrm{vol}) \le 2^{k+2} = 4\cdot 2^k$. This differs only by a constant from the lower bound $2^k$.

7.4. **Icosahedron.** Let $P$ be a *regular icosahedron* and let G be the corresponding plane triangulation with a constant (squared) length function $L$. This is our first example where computing coordinates of vertices of $P$ is a nontrivial task (see e.g. [Had, §564]).

The "longest diagonal" in this case is two times the radius of a sphere circumscribed around the icosahedron. Its length, is linear, of course, in the edge length. Of course, this does not imply that $\eta(\mathcal{W},\mathfrak{I}) = 1$ since the minimal degree $\eta(\mathcal{W},\mathfrak{I})$ is for a polynomial relation for polynomials in the coordinates of the vertices, not the real numbers which evaluate the squared edge lengths. The lower bound immediately gives $\eta(\mathcal{W},\mathfrak{I}) \ge 3$ as either of both of the longest diagonal vertex links can be bent inside. We skip the details.

Of course, the degree $\eta(\mathrm{G},\mathrm{vol})$ (of "generic icosahedra") would be very interesting to compute. The icosahedron has $m = 30$ edges, so the upper bound $\eta(\mathrm{G},\mathrm{vol}) \le 2^{30} \approx 1.07\cdot 10^9$ is rather discouraging. The second part of the upper bound is probably much smaller but hard to estimate. See [Mi] for a detailed analysis[5] of interesting length families with degree of freedom $d = 3$, and [Kl] for general background on icosahedron.

7.5. **Nearly equilateral bipyramid.** Let $\mathrm{G}_n$ be as in Section 2.2, a graph of a bipyramid with the set of vertices $V = \{v_1,\ldots,v_n,u_1,u_2\}$. Suppose the length of edges $(v_1,v_2),\ldots,(v_{n-1},v_n)$ is $a$, of $(v_1,v_n)$ is $b$, and of edges $(v_i,u_j)$ is $c$. Denote by $\mathcal{W}$ the corresponding length family with degree of freedom 3. Note that the corresponding bipyramid $P$ is regular, and is equilateral when $a = b$.

The squared length $d^2$ of the 'main diagonal' $(u_1,u_2)$ is a polynomial invariant. Thus, so is the squared radius $r^2 = c^2 - d^2/4$ of the circle circumscribed the $n$-gon $(v_1,\ldots,v_n)$. The number of complex realizations of $(\mathrm{G},\mathcal{W})$ is exponential in $n$, which gives an exponential upper bound for the degree $\eta(\mathrm{G},\mathcal{W})$ of the minimal polynomial relation for $r^2$. On the other hand, one can obtain the equation $(*)$ as in Corollary 1.2 of degree $2(n-1)$.

Denote $O$ the center of the circumscribed circle, and let $2\alpha := \angle(v_1Ov_2) = \ldots = \angle(v_{n-1}Ov_n)$, and $2\beta := \angle(v_1Ov_n)$. We have: $\sin\alpha = a/2r$, $\sin\beta = b/2r$, $\cos^2\alpha = 1-(a/2r)^2$, and $\cos^2\beta = 1-(b/2r)^2$. Since $(n-1)\alpha+\beta = \pi$, we have the following equation:

$$(\ddagger) \qquad \cos^2(n-1)\alpha \;=\; \cos^2\beta.$$

Using Chebyshev polynomials of the first kind $T_k(\cos x) = \cos nx$, we rewrite both sides of ($\ddagger$) as polynomials of $1/r^2$:

$$(\ddagger\ddagger) \qquad T_{n-1}^2\left(\frac{a}{2r}\right) \;=\; 1-\frac{b^2}{4r^2}.$$

Multiplying both sides of ($\ddagger\ddagger$) by $r^{2(n-1)}$ we obtain the desired equation for $r$ of degree $2(n-1)$.

[5] The reader might want to compare our proof of Theorem 5.1 with the proof of Theorem 4.2 in [Mi].

7.6. **Regular bipyramid.** In the notation above, let $\mathrm{G}_n$ be the graph of a bipyramid with the set of vertices $V = \{v_1, \ldots, v_n, u_1, u_2\}$. Let $\mathcal{W}_n$ be the corresponding length family. Denote the lengths of edges $\ell(v_1, v_2) = a_1, \ldots, \ell(v_{n-1}, v_n) = a_{n-1}$, $\ell(v_1, v_n) = a_n$, and $\ell(v_i, u_j) = b$. Denote by $d$ the length of the 'main diagonal' $(u_1, u_2)$. Recall that vertices $v_1, \ldots, v_n$ lie on a circle, and denote by $r$ the radius of this circle. Clearly, $r = r(a_1, \ldots, a_n)$ is independent of $b$, even though $r^2 = b^2 - d^2/4$.

To emphasize the difference, denote by $f_i$ quadratic polynomials giving the squared edge lengths $a_i^2$, by $h$ the squared edge length $b^2$, by $g$ a polynomial giving the squared diagonal length $d^2$, and by $R = h - g/4$ the squared radius length $r^2$.

From above, $R$ is a polynomial invariant and by Theorem 1.1 we conclude that $R$ is algebraic over $\mathbb{R}[f_1, \ldots, f_n]$. In other words $R$ satisfies the minimal polynomial relation

$$F_n = F_{\mathcal{W}_n, R}(x, f_1, \ldots, f_n) \,=\, 0.$$

Below we give an explicit formula for $F_n$.

Let $O$ denote the center of the circle, and let $2\alpha_1 = \angle(v_1 O v_2), \ldots, 2\alpha_{n-1} = \angle(v_{n-1} O v_n)$, $2\alpha_n = \angle(v_n O v_1)$. From geometric considerations, we have: $a_i = 2r \sin\alpha_i$, and $\alpha_1 + \ldots + \alpha_n = \pi$. Consider the following product:

$$(\circledast) \qquad \prod_{\varepsilon_2 \in \{\pm 1\}} \cdots \prod_{\varepsilon_n \in \{\pm 1\}} \sin\bigl(\alpha_1 + \varepsilon_2 \alpha_2 + \ldots + \varepsilon_n \alpha_n\bigr).$$

Use $\sin(\beta + \gamma) = \sin\beta \cos\gamma + \cos\beta \sin\gamma$ and $\cos(\beta + \gamma) = \cos\beta \cos\gamma - \sin\beta \sin\gamma$ formulas repeatedly to obtain the sum of products of $x_i = \sin\alpha_i$ and $y_i = \cos\alpha_i$. Observe that for $n \geq 2$, the product $(\circledast)$ is invariant under substitution $\alpha_i \leftarrow (\pm\alpha_i \pm \pi)$. Thus $(\circledast)$ is an even polynomial in $x_i$ and $y_i$, for all $1 \leq i \leq n$. Use the equality $x_i^2 + y_i^2 = 1$ to write the product $(\circledast)$ as follows:

$$(\circledast') \quad \prod_{\varepsilon_2 \in \{\pm 1\}} \cdots \prod_{\varepsilon_n \in \{\pm 1\}} \sin\bigl(\alpha_1 + \varepsilon_2 \alpha_2 + \ldots + \varepsilon_n \alpha_n\bigr) \,=\, H_n(\sin^2\alpha_1, \ldots, \sin^2\alpha_n).$$

From $\alpha_1 + \ldots + \alpha_n = \pi$, we obtain $H_n(\sin^2\alpha_1, \ldots, \sin^2\alpha_n) = 0$. Making the substitution $\sin^2\alpha_i = a_i^2/4r^2$ we obtain

$$H_n(a_1^2/4r^2, \ldots, a_n^2/4r^2) \,=\, 0.$$

Multiplying both sides by the smallest power of $(r^2)$, we see that $r^2$ is a root of the polynomial equation

$$(\circledast\circledast) \qquad \widetilde{H}_n(r^2, a_1^2, \ldots, a_n^2) \,=\, 0.$$

Note that even though we derived $(\circledast\circledast)$ for the real lengths $a_i$ and $r$, the argument immediately gives a polynomial relation $F_n = \widetilde{H}_n(x, f_1, \ldots, f_n)$ for the polynomial invariant $R$.

Recall the integer sequence $\Delta_k$ defined by ($\clubsuit$) in the Introduction. The following result says that $\widetilde{H}_n(\cdot)$ is minimal and computes its degree in terms of $\Delta_k$. The result is of independent interest and will be used in the next section to prove the Robbins Conjecture (Theorem 1.4).

**Theorem 7.1.** *The polynomial relation $\widetilde{H}_n(x, f_1, \ldots, f_n)$ is the minimal polynomial relation for the polynomial invariant $R$. Furthermore, the minimum degree $\nu(n) := \eta(\mathcal{W}_n, R)$ satisfies: $\nu(2k+1) = \Delta_k$, $\nu(2k+2) = 2\Delta_k$, for all $k \geq 1$.*

*Proof.* It was shown by Robbins [R1, R2] that $\Delta_k$ is the number of circumscribed (not necessarily convex) combinatorial types of polygons. Observe that for general lengths of the sides of the polygons, the radii take different values. This gives the lower bound for $\nu(n)$:

$$\nu(2k+1) = \Delta_k, \quad \nu(2k+2) = 2\Delta_k.$$

From above, $\widetilde{H}_n$ is a polynomial relation for $R$. Now the claim follows immediately from the following technical result:

**Lemma 7.2.** *The degree $\nu(n) := \deg \widetilde{H}_n$ of the polynomial relation $\widetilde{H}_n(x, f_1, \ldots, f_n)$ satisfies: $\nu(2k+1) = \Delta_k$, $\nu(2k+2) = 2\Delta_k$, for all $k \geq 1$.*

The lemma will be proved by a direct argument in Section 9. □

We conclude with the following corollary which follows from Theorem 7.1 and the results above.

**Corollary 7.3.** *Let $\nu(n) := \eta(\mathcal{W}_n, g)$ be the degree of the minimal polynomial relation for the polynomial invariant $g$ given by the squared length of the 'main diagonal' $(u_1, u_2)$. Then: $\nu(2k+1) = \Delta_k$ and $\nu(2k+2) = 2\Delta_k$, for all $k \geq 1$.*

7.7. **General bipyramid.** Consider the degree $\zeta(n) := \eta(\mathrm{G}_n, \mathfrak{I})$ for a general bipyramid $P_n$, and the polynomial invariant $\mathfrak{I}$ given by the squared length of the 'main diagonal'. By Proposition 6.1, we obtain the lower bound $\zeta(n) \geq \nu(n) = \theta(\sqrt{n}\, 2^n)$, where $\nu(n)$ are the same as in Corollary 7.3. Alternatively, the upper bound (the first part) gives $\zeta(n) \leq 2^m = 8^n$, where $m = 3n$ is the number of edges in $P_n$. As in the case of the icosahedron, using the second part of the upper bound is much harder here. It would be interesting to find the exact asymptotic behavior of $\zeta(n)$.

## 8. Proof of the Robbins conjecture.

Recall the problem from the introduction. Given a real positive $n$-tuple $(a_1, \ldots, a_n)$ which satisfies

$$\max\{a_1, \ldots, a_n\} < \frac{a_1 + \ldots + a_n}{2},$$

there exists exactly one convex $n$-gon inscribed into a circle whose sides are $a_1, \ldots, a_n$, in this order. The uniqueness and existence is clear, because we can take a circle with large enough radius, e.g. $r > a_1 + \ldots + a_n$ will suffice, and points $v_0, v_1, \ldots, v_n$ on a circle such that the length $\ell(v_{i-1}, v_i) = a_i$. Then decrease the radius $r$ of the circle until the first time $v_0 = v_n$, and we obtain the desired $n$-gon.

In the notation of Section 7.6, consider a regular bipyramid with vertices $v_1, \ldots, v_n, u_1, u_2$, and let $a_i$ denote the lengths of the edges of $n$-gon, while $b$ denote the length of the side edges $(v_i, u_j)$. Denote by $\mathcal{W}_n$ the corresponding length family with degree of freedom $n+1$.

Now observe that the squared area $\mathrm{A} = \mathtt{S}^2(a_1, \ldots, a_n)$ is a ratio of two polynomial invariants. Indeed, observe that $\frac{1}{3}(\mathtt{S} \cdot c) = \mathrm{vol}$, where $c = \ell(u_1, u_2)$ is the length of the 'main diagonal'. Thus $\mathrm{A} = 9\,\mathrm{vol}^2/c^2$, as desired. We conclude that the squared area is algebraic over $f_i$, and is a solution of a relation $(\mathfrak{s})$ as in Theorem 1.4. It remains to compute the minimal degree $\eta(\mathcal{W}_n, A)$ of such relation.

By Theorem 7.1, the minimal polynomial relation $\widetilde{H}(R, f_1, \ldots, f_n)$ for the squared radius $R$ has degree $\nu(n)$, where

$$\nu(2k+1) = \Delta_k, \qquad \nu(2k+2) = 2\Delta_k,$$

as in the theorem. While we do not compute the generalized Heron polynomials $(\mathfrak{s})$ in the same way as we computed polynomial relations $\widetilde{H}_n(\cdot)$, we can still determine $\eta(\mathcal{W}_n, \mathrm{A})$. We need the following technical result.

**Lemma 8.1.** *Given squared lengths $(a_1, \ldots, a_n, b)$ of sides of a regular bipyramid with $b$ large enough, there are exactly $\nu(n)$ preimages of $(a_1, \ldots, a_n, b)$ under the characteristic map $\mathbf{F}$.*

*Proof.* Given real $a_i$ and $b$, with $b$ large enough, there are exactly $\nu(n)$ real regular bipyramids. They all have different radius and different length of the 'main' diagonal. Moreover, from Corollary 7.3 the length of a diagonal cannot take more than $\nu(n)$ values for different realization of a single length diagram. Hence, all complex realizations of a regular bipyramid have real main diagonal lengths.

From above, we can assume that point $u_1 = (0,0,0)$ and $u_2 = (c,0,0)$ where $c$ is real. For the vertex $v_1 = (x,y,0)$ we obtain $x^2 + y^2 = (x-c)^2 + y^2 = b^2$. Therefore, $x = c/2$ and $y^2 = b^2 - (c/2)^2$, which implies that $x$ and $y$ are real.

Now, if $v_2 = (x_2, y_2, z_2)$ we similarly conclude that $x_2 = c/2$ is real and

$$(y-y_2)^2 + (z-z_2)^2 \,=\, a_1, \qquad y_2^2 + z_2^2 \,=\, b^2 - (c/2)^2.$$

Therefore,

$$y_2^2 + z_2^2 \,=\, b^2 - c^2/4, \qquad y y_2 + z z_2 \,=\, 2(b^2 - c^2/4) - a_1.$$

This system allows at most two solutions for a pair $(y_2, z_2)$. We know that there are two distinct real points on the circle $y^2 + z^2 = b^2 - c^2/4$ which give two distinct solutions. Therefore, both $x_2$ and $y_2$ are real numbers. Proceeding in a similar fashion, we conclude that all coordinates $x_i$ and $y_i$ are real. □

**Remark 8.2.** Notice that we implicitly assumed that a general point in the length family of a regular bipyramid has real coordinates. This assumption is valid, as real points are dense in a complex vector space $W_{\mathbb{C}}$, corresponding to the length family of a regular bipyramid.

By Theorem 7.1, $\deg(F_R) = \nu(n)$ is at most the number of complex bipyramid realizations. From Lemma 8.1, there are $\nu(n)$ complex realizations of a bipyramid with general real sides. Thus, by Lemma 4.1 the degree $\eta(\mathcal{W}_n, \mathrm{A})$ of the minimal polynomial relation for the (squared) area $\mathrm{A} = \mathtt{S}^2(\cdot)$ is at most $\nu(n)$. In a different direction, it is easy to see that for generic values of the side length $a_i$ the (signed) areas of these polygons is different. Thus, by the lower bound (Lemma 6.3) the degree $\eta(\mathcal{W}_n, \mathrm{A})$ is at least $\nu(n)$. This completes the proof of the Robbins Conjecture (Theorem 1.4).

**Example 8.3.** For $n = 3$, the Heron formula ($\triangle$) in Section 2.8 gives a linear relation for the squared area $\mathrm{A} = \mathtt{S}^2(a,b,c)$ of a triangle:

$$\mathrm{A} - \frac{1}{16}\bigl(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\bigr) = 0$$

This corresponds to the case $\nu(3) = \Delta_1 = 1$.

Similarly, for $n = 4$, the Brahmagupta formula ($\boxplus$) in Section 2.9 gives a linear relation for the squared area $\mathrm{A} = \mathtt{S}^2(a,b,c,d)$ of an inscribed convex quadrilateral:

$$(\boxtimes) \quad \mathrm{A} - \frac{1}{8}\bigl(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2\bigr) + \frac{1}{16}\bigl(a^4 + b^4 + c^4 + d^4\bigr) + \frac{1}{2}\,abcd = 0.$$

Note that this equation is not of the type ($\mathfrak{s}$) as the RHS of ($\boxtimes$) is not a polynomial in squared edge lengths. To correct this, let

$$g := \frac{1}{8}\bigl(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2\bigr) - \frac{1}{16}\bigl(a^4 + b^4 + c^4 + d^4\bigr) \in \mathbb{C}[a^2, b^2, c^2, d^2].$$

Now write equation ($\boxtimes$) as $\mathrm{A} - g + abcd/2 = 0$. This gives the desired quadratic relation for the squared area A:

$$\bigl(\mathrm{A} - g + abcd/2\bigr)\bigl(\mathrm{A} - g - abcd/2\bigr) \,=\, \mathrm{A}^2 - 2g\mathrm{A} + \bigl(g^2 - a^2b^2c^2d^2/4\bigr) \,=\, 0,$$

which corresponds to the case $\nu(4) = 2\Delta_1 = 2$. Let us note that the $\bigl(\mathrm{A} - g - abcd/2\bigr)$ term corresponds to the area of a self-intersecting inscribed quadrilateral. The cases $n = 5, 6$ are presented in [R2][6].

[6] The case $n = 5$ was rediscovered in [V1] (see also [V2]). Most recently concise formulas for the cases $n = 7, 8$ were presented in [MRR]. We refer to [P] for a short survey and history of these developments.

## 9. Proof of Lemma 7.2.

We consider the cases of odd and even $n$ separately. The even case will be, in effect, reduced to the odd case.

**Odd Case:** $n = 2k+1$. Recall the polynomials $H_n(\cdot)$ defined in $(\circledast')$ in Section 7.6. We write

$$H_n(\sin^2\alpha_1, \ldots, \sin^2\alpha_n) \,=\, \sum_{\mathbf{i}} H^{\mathbf{i}}(\sin^2\alpha_1, \ldots, \sin^2\alpha_n),$$

where $H^{\mathbf{i}}$ is a homogeneous part of degree $\mathbf{i} = (i_1, \ldots, i_n)$.

For all $1 \le i \le n$ let $\alpha_i = m_i\omega$, where $\{m_i\}$ are odd integers which satisfy the inequalities

$$(\gimel) \qquad m_1 \ge m_2 \ge \ldots \ge m_n > 0 \quad \text{and} \quad m_1 + m_2 + \ldots + m_k < m_{k+1} + \ldots + m_n\,.$$

To simplify the product $(\circledast)$, let $\overline{\varepsilon} = (1, \varepsilon_2, \ldots, \varepsilon_n)$ and $\mathcal{E} = \bigl\{\overline{\varepsilon} = (1, \varepsilon_2, \ldots, \varepsilon_n) \mid \varepsilon_i \in \{\pm 1\}\bigr\}$. Similarly, denote $\overline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, and let

$$(\overline{\varepsilon}, \overline{m}) \,:=\, m_1 + \varepsilon_2 m_2 + \ldots + \varepsilon_n m_n\,, \qquad \|\overline{\varepsilon}, \overline{m}\| := |m_1 + \varepsilon_2 m_2 + \ldots + \varepsilon_n m_n|\,.$$

The inequalities $(\gimel)$ and $n$ being odd easily imply that $(\overline{\varepsilon}, \overline{m})$ has the same sign as $(1 + \varepsilon_2 + \ldots + \varepsilon_n)$. In other words,

$$(\daleth) \qquad \|\overline{\varepsilon}, \overline{m}\| \,=\, \operatorname{sign}(1 + \varepsilon_2 + \ldots + \varepsilon_n) \cdot (\overline{\varepsilon}, \overline{m}), \quad \text{where } \operatorname{sign}(\varkappa) \,=\, \begin{cases} 1, & \text{if } \varkappa > 0 \\ -1, & \text{if } \varkappa < 0 \end{cases}$$

(note that $1 + \varepsilon_2 + \ldots + \varepsilon_n \ne 0$ here since $n$ is odd.)

In this notation, we have:

$$\sin(\alpha_1 + \varepsilon_2\alpha_2 + \ldots + \varepsilon_n\alpha_n) \,=\, \sin\bigl(m_1\omega + \varepsilon_2 m_2\omega + \ldots + \varepsilon_n m_n\omega\bigr) \,=\, \sin\bigl((\overline{\varepsilon}, \overline{m})\,\omega\bigr).$$

Since $n$, $m_i$, and $\varepsilon_i \in \{\pm 1\}$ are odd, the integer $(\overline{\varepsilon}, \overline{m})$ is also odd. Using properties of the Chebyshev polynomials (see Section 7.3) or by induction, it is easy to see that for all odd $m \in \mathbb{Z}$ a polynomial

$$(\blacklozenge) \qquad \sin(m\,\omega) \,=\, \pm\sin^{|m|}\omega \,+\, \ldots \,+\, m\sin\omega$$

has only odd degree terms. Here the sign $\pm$ is given by the parity of $(m-1)/2$; to simplify calculations we omit the exact formula for the signs throughout this section. Now use $(\blacklozenge)$ for each term of the product $(\circledast)$ in Section 7.6 to expand $H_n$ as follows:

$$\begin{aligned} H_n\bigl(\sin^2(m_1\omega), \ldots, \sin^2(m_n\omega)\bigr) \,&=\, \prod_{\overline{\varepsilon}\in\mathcal{E}} \sin(\alpha_1 + \varepsilon_2\alpha_2 + \ldots + \varepsilon_n\alpha_n) \\ (\maltese) \qquad &=\, \prod_{\overline{\varepsilon}\in\mathcal{E}} \sin\bigl(\|\overline{\varepsilon}, \overline{m}\|\,\omega\bigr) \,=\, \prod_{\overline{\varepsilon}\in\mathcal{E}} \Bigl(\pm(\sin\omega)^{\|\overline{\varepsilon}, \overline{m}\|} + \ldots + \|\overline{\varepsilon}, \overline{m}\|\sin\omega\Bigr) \\ &=\, \pm(\sin\omega)^{\sum_{\overline{\varepsilon}\in\mathcal{E}} \|\overline{\varepsilon}, \overline{m}\|} \,+\, \ldots \,+\, \prod_{\overline{\varepsilon}\in\mathcal{E}} (\overline{\varepsilon}, \overline{m}) \cdot \bigl(\sin\omega\bigr)^{2^{n-1}}. \end{aligned}$$

Observe that the highest power of $(\sin\omega)$ in $(\maltese)$ is

$$\sum_{\overline{\varepsilon}\in\mathcal{E}} \|\overline{\varepsilon}, \overline{m}\| \,=\, \sum_{\overline{\varepsilon}\in\mathcal{E}} \bigl|m_1 + \varepsilon_2 m_2 + \ldots + \varepsilon_n m_n\bigr| \,=\, \beta_1 m_1 + \beta_2 m_2 + \ldots + \beta_n m_n,$$

for some integer coefficients $\beta_1, \ldots, \beta_n \in \mathbb{Z}$. Summing equations $(\daleth)$ over all $\overline{\varepsilon} \in \mathcal{E}$, we have:

$$\beta_i \,=\, \sum_{\overline{\varepsilon}\in\mathcal{E}} \operatorname{sign}(1 + \varepsilon_2 + \cdots + \varepsilon_n) \cdot \varepsilon_i\,,$$

Therefore, the coefficients $\beta_i$ are independent of the values of $\{m_j\}$ whenever they satisfy $(\gimel)$. Varying $\{m_j\}$ and using $(\blacklozenge)$ for each variable $z_i = \sin^2(m_i\omega)$, we conclude that the polynomial $H_n(z_1, \ldots, z_i) \,=\, \sum_{\mathbf{i}} H^{\mathbf{i}}(z_1, \ldots, z_i)$ has the maximal degree $\mathbf{i} \,=\, \frac{1}{2}(\beta_1, \ldots, \beta_n)$. On the other hand, from the (intermediate) product formula (✠), all terms in $H_n$ have degree at least $\frac{1}{2}2^{n-1}$. We have

$$\beta_1 + \beta_2 + \ldots + \beta_n \,=\, \sum_{i=1}^{n} \sum_{\overline{\varepsilon}\in\mathcal{E}} \operatorname{sign}(1+\varepsilon_2+\ldots+\varepsilon_n)\cdot\varepsilon_i$$

$$= \sum_{\overline{\varepsilon}\in\mathcal{E}} \operatorname{sign}(1+\varepsilon_2+\ldots+\varepsilon_n)\cdot(1+\varepsilon_2+\ldots+\varepsilon_n) \,=\, \sum_{\overline{\varepsilon}\in\mathcal{E}} \bigl|1+\varepsilon_2+\ldots+\varepsilon_n\bigr|$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} \bigl|n-2j\bigr| \,=\, \sum_{j=0}^{k} \binom{2k}{j} \bigl(2k+1-2j\bigr) \,-\, \sum_{j=k+1}^{2k} \binom{2k}{j} \bigl(2k+1-2j\bigr)$$

$$= \sum_{j=0}^{k} \left[\binom{2k}{j} + \binom{2k}{j-1}\right] \bigl(2k+1-2j\bigr) \,=\, \sum_{j=0}^{k-1} \binom{2k+1}{j} \bigl(2k+1-2j\bigr)$$

$$= \sum_{j=0}^{k-1} 2\bigl(k-j\bigr)\binom{2k+1}{j} \,+\, \sum_{j=0}^{k-1} \binom{2k+1}{j} \,=\, 2\sum_{j=0}^{k-1} (k-j)\binom{2k+1}{j} + 2^{2k}$$

Therefore, the degree of $x$ in $\widetilde{H}_n(x, f_1, \ldots, f_n)$ is equal to

$$\nu(2k+1) \,=\, \frac{1}{2}\sum_{i=1}^{n}\beta_i \,-\, \frac{1}{2}\,2^{n-1} \,=\, \sum_{j=0}^{k-1}(k-j)\binom{2k+1}{j} + 2^{2k-1} - 2^{2k-1} \,=\, \Delta_k\,,$$

where the last equality follows from Robbins' formula (♣) in the introduction. This completes the proof of the odd case of the lemma.

**Even Case:** $n = 2k+2$. Let $\alpha_1 = 2\,m_1\omega, \ldots, \alpha_{2k+1} = 2\,m_{2k+1}\omega$, where $\{m_i\}$ are positive integers which satisfy (ℶ). Also, let $m_n = 1$ and $\alpha_{2k+2} = \omega$. For such $n$-tuples $(\alpha_1, \ldots, \alpha_n)$ we have:

$$\sin(\alpha_1 + \varepsilon_2\alpha_2 + \ldots + \varepsilon_n\alpha_n) \,=\, \sin\bigl[\bigl(2(m_1 + \varepsilon_2 m_2 + \ldots + \varepsilon_{2k+1}m_{2k+1}) + \varepsilon_{2k+2}\bigr)\,\omega\bigr].$$

Clearly $M = 2(m_1 + \varepsilon_2 m_2 + \ldots + \varepsilon_{2k+1}m_{2k+1}) \neq 0$. Note that for every even integer $M \neq 0$ the equation (♦) gives:

$$\sin(M+1)\omega\cdot\sin(M-1)\omega \,=\, \Bigl[\sin^{|M|+1}(\omega) \,+\, \ldots\Bigr]\cdot\Bigl[\sin^{|M|-1}(\omega) \,+\, \ldots\Bigr] \,=\, \sin^{2|M|}(\omega) \,+\, \ldots$$

Use this equation to obtain the following version of (✠) in the even case:

$$(✠✠) \qquad \begin{aligned} H_n\bigl(\sin^2(m_1\omega), \ldots, \sin^2(m_n\omega)\bigr) \,&=\, \pm(\sin\omega)^{\sum_{\overline{\varepsilon}\in\mathcal{E}}(\|\overline{\varepsilon},\overline{m}\|+1) + \sum_{\overline{\varepsilon}\in\mathcal{E}}(\|\overline{\varepsilon},\overline{m}\|-1)} \,+\, \ldots \\ &=\, \pm(\sin\omega)^{2\sum_{\overline{\varepsilon}\in\mathcal{E}}(\|\overline{\varepsilon},\overline{m}\|)} \,+\, \ldots \end{aligned}$$

where $\overline{m} = (2m_1, \ldots, 2m_{2k+1})$ and $\mathcal{E} = \bigl\{\overline{\varepsilon} = (1, \varepsilon_2, \ldots, \varepsilon_{2k+1}), \varepsilon_i \in \{\pm 1\}\bigr\}$ is as above. By analogy with the odd case we conclude:

$$\nu(2k+2) \,=\, \frac{1}{2}\cdot 2\,\bigl(\beta_1 + \ldots + \beta_{n-1}\bigr) \,-\, 2^{n-1} \,=\, 2\Delta_k.$$

This completes the proof of Lemma 7.2.

## 10. Final Remarks

10.1. **Rigidity theory.** While our interests lie in the study of realization spaces (see below) much of this work is related to Rigidity Theory which was under intense development in the last several decades. In the most general setting, it studies a general (not necessarily planar or triangulated) graph G and edge lengths (or stresses) and asks about infinitesimal and continuous deformations of realizations in $\mathbb{R}^n$ (see e.g. [C2, CW, Ko, Ro]). In case of polyhedra homeomorphic to a sphere these results extend the Cauchy Rigidity Theorem to more general families of polyhedra (see e.g. [G, RR]). We refer to [C4, GSS, W] for extensive surveys of known results.

As we learned only after this paper was finished, the algebraic geometry approach developed here for the study of polynomial invariants is strongly related to that in [AR1, G] (see also [AR2, RW]) in the study of rigidity. While our motivation is different, our proof of Proposition 3.1 and the Cauchy Rigidity Theorem (Corollary 5.3) resembles closely that given in [AR1, Ro]. The heart of our argument, Generic Freeness Lemma 4.1 seems to be new even if motivated by known ideas in algebraic geometry. It is perhaps of independent interest.

10.2. **Flexible polyhedra.** As we mentioned in the introduction, existence of flexible polyhedra (simplicial polyhedral complexes homeomorphic to a sphere and not self-intersecting) was established by Connelly, who also extensively studied them, and introduced a number of interesting conjectures [C1, C3, C4]. The Bellows Conjecture on invariance of the volume under continuous deformations was resolved by Sabitov in a series of papers with the same general idea but slightly varying proofs and conditions [AS, LNS, S1, S2, S3, S4] (see also [D, S5]). The similar problem for the *Denh invariant* remains open [Sc]. By Sydler's theorem this would imply scissor equivalence (as in Hilbert's Third Problem) of continuous deformations of flexible polyhedra [C3, Sc].

Let us mention that Corollaries 1.2 and 1.3 while stated for convex polytopes, in fact, extend verbatim to nonconvex simplicial polyhedra $Q$ homeomorphic to a sphere. Indeed, the Steinitz Theorem (see Section 2.5) implies that the corresponding plane triangulation G has an edge lengths function which gives a convex polytope $P$. Since $Q$ now is one of the complex realizations of G with this length function, its diagonals (or volume) must satisfy the equations $(*)$ and $(**)$. To see why this does not imply rigidity, recall the observation in the introduction that values of coefficients in $(*)$ can become zero for some particular edge length, and thus the realization space is no longer finite. It would be interesting to find general sufficient conditions on lengths of edges under which a polyhedron can be a flexor (cf. [G]).

10.3. **Realization spaces.** The study of realization (moduli) spaces of various geometric and combinatorial structures goes back to ancient times under the name *kinematic* [HC]. In the plane, the class of algebraic sets produced by linkages has been an open problem ever since Kempe's classical construction of a linkage "drawing a straight line" [Ke]. Most recently this problem was completely resolved in a remarkable paper [KM]; roughly, the authors showed that *every* algebraic set can be "drawn" in such a way (see also [Ki]). See also [M1, M2] for a different study of realization spaces of graphs.

The results of this kind are called *universality theorems*, after the celebrated theorem of Mnëv [Mn] on realization spaces of matroids and convex polyhedra (with combinatorial rather than metric conditions). This was our original motivation for study of realization spaces of length diagrams and, more generally, length families. Although we believe that universality type results exists in this case, we believe it is premature to formulate it as a formal conjecture.

To underscore the difference with the linkages, let us emphasize that linkages have no restriction on graph structure, while graphs we consider are plane triangulations. The high degrees of complex realization spaces of polytopes give a weak evidence in support of the universality claim. We plan to continue our study of realization spaces $\mathfrak{M}(\mathrm{G}, W)$ in the future.

10.4. **Computational aspects.** To see the relationship of this work and the computational problem of constructing convex polytopes from the graph and length function [E], consider the following straightforward algorithm: solve $m = 3n - 6$ quadratic equations in $3n$ variables, with 6 extra conditions to make the polytope planted. Now, if the space of realizations can be made as 'complicated' as desired with relatively small number $n$ of vertices, then so is the construction of convex polytopes. This *algebraic complexity* approach to lower bounds seem to be novel in this context and is waiting to be explored. We refer to [BCS] for background on algebraic complexity, numerous examples and references.

10.5. **Other Robbins conjectures.** In addition to what we call the Robbins Conjecture (Theorem 1.4), David Robbins made further conjectures on the behavior of the generalized Heron polynomials [R1, R2]. Most recently, Connelly [C5] and Varfolomeev [V1] established one of them related to the first coefficient of generalized Heron polynomials. We refer to a recent preprint [P] outlining these and other developments.

**Acknowledgements.** Let us start by saying that this work was partly motivated by the ideas of David Robbins [R1, R2] and his recently renewed interest in the area of polygons [L]. His untimely death came just weeks prior to our discoveries. We dedicate this paper to David Robbins for his foresight, intuition and love of mathematics.

We would like to thank Victor Aleksandrov, Lynne Butler, Nikolai Dolbilin, Joe Harris, Johan de Jong, Misha Kapovich, Julie Roskies, Izhad Sabitov, Jean-Michel Schlenker, Jason Starr, Vitalij Varfolomeev and Walter Whiteley for comments and helpful remarks. We are very grateful to Jeremy Martin and Ezra Miller for reading the previous version of the paper, and to Bob Connelly and Richard Stanley for encouragement.

This work was initiated as a part of the UROP program for MIT undergraduates. The first author was supported by the Paul E. Gray (1954) Endowed Fund for UROP and worked under supervision of the second author. We are thankful to Victor Guillemin for his help in making this possible. The second author was partially supported by the National Security Agency and the National Science Foundation.

## References

[A1] A. D. Aleksandrov (А. Д. Александров), *Внутренняя геометрия выпуклых поверхностей* (in Russian), Гостехиздат, Москва–Ленинград, 1948.

[A2] A. D. Aleksandrov (А. Д. Александров), *Выпуклые многогранники* (in Russian), Гостехиздат, Москва , 1950.

[AR1] L. Asimow, B. Roth, The rigidity of graphs. *Trans. Amer. Math. Soc.* **245** (1978), 279–289.

[AR2] L. Asimow, B. Roth, The rigidity of graphs. II. *J. Math. Anal. Appl.* **68** (1979), 171–190.

[AS] A. V. Astrelin, I. Kh. Sabitov, A canonical polynomial for the volume of a polyhedron, *Russian Math. Surveys* **54** (1999), 430–431.

[Be] M. Berger, *Géométrie*, Vol. 3, *Convexes et polytopes, polyèdres réguliers, aires et volumes* (in French), Nathan, CEDIC, Paris, 1977.

[Bo] B. Bollobás, *Modern Graph Theory*, Springer, New York, 1998.

[BGP] Yu. Burago, M. Gromov, G. Perelman, A.D. Alexandrov spaces with curvature bounded below, *Russian Math. Surveys* **47** (1992), no. 2, 1–58.

[BCS] P. Bürgisser, M. Clausen, M. A. Shokrollahi, *Algebraic Complexity Theory*, Grundlehren der Mathematischen Wissenschaften, Vol. 315, Springer, Berlin, 1997.

[C1] R. Connelly, Conjectures and open questions in rigidity, in *Proc. ICM Helsinki (1978)*, 407–414, Acad. Sci. Fennica, Helsinki, 1980.

[C2] R. Connelly, The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces, *Adv. in Math.* **37** (1980), 272–299

[C3] R. Connelly, Flexing surfaces, in *The mathematical Gardner* (Edited by D. A. Klarner), 79–89, PWS Publishers, Boston, MA, 1981.

[C4] R. Connelly, Rigidity, in *Handbook of Convex Geometry*, Vol. A, B, 223–271, North-Holland, Amsterdam, 1993.

[C5] R. Connelly, Comments on generalized Heron polynomials and Robbins' conjectures, preprint (2004), available at `http://www.math.cornell.edu/~connelly`

[CSW] R. Connelly, I. Sabitov and A. Walz, The bellows conjecture, *Beiträge Algebra Geom.* **38** (1997), 1–10.

[CW] H. Crapo, W. Whiteley, Statics of frameworks and motions of panel structures, a projective geometric introduction, *Structural Topology* **6** (1982), 43–82.

[D] N. P. Dolbilin (Н. П. Долбилин), *Жемчужины Теории Многогранников* (in Russian), МЦНМО, Москва, 2000, 40 pp.

[E] H. Edelsbrunner, in *Open Problems Presented at SCG'98* (P. K. Agarwal, J. O'Rourke, Eds.), available at `http://www.cs.duke.edu/~pankaj/scg98-openprobs`

[G] H. Gluck, Almost all simply connected closed surfaces are rigid, *Lecture Notes in Math.* **438**, 225–239 Springer, Berlin, 1975

[GSS] J. Graver, B. Servatius, H. Servatius, *Combinatorial rigidity*, Graduate Studies in Mathematics **2**, AMS, Providence, RI, 1993.

[GK] P. Gritzmann, V. Klee, On the complexity of some basic problems in computational convexity. II. Volume and mixed volumes, in *Polytopes: abstract, convex and computational*, 373–466, Kluwer, Dordrecht, 1994.

[HC] D. Hilbert, S. Cohn-Vossen, *Geometry and the imagination* (translation from German original), Chelsea, New York, 1952.

[Had] J. Hadamard, *Leçons de géométrie* II, *Géométrie dans l'espace* (in French), Reprint of the eighth (1949) edition, Éditions Jacques Gabay, Sceaux, 1988.

[Har] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer, New York, 1977.

[KM] M. Kapovich, J. J. Millson, Universality theorems for configuration spaces of planar linkages, *Topology* **41** (2002), 1051–1107.

[Ke] A. B. Kempe, How to draw a straight line, *Nature* XVI (1877), 65–67, 86–89.

[Ki] H. C. King, Planar linkages and algebraic sets, in Proc. 6-th Gökova Geometry-Topology Conf., *Turkish J. Math.* **23** (1999), 33–56.

[Kl] F. Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* (in German), Reprint of the (1884) original, Birkhäuser, Basel, 1993.

[Ko] M. D. Kovalev, Geometric theory of linkages, *Russian Acad. Sci. Izv. Math.* **44** (1995), 43–68.

[L] P. Landers, Dying Mathematician Spends Last Days on Area of Polygon, *WSJ* July 29, 2003, p. 1.

[LNS] S. Lawrencenko, S. Negami, I. Kh. Sabitov, A simpler construction of volume polynomials for a polyhedron, *Beiträge Algebra Geom.* **43** (2002), 261–273.

[MRR] F. M. Maley, D. P. Robbins and J. Roskies, On the areas of cyclic and semicyclic polyons, `arXiv:math.GM/0407300`

[M1] J. Martin, Geometry of graph varieties, Trans. AMS **355** (2003), 4151–4169.

[M2] J. Martin, On the topology of multigraph picture spaces, `arXiv:math.CO/0307405`, 20 pp.

[Mi] E. Miller, Icosahedra constructed from congruent triangles, *Grünmbaum birthday issue. Discrete Comput. Geometry* **24** (2000), 437–451.

[Mn] N. Mnëv, The universality theorems on the classification problem of configuration varieties and convex polytopes varieties, in *Lecture Notes in Mathematics*, **1346**, Springer, Berlin, 1988.

[P] I. Pak, The area of cyclic polygons: Recent progress on Robbins' Conjectures, to appear in *Adv. Applied Math.* (special issue in memory of David Robbins), available at `http://www-math.mit.edu/~pak/research.html`

[R1] D. P. Robbins, Areas of polygons inscribed in a circle, *Discrete Comput. Geom.* **12** (1994), 223–236.

[R2] D. P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly* **102** (1995), 523–530.

[Ro] B. Roth, Rigid and flexible frameworks, *Amer. Math. Monthly* **88** (1981), 6–21.

[RW] B. Roth, W. Whiteley, Tensegrity frameworks, *Trans. AMS* **265** (1981), 419–446.

[RR] L. Rodríguez, H. Rosenberg, Rigidity of certain polyhedra in $\mathbb{R}^3$, *Comment. Math. Helv.* **75** (2000), 478–503.

[S1] I. Kh. Sabitov, The generalized Heron-Tartaglia formula and some of its consequences, *Sbornik Math.* **189** (1998), 1533–1561

[S2] I. Kh. Sabitov, The volume as a metric invariant of polyhedra, *Discrete Comput. Geom.* **20** (1998), 405–425.

[S3] I. Kh. Sabitov (И. Х. Сабитов), Решение многогранников (in Russian), *Dokl. Akad. Nauk* **377** (2001), no. 2, 161–164.

[S4] I. Kh. Sabitov, Algorithmic solution of the problem of isometric realization for two dimensional polyhedral metrics, *Izvestiya: Mathematics* **66** (2002), 377–391.

[S5] I. Kh. Sabitov (И. Х. Сабитов), *Объемы многогранников* (in Russian), МЦНМО, Москва, 2002, 32 pp., avaliable from `http://www.mccme.ru`

[Sc] J.-M. Schlenker, La conjecture des soufflets (d'après Sabitov), *Séminaire Bourbaki* **55** (2002-3), No. 912.

[V1] V. V. Varfolomeev, Inscribed polygons and Heron polynomials, *Sbornik Math.* **194** (2003), 311–331.

[V2] V. V. Varfolomeev, Galois groups of Heron-Sabitov polynomials for pentagons inscribed in a circle, *Sbornik Math.* **195** (2004), 3–16.

[W] W. Whiteley, Matroids and rigid structures. Matroid applications, 1–53, in *Encyclopedia Math. Appl.* 40, Cambridge U. Press, Cambridge, 1992.

[Z] G. M. Ziegler, *Lectures on polytopes*, Springer, New York, 1995.