PERCOLATION ON GRIGORCHUK GROUPS

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Abstract

Let $p_c(G)$ be the critical probability of the site percolation on the Cayley graph of group G. In [2] of Benjamini and Schramm conjectured that$p_c < 1$, given the group is infinite and not a finite extension of $\mathbb{Z}$. The conjecture was proved earlier for groups of polynomial and exponential growth and remains open for groups of intermediate growth.

In this note we prove the conjecture for a special class of *Grigorchuk groups*, which is a special class of groups of intermediate growth. The proof is based on an algebraic construction. No previous knowledge of percolation is assumed.

1 Introduction

While percolation on $\mathbb{Z}^d$ has been studied for decades, percolation on general Cayley graphs became popular only in the past several years. This direction of research was outlined in an important paper [2] of Benjamini and Schramm. The authors conjectured that the critical probability p_c for the site percolation satisfies

$$p_c < 1$$

given the group is infinite and not a finite extension of $\mathbb{Z}$.

Since appearance of [2], a number of interesting results has been obtained. The above conjecture has been established for groups of polynomial or exponential growth, as well as for finitely presented groups (see

[1, 3, 23]). In this note we prove the conjecture for a class of so called *Grigorchuk groups*, which are defined as groups of Lebesgue–measure–preserving transformations on the unit interval. Recently other examples of groups of intermediate growth were found, notably Gupta–Sidki groups and Grigorchuk p-groups (see [5, 9, 16, 17]). These are the only known examples of groups of intermediate growth[1].

Let G be an infinite group generated by a finite set S, $S = S^{-1}$, and let $\Gamma = \Gamma(G, S)$ be the corresponding Cayley graph. Consider *Bernoulli site percolation* in which all vertices are independently open with probability p (closed with probability $1-p$). By $\theta(p)$ denote the probability $P(id \leftrightarrow \infty)$, i.e. the probability that the open cluster containing id is infinite. Define the *critical probability*

$$p_c(G) = \sup\{p : \theta(p) = 0\}$$

It is known that $p_c(G) > 1/|S|$, $p_c(\mathbb{Z}^2) = 1/2$, and that $p_c(G_1) \le p_c(G_2)$ if G_2 is a subgroup or a quotient group of G_1. We refer to [12, 13] for a thorough treatment of classical percolation, and to [2, 3] for the many interesting questions and results for percolation on Cayley graphs.

Conjecture (Benjamini and Schramm)
If Γ is the Cayley graph of an infinite (finitely generated) group G, which is not a finite extension of $\mathbb{Z}$, then $p_c < 1$.

The conjecture is one of the central unsolved questions of the percolation theory on Cayley graphs. While it was established for groups of exponential and polynomial growth (see [3, 23]), the conjecture remains open for groups of intermediate growth. The conjecture has been also confirmed for finitely presented groups (see [1]).

Let us remark here that the analogous conjecture for bond percolation is equivalent to Conjecture (see e.g. [12, 13]). Recently Häggström also showed equivalence of the conjecture with having a phase transition for Ising and Widom–Rowlinson models (see [19]).

Let $\omega = (\omega_1, \omega_2, \dots)$ be an infinite sequence of elements in the set $\{0, 1, 2\}$. *Grigorchuk group* G_ω is a infinite profinite 2-group whose construction depends on ω (see [8, 11]). Groups G_ω are generated by 4 involutions, while the structure and even the growth is different for different ω. We postpone definition of G_ω till the next section.

Theorem 1. *The conjecture holds for Grigorchuk groups G_ω for all ω.*

[1]Of course, other groups of intermediate growth can be constructed from these groups. Study of percolation then can be reduced to the latter.

The proof relies on growth considerations and a fractal structure of Grigorchuk groups. Essentially, we find $\mathbb{Z}_+^2$ in the corresponding Cayley graphs. More precisely, we prove that G_ω contains a subgroup of finite index which is isomorphic to $A_\omega \times B_\omega$, where A_ω, B_ω are infinite finitely generated groups. From here we show that conjecture holds for $A_\omega \times B_\omega$, and therefore for G_ω.

Let us conclude by saying that in a sequel paper [9] Grigorchuk introduced a wider class of groups which correspond to sequences of $0, 1, \ldots, p$, where p is a prime. Analogously, Gupta and Sidki in [16, 17] constructed different series of p-groups, some of them later proved to have intermediate growth (see [5]). While we do not consider these groups, it is not hard to see that our analysis can be directly translated to these cases.

2 Growth of groups and percolation

Before we define the G_ω and prove the theorem, let us comment on the growth of groups and it's relevance to percolation problems.

Let G be an infinite group generated by a finite set S, $S = S^{-1}$, and let $\Gamma = \Gamma(G, S)$ be the corresponding Cayley graph. Let $B(n)$ be the set of elements $g \in G$ at a distance $\leq n$ from *id* in graph Γ. The growth function of G with respect to the set of generators S is defined as $\gamma(n) = |B(n)|$.

We say that a function $f : \mathbb{N} \to \mathbb{R}$ is *dominated* by a function $g : \mathbb{N} \to \mathbb{R}$, denoted by $f \preccurlyeq g$, if there is a constant $C > 0$ such that $f(n) \leq g(C \cdot n)$ for all $n \in \mathbb{N}$. Two functions $f, g : \mathbb{N} \to \mathbb{R}$ are called *equivalent*, denoted by $f \sim g$, if $f \preccurlyeq g$ and $g \preccurlyeq f$. It is known that for any two finite sets of generators S_1, S_2 of a group G, the corresponding two growth functions are equivalent (see e.g. [21, 28]). Note also that if $|S| = k$, then $\gamma(n) \leq k^n$.

Growth of group G is called *exponential* if $\gamma(n) \sim e^n$. Otherwise the growth is called subexponential. For example, all nonamenable groups have exponential growth (but not vice versa.) Growth of group G is called *polynomial* if $\gamma(n) \sim n^c$ for some $c > 0$. Otherwise the growth is called superpolynomial.

We call a group G *almost* nilpotent (solvable, etc.) if it contains a nilpotent (solvable, etc.) subgroup of finite index. The celebrated result of Gromov implies that groups of polynomial growth must be virtually nilpotent (see [15]). We add that by the Tits alternative all linear groups must be either almost solvable or contain a free group F_2. Finally, the Milnor–Wolf theorem states that all solvable groups have either polynomial or exponential growth (see [21, 28]).

If the growth of G is subexponential and superpolynomial, it is called *intermediate*. In a pioneer paper [7] Grigorchuk disproved Milnor's conjecture by exhibiting a group of intermediate growth. In [8] he defined a class of groups and proved that they have growth

$$\exp(n^{\alpha_1}) \preccurlyeq \gamma(n) \preccurlyeq \exp(n^{\alpha_2})$$

for some $1/2 \le \alpha_1, \alpha_2 < 1$.

A sequence $\omega = (\omega_1, \omega_2, \dots)$ is called *constant* if $\omega_1 = \omega_2 = \dots$. Otherwise ω is called *non–constant*. It is not hard to see that if ω is constant, then $G_\omega \cong \{a, b \,|\, a^2 = b^2 = id\}$ (see section 3). Thus G_ω is $\mathbb{Z}_2 * \mathbb{Z}_2$ and the conjecture holds. If ω is non–constant we will show that the natural Cayley graph in this case contains $\mathbb{Z}_+^2$ as a subgraph, which proves the theorem. Note that under mild conditions G_ω is a torsion group, so it does not contain $\mathbb{Z}^2$ as a subgroup.

A sequence $\omega = (\omega_1, \omega_2, \dots)$ is called *stabilizing* if $\omega_N = \omega_{N+1} = \dots$ for some N. We remark here that if ω is stabilizing sequence, then Grigorchuk group G_ω has an intermediate growth (see [8, 20, 27]).

Let us return to percolation on Cayley graphs. Let us first note that if the critical percolation p_c is strictly less than 1 for some generating set, the same is true for *all* generating sets. While we could not find this precise result in the literature, we believe it to be known and refer to [25] for similar coupling arguments.

Now recall that when the growth of G is polynomial, G is virtually nilpotent. We have the following result.

Theorem 2. (Benjamini) *If Γ is a Cayley graph of an infinite group G of polynomial growth which is not a finite extension of $\mathbb{Z}$, then $p_c(G) < 1$.*

The case when the growth of G is exponential was solved in [23]. First, observe that the natural Cayley graph of a free group F_k, $k \ge 2$ is isomorphic to regular $2k$-ary tree, where the percolation is well understood (see e.g. [24]). While the general exponential groups do not have to contain F_k, their Cayley graphs have been shown to contain a tree with positive branching number (see [22, 23]), which proves that conjecture in this case.

Let us briefly outline the Lyons' argument. Fix a lexicographic ordering on generators in S and connect each element $g \in G$ with id by a path which corresponds to the reduced words of minimum length. This gives a spanning subtree T in the Cayley graph. Now, the exponential growth function of G gives the positive Minkowski dimension of T, which by the famous result of Furstenberg implies positive Hausdorff dimension of T (see [6, 22]). The latter implies that T has critical probability < 1 and proves the following theorem (see [22, 23]).

Theorem 3. (Lyons) *If G has exponential growth, then $p_c(G) < 1$.*

As a corollary we obtain the $p_c(G) < 1$ for almost solvable and linear group, which are not almost $\mathbb{Z}$, as well as for Burnside group $B(n,p)$, where $p > 661$ (see [21, 28]). Let us also mention here a different result of Grigorchuk that if $\gamma(n) \preccurlyeq e^{\sqrt{n}}$, and G is residually finite p-group, then G has polynomial growth (see [10]). This confirms Conjecture 1 for such groups as well.

A few words about group presentation. First, it is known that when ω is non–stabilizing, the Grigorchuk group G_ω is not finitely presented (see [8, 26]). Thus Theorem 1 does not follow from results in [1] for finitely presented groups. Moreover, it was conjectured by Adian that all finitely presented groups must have either polynomial or exponential growth (see [11]). Further, it was conjectured by Grigorchuk that all finitely presented groups must either contain F_2 or be virtually nilpotent. Even if (weaker) Adian conjecture holds, this would prove that results of [1] are inapplicable to groups of intermediate growth.

3 Grigorchuk Group

In this section we will describe a construction of Grigorchuk's 2-group. For a complete description and further results see [8].

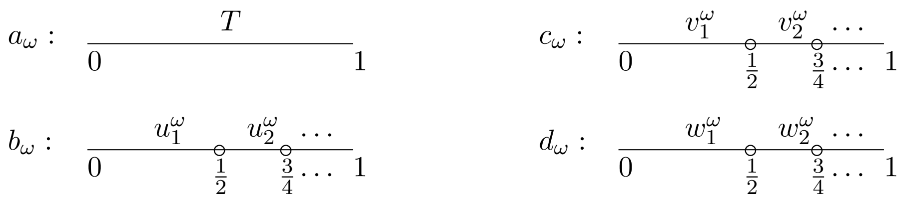
Let Δ be an interval. Denote by I an identity transformation on Δ and by T a transposition of two halves of Δ.

Let Ω be a set of infinite sequences $\omega = (\omega_1, \omega_2, \dots)$ of elements of the set $\{0, 1, 2\}$. For each $\omega \in \Omega$ define a $3 \times \infty$ matrix $\overline{\omega}$ by replacing ω_i with columns $\overline{\omega}_i$ where

$$\bar{0} = \begin{pmatrix} T \\ T \\ I \end{pmatrix}, \quad \bar{1} = \begin{pmatrix} T \\ I \\ T \end{pmatrix}, \quad \bar{2} = \begin{pmatrix} I \\ T \\ T \end{pmatrix}$$

By $U^\omega = (u_1^\omega, u_2^\omega, \dots), V^\omega = (v_1^\omega, v_2^\omega, \dots), W^\omega = (w_1^\omega, w_2^\omega, \dots)$ denote the rows of $\overline{\omega}$. Think of them as of infinite words in the alphabet $\{T, I\}$.

Define transformations $a_\omega, b_\omega, c_\omega, d_\omega$ of an interval $\Delta = [0,1] \setminus \mathbb{Q}$ as follows:



Observe that a_ω is independent of ω, and will be further denoted by a. Let G_ω be a group of transformations of the interval Δ generated by $a, b_\omega, c_\omega, d_\omega$. This family of groups was introduced and analyzed by Grigorchuk in [8] (see also [20] for further references). We refer to G_ω as Grigorchuk groups.

Observe that the generators of G_ω satisfy the following relations:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = 1,$$

$$c_\omega b_\omega = b_\omega c_\omega = d_\omega,$$
$$d_\omega b_\omega = b_\omega d_\omega = c_\omega,$$
$$c_\omega d_\omega = d_\omega c_\omega = b_\omega,$$

We call these *simple relations*. Under mild conditions, the groups G_ω are known to be not finitely presented (see [8, 20]).

Denote by Γ_ω a Cayley graph of the group G_ω with respect to the generators $a, b_\omega, c_\omega, d_\omega$. For every element $g \in G_\omega$ by $\partial(g)$ denote the smallest distance between g and id in Γ_ω. The paths in Γ_ω correspond to words in the alphabet $\{a, b_\omega, c_\omega, d_\omega\}^*$. The shortest paths (there could be many of them between two given elements) correspond to the reduced words in the alphabet. Recall that the balls in the Cayley graph Γ_ω are defined as $B_\omega(n) = \{g \in G_\omega | \partial(g) \le n\}$.

Let $\sigma : \Omega \to \Omega$ be a right shift operator acting on the infinite sequences as follows

$$\sigma : (\omega_1, \omega_2, \omega_3, \dots) \to (\omega_2, \omega_3, \dots)$$

Denote by Δ_0 and Δ_1 the half intervals $\Delta \cap [0, 1/2]$ and $\Delta \cap [1/2, 1]$. Define $H_\omega \subset G_\omega$ to be a stabilizer of Δ_0. Clearly, $g : \Delta_0 \to \Delta_0$ and $g : \Delta_1 \to \Delta_1$ for all $g \in H_\omega$.

Define $\phi_0^\omega : H_\omega \to G_{\sigma\omega}$ by restricting $h \in H_\omega$ to Δ_0. Formally, while H_ω acts on Δ_0 rather than Δ we can rescale the interval to obtain transformations in $G_{\sigma\omega}$. Similarly define $\phi_1^\omega : H_\omega \to G_{\sigma\omega}$ by restricting to Δ_1 and then rescaling to the unit interval.

It is easy to see that H_ω is a normal subgroup of index 2, which is generated by 6 elements $b_\omega, c_\omega, d_\omega, ab_\omega a, ac_\omega a, ad_\omega a$.

We will omit superscript ω in $\phi^\omega_{0,1}$ when it is clear on which H_ω the map ϕ^ω_i acts. The following table summarizes the images of homomorphisms of ϕ_0, ϕ_1 on the generators of subgroup H_ω.

TABLE

	b_ω	c_ω	d_ω	$ab_\omega a$	$ac_\omega a$	$ad_\omega a$
ϕ_0	u^ω_1	v^ω_1	w^ω_1	$b_{\sigma\omega}$	$c_{\sigma\omega}$	$b_{\sigma\omega}$
ϕ_1	$b_{\sigma\omega}$	$c_{\sigma\omega}$	$b_{\sigma\omega}$	u^ω_1	v^ω_1	w^ω_1

4 Lifted subgroups

Let $F_1, F_2 \subset H_\omega$ be subgroups which act trivially on Δ_0 and Δ_1 respectively:

$$F_1 = \{h \in H_\omega \,|\, h \equiv I \text{ on } \Delta_0\}$$

$$F_2 = \{h \in H_\omega \,|\, h \equiv I \text{ on } \Delta_1\}\,.$$

Define a subgroup $D_\omega = F_1 \cdot F_2$. Observe that F_1 commutes with F_2, $F_1 \cap F_2 = \{id\}$, and $D_\omega \simeq F_1 \times F_2$. Also, let $F \simeq F_1 = aF_2a \simeq F_2$.

Lemma 1.[2] D_ω is a normal subgroup of G_ω.

Proof. Observe that

$$a(F_1 \cdot F_2)a = aF_1a \cdot aF_2a = F_2 \cdot F_1 = F_1 \cdot F_2,$$

$$b_\omega(F_1 \cdot F_2)b_\omega = b_\omega F_1 b_\omega \cdot b_\omega F_2 b_\omega = F_1 \cdot F_2,$$
$$c_\omega(F_1 \cdot F_2)b_\omega = c_\omega F_1 c_\omega \cdot c_\omega F_2 c_\omega = F_1 \cdot F_2,$$
$$d_\omega(F_1 \cdot F_2)b_\omega = d_\omega F_1 d_\omega \cdot d_\omega F_2 d_\omega = F_1 \cdot F_2$$

Since $a, b_\omega, c_\omega, d_\omega$ are the generators in G_ω, we obtain $g\, D_\omega\, g^{-1} = D_\omega$ for all $g \in G$. This proves the result. $\square$

Lemma 2. If ω is non–constant, D_ω has a finite index in G_ω.

[2]It was pointed out to us by the referee that in [18] a related result on subgroups was proved. The referee suggests that this might lead to a generalization of our main result. We challenge the reader to obtain such a generalization.

Proof. Since ω is non–constant, there exist an integer k such that $\omega_k \neq \omega_1$. Without loss of generality assume that $\omega_1 = 0$, $\omega_k = 1$, and k is the first 1 in ω. To simplify the notation denote $s = k - 1$.

Since D_ω is a normal subgroup in G_ω, consider a natural map to the quotient group

$$\pi : G_\omega \to G_\omega / D_\omega$$

Observe that $\pi(d_\omega) = id$, $\pi(c_\omega) = \pi(b_\omega d_\omega) = \pi(b_\omega)$. Therefore $\pi(G) = \pi(\langle a, c_\omega \rangle)$. We will prove below that the subgroup $\langle a, c_\omega \rangle$ has a finite order. This immediately implies that the quotient group has a finite order. Thus D_ω has a finite index which proves the Lemma.

We claim that the element $ac_\omega \in G_\omega$ has a finite order. This immediately implies that the order of subgroup $\langle a, c_\omega \rangle \subset G$ is finite.

The proof by induction on k any for any sequence τ, such that $\tau_k = 1$. For $k = 1$, from the TABLE we have

$$\phi_0(ac_\tau ac_\tau) = c_{\sigma\tau} \quad \text{and} \quad \phi_1(ac_\tau ac_\tau) = c_{\sigma\tau} \, .$$

By definition the element $c_{\sigma\tau}$ has order 2. Therefore ac_τ has order 4.

Assume ac_τ has finite order for any $k \leq n$, where $\tau_k = 1$. For $k = n + 1$, we have

$$\phi_0(ac_\tau ac_\tau) = c_{\sigma\tau} a \quad \text{and} \quad \phi_1(ac_\tau ac_\tau) = ac_{\sigma\tau} \, .$$

By induction hypothesis for $\sigma\tau$ we conclude that each element on the right hand side has a finite order. Therefore ac_τ has a finite order. This finishes step of induction and proves the claim. □

Corollary 1. If ω is non–constant, then $F_1 \simeq F_2$ is infinite.

Proof. By Lemma 2 $D_\omega \simeq F_1 \times F_2$ has a finite index in an infinite group G_ω. This implies the result. □

5 Proof of Theorem 1.

By results in section 3, we need to consider only the case when ω is non-constant. We will prove that $p_c(G_\omega) < 1$. The proof of this claim follows from the results in the previous section.

Indeed, consider a percolation on $D = D_\omega$. We claim that $p_c(D) < 1$. Recall from the previous section that $D \simeq F_1 \times F_2$ and both F_1 and F_2 are infinite. We need the following simple result.

Lemma 3. *Let A and B be two infinite finitely generated groups. Then $p_c(A \times B) < 1$.*

The lemma immediately proves Theorem 1. Indeed, $p_c(G) \le p_c(D) < 1$ which finishes the proof. $\square$

Proof of Lemma 3. Let $S_1 = \{a_1, \dots, a_n\}$ and $S_2 = \{b_1, \dots, b_m\}$ be any generating sets of A and B respectively. Assume that $S_1 = S_1^{-1}$, $S_2 = S_2^{-1}$. Denote by Γ_1, Γ_2 the corresponding Cayley graphs.

Observe that the set $\{(a_1, 1), \dots, (a_n, 1), (1, b_1), \dots, (1, b_m)\}$ generates $A \times B$. The corresponding Cayley graph Γ is isomorphic to a direct product $\Gamma_1 \times \Gamma_2$.

Since A is infinite group we can choose an infinite self–avoiding path $(x_1, x_2, \dots) \subset \Gamma_1$ (i.e. path with distinct vertices.) Analogously, we can choose an infinite self–avoiding path $(y_1, y_2, \dots) \subset \Gamma_2$.

Now consider a spanning subgraph Ξ in Γ with vertices $(x_i, y_j) \in A \times B$. By construction $\Xi \supset \mathbb{Z}_+^2$. Therefore $p_c(A \times B) \le p_c(\Xi) \le p_c(\mathbb{Z}_+^2)$. Recall that $p_c(\mathbb{Z}_+^2) < 1$ (see [12]). This finishes the proof. $\square$

6 Odds and ends

Using a weak version of the axiom of choice one can conclude that the Cayley graph of G_ω contains $\mathbb{Z}^2$. Simply choose bi-directed paths in both Cayley graphs Γ_1, Γ_2 and let the length go to infinity.

In a different direction, a beautiful proof of Burton and Keane in [4] implies that for all amenable groups the infinite percolation cluster is almost surely (a.s.) unique. Now, using a more complicated construction of lifted subgroups (going to the next level) one can show that the Cayley graph of G_ω contains $\mathbb{Z}_+^4$. While similar, the proof is somewhat more involved and will be omitted. As a corollary we conclude that the (a.s. unique) percolation cluster is a.s. transient when $p > p_c$. This follows from the transience of the percolation cluster on $\mathbb{Z}^3$ (see [13, 14]). Note also that for Gupta–Sidki groups a similar construction of lifted subgroups gives immediately $\mathbb{Z}_+^3$. We leave the details to the reader.

As an ultimate challenge to the reader we ask about the behavior of the critical percolation on Grigorchuk groups. We refer to [2, 3, 13] for the references and details.

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