Oscillating Tableaux, $S_p \times S_q$-modules, and Robinson-Schensted-Knuth correspondence

Extended Abstract

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1 Introduction

The Robinson-Schensted-Knuth correspondence (RSK, see [8] and Corollary 2.5 below) is a bijection between pairs of semi-standard Young tableaux of the same shape and matrices with nonnegative integer entries with prescribed column and row sums. This correspondence plays an important role in the representation theory of the symmetric group and general linear groups, and in the theory of symmetric functions.

It is possible (see [2, 3, 4, 5, 10]) to construct an analogue of the RSK for oscillating tableaux, i.e., sequences of Young diagrams $\alpha = (\alpha_{(0)}, \ldots, \alpha_{(k)})$ such that each $\alpha_{(i)}$ and $\alpha_{(i+1)}$ differ by a horizontal strip.

We present a new approach to the RSK correspondence for oscillating tableaux. First, we show that the number of oscillating tableaux of a given weight and shape is equal to the multilplicity of the corresponding irreducible representation in a certain naturally defined $S_p \times S_q$-module. This allows us to recover the enumerative results from [4, 10, 11, 12] (see Section 4). In Section 5, we extend this construction to oscillating supertableaux. In

Section 6, we discuss commutation relations for the operators which add or delete horizontal or vertical strips (cf. [5, 6]) and give a generalization of these relations.

In Section 7, we introduce a piecewise-linear analogue of RSK for oscillating tableaux in the spirit of [1]. We construct a continuous piecewise-linear map which establishes a bijection between two convex polyhedra. The restriction of this map to integer points gives the the RSK correspondence for oscillating tableaux.

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2 Oscillating tableaux

We can view tableaux as paths in certain graph $\mathcal{Y}$. The vertices of $\mathcal{Y}$ are Young diagrams and diagrams λ and μ are connected by an edge in $\mathcal{Y}$ if λ/μ or μ/λ is a horizontal strip. We call $\mathcal{Y}$ the *extended Young graph* because it is obtained from the Young graph by adding some edges connecting non-adjacent levels. It is clear that Young tableaux correspond to decreasing paths in the graph $\mathcal{Y}$. An oscillating tableau is an arbitrary path in $\mathcal{Y}$.

Definition 2.1 *Let λ, μ be partitions and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathrm{Z}^k$. An oscillating tableau α of shape (λ, μ) and weight β is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that for all $i = 1, 2, \ldots, k$ the following conditions hold:*

1. *If $\beta_i \geq 0$ then $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-strip,*
2. *If $\beta_i < 0$ then $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a horizontal $(-\beta_i)$-strip.*

By $OT(\lambda, \mu, \beta)$ denote the set of all oscillating tableaux of shape (λ, μ) and weight β. If $|\beta_i| = 1$ for all i then an oscillating tableau of weight β is called standard.

Analogous definition was given in [10]. Standard oscillating tableaux were earlier considered in [12].

Definition 2.2 *Let $\delta = (\delta_1, \ldots, \delta_k) \in \mathrm{Z}^k$ be a sequence such that $\sum_i \delta_i = 0$. An intransitive graph of type δ is an oriented graph γ on the vertices $\{1, 2, \ldots, k\}$ (multiple edges allowed) such that:*

1. *If (i,j) is an edge of γ then $i<j$.*
2. *If $\delta_i \geq 0$ then indegree of i is δ_i and outdegree of i is 0.*
3. *If $\delta_i \leq 0$ then outdegree of i is $-\delta_i$ and indegree of i is 0.*

Denote by $G(\delta)$ the set of all intransitive graphs of type δ.

Theorem 2.3 *Let $\beta \in \mathrm{Z}^k$ be such that $\sum_i \beta_i = 0$. Then the number of oscillating tableaux of shape $(\hat{0},\hat{0})$ and weight β is equal to the number of intransitive graphs of type β*

$$|OT(\hat{0},\hat{0},\beta)| = |G(\beta)|.$$

This theorem in slightly different notation was proven by T. W. Roby [10] who generalized S. Fomin's results [3, 4, 5]. The following result was found in [12].

Corollary 2.4 *The number of paths in the Young graph from $\hat{0}$ to $\hat{0}$ of length $2k$ is equal to $(2k-1)!! = (2k-1)(2k-3)\dots 1$.*

Show how oscillation tableaux and intransitive graphs are connected with classical Robinson-Schensted-Knuth correspondence [8]. For weight $\beta = (\beta_1,\beta_2,\dots,\beta_k)$ such that $\beta_1,\dots,\beta_p \leq 0$, $\beta_{p+1},\dots,\beta_k \geq 0$ we get the following

Corollary 2.5 *Let $\beta' \in \mathrm{N}^s$ and $\beta'' \in \mathrm{N}^t$. Then the number of pairs (P,Q) of Young tableaux of the same shape and with weights β' and β'' respectively is equal to the number of $s\times t$-matrices $A=(a_{ij})$ such that*

1. *$a_{ij} \in \mathrm{N}$ for $i=1,2,\dots,s,\ j=1,2,\dots,t$,*
2. *$\sum_j a_{ij} = \beta'_i$ for $i=1,2,\dots,s$,*
3. *$\sum_i a_{ij} = \beta''_j$ for $j=1,2,\dots,t$.*

3 $S_p\times S_q$-module $M(p,\beta,q)$

We consider a permutational representation of $S_p\times S_q$ in the linear space generated by intransitive graphs. Multiplicities of irreducible components in this representation are given by the numbers of oscillating tableaux.

Let $p, q \in \mathrm{N}$, $\beta = (\beta_1, \dots, \beta_k) \in \mathrm{Z}^k$ such that $p - q = \sum_i \beta_i$, $r = p + k$, and $n = p + k + q$. Let $G(p, \beta, q)$ be the set of intransitive graphs of type $\delta = (\delta_1, \delta_2, \dots, \delta_n)$, where

$$\delta_i = \begin{cases} -1 & \text{for } i = 1, \dots, p, \\ \beta_{i-p} & \text{for } i = p+1, \dots, r, \\ 1 & \text{for } i = r+1, \dots, n. \end{cases}$$

The direct product of two symmetric groups $S_p \times S_q$ acts on the graphs $\gamma \in G(p, \beta, q)$ as follows: the group S_p permutes the first p vertices in γ and the group S_q permutes the last q vertices in γ.

Let $M(p, \beta, q)$ be the linear space over C with basis $\{v_\gamma\}$, $\gamma \in G(p, \beta, q)$. The action of the group $S_p \times S_q$ on $G(p, \beta, q)$ gives a linear representation $M(p, \beta, q)$ of $S_p \times S_q$.

Let π_λ be the irreducible S_n-module associated with a partition $\lambda \vdash n$ (see [7, 9]). Every irreducible representation of the group $S_p \times S_q$ is of the form $\pi_\lambda \otimes \pi_\mu$, where $|\lambda| = p$ and $|\mu| = q$.

Theorem 3.1

$$M(p, \beta, q) \simeq \sum |OT(\lambda, \mu, \beta)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the sum is over all partitions $\lambda \vdash p$ and $\mu \vdash q$.

Clearly, Theorem 2.3 is a special case of Theorem 3.1 for $p = q = 0$.

Example 3.2 *Let $p = q$ and $\beta = \emptyset$ be the empty sequence. Then graphs from $G(p, \emptyset, p)$ can be identified with permutations in S_p. In this case $M(p, \emptyset, p)$ is the regular representaion $Reg(S_p)$ of $S_p \times S_p$, i.e., the group algebra* $\mathrm{C}[S_p]$ *on which one copy of S_p acts by left multiplications and the other copy of S_p acts by right multiplications. Theorem 3.1 gives the following well-known identity.*

$$Reg(S_p) = \sum_{\lambda \vdash p} \pi_\lambda \otimes \pi_\lambda.$$

Example 3.3 *Let $q = 0$ and $\beta_i \geq 0$ for all $i = 1, 2, \dots, k$. Then a graph $\gamma \in G(p, \beta, 0)$ can be identified with the word $w = w_1 w_2 \dots w_p$ with β_1 1's, β_2 2's, etc. The symmetric group S_p acts on such words w by permutation of* *letters w_i. The representation $M_\beta = M(p,\beta,0)$ is the well-known monomial representation, see* [7], *i.e., $M_\beta = Ind_{S_{\beta_1}\times\ldots\times S_{\beta_k}}^{S_p} Id$. By Theorem 3.1 we get*

$$M_\beta = M(p,\beta,0) = \sum_{\lambda\vdash p} |YT(\lambda,\beta)| \cdot \pi_\lambda$$

This is the classical Young's rule for decomposition of monomial representations M_β, see [7, 9].

4 Combinatorial theorem

A sequence $\tau = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathrm{Z}^k$ is called *normal* if there exist $0 \le r \le l \le k$ such that $\tau_1, \tau_2, \ldots, \tau_r > 0$; $\tau_{r+1} = \ldots = \tau_l = 0$; $\tau_{l+1}, \ldots, \tau_k < 0$. For a sequence $\beta \in \mathrm{Z}^k$, let $\mathrm{nor}(\beta)$ denote the normal sequence obtained from β by shuffling all positive entries of β into the beginning and all negative entries into the end. For example, $\mathrm{nor}(0,-3,1,-1,0,-2,0,1,3) = (1,1,3,0,0,0,-3,-1,-2)$.

For $\beta, \delta \in \mathrm{Z}^k$ the expression $\delta \prec \beta$ means that for all $i = 1, 2, \ldots, k$ either $0 \le \delta_i \le \beta_i$ or $0 \ge \delta_i \ge \beta_i$.

It is not difficult to deduce from Theorem 3.1 the following result.

Theorem 4.1 *Let λ, μ be some partitions, $\beta \in \mathrm{Z}^k$. Then*

$$|OT(\lambda,\mu,\beta)| = \sum |G(\delta)| \cdot |OT(\lambda,\mu,nor(\beta-\delta))|,$$

where the sum is over all $\delta \in \mathrm{Z}^k$ such that $\sum_i \delta_i = 0$ and $\delta \prec \beta$.

An analogous result but in different notation was obtained in [10]. Clearly, Theorem 2.3 is a special case of Theorem 4.1 for $\lambda = \mu = \hat{0}$.

It is possible (see [10]) to construct a bijection $\Phi_{\lambda\mu\beta}$ between two sets in Theorem 4.1. This construction is based on certain local operations (see Section 6).

5 Superanalogue

In this section we give superanalogues of definitions and theorems from Sections 2–4.

Let $\beta \in \mathrm{Z}^k$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{1, -1\}^k$. By β^ε denote the sequence $b = (b_1, b_2, \ldots, b_k)$ in the alphabet $\{m, \overline{m} \mid m \in \mathrm{Z}\}$ such that $b_i = \beta_i$ (respectively $b_i = \overline{\beta}_i$) if $\varepsilon_i = 1$ (respectively, $\varepsilon_i = -1$).

Definition 5.1 *Let λ, μ be partitions. An oscillating supertableau of shape (λ, μ) and weight $b = \beta^\varepsilon$ is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \ldots, \alpha_{(k)} = \mu)$ such that for all $i = 1, 2, \ldots, k$ the following conditions hold.*

1. *If $\varepsilon_i = 1$ then (a) for $\beta_i \geq 0$ we have $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-strip;*

   (b) for $\beta_i < 0$ we have $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a horizontal $(-\beta_i)$-strip;

2. *If $\varepsilon_i = -1$ then (a) for $\beta_i \geq 0$ we have $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a vertical β_i-strip;*

   (b) for $\beta_i < 0$ we have $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a vertical $(-\beta_i)$-strip.

The set of all oscillating supertableaux of shape (λ, μ) and weight $b = \beta^\varepsilon$ is denoted by $OST(\lambda, \mu, b)$.

Definition 5.2 *Let $\delta \in \mathrm{Z}^k$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{1, -1\}^k$. An intransitive graph of type $d = \delta^\epsilon$ is an oriented graph γ on the set of vertices $\{1, 2, \ldots, k\}$ satistying the conditions 1–3 of Definition 2.2 and also the condition:*

4. *If $\epsilon_i \neq \epsilon_j$ then γ contains at most one edge (i, j).*

Let $SG(\delta^\epsilon)$ be the set of all such graphs.

The following algebra $\mathcal{A}(\epsilon)$ is closely related to Definition 5.2.

Definition 5.3 *Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{1, -1\}^k$. The algebra $\mathcal{A}(\epsilon)$ generated by variables $x_{ij}, 1 \leq i < j \leq k$ with the following relations.*

1. $x_{ij}\, x_{jr} = 0$ *for any* $1 \leq i < j < r \leq k$,

2. $x_{ij}\, x_{lm} = (-1)^{\sigma_{ij}\sigma_{lm}} x_{lm}\, x_{ij}$, *where*

$$\sigma_{ij} = \begin{cases} 0 & \epsilon_i = \epsilon_j, \\ 1 & \epsilon_i \neq \epsilon_j. \end{cases}$$

Let m_γ denote the product of x_{ij} over all edges (i,j) of a graph γ. Let $\mathcal{A}_\delta(\epsilon)$ denote the subspace of $\mathcal{A}(\epsilon)$ which is generated (as a linear space) by monomials m_γ for $\gamma \in SG(\delta^\epsilon)$. It is clear that $\mathcal{A}(\epsilon) = \bigoplus_\delta \mathcal{A}_\delta(\epsilon)$. Let $p, q \in \mathrm{N}$, $\beta = (\beta_1, \ldots, \beta_k)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{1, -1\}^k$, $b = \beta^\varepsilon$, and $\psi, \omega \in \{1, -1\}$. Suppose that $\delta^\epsilon = (-1^\psi, \ldots, -1^\psi, b_1, \ldots, b_k, 1^\omega, \ldots, 1^\omega)$ (we have -1^ψ p times and 1^ω q times.

Let $SG(\mathrm{p}, \beta^\varepsilon, \mathrm{q})$ be the set of intransitive graphs of type δ^ϵ. Denote by $M(\mathrm{p}, \beta^\varepsilon, \mathrm{q})$ the subspace $\mathcal{A}_\delta(\epsilon)$, where $\mathrm{p} = p^\psi$ and $\mathrm{q} = q^\omega$. Then $\{m_\gamma : \gamma \in SG(\mathrm{p}, \beta^\varepsilon, \mathrm{q})\}$ is a basis of the space $M(\mathrm{p}, \beta^\varepsilon, \mathrm{q})$.

The group $S_p \times S_q$ acts on this space, cf. Section 3. The symmetric group S_p permutes the first index of variables x_{ij} with $i = 1, 2, \ldots, p$ and S_q permutes the second index of variables x_{ij} with $j = p+k+1, \ldots, p+k+q$.

The following example gives an odd analogue of the regular representation of S_p (see Example 3.2).

For a partition $\lambda \in \mathcal{P}$ and $\psi \in \{1, -1\}$, $\lambda^\psi = \lambda$ if $\psi = 1$ and $\lambda^\psi = \lambda'$ (the conjugate partition) if $\psi = -1$. Now we can present a superanalogue of Theorem 3.1.

Theorem 5.4

$$M(p^\psi, \beta^\varepsilon, q^\omega) \simeq \sum |OST(\lambda^\psi, \mu^\omega, \beta^\varepsilon)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the sum is over all partitions $\lambda \vdash p$ and $\mu \vdash q$.

Example 5.5 *Let $\beta^\varepsilon = \emptyset$ be the empty sequence, $\mathrm{p} = p$ and and $\mathrm{q} = \overline{p}$, $p \in \mathrm{N}$. Then $Alt_p = M(p, \emptyset, \overline{p})$ is the representation of $S_p \times S_p$ on the group algebra $\mathrm{C}[S_p]$ such that for $(\sigma, \pi) \in S_p \times S_p$ and $f \in \mathrm{C}[S_p]$ we have $(\sigma, \pi) \cdot f = sgn(\sigma\pi^{-1})\, \sigma f \pi^{-1}$. By Theorem 5.4 we have $Alt_p = \sum_{\lambda \vdash p} \pi_\lambda \otimes \pi_{\lambda'}$. This is an odd analogue of Example 3.3. Of course this formula easily follows from definition of Alt_p.*

Now we give a superanalogue of Theorem 4.1. Let $b = (b_1, b_2, \ldots, b_k) = \beta^\varepsilon$ Let $\mathrm{nor}(b)$ denote the word obtained from the word $b = (b_1, b_2, \ldots, b_k)$ by shuffling negative entries into the beginning and positive entries into the end. For example, $\mathrm{nor}(0, \overline{3}, -1, \overline{1}, 0, 2, \overline{0}, -\overline{1}, -3) = (-1, -\overline{1}, -3, 0, 0, \overline{0}, \overline{3}, \overline{1}, 2)$.

Theorem 5.6 *Let $\lambda, \mu \in \mathcal{P}$ be some partitions, $\beta \in \mathrm{Z}^k$, $\varepsilon \in \{1, -1\}^k$. Then*

$$|OST(\lambda, \mu, \beta^\varepsilon)| = \sum_{\delta \prec \beta} |SG(\delta^\varepsilon)| \cdot |OST(\lambda, \mu, nor((\beta - \delta)^\varepsilon))|.$$

This theorem can be deduced from Theorem 5.4 in the same way as Theorem 4.1 from Theorem 3.1.

It is possible to construct a bijection $\Phi^{super}_{\lambda\mu b}$ between two set from Theorem 5.6 using local operations ψ_3 and ψ_4 from Section 6.

If $\lambda = \mu = \hat{0}$ then Theorem 5.6 implies the following

Corollary 5.7 *Let $\beta \in \mathrm{Z}^k$ and $\varepsilon \in \{1, -1\}^k$. Then the number of oscillating tableaux of shape $(\hat{0}, \hat{0})$ and weight $b = \beta^\varepsilon$ is equal to the number of intransitive graphs of type b*

$$|OST(\hat{0}, \hat{0}, b)| = |G(b)|.$$

Corollary 5.8 *Let $\beta' \in \mathrm{N}^s$ and $\beta'' \in \mathrm{N}^t$. Then the number of pairs of tableaux (P, Q) with conjugated shapes and with weights β' and β'' respectively is equal to the number of $s\times t$-matrices satisfying the conditions 1–3 of Corollary 2.5 with all entries equal to 0 or 1.*

Knuth in [8] constructed also a variant of RSK which gives a bijection between the set of such $s\times t$-matrices and the set of such pairs of tableaux (P, Q). In this case the bijection $\Phi^{super}_{\lambda\mu b}$ coincides with Knuth's correspondence.

6 Local operators

Let $n \in \mathrm{N}$. Consider the operators $I(n), I(\overline{n}), D(n), D(\overline{n})$ in the infinite-dimensional space R of formal linear combinations of partitionssuch that $I(n)$ (respectively, $I(\overline{n})$) deletes a horizontal (respectively, vertical) n-strip and $D(n)$ (respectively, $D(\overline{n})$) add a horizontal (respectively, vertical) n-strip. These operators were considered by I. Gessel [6].

Let $b \in \{n, \overline{n} \mid n \in \mathrm{Z}\}$. If $b \geq 0$ denote by $\langle b\rangle$ the operator $I(b)$. If $b \leq 0$ denote by $\langle b\rangle$ the operator $D(-b)$. It is clear that $(\langle b_1\rangle \cdot \langle b_2\rangle \cdot \ldots \cdot \langle b_k\rangle)_{\lambda\mu} = |OST(\lambda, b, \mu)|$.

Theorem 6.1 *Let $m, n \in \mathrm{N}$. The following relations hold.*

1. $[I(m), I(n)] = [I(\overline{m}), I(\overline{n})] = [D(m), D(n)] = [D(\overline{m}), D(\overline{n})] = 0.$
2. $[I(m), I(\overline{n})] = [D(m), D(\overline{n})] = 0.$
3. $[I(m+1), D(n+1)] = I(m)D(n)$, $[I(\overline{m+1}), D(\overline{n+1})] = I(\overline{m})D(\overline{n}).$

4. $[I(m+1), D(\overline{n+1})] = D(\overline{n})I(m)$, $[I(\overline{m+1}), D(n+1)] = D(n)I(\overline{m})$.

Clearly, this theorem follows from

Proposition 6.2 *Let $m, n \geq 1$. There exist bijections between the following sets*

1. $\psi_1 : YT(\lambda/\nu, (m, n)) \to YT(\lambda/\nu, (n, m))$,
2. $\psi_2 : ST(\lambda/\nu, (m, \overline{n})) \to ST(\lambda/\nu, (\overline{n}, m))$,
3. $\psi_3 : OT(\lambda, \nu, (-m, n)) \to \coprod_{0 \leq k \leq \min(m,n)} OT(\lambda, \nu, (n-k, -m+k))$,
4. $\psi_4 : OST(\lambda, \nu, (-m, \overline{n})) \to \coprod_{k=0,1} OST(\lambda, \nu, (\overline{n-k}, -m+k))$.

Here $YT(\lambda/\nu, \beta)$ and $ST(\lambda/\nu, \beta)$ denote the set of Young tableaux and supertableaux, resp., of weight β

It is not difficult to construct these bijections. Here we construct bijection ψ_3 which is analogous to a bijection given [6].

Let $\alpha = (\lambda, \mu, \nu) \in OT(\lambda, \mu, (-m, n)$, $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$, and $\nu = (\nu_1, \nu_2, \ldots)$. On the following diagram arrow $x \to y$ denotes the inequality $x \geq y$.

$$\begin{array}{cccccccccccccccccc}
 & & \lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \dots & \lambda_i & & & & \lambda_{i+1} & \dots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & \searrow & & \nearrow & & \\
\mu_1 & & & & \mu_2 & & & & \mu_3 & & & \dots & & & \mu_{i+1} & & & \dots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & \nearrow & & \searrow & & \\
 & & \nu_1 & & & & \nu_2 & & & & \nu_3 & \dots & \nu_i & & & & \nu_{i+1} & \dots
\end{array}$$

Let $a_i = \min(\lambda_i, \nu_i)$ and $b_i = \max(\lambda_{i+1}, \nu_{i+1})$, $i = 1, 2 \ldots$. Set $\widetilde{\mu}_i = a_i + b_i - \mu_{i+1}$, $i = 1, 2, \ldots$ and $k = \mu_1 - \min(\lambda_1, \nu_1)$. Clearly, $0 \leq k \leq \min(n, m)$. Now $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ is a partition and $\widetilde{\alpha} = (\lambda, \widetilde{\mu}, \nu) \in OT(\lambda, \mu, (n - k, -m + k))$. Define $\psi_3 : \alpha \mapsto \widetilde{\alpha}$. Then ψ_3 gives a bijection between the sets $OT(\lambda, \mu, (-m, n))$ and $\coprod_k OT(\lambda, \mu, (n - k, -m + k))$, $0 \leq k \leq \min(m, n)$. Indeed, if we have a partition $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ and $0 \leq k \leq \min(m, n)$ then we can reconstruct μ setting $\mu_1 = k + \min(\lambda_1, \nu_1)$ and $\mu_{i+1} = a_i + b_i - \widetilde{\mu}_i$, $i = 1, 2, \ldots$.

Remark 6.3 *Note that in this construction we can assume that λ_i, μ_i, ν_i, m, n, and k are arbitary real numbers. So we can give a continuous analogue of bijection ψ_3.*

In the end of this section we give a generalization of Theorem 6.1. Let Λ be the ring of symmetric function of infinite many variables $x_1, x_2, \ldots$, see [9]. Then Λ has a basis of Schur functions $s_\lambda(x)$ with the norm such that $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$. Consider the linear operator on Λ given by $S_{\lambda/\mu} : f \to s_{\lambda/\mu} \cdot f$. Let $S^*_{\lambda/\mu}$ be the conjugate operator. Now we can view operators $D(n)$, $D(\overline{n})$, $I(n)$, and $I(\overline{n})$ as S_n, S_{1^n}, S^*_n, and $S^*_{1^n}$, respectively. We have the following relation for operators S_λ and S^*_ν.

Theorem 6.4

$$S^*_\nu\, S_\lambda = \sum_{\mu\subset\lambda\cap\nu} S_{\lambda/\mu}\, S^*_{\nu/\mu}.$$

7 Continuous analogue

In this section we sketch a continuous piecewise-linear analogue of RSK for osclillating tableaux

Using operations π_3 from the previous section (see Remark 6.3) it is possible to construct a continuous piecewise-linear volume-preserving map $\Phi : A \to B$ between two convex polyhedra. Rather than state the theorem in it full generality we give an example.

Consider some array $\{p_{ij}\}$ whose shape is a Young diagram

p_{11}	p_{12}	p_{13}	p_{14}
p_{21}	p_{22}	p_{23}	
p_{31}	p_{32}		

where all entries p_{ij} are nonnegative real numbers weakly increasing along from left to right and from top to bottom. Consider the polyherdon A consisting of all such arrays with fixed diagonal-sums: $p_{31} = \gamma_1,\ p_{21} + p_{32} = \gamma_2,\ p_{11} + p_{22} = \gamma_3,\ p_{12} + p_{23} = \gamma_3,\ p_{13} = \gamma_5,\ p_{14} = \gamma_6$.

Consider another array $\{q_{ij}\}$ of the same shape where all entries q_{ij} are nonnegative real numbers. Let B be the polyhedron of all such arrays with fixed row and column sums $q_{11}+q_{21}+q_{31} = \alpha_1,\ q_{12}+q_{22}+q_{32} = \alpha_2,\ q_{13}+q_{23} = \alpha_3,\ q_{14} = \alpha_4,\ q_{31} + q_{32} = \beta_1,\ q_{21} + q_{22} + q_{23} = \beta_2,\ q_{11} + q_{12} + q_{13} + q_{14} = \beta_3$.

Suppose that $\gamma_1 = \alpha_1,\ \gamma_2 = \gamma_1 + \alpha_2,\ \gamma_3 = \gamma_2 - \beta_1,\ \gamma_4 = \gamma_3 + \alpha_3,\ \gamma_5 = \gamma_4 - \beta_2,\ \gamma_6 = \gamma_5 + \alpha_4$.

Repeting operations ψ_3 from the previuos section, we can construct a continuous piecewise linear bijection Φ between A and B. If all p_{ij} and q_{ij} are integer we get the RSK for osclillating tableaux.

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