VOLUME-PRESERVING PL-MAPS BETWEEN POLYHEDRA

ANDRE HENRIQUES⋆ AND IGOR PAK⋆

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Abstract. We prove that for every two convex polytopes $P, Q \in \mathbb{R}^d$ with $\mathrm{vol}(P) = \mathrm{vol}(Q)$, there exists a continuous piecewise-linear (PL) volume-preserving map $f : P \to Q$. The result extends to general PL-manifolds. The proof is inexplicit and uses the corresponding fact in the smooth category, proved by Moser in [Mo]. We conclude with various examples and combinatorial applications.

Introduction

The study of piecewise-linear (PL–) manifolds and PL-homeomorphisms goes back to the early days of topology, and blossomed in recent years in part due to modern advances in combinatorics. The main result of this paper is the following theorem:

Theorem 1. *Let $M_1, M_2 \subset \mathbb{R}^d$ be two PL-manifolds, possibly with boundary, which are PL-homeomorphic and equipped with piecewise-constant volume forms ω_1 and ω_2. Suppose M_1 and M_2 have equal volume: $\int_{M_1} \omega_1 = \int_{M_2} \omega_2$. Then there exists a volume-preserving PL-homeomorphism $f : M_1 \to M_2$, i.e. a map f satisfying $f^*(\omega_2) = \omega_1$.*

Here by a PL-manifold we mean a topological space, obtained by gluing polytopes in $\mathbb{R}^d$ along some of their facets, and which is locally PL-homeomorphic to $\mathbb{R}^d$ [Br, RS]. We may assume that the volume form is inherited from the standard volume form in $\mathbb{R}^d$. The result also holds for PL-pseudomanifolds. Note that it is nontrivial even for $d = 2$. We were motivated by the following application to convex polytopes:

Theorem 2. *Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes of equal volume: $\mathrm{vol}(P) = \mathrm{vol}(Q)$. Then there exists a one-to-one map $f : P \to Q$, which is continuous, piecewise-linear and volume-preserving. Moreover, if both P and Q are rational, then f can be also made rational.*

Here by *rational polytopes* and *rational maps* we mean polytopes and maps defined over $\mathbb{Q}$.

One can think of Theorem 2 as of a modified version of Hilbert's Third Problem, which asks whether every two polytopes $P, Q \subset \mathbb{R}^d$ of equal volume are *scissor-equivalent*, i.e. whether there exists polyhedral subdivisions $P = \cup_{i=1}^k P_i$ and $Q = \cup_{i=1}^k Q_i$ such that P_i can be moved into Q_i by a Euclidean motion [Bo]. The problem was resolved negatively by Dehn who proposed an invariant of polytopes for $d = 3$. In a special case the invariant implies that cube and regular tetrahedron of equal volume are not scissor-equivalent. Much later Sydler proved that Dehn's invariant is in fact the only obstacle. Namely, any two convex polytopes with same volume and same Dehn invariant are scissor equivalent. While the situation remains similar for $d = 4$, in higher dimension there are additional (Hadwiger) invariants and proving the analogue of Sydler's result remains an important open problem [Car].

Now, the maps considered in Theorem 2 are less restrictive in one direction, and more restrictive in the other. We allow here all volume-preserving affine linear transformations between P_i

⋆Department of Mathematics, MIT, Cambridge, MA, 02139. Email: {andrhenr,pak}@math.mit.edu.

and Q_i, not just Euclidean motions. On the other hand, we add the condition of continuity on the map f, which amounts to saying that the common faces $P_i \cap P_j$ have to be mapped into faces $Q_i \cap Q_j$.

It is instructive to ask what happens when either of the three conditions in Theorem 2 is omitted: the map f is continuous, piecewise-linear, and volume-reserving. Without the continuity the result is straightforward. Indeed, without loss of generality we can assume that $\mathrm{vol}(P) = \mathrm{vol}(Q) = 1$. Consider any two simplicial subdivisions $P = \cup_{i=1}^m P_i$, $Q = \cup_{j=1}^n Q_j$, and let $\alpha_i = \mathrm{vol}(P_i)$, $\beta_j = \mathrm{vol}(Q_j)$. Subdivide further each of these simplices into smaller simplices: $P_i = \cup_{j=1}^n P_{ij}$, $Q_j = \cup_{i=1}^m Q_{ij}$, such that $\mathrm{vol}(P_{ij}) = \mathrm{vol}(Q_{ij}) = \alpha_i \beta_j$. Since every simplex P_{ij} can be mapped into simplex Q_{ij} by a volume-preserving map, this implies the claim.

When the volume-preserving condition is omitted, the result is the starting point of our proof; this simplified version is established in Lemma 1.1. On the other hand, if piecewise-linearity is substituted with smoothness, the claim in the Theorem 2 becomes a corollary of a well known result of Moser [Mo]. This result is another ingredient in our proof, and will be stated in Section 2. The use of Moser's theorem is one of the inexplicit parts in our otherwise rather explicit construction of the desired map f between polytopes. Let us also emphasize the importance of smoothness in the proof of Moser's theorem—it is used later in the construction of the desired map.

Notation. Throughout the paper we refer to simplices by their vertices, i.e. $(v_0, \ldots, v_d) \subset \mathbb{R}^d$ is a d-dimensional simplex.

1. Piecewise-linear maps

We begin with a preliminary result which reduces the first part of Theorem 2 to Theorem 1. The following lemma is very natural, but we were unable to find it in the literature.

Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes. Recall that a map $f : P \to Q$ is *piecewise-linear* (PL) if there exists a simplicial subdivisions $P = \cup_{r=1}^n P_r$ and $Q = \cup_{r=1}^n Q_n$, such that $Q_r = f(P_r)$ and the map f is linear on each P_r.

Lemma 1.1. *For any two convex polytopes $P, Q \subset \mathbb{R}^d$ there exists a PL-homeomorphism $f : P \to Q$.*

Proof. We can assume that P, Q are simplicial; otherwise subdivide each facet into simplices. We can also assume that the origin $O \in \mathbb{R}^d$ lies in the interior of both polytopes: $O \in P, Q$.

Now, consider the simplicial fan $F = \cup_{i=1}^m F_i \in \mathbb{R}^d$ defined as the union of infinite cones F_i which start at O and span over the facet simplices. Similarly, consider a fan $G = \cup_{j=1}^n G_i \in \mathbb{R}^d$ over facet simplices of Q. Let C be the 'union fan' which consists of cones $F_i \cap G_j$, and denote by $\widetilde{C} = \cup_r \widetilde{C}_r$ a simplicial subdivision of C. Finally, define simplicial subdivisions $P = \cup_r P_r$, $Q = \cup_r Q_r$ by intersecting the fan $\widetilde{C}$ with the polytopes P and Q: $P_r = P \cap \widetilde{C}_r$, $Q_r = Q \cap \widetilde{C}_r$.

Fix r and consider the simplices P_r and Q_r. Denote by $v_1, \ldots, v_d$ and $w_1, \ldots, w_d$ their vertices other than O. From above, $w_i = \alpha_i v_i$, for some $\alpha_1, \ldots, \alpha_d > 0$. Now define a piecewise-linear map $f : P \to Q$, which is linear on each P_r and maps P_r into Q_r:

$$f : \sum x_i v_i \mapsto \sum x_i w_i \,, \quad \text{where } \; x_i \geq 0.$$

The map f is clearly continuous and invertible. $\square$

The 2-dimensional example in Figure 1 illustrates the construction of the PL-map in Lemma 1.1.

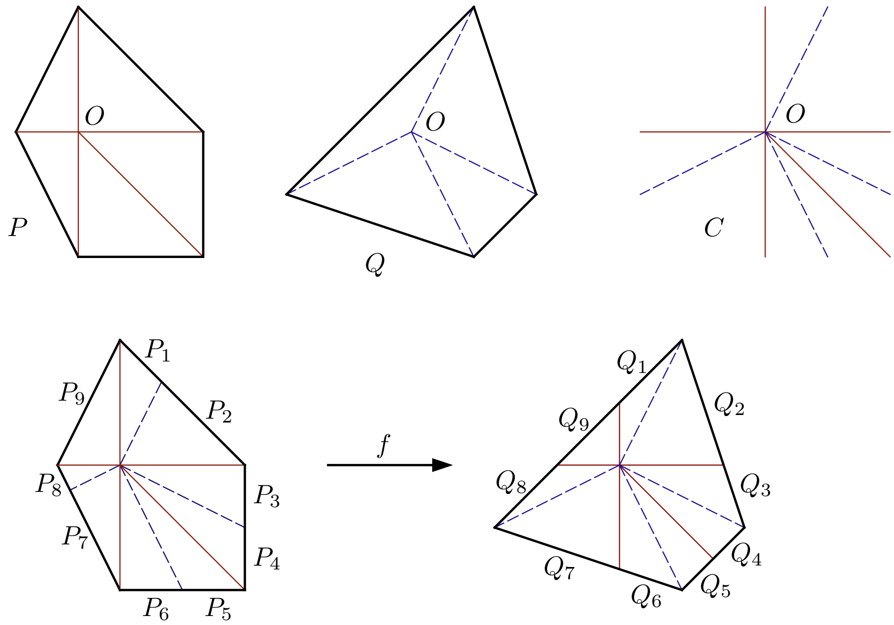

FIGURE 1. The polytope P with its fan F, the polytope Q with its fan G, the union fan $C = \widetilde{C}$, and the PL-homeomorphism $f : P \to Q$.

Remark 1.2. Let us mention here that there exist homeomorphic polyhedra which are not PL-homeomorphic. The first such example was constructed by Milnor [Mi]. We refer to [Br, §9] for further references and related results.

2. Volume-preserving maps

In this section we 'sacrifice' the linearity condition (by 'downgrading' it to smoothness) in favor of the volume-preserving condition; we will get back piecewise-linearity in section 4. We start with a number of definitions and an important technical result.

Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes. We say that a homeomorphism $f : P \to Q$ is *piecewise-smooth* if there exists a simplicial subdivision $P = \cup_{r=1}^n P_r$, such that map f is continuous and smooth on each P_r. Note that this definition is asymmetric: if $f : P \to Q$ is piecewise-smooth, this does not necessarily imply that $f^{-1} : Q \to P$ is piecewise-smooth.

Consider the standard volume form $\omega_\circ = dx_1 \wedge \cdots \wedge dx_d$, so that $\mathrm{vol}(X) = \int_X \omega_\circ$ for all $X \subset P$. A general *volume form* can written as

$$(\star) \qquad \omega = \xi(x_1, \ldots, x_d)\, dx_1 \wedge \cdots \wedge dx_d, \quad \text{where } \xi(\cdot) > 0.$$

To every piecewise-smooth map $f : P \to Q$ as above corresponds a *pull back* volume form $f^*(\omega_\circ)$ on P defined by $(\star)$, with

$$\xi(x_1, \ldots, x_d) := \left| \det \left(\frac{\partial f_i}{dx_i} \right) \right|.$$

We say that a smooth map $f = (f_1, \ldots, f_d) : X \to Y$, where $X, Y \subset \mathbb{R}^d$, is *volume-preserving at* $x = (x_1, \ldots, x_d) \in X$ if the Jacobian $\xi(x_1, \ldots, x_d)$ is equal to 1. We say that f is *volume-preserving* if f is volume-preserving at every $x \in X$ where the above Jacobian is defined.

Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes, and let ∂P denote the boundary of P. Denote by $(\partial P)_\varepsilon$ the set of points $x \in P$ which are at a distance $\le \varepsilon$ from ∂P. We say that two functions

f, g on P *are equal near the boundary* ∂P if there exists $\varepsilon > 0$ such that $f(x) = g(x)$ for all $x \in (\partial X)_\varepsilon$. Similarly, a (piecewise-) smooth map $f : P \to Q$ is *volume-preserving near the boundary* if there exists $\varepsilon > 0$ such that $f^*(\omega_\circ) = \omega_\circ$ in $(\partial X)_\varepsilon$. In other words, the function f is volume preserving at all $x \in (\partial X)_\varepsilon$.

Lemma 2.1. *Let $\Delta_1, \Delta_2 \subset \mathbb{R}^d$ be two simplices. Then there exists a continuous piecewise-smooth homeomorphism $f : \Delta_1 \to \Delta_2$ which is volume-preserving near the boundary, linear on the boundary, and such that $f^*(\omega_\circ)$ is smooth.*

The proof of the lemma is technical and somewhat involved. It is postponed until Section 6 so as to separate it from the main ideas of the paper.

Lemma 2.2. *Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes. Then there exists a continuous piecewise-smooth homeomorphism $f : P \to Q$ which is volume-preserving near the boundary ∂P, and such that $f^*(\omega)$ is a smooth everywhere non-zero volume form.*

Proof. By Lemma 1.1 there exists a simplicial subdivision $P = \cup_r P_r$ and a piecewise-linear map $f : P \to Q$ which is linear on P_r. By Lemma 2.1 each linear map $f = f_r : P_r \to Q_r$ can be replaced by a continuous piecewise-smooth map $g_r : P_r \to Q_r$ which is volume-preserving near the boundary ∂P_r. Define $g : P \to Q$ by g_r on P_r. By continuity and since $\partial P \subset \cup_r \partial P_r$, we conclude that g is as desired. $\square$

The following is the key result which enables us to extend the volume-preserving condition to the whole domain. It is applied to polytopes in Lemma 2.4 below, to obtain a continuous piecewise-smooth volume-preserving map $f : P \to Q$.

Lemma 2.3 (Moser's Theorem). *Let $P \subset \mathbb{R}^d$ be a convex polytope, let $\omega_\circ$ be the standard volume form, and let ω be a volume form which satisfies $\int_P \omega = \mathrm{vol}(P)$. Assume that $\omega = \omega_\circ$ near the boundary ∂P. Then there exists a smooth map $g : P \to P$ which is equal to identity map Id_P near the boundary ∂P, and such that $\omega = g^*(\omega_\circ)$.*

The lemma is a special case of the the main theorem in [Mo]. For completeness and for the reader's convenience, we present a short proof below.

Proof. Let us prove a slightly more general statement: for every two volume forms ω_0, ω_1 such that $\int_P \omega_0 = \int_P \omega_1$ and $\omega_0 = \omega_1$ near ∂P, there exists a smooth map $g : P \to P$ such that $\omega_1 = g^*(\omega_0)$ and $g = \mathrm{Id}$ near the boundary ∂P.

Since $\int_P (\omega_1 - \omega_0) = 0$, the form $\omega_1 - \omega_0$ represents zero in the relative de Rham cohomology group $\mathrm{H}^d_{\mathrm{DR}}(P, \partial P) \simeq \mathbb{R}$. Take a form $\alpha \in \Omega^{d-1}(P, \partial P)$ satisfying $d\alpha = \omega_1 - \omega_0$. For $t \in [0, 1]$, let $\omega_t := t\,\omega_1 + (1-t)\,\omega_0$, and let v_t be the unique vector field satisfying $\iota_{v_t}\omega_t = \alpha$. Since α is zero near the boundary ∂P, then so is the vector field v_t. Integrating (the time dependent) vector field v_t we obtain a flow $\Phi : P \times [0, 1] \to P$. Let us show that

$$\Phi_t^*(\omega_0) \,=\, \omega_t, \quad \text{where} \;\; \Phi_t := \Phi(\cdot, t).$$

Indeed,

$$\frac{d}{dt}\omega_t \,=\, \omega_1 - \omega_0 \,=\, d\alpha \,=\, d\iota_{v_t}\omega_t$$

and

$$\frac{d}{dt}\Phi_t^*(\omega_0) \,=\, \mathcal{L}_{v_t}\Phi_t^*(\omega_0) \,=\, d\iota_{v_t}\Phi_t^*(\omega_0) + \iota_{v_t}d\,\Phi_t^*(\omega_0) \,=\, d\iota_{v_t}\Phi_t^*(\omega_0).$$

Since both $\Phi_t^*(\omega_0)$ and and ω_t are solutions to the same differential equations $\frac{d}{dt}\nu_t = d\iota_{v_t}\nu_t$ and satisfy the same initial conditions $\nu_0 = \omega_0$, they are equal. Since $v_t = 0$ near the boundary ∂P, we have $\Phi_t = \mathrm{Id}$ near ∂P. Letting $g := \Phi_1$ gives the desired map. $\square$

Lemma 2.4. *Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes of equal volume:* $\mathrm{vol}(P) = \mathrm{vol}(Q)$. *Then there exists a piecewise-smooth volume-preserving homeomorphism $f : P \to Q$.*

Proof. Lemma 2.2 gives a continuous map $f : P \to Q$ which is piecewise-smooth and volume-preserving near the boundary ∂P. Since $\mathrm{vol}(P) = \mathrm{vol}(Q)$, the corresponding volume form $\omega_1 = (f^{-1})^*(\omega_\circ)$ on Q satisfies the conditions of Lemma 2.3. The lemma now gives a smooth map $g : Q \to Q$ such that $\omega_1 = g^*(\omega_\circ)$ on P. Therefore, the composition $g \circ f : P \to Q$ is the desired piecewise-smooth volume-preserving map. □

3. Smooth triangulations

In this section we refine the smooth triangulation obtained as the image of the piecewise-smooth map and construct a PL-map which is 'nearly volume preserving' (see Lemma 3.1 below).

A *smooth simplex* $T \subset \mathbb{R}^d$ is defined as the image of a simplex $\Delta = (u_0, u_1, \ldots, u_d) \subset \mathbb{R}^d$ under a smooth embedding σ. Usually, we will take σ to be part of the data of a smooth simplex. We shall speak of i-dimensional faces of T, defined as images of the i-dimensional faces in Δ. As in the case of simplices, faces correspond to subsets of the vertices of T.

By analogy with simplicial subdivisions, we define a *smooth triangulation* of a polytope P to be a subdivision $P = \cup_r T_r$, where T_r are smooth simplices, and $T_r \cap T_j$ are faces of T_r, T_j. We also require that $\sigma_r \sigma_j^{-1}$ be linear on the face of Δ on which it's defined. We say that a triangulation $P = \cup_{j=1}^N T'_j$ *refines* a triangulation $P = \cup_r T_r$ if every smooth simplex T_r has a triangulation $T_r = \cup_{j \in S(r)} T'_j$, for some subset $S(r) \subset \{1, \ldots, N\}$ of the set of smooth simplices.

We say that a smooth triangulation $P = \cup_{r=1}^n T_r$ and a simplicial subdivision $P = \cup_{r=1}^n P_r$ are *aligned* if the vertices of T_r coincide with the vertices of P_r, for all $1 \le r \le n$. An example is given in Figure 2.

Now consider a homotopy $h : P \times [0,1] \to P$ such that $h(\cdot, 0)$ is the identity map and $h(\cdot, 1)$ maps the faces of P_r into the corresponding faces of T_r, for all r. One can think of h as of a homotopy between the above simplicial subdivision and the smooth triangulation. Of course, the homotopy h is not uniquely defined; to avoid this ambiguity we will use the following construction. By definition, for every r, $1 \le r \le n$, we have a smooth map $\sigma_r : \Delta \to T_r$, where $\Delta = (u_0, \ldots, u_d)$ is a d-dimensional simplex. We let $\tau_r : \Delta \to P_r$ be the unique linear map that sends u_i to $\sigma(u_i)$. The maps $\gamma_r = \sigma_r \circ \tau_r^{-1}$ assemble to a map $\gamma : P \to P$ that maps P_r into T_r. Now set h to be the *straight line homotopy*:

$$h(x,t) \,:=\, (1-t)\,x \,+\, t\,\gamma(x), \quad \text{for all} \quad x \in P,\ t \in [0,1].$$

Fix an orientation on all $(d-1)$-dimensional faces F in the subdivision $P = \cup_r P_r$. Given our homotopy $h : P \times [0,1] \to P$ as above, we let $A_F := h\big(F, [0,1]\big)$ be the segment between the face F and the corresponding face of the smooth triangulation.

Denote by $a_F := \mathrm{vol}(A_F)$ the algebraic volume of the segment, i.e. the volume taken with sign depending on the orientation. Note that the volume a_F becomes $-a_F$ if one changes the orientation of F. For example, if $F = (v_1, v_3)$ as in Figure 3, then a_F is the area of the shaded segment between v_1 and v_3. Similarly, if $F = (v_3, v_5)$, then a_F is the sum of the areas of the first and third shaded segments between v_3 and v_5, minus the middle segment. Clearly,

$$\mathrm{vol}(T_r) \,=\, \mathrm{vol}(P_r) \,+\, \sum_{F \subset P_r} \varepsilon(F, P_r)\, a_F\,,$$

where $\varepsilon(F, P_r) = 1$ if the orientation of $F \subset P_r$ is induced by that of P_r, and $\varepsilon(F, P_r) = -1$ otherwise.

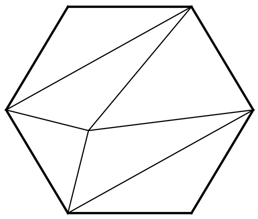

FIGURE 2. Aligned smooth triangulation and simplicial subdivision of a hexagon.

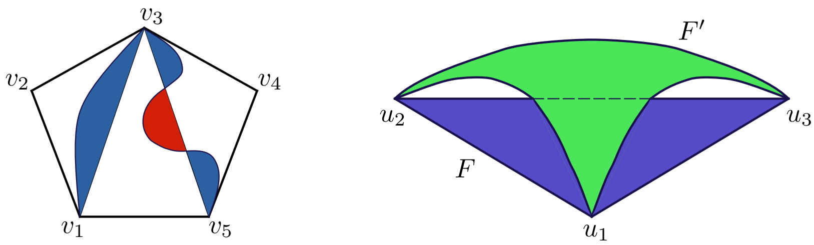

FIGURE 3. The straight line homotopy between a smooth triangulation and the corresponding aligned simplicial subdivision in two and three dimensions.

Lemma 3.1. *Let $P = \cup_{r=1}^m T_r$ be a smooth triangulation of a convex polytope P. Then there exists a smooth triangulation $P = \cup_{j=1}^N R_j$ and a simplicial subdivision $P = \cup_{j=1}^N P_j$ such that:*

i) *the triangulation $P = \cup_j R_j$ is aligned with the subdivision $P = \cup_j P_j$,*

ii) *the triangulation $P = \cup_j R_j$ refines the triangulation $P = \cup_r T_r$,*

iii) $|a_F| < \frac{1}{d+1}\operatorname{vol}(P_r)$, *for all $F \subset P_r$ and $1 \le r \le N$.*

Before proving Lemma 3.1, we will need the following technical result which will enable us to bound the volume $|a_F|$.

Let $P = (u_0, \dots, u_d) \subset \mathbb{R}^d$ be a simplex and let $f : P \to \mathbb{R}^n$ be a continuous map. Denote by $\|f\|_\infty = \max_{x\in P} |f(x)|$ the *sup norm* of f. Define the *aligned linear map* g to be the unique linear map $g : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $g(u_i) = f(u_i)$, for all $0 \le i \le d$.

Lemma 3.2. *Let $P = (u_0, \dots, u_d) \subset \mathbb{R}^d$ be a simplex, let $f : P \to \mathbb{R}^n$ be a smooth map, and let g be the corresponding aligned linear map. Then there exists a constant $C = C(d, n)$ such that the distance $|f(x) - g(x)|$ in $\mathbb{R}^n$ is smaller than $C \cdot \|f''\|_\infty \cdot \delta(P)^2$, where $\delta(P)$ denotes the diameter of P.*

Proof. We show that $C = d\,n$ satisfies conditions of the lemma. If $d = n = 1$, then $P = [a, b]$, and after replacing of f by $f - g$ and g by 0, we may assume that $f(a) = f(b) = 0$. We need to show that $|f(x)| \le \|f''\|_\infty \cdot (b - a)^2$. Let $x_\circ \in [a, b]$ be such that $|f(x_\circ)|$ is maximal. Clearly, $f'(x_\circ) = 0$. Given another point $x \in [a, b]$, we can bound

$$|f'(x)| \le \|f''\|_\infty \cdot |x - x_\circ| \le \|f''\|_\infty \cdot (b - a).$$

Therefore, $\|f'\|_\infty \le \|f''\|_\infty \cdot (b - a)$. Repeating this a second time we have:

$$|f(x)| \le \|f'\|_\infty \cdot (x - a) \le \|f''\|_\infty \cdot (b - a)^2,$$

exactly what we wanted to show.

If $d = 1$ and n is arbitrary, then

$$|f(x) - g(x)|_{\mathbb{R}^n} \le \sum_{i=1}^{n} |f_i(x) - g_i(x)|_{\mathbb{R}} \le n \cdot \|f''\|_\infty \cdot (b-a)^2.$$

Therefore, $C = n$ is as desired.

Finally, if d is arbitrary, we proceed by induction. Let x be a point in P, and let y be the point of intersection of the line (x, u_d) with the facet $F = (u_0, \ldots, u_{d-1})$. Denote by ℓ the segment (u_0, y) and by $\widetilde{g}$ the linear map $\widetilde{g} : \ell \to \mathbb{R}^n$ defined by $\widetilde{g}(u_0) = f(u_0)$ and $\widetilde{g}(y) = f(y)$. By the triangle inequality, we have

$$|f(x) - g(x)| \le |f(x) - \widetilde{g}(x)| + |\widetilde{g}(x) - g(x)|.$$

Applying the $d = 1$ case to ℓ, the first term is bounded as $|f(x) - \widetilde{g}(x)| \le n \cdot \|f''\|_\infty \cdot \delta(\ell)$. Now, applying the induction hypothesis to F, we get

$$|\widetilde{g}(y) - g(y)| = |f(y) - g(y)| \le (d-1)\, n \cdot \|f''\|_\infty \cdot \delta(F).$$

But clearly $|\widetilde{g}(x) - g(x)| \le |\widetilde{g}(y) - g(y)|$ since both $\widetilde{g}$ and g are linear maps, and they agree on one of the endpoints of the interval ℓ. Putting all these inequalities together, we conclude:

$$|f(x) - g(x)| \le n \cdot \|f''\|_\infty \cdot \delta(\ell) + (d-1)\, n \cdot \|f''\|_\infty \cdot \delta(F) \le d\, n \cdot \|f''\|_\infty \cdot \delta(P).$$

This completes the proof of the lemma. □

An illustration of Lemma 3.1 when $d = 2$ is given in Figure 4. The second picture in the figure shows a special case of Lemma 3.2 when $d = 1$, $n = 2$, and the image $f(P)$ is an arc of a circle of radius R. Here we can compute $\varepsilon = R\,\big(1 - cos(\theta)\big) < R\,\theta^2$ and $\delta(P) = 2\,R\,\sin(\theta) > R\,\theta$. Now we check that

$$\varepsilon < R\theta^2 = \frac{1}{R}\,(R\,\theta^2) < 1 \cdot \frac{1}{R} \cdot \delta(P)^2.$$

Therefore, we can take the constant $C = 1$ in this case.

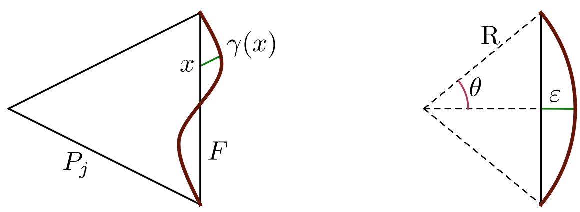

FIGURE 4. The distance between a smooth simplex and the corresponding aligned simplex.

Proof of Lemma 3.1. Note first that given the smooth triangulation $P = \cup_{j=1}^{N} R_j$, there exists a unique aligned simplicial subdivision $P = \cup_{j=1}^{N} P_j$ simply because they share the same vertices. Similarly, the smooth triangulation determines the straight line homotopy h, which determine the values a_F as above. Below we construct a family of smooth triangulations $P = \cup_j R_j^{(n)}$, for all $n \ge 1$, and let $P = \cup_j P_j^{(n)}$ and h_n be the corresponding simplicial subdivisions and straight line homotopies. We then show that for for n large enough the resulting smooth triangulations $P = \cup_j R_j^{(n)}$ and simplicial subdivision $P = \cup_j P_j^{(n)}$ satisfy the conditions of the lemma.

We need a preliminary construction. Start with the family of hyperplanes in $\mathbb{R}^{d+1}$ given by the equations $x_i = \frac{a}{n}$, for all $a \in \mathbb{Z}$ and $0 \le i \le d$. They define a polyhedral subdivision of the standard simplex Δ^d (see above), where each polyhedron is a $\frac{1}{n}$ scaled copy of a *hypersimplex*:

$$\Delta^d_k = \big\{x \in \mathbb{R}^{d+1} \mid 0 \le x_0, \ldots, x_d \le 1,\, x_0 + \ldots + x_d = k\big\}, \quad \text{where } 1 \le k \le d.$$

When $k = 1$ and $k = d$ we obtain the (usual) simplices. Denote by Σ_n the simplicial subdivision of Δ^d given by the barycentric subdivision of these hypersimplices. From above, these simplicial subdivisions comprise of copies of a finite number $B = B(d)$ different *types* of simplices, each of them scaled by $\frac{1}{n}$.

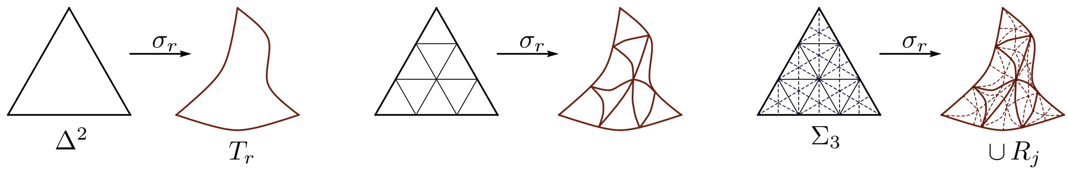

FIGURE 5. The step-by-step construction of the subdivision Σ_3 for $d = 2$, and its image under σ_r.

Now, take each simplex $\sigma_r : \Delta^d \to T_r$ in the given smooth triangulation and subdivide the standard simplex Δ^d according to Σ_n. The image of Σ_n by σ_r gives a smooth triangulation of T_r; the union of these triangulations over all r is the desired smooth triangulation $P = \cup_{j=1}^{N(n)} R_j^{(n)}$. The conditions $i)$ and $ii)$ follow from our construction, and we only need to check the inequalities $iii)$ which hold for n large enough. Denote $\alpha_{j,F} := |a_F|/\mathrm{vol}(P_j^{(n)})$, and let us show that

$$(\circledast) \qquad \max_{(j,F)} \alpha_{j,F} \;\to\; 0 \quad \text{as} \quad n \to \infty.$$

Since the dimension d is fixed, for n large enough this maximum is $< \frac{1}{d+1}$, as desired.

Recall that there exists only a finite number of types of d-dimensional simplices in Σ_n. Since P is compact and the maps σ_r are smooth, we conclude that there exists a constant $C > 0$ independent on n, such that

$$\mathrm{vol}(P_j^{(n)}) \;>\; C \left(\frac{1}{n}\right)^d, \quad \text{for all } \; 1 \le j \le N(n).$$

In a different direction, there exists a constant $c > 0$ independent on n, such that

$$\mathrm{vol}(F) \;<\; c \left(\frac{1}{n}\right)^{d-1}, \quad \text{for all facets } \; F \subset P_j^{(n)} \;\text{ and }\; 1 \le j \le N(n).$$

Since h is the straight line homotopy, observe that for every such F we have:

$$|a_F| \;\le\; \mathrm{vol}(F) \,\cdot\, \max_{x\in F} \big|\gamma(x) - x\big|,$$

where $\gamma(x) = h(x, 1)$ is as above. Since the curvature of $R_j^{(n)}$ is bounded and independent on n, by Lemma 3.2, we can bound the distance on the r.h.s. by D/n^2, for some constant $D > 0$ independent on n. We conclude:

$$\max_{(j,F)} \alpha_{j,F} \;\le\; \max_{(j,F)} \frac{\mathrm{vol}(F) \cdot \max_{x\in F} \big|\gamma(x) - x\big|}{\mathrm{vol}(P_j^{(n)})} \;<\; \frac{c\left(\frac{1}{n}\right)^{d-1} \cdot D\left(\frac{1}{n^2}\right)}{C\left(\frac{1}{n}\right)^d} \;=\; O\left(\frac{1}{n}\right).$$

This implies $(\circledast)$ and completes the proof. $\square$

Remark 3.3. The reasoning behind the technical conditions in Lemma 3.1, especially property *iii*) will be clear in the next section. Of course, one can sharpen *iii*) to any fraction independent on n, but the bound in the lemma suffices for our arguments.

Let us note also that the subdivisions Σ_n cannot be replaced with the more natural choice of the iterated barycentric subdivision since the latter contains a number of of different simplices which grows with n. In the case $d = 2$, all hypersimplices are in fact simplices and we could have spared the barycentric subdivision, but it will definitely be needed for $d \geq 3$.

4. Construction of the map

We start with two lemmas which are ingredients in the construction of the desired map. The map itself is explicitly constructed in Lemma 4.3 out of the smooth triangulation produced in Lemma 3.1.

Lemma 4.1. *Let $\Delta = (v_0, \ldots, v_d) \subset \mathbb{R}^d$ be a d-dimensional simplex, and let $\alpha_0, \ldots, \alpha_d > 0$ satisfy $\alpha_0 + \ldots + \alpha_d = \mathrm{vol}(\Delta)$. Then there exists a point $z \in \Delta$ such that the volumes of simplices*

$$\mathrm{vol}(v_0, \ldots, v_{i-1}, z, v_{i+1}, \ldots, v_d) \, = \, \alpha_i, \quad \textit{for all} \;\; 0 \leq i \leq d.$$

Proof. In vector notation, let $z := (\alpha_0 v_0 + \ldots + \alpha_d v_d)/\mathrm{vol}(\Delta)$ be the *weighted barycenter* of Δ. It is straightforward to check that z is as desired. □

A d-dimensional *bipyramid* is a union of two d-dimensional simplices joined by a facet. A vertex is called simple if it has exactly d edges leaving it. Clearly, a bipyramid has exactly two simple vertices. Note also that a bipyramid is not necessarily convex.

Lemma 4.2. *Let $P, Q \subset \mathbb{R}^d$ be two d-dimensional bipyramids of equal volume:* $\mathrm{vol}(P) = \mathrm{vol}(Q)$. *Let $u_1, u_2, x_1, \ldots, x_d$ and $v_1, v_2, y_1, \ldots, y_d$ be the vertices of P and Q respectively, where u_1, u_2 and v_1, v_2 are the simple vertices. Then there exists a continuous piecewise-linear volume-reserving map $f : P \to Q$ which is linear on the facets and which sends u_i to v_i and x_i to y_i.*

Proof. Use a volume-preserving linear transformation of $\mathbb{R}^d$ to map the vertices $x_1, \ldots, x_d$ into the vertices $x'_1, \ldots, x'_d$ of a regular $(d-1)$-dimensional simplex S. Denote by z the barycenter of S, and by ℓ the line going through z and orthogonal to S. Let u'_1 and u'_2 be the orthogonal projections of u_1 and u_2 onto ℓ. Apply the volume-preserving linear map $(u_1, x'_1, \ldots, x'_d) \to (u'_1, x'_1, \ldots, x'_d)$ which fixes S and sends u_1 to u'_1. Similarly, map $(u_2, x'_1, \ldots, x'_d)$ to $(u'_2, x'_1, \ldots, x'_d)$. The resulting bipyramid is now symmetric with respect to the diagonal $\ell = (u'_1, u'_2)$. Therefore, the simplices $A_1 = (u_1, u_2, x_2, \ldots, x_d)$, $A_2 = (u_1, u_2, x_1, x_3 \ldots, x_d)$, $\ldots$, $A_d = (u_1, u_2, x_1, \ldots, x_{d-1})$ are all of equal volumes. They each contain the edge ℓ, and form a simplicial subdivision $P' = \cup_{i=1}^d A_i$.

Now, apply the analogue piecewise-linear transformations to Q, to obtain a simplicial subdivision $Q' = \cup_{i=1}^d B_i$, where all simplices B_i contain an edge $\ell' = (v'_1, v'_2)$. There is a natural linear transformation $f_i : A_i \to B_i$ which maps $\ell \to \ell'$ by sending u'_1 to v'_1 and u'_2 to v'_2, and maps the boundary of P' into the boundary of Q'. Since $\mathrm{vol}(A_i) = \mathrm{vol}(B_i)$, these maps are volume-preserving and combine into a continuous piecewise-linear volume-preserving map $f : P' \to Q'$ (see Figure 6). Composing f with the maps $P \to P'$ and $Q' \to Q$ gives the desired map. □

We are now ready to present the final construction of our map. Roughly speaking, we take the 'nearly volume-preserving' map obtained from Lemma 3.1 and 'correct' it using the 'bipyramid maps' of Lemma 4.2 to obtain an 'honest' volume-preserving, piecewise-linear map.

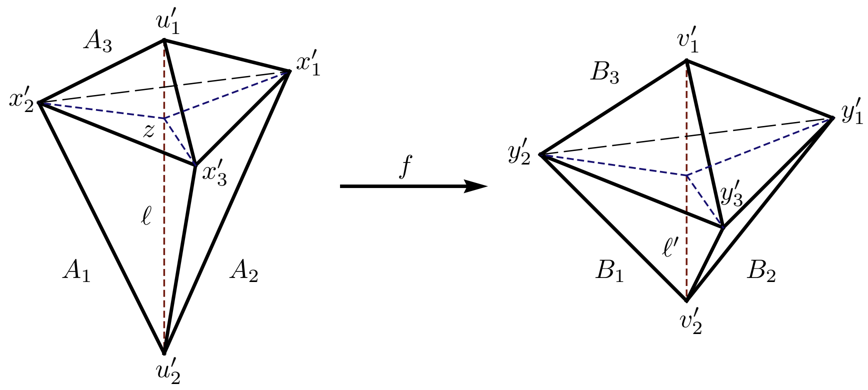

FIGURE 6. The map $f : P' \to Q'$, where $P = A_1 \cup A_2 \cup A_3$ and $Q' = B_1 \cup B_2 \cup B_3$.

Lemma 4.3. *Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes, and let $P = \cup_{r=1}^n P_r$, $Q = \cup_{r=1}^n Q_r$ be simplicial subdivisions. Suppose $f : P \to Q$ is a PL-homeomorphism, which is linear on the simplices: $f : P_r \to Q_r$, for all $1 \le r \le n$. In addition, assume there exist numbers $a_F \in \mathbb{R}$, for all oriented $(d-1)$-dimensional faces $F \subset P_r$, which satisfy:*

i) $a_F = -a_{\overline{F}}$, *where $\overline{F}$ is obtained from F by changing the orientation,*
ii) $a_F = 0$ *for all $F \subset \partial P$,*
iii) $\mathrm{vol}(Q_r) = \mathrm{vol}(P_r) + \sum_{F \subset P_r} a_F$, *where the orientation of F is induced by the orientation of P_r,*
iv) $|a_F| < \frac{1}{d+1}\,\mathrm{vol}(P_r)$, *for all $F \subset P_r$ and $1 \le r \le n$.*

Then there exists a volume-preserving PL-homeomorphism $g : P \to Q$.

Proof. Let z_r denote the barycenter of P_r. Let $P = \cup_{r,F} P_{r,F}$ be a simplicial subdivision obtained by letting $P_{r,F}$ be the convex hull of z_r and F, where F is a facet in P_r. For each P_r, we have a decomposition $P_r = \cup_F P_{r,F}$ in $d+1$ simplices of equal volume. Similarly, let $Q = \cup_{r,G} Q_{r,G}$ be a simplicial subdivision obtained by using *some point* $z'_r \in Q_r$ and letting $Q_{r,G}$ be the convex hull of z'_r and G, where G is a facet in Q_r. We again have $Q_r = \cup_G Q_{r,G}$ but this time the $Q_{r,G}$ are not of equal volume. Instead we ask that

$$(\#) \qquad \mathrm{vol}(Q_{r,G}) = \frac{1}{d+1}\,\mathrm{vol}(P_r) + a_F\,,$$

where G is a facet in Q_r, and $F = f^{-1}(G)$ is its preimage under f. The existence of a point z'_r that guarantees (#) is given by Lemma 4.1. Indeed, condition *iv*) implies that this volume is nonnegative, and condition *iii*) gives

$$\sum_{G \subset Q_r} \mathrm{vol}\big(Q_{r,G}\big) \;=\; \mathrm{vol}(Q_r).$$

For every facet $F \subset P_r \cap \partial P$ and $G = f(F) \subset Q_r \cap \partial Q$, condition *ii*) implies that $\mathrm{vol}(P_{r,F}) = \mathrm{vol}(Q_{r,G})$. There is a natural linear volume-preserving map $g : P_{r,F} \to Q_{r,G}$, such that $G = f(F)$, and the map g maps facets $R \subset P_{r,F}$ into the corresponding facets $f(R) \subset Q_{r,G}$.

Now, for every $(d-1)$-dimensional face F which is a facet of two simplices P_r, P_s, i.e. $F = P_r \cap P_s$, consider the bipyramid $\widehat{P}_F := P_{r,F} \cup P_{s,F}$. Similarly, for a facet $G = f(F)$ consider the bipyramid $\widehat{Q}_G := Q_{r,G} \cup Q_{s,G}$. Condition *i*) implies that the volumes of these bipyramids are equal:

$$\begin{aligned}\operatorname{vol}\big(\widehat{Q}_G\big) &= \operatorname{vol}\big(Q_{r,G}\big)+\operatorname{vol}\big(Q_{s,G}\big) = \big(\operatorname{vol}\big(P_{r,F}\big)+a_F\big)+\big(\operatorname{vol}\big(P_{s,F}\big)-a_F\big)\\ &= \operatorname{vol}\big(P_{r,F}\big)+\operatorname{vol}\big(P_{s,F}\big) = \operatorname{vol}\big(\widehat{P}_F\big).\end{aligned}$$

Therefore, by Lemma 4.2, there exists a continuous piecewise-linear volume-reserving map $g : \widehat{P}_F \to \widehat{Q}_G$ which is linear on the corresponding facets. Taking all these maps g together, we obtain the desired construction of a continuous piecewise-linear volume-preserving map $g : P \to Q$. □

Example 4.4. Consider the two pentagons P, Q shown in Figure 7. Here $\operatorname{vol}(P_1) = 5$, $\operatorname{vol}(P_2) = 8$, $\operatorname{vol}(P_3) = 7$, and $\operatorname{vol}(Q_1) = 6$, $\operatorname{vol}(Q_2) = 8$, $\operatorname{vol}(Q_3) = 6$. Thus the pentagons have equal area: $\operatorname{vol}(P) = \operatorname{vol}(Q) = 20$. Fix orientations of the facets $F, F' \subset P_2$ according to P_2. We have $a_F = -1$, while $a_{F'} = 1$, and all conditions of Lemma 4.3 are satisfied. We first construct the subdivisions of the P_i using their barycenter, and the subdivisions of the Q_i using a weighted barycenter (see Figure 7). Following the proof of Lemma 4.2, construct separately the piecewise-linear maps between bipyramids $g : \widehat{P}_F \to \widehat{Q}_F$ and $g : \widehat{P}_{F'} \to \widehat{Q}_{F'}$. The final map $g : P \to Q$ constructed in the proof above is shown in the Figure 7. Here the corresponding bipyramids (which are 4-gons on a plane) and boundary simplices are shaded similarly.

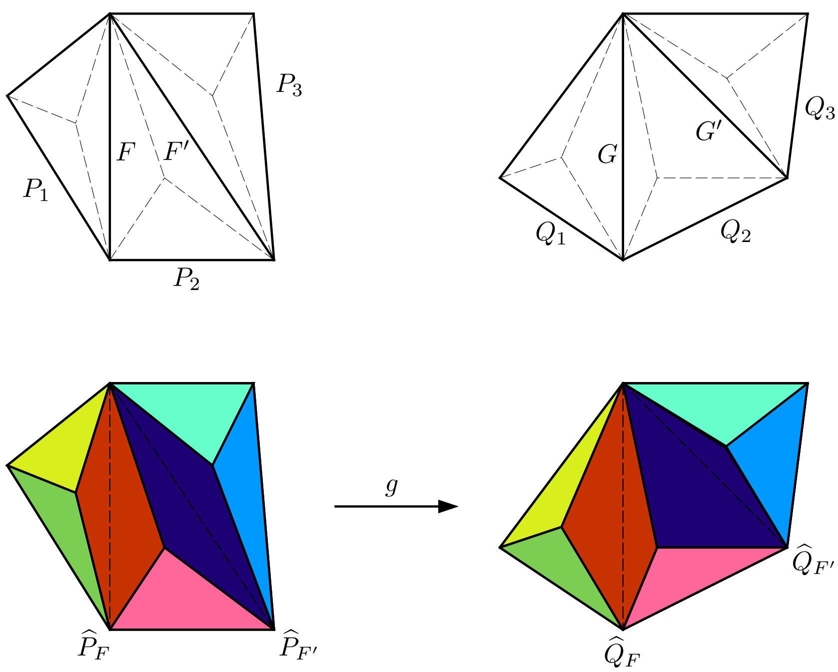

FIGURE 7. A continuous piecewise-linear volume-preserving map $g : P \to Q$, where $P = P_1 \cup P_2 \cup P_3$, $Q = Q_1 \cup Q_2 \cup Q_3$, which maps the bipyramids $\widehat{P}_F, \widehat{P}_{F'}$ into the bipyramids $\widehat{Q}_F, \widehat{Q}_{F'}$, respectively.

5. Proof of Theorems

Proof of Theorem 2. The first part of the theorem can be deduced from the lemmas as follows. Given two polytopes $P, Q \subset \mathbb{R}^d$, use Lemma 2.4 to construct a piecewise-smooth volume-preserving map $g : P \to Q$. The map produces a smooth triangulation $Q = \cup_i T_i$ of the polytope Q. Use Lemma 3.1 to obtain a refinement $Q = \cup_j R_j$ such that the aligned polyhedral subdivision $Q = \cup_j Q_j$ satisfies inequalities *iii*) in the lemma. Now consider the simplicial subdivision $P = \cup_j P_j$ by taking preimages: $P_j = g^{-1}(R_j)$. Define a continuous piecewise-linear map $f : P \to Q$ by letting $f(P_j) = Q_j$, for all j. By Lemma 3.1, the map f satisfies conditions in the Lemma 4.3, which implies the result.

For the second part, observe that the construction in Lemma 4.3 gives a rational map g, if the simplicial subdivisions and the map f are rational. One the other hand, the simplicial subdivision produced by Lemma 3.1 is not necessarily rational. To correct this, simply move all its vertices within a small neighborhood to make them rational; whenever they lie on faces of the polytope, use the fact that the polytope is rational to make sure these new vertices are rational and still lie in these faces. We can always do this so that the inequalities *iii*) remain valid. Putting everything together as above, implies the second part of the theorem. □

Proof of Theorem 1. Much of the proof follows verbatim the proof of of Theorem 2, by substituting PL-manifolds (or pseudomanifolds) M_1, M_2 in place of convex polytopes P, Q.

Let us first mention that we no longer need Lemma 1.1 since we already assume that the PL-manifolds M_1, M_2 are PL-homeomorphic. Also, importantly, before applying Moser's Theorem, we need to pick a smooth structure on the complement of the $(d-2)$-skeleton of our PL-manifold. The rest of the proof of the theorem follows verbatim and the changes are straightforward. □

6. Proof of Lemma 2.1

We deduce the lemma from the following stronger result, which in turn is proved by induction on the dimension d. Let $B^\circ = B - \partial B$ denote the interior of B.

Lemma 6.1. *Let $B \subset \mathbb{R}^n$ be a convex polytope, and let $\lambda : B \to \mathbb{R}_{>0}$ be a smooth function which is equal to 1 near the boundary ∂B. Let $\pi : \mathbb{R}^{d+n} \to \mathbb{R}^n$ be the projection, and let $E \subset \mathbb{R}^{d+n}$ be a convex polytope such that π restricts to a surjective map, $\pi : E \twoheadrightarrow B$ mapping faces onto faces. Assume that the fibers $\pi^{-1}(b)$ over interior points $b \in B^\circ$ are d-dimensional simplices and that $d \geq 1$. Then there exists a continuous piecewise-smooth homeomorphism $f : E \to E$ such that:*

1) *f preserves the fibers of π,*
2) *f fixes the boundary ∂E,*
3) *$f = \mathrm{Id}$ near the fibers $\pi^{-1}(\partial B)$,*
4) *$f^*(\omega_\circ) = \lambda(b) \cdot \omega_\circ$ near ∂E,*
5) *$f^*(\omega_\circ)$ is smooth and everywhere non-zero.*

The conditions of the lemma are illustrated in Figure 8 below.

Let us first deduce Lemma 2.1 from Lemma 6.1, and then prove Lemma 6.1.

Proof of Lemma 2.1. In Lemma 6.1, let $n = 0$, and the polytope B consist of a single point p. Note that $\partial B = \emptyset$. Let $E = P$, and $\lambda(p) = \mathrm{vol}(P)/\mathrm{vol}(Q)$. The lemma now gives a map $g : P \to P$ such that

$$g^*(\omega_\circ) = \frac{\mathrm{vol}(P)}{\mathrm{vol}(Q)} \cdot \omega_\circ \quad \text{near } \partial P.$$

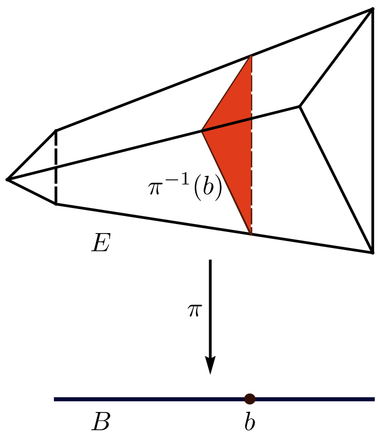

FIGURE 8. The projection $\pi : E \to B$.

Composing g with a natural linear map $h : P \to Q$ gives the desired map $f = h \circ g : P \to Q$, which from above is volume-preserving near the boundary ∂P. The smoothness of $f^*(\omega_\circ)$ follows immediately from the corresponding property of g. $\square$

A careful analysis of the proof of Lemma 6.1 shows that the map f is smooth on the simplices of the barycentric subdivision of P.

Proof of Lemma 6.1. We use induction on d. First, let us reduce the problem to the case when E is a product $P \times B$ and $P \subset \mathbb{R}^d$ is a simplex containing the origin O in its interior.

Let $\varphi : P \times B \to E$ be a continuous surjective maps such that the restriction to the fiber $\varphi|_{\{b\}\times P} \to \pi(b)$ is linear, for every $b \in B$. It is smooth and admits an inverse, defined over the interior of B, namely $\varphi^{-1} : \pi^{-1}(B^\circ) \to P \times B$. Assume that we have built a map $f_1 : P \times B \to P \times B$ satisfying 1) to 5). Then it is easy to see that the unique continuous extension f of the composition $\varphi \circ f_1 \circ \varphi^{-1}$ will also satisfy them.

We now assume that $E = P \times B$, and $O \in P$. Let $\rho : \mathbb{S}^{d-1} \to \mathbb{R}_+$ be the function whose graph (in spherical coordinates) is the boundary of P. Formally, let $\rho(\theta) := |x_\theta|$ be the distance from O to the point $x_\theta \in \partial P$ whose projection on $\mathbb{S}^{d-1}$ is θ.

Pick a diffeomorphism $g : [0,1] \times B \to [0,1] \times B$ satisfying:

i) $g(r, b) = (r, b)$ near $b \in \partial B$,

ii) $g(r, b) = \big(\sqrt[d]{\lambda(b)}\, r, b\big)$ near $r = 0$,

iii) $g(r, b) = \big(\sqrt[d]{\lambda(b) \cdot (r^d - 1) + 1}, b\big)$ near $r = 1$.

We also assume that g preserves the fibers of $[0,1] \times B \to B$ and write $g(r, b) = (g_b(r), b)$. We then let $h_b : P \to P$ be defined by $h_b(r, \theta) = \rho(\theta)\, g_b\big(r/\rho(\theta)\big)$, where $\theta \in \mathbb{S}^{d-1}$ and $r \in [0, \rho(\theta)]$ are the spherical coordinates as above.

Define a map $\widetilde{f} : E \to E$ by $\widetilde{f}(r, \theta, b) := \big(h_b(r, \theta), \theta, b\big)$. It is straightforward to check that $\widetilde{f}^*(\omega_\circ) = \lambda(b) \cdot \omega_\circ$ near the boundary ∂E. If the dimension d is 1, then the form $\widetilde{f}^*(\omega_\circ)$ is smooth and everywhere non-zero; this proves the base of our induction.

For the induction step, assume that $d \geq 2$. We have

$$\widetilde{f}^*(\omega_\circ) \; = \; \left[\frac{h_b(r, \theta)}{r}\right]^{d-1} \cdot \; \frac{\partial}{\partial r}\, h_b(r, \theta) \; \cdot \; \omega_\circ,$$

but $h_b(r, \theta)$ is not smooth in θ. The following argument shows how to 'correct' this problem.

Let $\{F_i\}$ be set of facets of P, and let $P = \cup_i P_i$ be the simplicial subdivision of the polytope P, where each P_i is obtained as the convex hull of O and F_i (cf. proof of Lemma 4.3).

Similarly, consider a polyhedral decomposition $E = \cup_i E_i$, where $E_i = P_i \times B$. Observe that the map $q_i : E_i \to [0,1] \times B$ defined by $q_i(r,\theta,b) = \big(r/\rho(\theta), b\big)$ is linear and its fibers are $(d-1)$-simplices. Indeed, restricted to P, the map $(r,\theta) \mapsto r/\rho(\theta)$ is just the linear functional defining F_i. The fibers of q_i are the same as the fibers of $(r,\theta) \mapsto r/\rho(\theta) : P_i \to [0,1]$, they are the various homotetic images of F_i.

We can now apply our induction hypothesis to the case $\widehat{B} = [0,1] \times B$, $\widehat{E} = E_i$, $\widehat{\pi} = q_i$ and

$$\widehat{\lambda}(b,t) \; = \; \frac{\lambda(b)\, t^{d-1}}{g_b(t)^{d-1}\, g_b'(t)},$$

to obtain a map $\widehat{f}_i : E_i \to E_i$. By induction assumption, the map $\widehat{f}$ is piecewise-smooth and commutes with the projection to B and with the the function $r/\rho(\theta)$. Namely $\pi \circ \widehat{f} = \pi$ and $\widehat{f}^*(r/\rho(\theta)) = r/\rho(\theta)$. It fixes the boundary of E_i and satisfies

$$\left(\widehat{f}_i\right)^* (\omega_\circ) \; = \; \frac{\lambda(b)\,\big(r/\rho(\theta)\big)^{d-1}}{g_b\big(r/\rho(\theta)\big)^{d-1}\, g_b'\big(r/\rho(\theta)\big)} \cdot \omega_\circ$$

near the boundary ∂E_i. Putting together all maps $\widehat{f}_i$ we obtain a continuous piecewise-smooth homeomorphism $\widehat{f} : E \to E$. Clearly, the boundary ∂E consists of points $(x,b) \in P \times B$, $x \in P$, $b \in B$, such that either $b \in \partial B$ or (in the spherical coordinates of x) $r/\rho(\theta) = 1$. Therefore, the inductive assumption implies $\widehat{f} = \mathrm{Id}$ near the boundary ∂E.

Define $f := \widetilde{f} \circ \widehat{f}$. The map f preserves the fibers of π and fixes the boundary ∂E, since both maps $\widetilde{f}$ and $\widehat{f}$ do. Similarly, $f^*(\omega_\circ) = \lambda(b){\cdot}\omega_\circ$ near the boundary ∂E, since $\widetilde{f}^*(\omega_\circ) = \lambda(b){\cdot}\omega_\circ$ and $\widehat{f} = \mathrm{Id}$ near ∂E. The fact that $f^*(\omega_\circ)$ is everywhere non-zero also follows form the corresponding properties of $\widetilde{f}$ and $\widehat{f}$. It remains to check that $f^*(\omega_\circ)$ is smooth; we show this in the following argument.

Let $\Sigma \subset P$ be the cone over $(d-2)$-skeleton of P (see Figure 9). The subset $\Sigma \times B \subset E$ is exactly the locus where $\widetilde{f}^*(\omega_\circ)$ is not smooth. It is also the locus where the different maps $\widehat{f}_i$ were glued together. This implies that $f^*(\omega_\circ)$ is smooth outside of $\Sigma \times B$. For points in $\Sigma \times B$ we compute:

$$\begin{aligned}
f^*(\omega_\circ) \; &=_{(1)} \; \widehat{f}^* \left(\widetilde{f}^*(\omega_\circ)\right) \; =_{(2)} \; \widehat{f}^* \left(\left[\frac{h_b(r,\theta)}{r}\right]^{d-1} \cdot \frac{\partial}{\partial r}\, h_b(r,\theta) \cdot \omega_\circ \right) \\
&=_{(3)} \; \widehat{f}^* \left(\frac{g_b\big(r/\rho(\theta)\big)^{d-1}\, g_b'\big(r/\rho(\theta)\big)}{\big(r/\rho(\theta)\big)^{d-1}} \cdot \omega_\circ \right) \\
&=_{(4)} \; \frac{g_b\big(r/\rho(\theta)\big)^{d-1}\, g_b'\big(r/\rho(\theta)\big)}{\big(r/\rho(\theta)\big)^{d-1}} \cdot \widehat{f}^*(\omega_\circ) \\
&=_{(5)} \; \frac{g_b\big(r/\rho(\theta)\big)^{d-1}\, g_b'\big(r/\rho(\theta)\big)}{\big(r/\rho(\theta)\big)^{d-1}} \cdot \frac{\lambda(b) \cdot \big(r/\rho(\theta)\big)^{d-1}}{g_b\big(r/\rho(\theta)\big)^{d-1}\, g_b'\big(r/\rho(\theta)\big)} \cdot \omega_\circ \\
&=_{(6)} \; \lambda(b) \cdot \omega_\circ \,,
\end{aligned}$$

where the fourth equality holds because $\widehat{f}^*\big(r/\rho(\theta)\big) = r/\rho(\theta)$, and the fifth equality holds for points near $\Sigma \times B$. By assumption of the lemma, $\lambda(b)$ is smooth, which implies that $f^*(\omega_\circ)$ is smooth and completes the proof. $\square$

7. Two-dimensional case

It is instructive to consider what happens when $d = 2$ and obtain an explicit construction in this case.

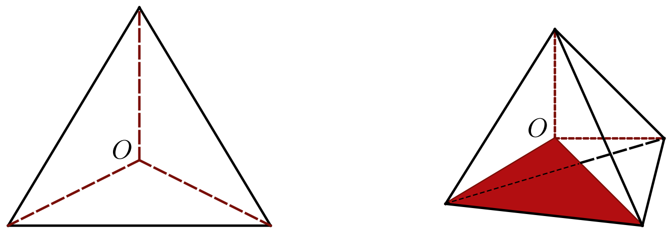

FIGURE 9. Examples of cone Σ in two and three dimensions. In the second case, only one of the six simplices is shaded.

For convex polygons P, Q the construction is easy. Given an n-gon P start with any vertex v and consider a triangle formed by v and its neighbors u and w. Let z be the second neighboring vertex of w (other than v). Now transform the triangle (uvw) into $(uv'w)$ by shifting v along the line parallel to (uw), such that v' now lies on a line containing (wz). Keep the rest of the polygon unchanged. We obtain a convex $(n-1)$-gon P' (see Figure 10).

Proceed in this manner until P is mapped into a triangle: $f_1 : P \to \Delta_1$. Proceed similarly with the polygon Q to obtain $f_2 : Q \to \Delta_2$. Since the resulting triangles Δ_1, Δ_2 have the same area, there exist a volume-preserving linear map $g : \Delta_1 \to \Delta_2$. Now, the composition $h := f_2^{-1} \circ g \circ f_1$ gives the desired continuous piecewise-linear map $h : P \to Q$.

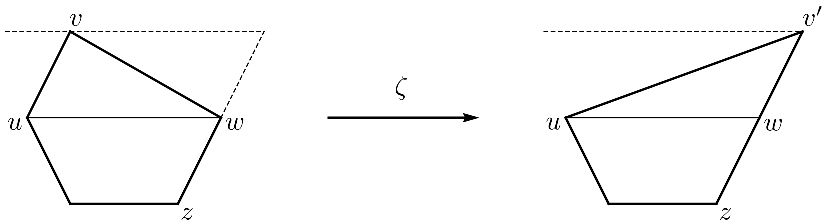

FIGURE 10. The map $\zeta : P \to P'$.

Interestingly, already for nonconvex and not simply connected polygons Theorem 2 becomes quite challenging. In fact, we do not know any easier solutions in these cases.

8. Final remarks and open problems

8.1. Variations on Hilbert's Third Problem with modified sets of linear maps have been studied before. See e.g. [Bo] for scissor equivalence under parallel translation (in two dimensions). However, the continuity condition is somewhat contrary to the spirit of 'scissor equivalence' and has never been considered in this context.

Similarly, a large part of PL-topology deals with continuous PL-maps between polyhedra, but it seems that the 'differential geometry style' idea of looking at volume-preserving maps has never been studied in this context (see [Br]).

8.2. In case of general convex bodies, continuous volume-preserving maps appear in connection with Monge's mass transportation (optimal transportation) problems and Monge-Ampère equations (see e.g. [Caf, FM]). The uniqueness of solutions in this case is especially intriguing but seems to have no analogue in our (polyhedral) situation. In fact, the proof has a large flexibility in the construction of the desired piecewise-linear map, which is not surprising in the absence of a minimization functional.

Let us also mention the important *Knothe map* which is volume-preserving and upper triangular [Kn]. It is used widely in convex analysis.

8.3. Symplectomorphisms have often been found to behave in a matter similar to volume-preserving diffeomorphisms (see for example [Mo]). It would be interesting to know if two PL-manifolds, equipped with piecewise constant symplectic forms, and which are symplectomorphic via a piecewise smooth map, are also symplectomorphic via a PL map.

Let us emphasize the distinction between manifolds with and without boundary, as in the former case Gromov showed the result to be negative [Gr]. In the language of polytopes, one should require additional conditions on their geometry.

8.4. The results in this paper were motivated by combinatorial applications. In recent years it was realized that many combinatorial bijections between Young tableaux, such as RSK correspondence, Hillman-Grassl and jeu-de-taquin bijection, etc., can in fact be extended to continuous piecewise-linear maps between convex polytopes. We refer to [KB, P, PV] for references and details.

Following the previous notation, let us denote such polytopes by P and Q, and by $f : P \to Q$ the map as above. The set of integral points $\mathcal{E}(X), \mathcal{E}(Q)$ in polytopes P, Q are in natural bijection with the corresponding sets of Young tableaux, and f maps $\mathcal{E}(X)$ to $\mathcal{E}(Q)$. In other words, the map f preserves the integral lattice, and thus satisfies the conditions of Theorem 2.

It is natural to ask how special are these 'combinatorial maps', i.e. whether their existence is in fact a delicate property of Young tableau bijections, or a general property of polytopes. The results of this paper suggest the latter.

On the other hand, let us point out the fundamental differences between the constructions of combinatorial maps and the maps produced in the proof of Theorem 2. The combinatorial maps are obtained as a small number of explicit and relatively simple 'large scale' piecewise-linear maps, each continuous and volume-preserving. This is in sharp contrast with the maps in this paper which are inexplicit and 'local', but are defined directly (no compositions are needed). In many ways, the combinatorial maps are closer in spirit to the maps defined in two-dimensional case (Section 7). It would be interesting to formalize these differences.

Let us mention also that the above mentioned 'combinatorial' maps preserve the Ehrhart (quasi-) polynomials of the polytopes. Extending the second part of Theorem 2 in this direction is an interesting question.

8.5. An important paper [S] gives an interesting example of a continuous piecewise-linear volume-preserving map between two polytopes associated with finite posets. The map in this case is defined explicitly and 'in one step', and a number of combinatorial applications are obtained by a 'transfer' of properties from one polytope to another.

8.6. For a piecewise-linear map $f : P \to Q$ between convex polytopes one can define the *complexity measure* to be the number of simplices in the corresponding simplicial subdivisions of P and Q (where the map f is linear). For any two polytopes P, Q of equal volume let $c(P, Q)$ be the minimum complexity measure over all continuous piecewise-linear volume-reserving maps $f : P \to Q$. Computing or at least finding sharp upper and lower bounds on $c(P, Q)$ is an interesting and challenging problem related to the geometry of polytopes and motivated by combinatorial and computational applications.

8.7. Finally, the existence of a continuous piecewise-linear volume-preserving map between polytopes of equal volume raises a natural question of explicitly constructing one. There are several ways to approach this problem: from a traditional 'mathematical' point of view as well as a question in *Computational Geometry* (see [GO]). We believe that the non-combinatorial proof of the Moser Theorem (Lemma 2.3) can be perhaps modified so that our construction becomes explicit. On the other hand, we reserve the judgement whether our proof is really the "proof from the book"—perhaps a radically different construction can avoid the complexity of our approach.

In the spirit of *Algebraic Complexity Theory*, one can also ask for the lower bounds on the algorithm producing the map as above. We refer to [BCS] for the related idea and references.

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