On growth of Grigorchuk groups

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Abstract

We present an analytic technique for estimating the growth for groups of intermediate growth. We apply our technique to Grigorchuk groups, which are the only known examples of such groups. Our estimates generalize and improve various bounds by Grigorchuk, Bartholdi and others.

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1 Introduction

In a pioneer paper [3] R. Grigorchuk discovered a family of groups of intermediate growth, which gave a counterexample to Milnor's Conjecture (see [3, 7, 13]). The groups are defined as groups of Lebesgue-measure-preserving transformations on the unit interval, but can be also defined as groups acting on binary trees, by finite automata, etc. While Grigorchuk was able to find both lower and upper bounds on growth, there is a wide gap between them, and more progress is desired.

In this paper we present a unified approach to the problem of estimating the growth. We introduce an analytic result we call *Growth Theorem*, which lies in the heart of our computations. This reduces the problem to combinatorics of words which is a natural language in this setting. We proceed to obtain both upper and lower bounds in several cases. This technique simplifies and improves the previous bounds obtained by various ad hoc approaches (see [2, 4, 5]). We believe that our Growth Theorem can be also applied to other classes of groups.

Let G be an infinite group generated by a finite set S, $S = S^{-1}$, and let Γ be the corresponding Cayley graph. Let $B(n)$ be the set of elements $g \in G$ at a distance $\leq n$ in graph Γ. The growth function of G with respect to the set of generators S is defined as $\gamma_G(n) = |B(n)|$.

We say that a function $f : \mathbb{N} \to \mathbb{R}$ is *dominated* by a function $g : \mathbb{N} \to \mathbb{R}$, denote by $f \preccurlyeq g$, if there is a constant $C > 0$ such that $f(n) \leq g(C \cdot n)$ for all $n \in \mathbb{N}$. Two functions $f, g : \mathbb{N} \to \mathbb{R}$ are called *equivalent*, denoted by $f \sim g$, if $f \preccurlyeq g$ and $g \preccurlyeq f$. It is known that for any two finite sets of generators S_1, S_2 of a group G, the corresponding two growth functions are equivalent (see e.g. [10, 13]). Note also that if $|S| = k$, then $\gamma(n) \leq k^n$.

Growth of group G is called *exponential* if $\gamma(n) \sim e^n$. Otherwise the growth is said to be *subexponential*. For example, all non–amenable groups[1] have an exponential growth, but not vice versa (see [1, 6]). Growth of group G is called *polynomial* if $\gamma(n) \sim n^c$ for some $c > 0$. The celebrated result of Gromov implies that c must be an integer, and G is almost nilpotent. See [8, 6] for details and references.

If $\gamma_G(n) \succcurlyeq n^c$ for all c, the growth of G is said to be *superpolynomial*. If the growth is subexponential and superpolynomial, it is called *intermediate*. This is a very interesting, but hardly understood class of groups.

Let ω be an infinite sequence of elements in the set $\{0, 1, 2\}$. Grigorchuk group G_ω is a infinite profinite 2-group whose construction depends on ω (see [4, 7]). Groups G_ω are generated by 4 involutions, while the structure and even the growth is different for different ω. We postpone definition of G_ω till section 3.

Since the original publication, much has been discovered regarding the Grigorchuk groups. Recent advancements include improved upper and lower bounds, solution of the word problem, abstract presentation, bond percolation, etc. (see [5, 9, 11, 12]). We refer to review articles [7, 9] for the references.

In this paper we present a new technique to estimate the growth of the Grigorchuk groups. First, we present a simple proof of the lower bound

$$\gamma(n) \succcurlyeq e^{\sqrt{n}}$$

for Grigorchuk group G_ω corresponding to non-flat sequences (see section 4). Using a different approach Grigorchuk showed in [5] that the result holds in greater generality. While neither our bound nor the idea of the proof is new, we believe that the technique may be proved useful in the future.

In section 5 we present an upper bound on the growth of Grigorchuk group G_ω such that every interval of ω of length k contains each element 0,

[1] Non–amenable group G can be defined as a group whose Cayley graphs have positive Cheeger constant $h > 0$, where

$$h = \inf_{X \in G} \frac{|\partial X|}{|X|}$$

1, 2 at least once. We prove that the growth $\gamma(n)$ in this case satisfies

$$\gamma(n) \preccurlyeq \exp(n^\alpha),$$

where $\alpha = \log_{2/\nu} 2$ and $\nu^k + \nu^{k-1} + \nu = 2$. In a special case when $k = 3$ we obtain the recent result of Bartholdi (see [2]). Interestingly, he gives the exactly same estimate while using a totally different approach. We would like to remark that one can try to improve our bounds if more information is known about frequencies of generators ρ_b, ρ_c, ρ_d (see Section 5.)

We conclude with a improved bound for Grigorchuk p-groups. Without going through the combinatorial estimates, we apply our Growth Theorem to inequalities proved by Grigorchuk to obtain sharper upper bounds on growth.

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2 Growth Theorem

In this section we present analytic estimates on growth of functions restricted by the recurrence inequalities. We refer to the following result as *Growth Theorem*.

Theorem 2.1 *Let $B : \mathbb{N} \to \mathbb{N}$ be an increasing integer function with $B(n) \to \infty$ as $n \to \infty$ and $B(n) \preccurlyeq e^n$. Let $s_j < 1$, c_j be fixed constants and let m_j be fixed integers, $1 \leq j \leq l$. Let $\pi_i^{(j)} : \mathbb{N} \to \mathbb{N}$, $1 \leq i \leq m_j$, $1 \leq j \leq l$ be integer functions such that*

$$\sum_{i=1}^{m_j} \pi_i^{(j)}(n) \leq s_j n + c_j$$

In addition, let $F_j : \mathbb{N} \to \mathbb{R}_+$, $1 \leq j \leq l$ be positive functions such that

$$\frac{\log(F_j(n))}{n^\epsilon} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0.$$

Assume that for all $n \in \mathbb{N}$, at least one inequality is satisfied in the following system:

$$\left\{ B(n) \leq F_j(n) \cdot \prod_{i=1}^{m_j} B\left(\pi_i^{(j)}(n)\right), \quad \text{where } j = 1, \ldots, l. \right.$$

Then $B(n) \preccurlyeq \exp(n^\alpha)$, where

$$\alpha = \max_{1 \leq j \leq l} \frac{\log(m_j)}{\log(m_j) - \log(s_j)}.$$

The proof requires the following technical result.

Lemma 2.2 *For any $0 < s < 1$, $c > 0$, $\epsilon > 0$ and sequence $(t_1, \ldots, t_n)$ of positive numbers such that*

$$\frac{c}{1-s} + 1 < t_1 < t_2 < \cdots < t_n$$

and $t_k \leq s \cdot t_{k+1} + c$ we have

$$\prod_{k=1}^{n} (1 + \frac{1}{t_k{}^\epsilon}) \leq \exp\left(\frac{1}{1 - s^\epsilon}\right).$$

Proof of Lemma 2.2 Observe that $\log(1+x) < x$ for all $x > 0$. This gives us

$$\sum_{k=1}^{n} \log\left(1 + \frac{1}{t_k{}^{\epsilon}}\right) \leq \sum_{k=1}^{n} \frac{1}{t_k{}^{\epsilon}}$$

Define a sequence $(j_1, \ldots, j_n)$ as follows: $j_i = t_i - c/(1-s)$, $1 \leq i \leq n$. Then

$$j_i = t_i - \frac{c}{1-s} \leq s \cdot t_{i+1} + c - \frac{c}{1-s} = s\left(t_{i+1} - \frac{c}{1-s}\right) = s \cdot j_{i+1}$$

Therefore $j_1 \leq s\, j_2 \leq \cdots \leq s^{i-1}\, j_i$. By definition we have $j_1 = t_1 - c/(1-s) > 1$ and $j_i \geq s^{1-i}$, $1 \leq i \leq n$.

From here for all $\epsilon > 0$ we obtain

$$\sum_{i=1}^{n} \frac{1}{t_i{}^{\epsilon}} \leq \sum_{i=1}^{n} \frac{1}{j_i{}^{\epsilon}} \leq \sum_{i=1}^{n} (s^{\epsilon})^{i-1} < \sum_{i=1}^{\infty} (s^{\epsilon})^{i-1} = \frac{1}{1-s^{\epsilon}}$$

We conclude

$$\prod_{i=1}^{n}\left(1 + \frac{1}{t_i{}^{\epsilon}}\right) = \exp\left(\sum_{i=1}^{n} \log\left(1 + \frac{1}{t_i{}^{\epsilon}}\right)\right) \leq \exp\left(\sum_{i=1}^{n} \frac{1}{t_i{}^{\epsilon}}\right) < \exp\left(\frac{1}{1-s^{\epsilon}}\right)$$

This proves the lemma. $\square$

Proof of Theorem 2.1 Let $f(n) = \log B(n)$. Then $f(n)$ satisfies at least one of the inequalities in the following system

$$\left\{ f(n) \leq \sum_{i=1}^{m_j} f(\pi_i^{(j)}(n)) + \log F_j(n)\ , \qquad j = 1, \ldots l \right.$$

Let $0 < v \leq 1$ be such that $\sup_n f(n)/n^{v} = \infty$. Let $h(n) = \frac{f(n)}{n^{v}}$. Clearly, $\sup_n h(n) = \infty$.

Define $\widehat{h}(n) = \max\{h(k) : 1 \leq k \leq n\}$. Clearly $\widehat{h}(n)$ is non-decreasing and $\widehat{h}(n) = h(n)$ for infinitely many n. Observe that $\widehat{h}(t) = h(t)$ if and only if $\widehat{h}(t) > \widehat{h}(t-1)$. Call an integer t of type (I) if $\widehat{h}(t) > \widehat{h}(t-1)$ and of type (II) if $\widehat{h}(t) = \widehat{h}(t-1)$. Clearly for any integer m of type (II) there exist an integer $n \leq m$ of type (I) such that $\widehat{h}(n) = \widehat{h}(m)$.

Take a large integer a of type (I). Then $f(a) = \widehat{h}(a)a^{v}$. Assume that for a inequality j holds.

$$a^{v}\widehat{h}(a) = a^{v}h(a) = f(a) \leq \sum_{i=1}^{m_j} f(\pi_i^{(j)}(a)) + \log F_j(a)$$

$$
\begin{aligned}
&\le \sum_{i=1}^{m_j} \left[\pi_i^{(j)}(a)\right]^{\upsilon} h(\pi_i^{(j)}(a)) + \log F_j(a) \\
&\le \sum_{i=1}^{m_j} \left[\pi_i^{(j)}(a)\right]^{\upsilon} \widehat{h}(\pi_i^{(j)}(a)) + \log F_j(a) \\
&\le \max_i(\widehat{h}(\pi_i^{(j)}(a))) \cdot \sum_{i=1}^{m_j} \left[\pi_i^{(j)}(a)\right]^{\upsilon} + \log F_j(a).
\end{aligned}
$$

Since the max $\left(\widehat{h}(\pi_i^{(j)}(a))\right)$ is taken over a finite set, there exist an integer b of type (I) such that $\widehat{h}(b) = \max\left(\widehat{h}(\pi_i^{(j)}(a))\right)$ and

$$
a^{\upsilon}\widehat{h}(a) \le \widehat{h}(b) \cdot \sum_{i=1}^{m_j} \left[\pi_i^{(j)}(a)\right]^{\upsilon} + \log F_j(a).
$$

Trivially $b \le s_j a + c_j$. Dividing by $a^{\upsilon}\widehat{h}(b)$ we get

$$
\frac{\widehat{h}(a)}{\widehat{h}(b)} \le \sum_{i=1}^{m_j} \left(\frac{\pi_i^{(j)}(a)}{a}\right)^{\upsilon} + \frac{\log F_j(a)}{a^{\upsilon}\widehat{h}(b)}
$$

Now since $0 < \upsilon < 1$, x^{υ} is convex up, by Jensen inequality we have

$$
\begin{aligned}
\frac{\widehat{h}(a)}{\widehat{h}(b)} - \frac{\log F_j(a)}{a^{\upsilon}\widehat{h}(b)} &\le \sum_{i=1}^{m_j} \left(\frac{\pi_i^{(j)}(a)}{a}\right)^{\upsilon} \le m_j \left(\frac{\sum_{i=1}^{m_j} \pi_i^{(j)}(a)}{a\, m_j}\right)^{\upsilon} \\
&\le m_j \left(\frac{s_j}{m_j} + \frac{c_j}{a\, m_j}\right)^{\upsilon}
\end{aligned}
$$

Since $0 < \upsilon \le 1$ we have

$$
\left(\frac{s_j}{m_j} + \frac{c_j}{a\, m_j}\right)^{\upsilon} \le \left(\frac{s_j}{m_j}\right)^{\upsilon} + \left(\frac{c_j}{a\, m_j}\right)^{\upsilon}
$$

Let us summarize what we have so far: for any integer a of type (I), there exists $j \in \{1, \ldots, l\}$ and $b \le s_j a + c_j$ of type (I) such that

$$
(*) \qquad \frac{\widehat{h}(a)}{\widehat{h}(b)} - \frac{\log F_j(a)}{a^{\upsilon}\widehat{h}(b)} - \left(\frac{c_j}{a\, m_j}\right)^{\upsilon} \le m_j \left(\frac{s_j}{m_j}\right)^{\upsilon}
$$

Let $s = \max(s_j) < 1$, $c = \max(c_j)$, and L is large enough constant (see below). Now recursively construct a sequence of integers of type (I) $t_1 > t_2 > \cdots > t_{n-1} > L \geq t_n$ as follows. Take $t_1 > L$ such that

$$\frac{\widehat{h}(t_1)}{\widehat{h}(L)} > \exp\left(\frac{1}{1 - s^{\upsilon/2}}\right)$$

By the process above, find t_2 such that $(*)$ holds for some j, where in $(*)$ $a = t_1$, $b = t_2$. Clearly, $t_2 \leq s \cdot t_1 + c$. Analogously find t_3 such that $(*)$ holds for some j', where $a = t_2$, $b = t_3$, etc. Proceed until we find the first n such that $t_n \leq L$. By construction, we have $t_{i+1} \leq s \cdot t_i + c$, $1 \leq i \leq n-1$.

Assume that

$$\frac{\widehat{h}(t_i)}{\widehat{h}(t_{i+1})} \leq 1 + \frac{1}{{t_i}^{\upsilon/2}}.$$

Multiplying all these inequalities we get

$$\frac{\widehat{h}(t_1)}{\widehat{h}(L)} \leq \frac{\widehat{h}(t_1)}{\widehat{h}(t_n)} \leq \prod_{i=1}^{n-1}\left(1 + \frac{1}{{t_k}^{\upsilon/2}}\right) \leq \exp\left(\frac{1}{1 - s^{\upsilon/2}}\right),$$

which gives a contradiction. Therefore there exists an integer a of type (I) such that

$$\frac{\widehat{h}(a)}{\widehat{h}(b)} \geq 1 + \frac{1}{a^{\upsilon/2}}.$$

We claim that

$$\frac{\widehat{h}(a)}{\widehat{h}(b)} - \frac{\log F_j(a)}{a^{\upsilon}\widehat{h}(b)} - \left(\frac{c_j}{a\, m_j}\right)^{\upsilon} > 1$$

for sufficiently large a. Indeed, since $\widehat{h}(b) \geq \widehat{h}(1)$, we have

$$\frac{\widehat{h}(a)}{\widehat{h}(b)} - \frac{\log F_j(a)}{a^{\upsilon}\widehat{h}(b)} - \left(\frac{c_j}{a\, m_j}\right)^{\upsilon} \geq 1 + \frac{1}{a^{\upsilon/2}} - \frac{\log F_j(a)}{a^{\upsilon}\widehat{h}(1)} - \left(\frac{c_j}{a\, m_j}\right)^{\upsilon}$$

$$\geq 1 + \frac{1}{a^{\upsilon/2}}\left(1 - \frac{\log F_j(a)}{a^{\upsilon/2}\widehat{h}(1)} - \left(\frac{c_j}{m_j}\right)^{\upsilon} \cdot \frac{1}{a^{\upsilon/2}}\right)$$

Now since the expression in parenthesis $\to 1$ as $a \to \infty$ the l.h.s. is strictly greater than 1 for a large enough. This proves the claim.

Therefore we have

$$1 < m_j\left(\frac{s_j}{m_j}\right)^{\upsilon}$$

for some j, $1 \leq j \leq l$. From here $\upsilon < \log_{m_j/s_j} m_j$ and therefore

$$\upsilon < \max_{1\leq j\leq l} \frac{\log(m_j)}{\log(m_j) - \log(s_j)}$$

Let $\alpha = \sup\{\beta \in [0,1] \mid \sup_n \frac{f(n)}{n^\beta} \to \infty\}$.

Now if $\sup \frac{f(n)}{n^\alpha} < M$, then take $\upsilon = \alpha - \epsilon$. By result above $\upsilon < \frac{\log(m_j)}{\log(m_j)-\log(s_j)}$. Taking $\epsilon \to 0$ we get $\alpha \leq \frac{\log(m_j)}{\log(m_j)-\log(s_j)}$. and the result of the theorem follows.

If $\sup \frac{f(n)}{n^\alpha} = \infty$ take $\upsilon = \alpha$. Again, from the discussion above

$$\alpha < \max_{1\leq j\leq p} \frac{\log(m_j)}{\log(m_j) - \log(s_j)}$$

and the theorem follows. $\square$

The following corollary states the result in a special case when the system contains just one inequality. This is probably the most useful case of all.

Corollary 2.3 *Let $B : \mathbb{N} \to \mathbb{N}$ be an increasing integer function such that $B(n) \to \infty$ as $n \to \infty$, and $B(n) \preccurlyeq e^n$. Also assume that for all n large enough,*

$$B(n) \leq F(n) \cdot \prod_{i=1}^{m} B\big(\pi_i(n)\big)$$

is satisfied, where m is a positive integer,

$$\limsup \frac{\pi_i(n)}{n} < \theta_i\,, \quad \textit{such that} \quad \sum_{i=1}^{m} \theta_i < 1$$

for some $\theta_i \geq 0$, and $\log(F(n))/\,n^\epsilon \to 0$ as $n \to \infty$ for all $\epsilon > 0$. Then

$$B(n) \preccurlyeq \exp(n^\alpha) \quad \textit{where} \quad \sum_{i=1}^{m} (\theta_i)^\alpha \;=\; 1.$$

Observe that everywhere in the proof of Theorem 2.1 we can always reverse signs and obtain a lower bound $B(n) \succcurlyeq \exp(n^\alpha)$. The proof is analogous up up slight changes of lim sup to lim inf, etc. Rather than state the whole result, we will symbolically indicate it as follows.

Corollary 2.4 *In the setup of Corollary 2.3 one can reverse signs.*

We will use this lower bound in the next section.

3 Grigorchuk Group

In this section we will describe a construction of Grigorchuk's 2-group. For a complete description and further results see [4].

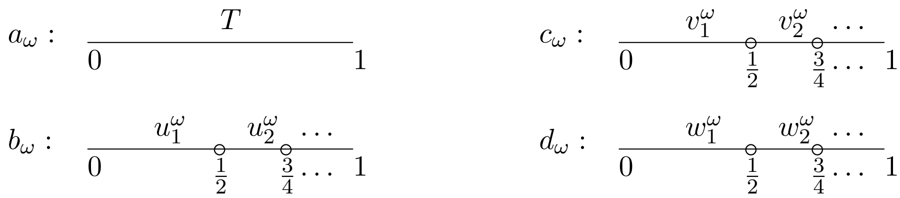
Let Δ be an interval. Denote by I an identity transformation on Δ and by T a transposition of two halves of Δ.

Let Ω be a set of infinite sequences $\omega = (\omega_1, \omega_2, \dots)$ of elements of the set $\{0, 1, 2\}$. For each $\omega \in \Omega$ define a $3 \times \infty$ matrix $\overline{\omega}$ by replacing ω_i with columns $\overline{\omega}_i$ where

$$\bar{0} = \begin{pmatrix} T \\ T \\ I \end{pmatrix}, \quad \bar{1} = \begin{pmatrix} T \\ I \\ T \end{pmatrix}, \quad \bar{2} = \begin{pmatrix} I \\ T \\ T \end{pmatrix}$$

By $U^\omega = (u_1^\omega, u_2^\omega, \dots), V^\omega = (v_1^\omega, v_2^\omega, \dots), W^\omega = (w_1^\omega, w_2^\omega, \dots)$ denote the rows of $\overline{\omega}$. Think of them as of infinite words in the alphabet $\{T, I\}$.

Define transformations $a_\omega, b_\omega, c_\omega, d_\omega$ of an interval $\Delta = [0, 1] \setminus \mathbb{Q}$ as follows:



Observe that a_ω is independent of ω, and will be further denoted by a. Let G_ω be a group of transformations of the interval Δ generated by $a, b_\omega, c_\omega, d_\omega$. This family of groups was introduced and analyzed by Grigorchuk in [4] (see also [9] for further references). We refer to G_ω as Grigorchuk groups.

Observe that the generators of G_ω satisfy the following relations:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = 1,$$

$$c_\omega b_\omega = b_\omega c_\omega = d_\omega,$$

$$d_\omega b_\omega = b_\omega d_\omega = c_\omega,$$

$$c_\omega d_\omega = d_\omega c_\omega = b_\omega,$$

We call these *simple relations*. Under mild conditions, the groups G_ω are known to be not finitely presented (see [4, 9]).

Denote by Γ_ω a Cayley graph of the group G_ω with respect to the generators $a, b_\omega, c_\omega, d_\omega$. For every element $g \in G_\omega$ by $\partial(g)$ denote the smallest distance between g and id in Γ_ω. The paths in Γ_ω correspond to words in the alphabet $\{a, b_\omega, c_\omega, d_\omega\}^*$. The shortest paths (there could be many of them between two given elements) correspond to the reduced words in the alphabet. Recall that the balls in the Cayley graph Γ_ω is defined as $B_\omega(n) = \{g \in G_\omega | \partial(g) \leq n\}$.

We define *almost reduced* words to be words obtained after application of contractions which correspond to simple relations. We call these *simple contractions*. It follows from the the simple relations that almost reduced word w must be of the form $\star\, a \star a \star a \ldots$ or $a \star a \star a \ldots$, where by $\star$ we denote any element of in $\{b_\omega, c_\omega, d_\omega\}$.

Let $\sigma : \Omega \to \Omega$ be a right shift operator acting on the infinite sequences as follows

$$\sigma : (\omega_1, \omega_2, \omega_3, \ldots) \to (\omega_2, \omega_3, \ldots)$$

Denote by Δ_0 and Δ_1 the half intervals $\Delta \cap [0, 1/2]$ and $\Delta \cap [1/2, 1]$. Define $H_\omega \subset G_\omega$ to be a stabilizer of Δ_0. Clearly, $g : \Delta_1 \to \Delta_1$ for all $g \in H_\omega$. Analogously, define Δ_i^k to be an interval $\Delta \cap [i/2^k, (i+1)/2^k]$, where $0 \leq i < 2^k$. Denote by H_ω^k a stabilizer of all Δ_i^k. Note also that G_ω acts on a set of Δ_i^k by permuting them. This defines a representation of G_ω as a permutation group S_{2^k}. By definition, H_ω^k is the kernel of this representation. Let $H_\omega^k(n) = H_\omega^k \cap B_\omega(n)$.

Define $\phi_0^\omega : H_\omega \to G_{\sigma\omega}$ by restricting $h \in H_\omega$ to Δ_0. Formally, while H_ω acts on Δ_0 rather than Δ we can rescale the interval to obtain transformations in $G_{\sigma\omega}$. Similarly define $\phi_1^\omega : H_\omega \to G_{\sigma\omega}$ by restricting to Δ_1 and then rescaling to the unit interval.

It is easy to see that H_ω is a normal subgroup of index 2, which is generated by 6 elements $b_\omega, c_\omega, d_\omega, ab_\omega a, ac_\omega a, ad_\omega a$. We will omit superscript ω in $\phi_{0,1}^\omega$ when it is clear on which H_ω the map ϕ_i^ω acts. The following table summarizes the images of homomorphisms of ϕ_0, ϕ_1 on the generators of subgroup H_ω.

TABLE

	b_ω	c_ω	d_ω	$ab_\omega a$	$ac_\omega a$	$ad_\omega a$
ϕ_0	u_1^ω	v_1^ω	w_1^ω	$b_{\sigma\omega}$	$c_{\sigma\omega}$	$b_{\sigma\omega}$
ϕ_1	$b_{\sigma\omega}$	$c_{\sigma\omega}$	$b_{\sigma\omega}$	u_1^ω	v_1^ω	w_1^ω

Define the following maps

$$\phi_{i_1, \ldots 1^k}^\omega = \phi_{i_k} \circ \phi_{i_{k-1}} \circ \cdots \circ \phi_{i_1},$$

where $i_1, \ldots, i_k \in \{0, 1\}$. While these maps are not defined on G_ω, they are defined on H^k_ω and $\phi^\omega_{i_1,\ldots 1^k} : H^k_\omega \to G_{\sigma^k\omega}$.

4 Lower bounds

Let $\omega \in \Omega$. We call ω *flat* if for any $k > 0$ there exits an i such that $\omega_i = \omega_{i+1} = \cdots = \omega_{i+k}$.

Theorem 4.1. *Let G_ω be a Grigorchuk group, and ω is not flat. Then for the growth $\gamma(n)$ of G_ω we have*

$$\gamma(n) \succcurlyeq e^{\sqrt{n}}$$

Proof: Denote by $b_n = b_{\sigma^n\omega}$, $c_n = c_{\sigma^n\omega}$, $d_n = d_{\sigma^n\omega}$ the generators of $G_{\sigma^k\omega}$. Since ω is not flat, there exists a k such that for any i there exists j such that $j - i < k$ and $\omega_j \neq \omega_i$. Without loss of generality assume that $\omega_1 = 0$ and $\omega_s = 1$ for $1 < s \leq k$.

Then $(ad_0)^4 = I$ and $(ac_0)^{(4\cdot 2^s)} = I$. This can be shown by restricting these group elements to subintervals Δ^k_i and checking that they act as identities.

Recall the maps $\phi_{0,1} : G_\omega \to G_{\sigma\omega}$. Let us show that every element $h \in B_{\sigma\omega}(n) \subset G_{\sigma\omega}$ can be *lifted* to $g \in B_\omega(2\,n + 1) \subset G_\omega$ such that $\phi_0(g) = h$, and $\phi_1(g)$ is in the finite subgroup $\langle a, c_1\rangle$.

Indeed, observe that $G_{\sigma\omega}$ is generated by elements $a\,b_1$, $a\,c_1$, $a\,d_1$, a. From the TABLE in section 3, we have $\phi_0(c_0\,a\,b_0\,a) = a\,b_1$, $\phi_0(c_0\,a\,c_0\,a) = a\,c_1$, $\phi_0(c_0\,a\,d_0\,a) = a\,d_1$, $\phi_0(c_0) = a$. Let $h \in B_{\sigma\omega}(n)$. Since ϕ_0 is a homomorphism there exists $g \in B_\omega(2n + 1)$ such that $\phi_0(g) = h$. Observe that since $\omega_1 = 0$ we have $\phi_1(g) \in \langle a, c_1\rangle$. On the other hand,

$$\langle a, c_1\rangle = \langle a, b\,|\,a^2 = b^2 = (a\,b)^{4\cdot 2^s}\rangle = D_{(4\cdot 2^s)}$$

is a dihedral group of order $2 \cdot (4 \cdot 2^s) \leq 2^{k+3}$ By the symmetry, a similar argument is valid for the lifting of ϕ_1.

Now take $h_0, h_1 \in B_{\sigma\omega}(n)$. There exist $g_0, g_1 \in B_{\sigma\omega}(2n + 1)$ such that $\phi_0(g_0) = h_0$ $\phi_1(g_1) = h_1$ and $\phi_1(g_0) = z_0$, $\phi_0(g_1) = z_1$, where $z_0, z_1 \in \langle a, c_1\rangle$. Now to each pair (h_0, h_1) we associate an element $g = g_0g_1 \in B_{\sigma\omega}(4n + 2)$. This g has the following property: $\phi_0(g) = h_0z_1$, $\phi_1(g) = z_0h_1$.

Now if (h_0, h_1) and (h'_0, h'_1) are associated to the same element g. Then $h_0^{-1}h_0{}', h'_1h_1{}^{-1} \in \langle a, c_1\rangle$, i.e there are at most $|\langle a, c_1\rangle|^2 \leq 4^{k+3}$ pairs that could be associated to the same element g in B_ω.

Thus we obtain an inequality $|B_{\sigma\omega}(n)|^2 \leq 4^{k+3}|B_\omega(4n+2)|$.

Now let $|B(n)| = \inf_s \ |B_{\sigma^s\omega}(n)|$. For a fixed n we have $|B_{\sigma^s\omega}(n)|$ is an integer number bounded from above by 4^n. Therefore, $|B(n)|$ is well-defined and there exists $s(n)$ such that $|B(n)| = |B_{\omega'}(n)|$, where $\omega' = \sigma^{s(n)}\omega$. Thus we obtain

$$4^{k+3}|B(4n+2)| = 4^{k+3}|B_{\omega'}(4n+2)| \geq |(B_{\sigma\omega'}(n))|^2 \geq |B(n)|^2$$

In particular, $|B(n)|$ satisfies $|B(n)|^2 \leq 4^{k+3}|B(4n+2)|$ and by Corollary 2.4 we have $|B_\omega(n)| \geq |B(n)| \geq 2^{M\sqrt{n}}$. This proves the result. $\square$

5 Upper bounds

Let $b_i, c_i, d_i \in G_{\sigma^i\omega}$ be as in previous section. Let w be any word representation of an element $g \in G_\omega$. Recall that w is *almost reduced* if w is reduced with respect to simple contractions.

Denote by $\rho(w)$ the length of the word w. For a word τ denote by $\rho_\tau(w)$ the number of times τ appears in word w. We will be working with $\rho_a(w)$, $\rho_{b_i}(w)$, $\rho_{c_i}(w)$, $\rho_{d_i}(w)$ in the group $G_{\sigma^i\omega}$. To simplify the notation, we will omit the index i whenever possible.

We will extend the definition of maps ϕ_0, ϕ_1 to almost reduced words w which correspond to elements $g \in H_\omega$. First, we apply ϕ_0, ϕ_1 to w by using the TABLE, and then apply simple contractions (cf. [9]). Similarly, from an almost reduced word w representing an element $g \in H_\omega^n$ we can obtain almost reduced word $\phi_{i_1,\ldots,i_n}(w)$ corresponding to $\phi_{i_1,\ldots,i_n}(g)$.

Denote by

$$\rho^n(w) = \sum_{i_1,\ldots,i_n \in \{0,1\}} \rho(\phi_{i_1,\ldots,i_n}(w))\,.$$

Similarly, for an element τ we can define

$$\rho_\tau^n(w) = \sum_{i_1,\ldots,i_n \in \{0,1\}} \rho_r(\phi_{i_1,\ldots,i_n}(w))\,.$$

Observe that in these notation $\rho_\tau^0(w) = \rho_\tau(w)$ As before, we will be working with $\rho_a^n, \rho_b^n, \rho_c^n, \rho_d^n$.

Lemma 5.1 *Let w represent an almost reduced word of $g \in H_\omega^{n+1}$ then*

$$\rho_b^{n+1}(w) + \rho_c^{n+1}(w) \leq \rho_b^n(w) + \rho_c^n(w),$$

$$\rho_d^{n+1}(w) + \rho_c^{n+1}(w) \leq \rho_d^n(w) + \rho_c^n(w),$$
$$\rho_b^{n+1}(w) + \rho_d^{n+1}(w) \leq \rho_b^n(w) + \rho_d^n(w).$$

Proof: Let us prove the first inequality. Observe that after application of maps $\phi_{i_1,\ldots,i_n}$ the number of b's and c's stays the same. In order to obtain a new b, one c should contract with d leaving $\rho_b^{n+1}(w) + \rho_c^{n+1}(w)$ unchanged. Similarly, to obtain a new c, at least one d should contract with d. This proves the first inequality. Note also that some simple contraction may actually reduce $\rho_b^{n+1}w((g)) + \rho_c^{n+1}(w)$.

Proof of the second and third inequality is similar and will be omitted. □

Lemma 5.2 *Let w represent an almost reduced word of $g \in H_\omega^s$, for some $s > 0$.*

If $w_s = 0$, then $\rho^s(w) \leq 2\,\rho_b(w) + 2\,\rho_c(w) + 2^s$.

If $w_s = 1$, then $\rho^s(w) \leq 2\,\rho_b(w) + 2\,\rho_d(w) + 2^s$.

If $w_s = 2$, then $\rho^s(w) \leq 2\,\rho_d(w) + 2\,\rho_c(w) + 2^s$.

Proof: Let us prove the first inequality. From the representation of the almost reduced word w we have $\rho(w) \leq 2\rho_a(w)+1$ for every almost reduced word w. Since $\rho^s(w)$ is a sum of the length of 2^s almost reduced word we have $\rho^s(w) \leq 2\rho_a^s(w) + 2^s$. Now for $w_s = 0$, every a is an image of some b_{s-1} or c_{s-1}. Hence $\rho_a^s(w) \leq \rho_b^{s-1}(w) + \rho_b^{s-1}(w)$. Now, by Lemma 5.1, $\rho_a^s(w) \leq \rho_b(w) + \rho_b(w)$. Hence $\rho^s(w) \leq 2\,\rho_b(w) + 2\,\rho_c(w) + 2^s$. This proves the first inequality.

The proof of two remaining inequalities in analogous and will be omitted. □

Lemma 5.3 *Let w represent an almost reduced word of $g \in H_\omega^{s+t}$. Then*

$$\rho^{s+t}(w) \leq \rho^s(w) + 2^{s+t} + 2^s.$$

Proof: Without loss of generality assume $w_{s+t} = 0$. Similarly, as in Lemma 5.2, we have

$$\begin{aligned}\rho^{s+t}(w) &\leq 2\,\rho_a^{s+t}(w) + 2^{s+t} \leq 2\,\rho_b^{s+t-1}(w) + 2\,\rho_b^{s+t-1}(w) + 2^{s+t}\\ &\leq 2\,\rho_b^s(w) + 2\,\rho_c^s(w) + 2^{s+t}\end{aligned}.$$

Using the representation of almost reduced word w, we have $2\,\rho_b(w) + 2\,\rho_c(w) \le \rho(w) + 1$. As $\rho^s(w)$ is a sum of the length of 2^s almost reduced words, we get $2\,\rho_b^s(w) + 2\,\rho_c^s(w) \le \rho^s(w) + 2^s$. This proves the result. $\square$

Denote by Ω^r the set of all $\omega \in \Omega$ such that for all $i > 0$ a subsequence $(\omega_{i+1}, \omega_{i+2}, \ldots, \omega_{i+r})$ contains all the elements $0, 1, 2$.

Notice that if w is a reduced word of g, then $\partial(g) = \rho(w)$. For each element $g \in G_\omega$ fix a lexicographically first reduced word $w = w(g)$ of g.

Proposition 5.4 *Let $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, $s \ge 1$. Then for any $g \in H_\omega^{s+t}$ we have*

$$\sum \partial(\phi_{i_1,\ldots,i_{s+t}}(g)) \le \partial(g) + 2^{s+t} + 2^{s+1} + 1 - \begin{cases} 2\,\rho_d(w(g)) & \textit{if } \ \omega_s = 0 \\ 2\,\rho_c(w(g)) & \textit{if } \ \omega_s = 1 \\ 2\,\rho_b(w(g)) & \textit{if } \ \omega_s = 2 \end{cases}$$

where the sum is over all $i_1, \ldots, i_{s+t} \in \{0, 1\}$.

Proof: Let $\omega_s = 1$. By definition of $\rho^k(w)$ we have

$$\sum_{i_1,\ldots,i_{s+t} \in \{0,1\}} \partial\big(\phi_{i_1,\ldots,i_{s+t}}(g)\big) \le \rho^{s+t}(w(g))$$

By Lemma 5.3 we have

$$\rho^{s+t}(w(g)) \le \rho^s(w(g)) + 2^s + 2^{s+t}$$

Lemma 5.2 gives us

$$\rho^s(w(g)) \le 2(\rho_b(w(g)) + \rho_d(w(g))) + 2^s$$

From the representation of reduced words we have

$$(*) \qquad 2\,\rho_b(w(g)) + 2\,\rho_d(w(g)) + 2\,\rho_c(w(g)) \le \partial(g) + 1$$

Here we use the condition that $w(g)$ is reduced.

Combining these three inequalities we obtain the middle case. The other two cases are similar. $\square$

Definition 5.5 For any $\epsilon > 0$ and any element $\tau \in \{b, c, d\}$ define a set

$$F_\tau^\epsilon = \left\{ g \in G_\omega \,|\, 2\,\rho_\tau(w(g)) + \frac{1}{3} \ge \epsilon\,\partial(g) \right\}.$$

Lemma 5.6 *Let $\epsilon+\epsilon'+\epsilon''=1$. Then*

$$G_\omega = F_b^{\epsilon} \cup F_c^{\epsilon'} \cup F_d^{\epsilon''}.$$

Proof: Assume that exists $g \in G_\omega \setminus F_b^{\epsilon} \cup F_c^{\epsilon'} \cup F_d^{\epsilon''}$. Then

$$2\,\rho_b(w(g))+\frac{1}{3} < \epsilon\cdot\partial(g),\ 2\,\rho_c(w(g))+\frac{1}{3} < \epsilon'\cdot\partial(g),\ 2\,\rho_d(w(g))+\frac{1}{3} < \epsilon''\cdot\partial(g).$$

Adding these inequalities we get

$$2\,\rho_b(w(g)) + 2\,\rho_c(w(g)) + 2\,\rho_d(w(g)) + 1 < \partial(g)$$

However from a representation of reduced word $w(g)$ we have opposite inequality. This gives a contradiction. □

Proposition 5.7 *Let $\epsilon+\epsilon'+\epsilon''=1$, $k\geq 1$. Then at least one of the following inequalities holds:*

$$|B_\omega(n)| \leq 3\,C\cdot\left|F_b^{\epsilon}\cap H_\omega^k(n+C)\right|,$$

$$|B_\omega(n)| \leq 3\,C\cdot\left|F_c^{\epsilon'}\cap H_\omega^k(n+C)\right|,$$

$$|B_\omega(n)| \leq 3\,C\cdot\left|F_d^{\epsilon''}\cap H_\omega^k(n+C)\right|,$$

where $C=(2^k)!$

Proof: As G_ω/H_ω^k is isomorphic to a subgroup of a group of permutations of Δ_i^k, $i=0,\ldots,2^k-1$, we have $|G_\omega/H_\omega^k| \leq (2^k)!$ Fix a Schreier system of representatives of the right cosets of H_ω^k in G_ω. Then every element $g\in G_\omega$ can be written as $g=l\,h$ where l is a Schreier representative and $h\in H_\omega^k$. As $\partial(l)\leq(2^k)!$ and $\partial(g)\leq n$. Thus $\partial(h)\leq n+(2^k)!$ In particular, $|B_\omega(n)|\leq (2^k)!\,|H_\omega^r(n+(2^k)!)|$. Let $C=(2^k)!$ Now Lemma 5.6 immediately implies the result. □

Theorem 5.8 *Let $\beta+\gamma+\delta=1$. Assume that each of elements 0,1,2 appears in the sequence $(\omega_1,\ldots,\omega_r)$. Then there exist polynomials $p_1(n),p_2(n),p_3(n)$ and functions $\pi_i^{(1)},\pi_i^{(2)},\pi_i^{(3)}:\mathbb{N}\to\mathbb{N}$ which satisfy the* *following conditions*

$$\sum_{i=1}^{2^r} \pi_i^{(1)}(n) \leq (1-\beta)n + 3 \cdot 2^r + 2,$$
$$\sum_{i=1}^{2^{r-1}} \pi_i^{(2)}(n) \leq (1-\gamma)n + 3 \cdot 2^{r-1} + 2,$$
$$\sum_{i=1}^{2} \pi_i^{(3)}(n) \leq (1-\delta)n + 8,$$

such that at least one of the following inequalities holds

$$|B_\omega(n)| \leq p_1(n) \prod_{i=1}^{2^r} \left| B_{\sigma^r \omega}\left(\pi_i^{(1)}(n)\right) \right|,$$
$$|B_\omega(n)| \leq p_2(n) \prod_{i=1}^{2^{r-1}} \left| B_{\sigma^{r-1} \omega}\left(\pi_i^{(2)}(n)\right) \right|,$$
$$|B_\omega(n)| \leq p_3(n) \prod_{i=1}^{2} \left| B_{\sigma \omega}\left(\pi_i^{(3)}(n)\right) \right|,$$

Proof: Without loss of generality we can assume that $w_1 = 0$, $w_s = 1$, and $w_t = 2$, where $1 < s < t \leq r$. Then if $g \in F_d^\delta \cap H_\omega^r$ then by Proposition 5.4 we have

$$\partial(\phi_0(g)) + \partial(\phi_1(g)) \leq \partial(g) - 2\,\rho_d(w(g)) + 7$$

Since $2\,\rho_d(w(g)) + 1 \geq \delta\,\partial(g)$, we get

$$\partial(\phi_0(g)) + \partial(\phi_1(g)) \leq (1-\delta)\,\partial(g) + 8$$

Observe that every $g \in H_\omega^r$ is uniquely defined by its restriction to subintervals $\Delta_{0,1}$. We claim that

$$\left| F_d^\delta \cap H_\omega^r(n) \right| \leq \sum_{i_1+i_2=\lfloor (1-\delta)\,n \rfloor + 8} \prod_{h=1}^{2} |B_{\sigma\omega}(i_h)|$$

Indeed, the sum counts how many pairs $(\phi_0(g), \phi_1(g))$ can possibly be images of g, where $g \in F_d^\delta \cap H_\omega^r$. Denote $m_3 = \lfloor (1-\delta)\,n \rfloor + 8$.

Therefore,

$$|F_d^\delta \cap H_\omega^r(n)| \leq \binom{m_3+2-1}{2-1} \max_{i_1+i_2=m_3} \prod_{h=1}^{2} |B_{\sigma\omega}(i_h)|$$

where the binomial coefficient gives the number of summands in the sum.

Denote by $\pi_h^{(3)}(n) : \mathbb{N} \to \mathbb{N}$ integer functions which satisfy $\pi_1^{(3)}(n) + \pi_2^{(3)}(n) = m_3$ and such that

$$\left|B_{\sigma\omega}(\pi_1^{(3)}(n))\right| \cdot \left|B_{\sigma\omega}(\pi_2^{(3)}(n))\right| = \max_{i_1+i_2=m_3} \prod_{h=1}^{2} |B_{\sigma\omega}(i_h)|$$

We conclude

$$|F_d^\delta \cap H_\omega^r(n)| \leq \binom{m_3+2-1}{2-1} \prod_{h=1}^{2} \left|B_{\sigma\omega}\left(\pi_h^{(3)}(n)\right)\right|,$$

where $\pi_1^{(3)}(n) + \pi_2^{(3)}(n) = m_3$.

Similarly we can prove

$$|F_c^\gamma \cap H_\omega^r(n)| \leq \binom{m_2+2^{r-1}-1}{2^{r-1}-1} \prod_{h=1}^{2^{r-1}} \left|B_{\sigma^{r-1}\omega}\left(\pi_h^{(2)}(n)\right)\right|$$

where $\pi_1^{(2)}(n) + \cdots + \pi_{2^{r-1}}^{(2)}(n) = m_2 = \lfloor (1-\gamma)\, n \rfloor + 3 \cdot 2^{r-1} + 2$. Analogously

$$|F_b^\beta \cap H_\omega^r(n)| \leq \binom{m_1+2^r-1}{2^r-1} \prod_{h=1}^{2^r} \left|B_{\sigma^r\omega}\left(\pi_h^{(1)}(n)\right)\right|$$

where $\pi_1^{(1)}(n) + \cdots + \pi_{2^r}^{(1)}(n) = m_1 = \lfloor (1-\beta)\, n \rfloor + 3 \cdot 2^r + 2$. Now apply Proposition 5.7 to obtain the result. □

In order to apply Theorem 2.1 we need to remove the dependence on ω in Theorem 5.8. We use the following result.

Theorem 5.9 *Let $\omega \in \Omega^r$. Then for the growth $\gamma(n)$ of G_ω we have*

$$\gamma(n) \preceq \exp(n^\alpha),$$

where

$$\alpha = \frac{\ln 2}{\ln 2 - \ln \nu_r},$$

and ν_r satisfies the equation

$$x^r + x^{r-1} + x = 2.$$

Proof: Observe that $\sigma^k\omega \in \Omega^r$. Let $|B(n)| = \max_{l\geq 0} |B_{\sigma^l\omega}(n)|$. Recall that for all n there exist $l = l(n)$ such that

$$|B(n)| = |B_{\sigma^l\omega}(n)|.$$

Since Theorem 5.8 holds for all $\sigma^k\omega \in \Omega^r$, it also holds for $B(n)$ (cf. section 4).

Fix $\beta + \gamma + \delta = 1$. By Theorem 2.1, $B_\omega(n) \leq |B(n)| \leq C \cdot 2^{d\,n^s}$, where $s = s(\beta, \gamma, \delta)$ is given by

$$s = \max\left\{ \frac{\log 2}{\log 2 - \log(1-\delta)}, \frac{(r-1)\log 2}{(r-1)\log 2 - \log(1-\gamma)}, \frac{r\log 2}{r\log 2 - \log(1-\beta)} \right\}$$

This holds for every $\beta + \gamma + \delta = 1$. Take

$$\alpha = \min_{\beta+\gamma+\delta=1} s(\beta, \gamma, \delta)$$

A direct computation shows that the minimum is achieved when

$$\alpha = \frac{\log 2}{\log 2 - \log(1-\delta)} = \frac{(r-1)\log 2}{(r-1)\log 2 - \log(1-\gamma)} = \frac{r\log 2}{r\log 2 - \log(1-\beta)}$$

In this case $(1-\delta)^r = 1-\beta$ and $(1-\delta)^{r-1} = 1-\gamma$. Since $\beta + \gamma + \delta = 1$, we get $\nu_r = 1 - \delta$ is a positive real root of the equation

$$x^r + x^{r-1} + x = 2$$

We conclude

$$\alpha = \frac{\log 2}{\log 2 - \log \nu_r}$$

This finishes the proof. $\square$

In a special case $r = 3$ we obtain a bound which was earlier and independently obtained by Bartholdi (see [2]).

Corollary 5.10 (Bartholdi) *Take $r = 3$ and $\omega = (012012012\ldots)$ we have $\gamma(n) \preccurlyeq \exp(n^\alpha)$, with $\alpha = \frac{\log 2}{\log 2 - \log nu_3)}$, where ν_3 is a solution of $x^3 + x^2 + x = 2$.*

6 Generalization to p-groups

In this section we will give an estimate on the growth of Grigorchuk p-group. These groups are defined similarly to Grigorchuk 2-groups defined in section 3. In this case the group G_ω is again generated by $a_\omega, b_\omega, c_\omega$, and $\omega \in \Omega$, where Ω is a space of all infinite sequences from the set $\{0, \ldots, p\}$. Now the transposition T is substituted by a p-permutation cycle of p equal subintervals. For a complete definition of these group we refer to [5].

Fix an integer $m \geq p$. Consider a sequence ω such that every symbol $\{0, \ldots, p\}$ occurs at least once in any subsequence $(\omega_n, \omega_{n+1}, \ldots, \omega_{n+m})$. It was proved by Grigorchuk in [5] (see Formula 12) that

$$|B(n)| \leq (p^m)! \sum |B_{\sigma^m \omega}(i_1)| \, |B_{\sigma^m \omega}(i_2)| \ldots |B_{\sigma^m \omega}(i_{p^m})| \, ,$$

where the summation on the right hand side runs over the multi-indices $(i_1, i_2, \ldots, i_{p^m})$ such that $i_1, i_2, \ldots, i_{p^m} > 0$ and

$$i_1 + i_2 + \cdots + i_{p^m} \leq \frac{3\,n}{4} + (p^m)! + p^m$$

Applying similar arguments as in the proof of Theorem 5.1, 5.2 we obtain

$$|B_\omega(n)| \leq 2^{n^\alpha},$$

where

$$\alpha = \frac{\log(p^m)}{\log(p^m) - \log(3/4)}$$

This implies the following result.

Theorem 6.1. *Let $\omega \in \Omega$ be as above, and let G_ω be the corresponding Grigorchuk p-group. Then growth γ of G_ω satisfies*

$$\gamma(n) \preccurlyeq \exp(n^\alpha)$$

where $\alpha = \alpha(m, p)$ is as above.

The result of Grigorchuk gives a bound $\gamma \preccurlyeq \exp(n^\beta)$, where

$$\beta < 1 + \inf_N \, \log\left(\frac{3}{4} + \frac{(p^m)! + p^m}{N}\right) + \epsilon$$

for any $\epsilon > 0$. A straightforward calculation shows that $\beta > \alpha$ for all ϵ, p and $m > p$. The gap between bounds is particularly wide for relatively small values of p. For example, when $p = 3$, $m = 4$ we have $\alpha \approx .9385574519$. On the other hand, Grigorchuk's result gives $\beta \approx .9989950236$.

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