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On Fine's Partition Theorems, Dyson, Andrews, and Missed Opportunities

History almost never works out the way you want it to, especially when you are looking at it after the dust settles. The same is true in mathematics. There are times when the solution of a problem is overlooked simply by accident, due to a combination of unfortunate circumstances. In a celebrated address [D4], Freeman Dyson described several *"missed opportunities,"* in particular his own advance glimpse of Macdonald's eta-function identities. I present here the history of Fine's partition theorems and their combinatorial proofs. As the reader will see, many of the results could and perhaps should have been discovered a long time ago. There was a whole string of "missed opportunities."

The central event is the publication of a short note [F1] by Nathan Fine. To quote George Andrews, "[Fine] announced several elegant and intriguing partition theorems. These results were marked by their simplicity of statement and [. . .] by the depth of their proof." [A7] Without taking anything away from the depth and beauty of the results, I will show here that most of them have remarkably simple combinatorial proofs, in a very classical style. Perhaps that's exactly how it should be with important results! Even a reader who prefers analytic methods may find that here the combinatorial approach fits the problem well.

Fine's partition theorems can be split into two (overlapping) categories: those dealing with partitions into odd and distinct parts, à la Euler, and those dealing with Dyson's rank. I shall separate these two stories, as they have relatively little to do with each other. The fortune and misfortune, however, had the same root in both stories, as you will see.

Fine's note [F1] didn't have any proofs; not even hints on complicated analytic formulae which were used to prove the results. It was published in a National Academy of Sciences publication, in a journal devoted to all branches of science. Thus the paper was largely overlooked by subsequent investigators. The note contained a promise to have complete proofs published in a journal "devoted entirely to mathematics." This promise was never fulfilled.

Good news came from a different quarter. In the sixties, George Andrews, while a graduate student at the University of Pennsylvania, took a course of Nathan Fine on basic hypergeometric series. As he writes in his mini biography [A8], "His course was based on a manuscript he had been perfecting for a decade; it eventually became a book [F2]." In fact, the book [F2] was published only in 1988, exactly 40 years after the publication of [F1]. It indeed contained the proofs of all partition results announced in [F1]. Meanwhile, Andrews kept the manuscript and used it on many occasions before [F2] appeared. Among other things, Andrews gave new analytic proofs of many results, found connections to the works of Rogers and Ramanujan, and, what's important for the subject of this paper, gave combinatorial proofs to some of the theorems. Much of the fame Fine's long-unpublished results now have is owing to Andrews's work and persistence (see [A1–A8]).

This is where the story splits into two. The rest of this article is largely mathematical, dealing separately with each of Fine's partition theorems. To simplify the presentation, I change their order and use different notation. I conclude the discussion with Dyson's proof of Euler's Pentagonal Theorem and a few more surprises.

A few words about the notation. I denote partitions of n by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, and I write $\lambda \vdash n$, or $|\lambda| = n$. Let λ' be the conjugate partition to λ. The largest part and the number of parts of λ are denoted by $a(\lambda)$ and $\ell(\lambda)$, respectively. Every partition λ may be represented graphically by its Young diagram $[\lambda]$; recall that one definition of λ' is the condition that $[\lambda']$ is the transpose of $[\lambda]$. See [A3] for standard references, definitions, and details.

Partitions Into Distinct Parts and Franklin's Involution

The following result is straight from [F1]:

THEOREM 1 (Fine) *Let $\mathcal{D}_n^0$ and $\mathcal{D}_n^1$ be the sets of partitions λ of n into distinct parts, such that the largest part $a(\lambda) = \lambda_1$ is even and odd, respectively. Then*

$$|\mathcal{D}_n^0| - |\mathcal{D}_n^1| = \begin{cases} 1, & \text{if } n = k(3k+1)/2 \\ -1, & \text{if } n = k(3k-1)/2 \\ 0, & \text{otherwise.} \end{cases}$$

It is perhaps suggestive to compare Theorem 1 with the similar-looking *Euler's Pentagonal Theorem*, which can be stated as follows:

THEOREM 2 (Euler) *Let $\mathcal{Q}_n^0$ and $\mathcal{Q}_n^1$ be the sets of partitions λ of n into distinct parts, such that the number of parts $\ell(\lambda) = \lambda_1'$ is even and odd, respectively. Then*

$$|\mathcal{Q}_n^0| - |\mathcal{Q}_n^1| = \begin{cases} (-1)^k, & \text{if } n = k(3k \pm 1)/2 \\ 0, & \text{otherwise.} \end{cases}$$

Of course, this similarity was not overlooked. Fine himself acknowledged that Theorem 1 "bears some resemblance to the famous pentagonal theorem of Euler, but we have not been able to establish any real connection between the two theorems." In the *Math. Reviews* article [L], Lehmer reiterates: "This result parallels a famous theorem of Euler."

As I shall show, Theorem 1 has a proof nearly identical to the famous involutive proof by Franklin of Theorem 2. Franklin was a student of Sylvester at Johns Hopkins University, active in Sylvester's exploration of the *"constructive theory of partitions."* He published his proof [Fr] right before the publication of a celebrated treatise [S1] by Sylvester (to which Franklin also contributed). These two papers laid the foundations of Bijective Combinatorics, a field which blossomed in the second half of the twentieth century.

Of course, it is hard to blame Fine for not discovering the connection. In those days bijections were rarely used to prove combinatorial results. Since the late sixties, however, the method became popular again, with a large number of papers proving partition identities by means of explicit bijections. Franklin's involution was far from forgotten, and was used on many occasions to prove various refinements of Euler's Pentagonal theorem [KP], and even most recently to prove a new partition identity [C]. It is a pity that an application to Fine's theorem remained unnoticed for so many years.

Proof. Denote by $\mathcal{D}_n = \mathcal{D}_n^0 \cup \mathcal{D}_n^1$ the set of all partitions into distinct parts. Let $\lambda \in \mathcal{D}_n$, and let $[\lambda]$ be the Young diagram corresponding to λ. Denote by $s(\lambda)$ the length of the smallest part in λ, and by $b(\lambda)$ the length of a maximal sequence of subsequent parts: $a, a-1, a-2, \ldots$, where $a = a(\lambda) = \lambda_1$. One can view $s(\lambda)$ and $b(\lambda)$ as the lengths of the *horizontal line* and *diagonal line* of squares of $[\lambda]$, as in Figure 1. Now, if $s(\lambda) \le b(\lambda)$, move the horizontal line to attach to the diagonal line. Similarly, if $s(\lambda) > b(\lambda)$, move the diagonal line to attach below the horizontal line. If we cannot make a move, stay put. This defines Franklin's involution $\alpha : \mathcal{D}_n \to \mathcal{D}_n$.

Note that α changes parity of the number of parts, except when λ is a fixed point. Observe that the only fixed points of the involution are the Young diagrams where the lines overlap, and $s(\lambda) - b(\lambda)$ is either 0 or 1 (see Figure 2). The number of squares in these diagrams are $m(3m \pm 1)/2$, which are called *pentagonal numbers*. Therefore, $|\mathcal{Q}_n^0| - |\mathcal{Q}_n^1|$ is 0 unless n is a pentagonal number, and is ± 1 in that case. This proves Theorem 2.

Similarly, note that α changes parity of the largest part. Thus again, we infer that $|\mathcal{D}_n^0| - |\mathcal{D}_n^1|$ is 0 unless n is a pentagonal number, and is ± 1 in that case. This completes the proof of Theorem 1. □

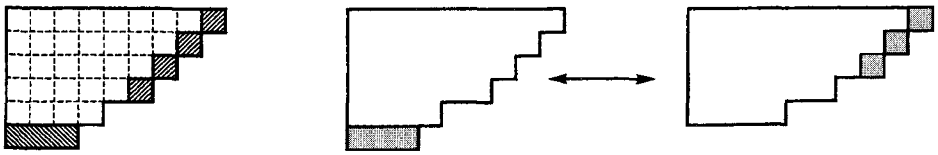

Figure 1. Young diagram [λ] corresponding to a partition $\lambda = (9,8,7,6,4,3)$. Here $s(\lambda) = 3$, $b(\lambda) = 4$, and $\alpha(\lambda) = (10,9,8,6,4)$.

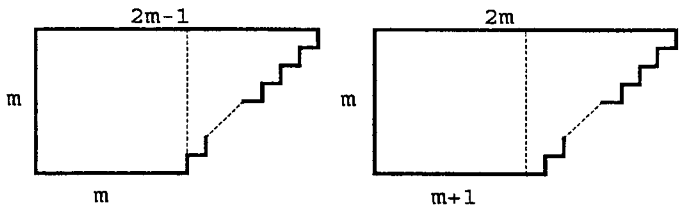

Figure 2. Fixed points of Franklin's involution.

Partitions Into Odd Parts and Sylvester's Bijection

Now I recall another famous theorem of Euler: that the number of partitions of n into odd numbers is equal to the number of partitions of n into distinct numbers. Here is another gem from [F1]:

THEOREM 3 (Fine) *Let* $\mathcal{O}_n^1$ and $\mathcal{O}_o^3$ *be the sets of partitions* λ *of* n *into odd parts such that the largest part* $a(\lambda)$ *is* 1 *and* 3 mod 4, *respectively. Then*

$$|\mathcal{O}_n^1| = |\mathcal{D}_n^0|, \quad |\mathcal{O}_n^3| = |\mathcal{D}_n^1|, \quad \textit{if } n \textit{ is even},$$
$$|\mathcal{O}_n^1| = |\mathcal{D}_n^1|, \quad |\mathcal{O}_n^3| = |\mathcal{D}_n^0|, \quad \textit{if } n \textit{ is odd}.$$

Clearly Fine's Theorem 3 is a refinement of Euler's theorem. As we shall see shortly, the following result of Fine [F2] is an extension:

THEOREM 4 (Fine) *For any* $k > 0$, *the number of partitions* $\mu \vdash n$ *into distinct parts such that* $a(\mu) = k$, *is equal to the number of partitions* $\lambda \vdash n$ *into odd parts such that* $a(\lambda) + 2\ell(\lambda) = 2k + 1$.

In his early paper [A1], Andrews proved Theorem 4 combinatorially, but never noticed that it implies Theorem 3. The reason could be that Theorem 3 was coupled with Theorem 1 in [F1], while the proofs use two different classical combinatorial arguments. The proofs of Theorem 3 and Theorem 4 follow from Sylvester's celebrated bijection, sometimes called a *fish-hook construction*. This bijection is a map between partitions into odd and distinct numbers, and gives a combinatorial proof of Euler's theorem (see [A1,A3]).

Sylvester's bijection is another fixture in the combinatorics of partitions. It has been restated in many different ways (e.g., using Frobenius coordinates and 2-modular diagrams [A6,B,PP]), and was used to prove other refinements of Euler's theorem [KY]. Had Theorem 3 been better known and not omitted in [L], the following proof could have been standard.

Proof. Denote by $\mathcal{O}_n = \mathcal{O}_n^1 \cup \mathcal{O}_n^3$ the set of all partitions into odd parts. Define *Sylvester's bijection* $\varphi : \mathcal{O}_n \to \mathcal{D}_n$ as shown in Figure 3. To show it is a bijection as alleged requires some work, and great astuteness on Sylvester's part. Observe that $a(\mu) = (a(\lambda) - 1)/2 + \ell(\lambda)$, for all $\mu = \varphi(\lambda)$. Now rewrite this formula as $a(\lambda) + 2\ell(\lambda) = 2a(\mu) + 1$. This proves Theorem 4.

Note that $\ell(\lambda) \equiv n \bmod 2$ for all $\lambda \in \mathcal{O}_n$. From the above equation, we conclude: $\varphi : \mathcal{O}_n^1 \to \mathcal{D}_n^0$, $\mathcal{O}_n^3 \to \mathcal{D}_n^1$, when n is even, and $\varphi : \mathcal{O}_n^1 \to \mathcal{D}_n^1$, $\mathcal{O}_n^3 \to \mathcal{D}_n^0$, when n is odd. This proves Theorem 3. □

Rank and Dyson's Map

This story started in 1944 with Dyson's paper [D1], which appeared in *Eureka*, a publication of mathematical students in Cambridge. Motivated by Ramanujan's identities for divisibility of the partition function, Dyson introduced the *rank* of a partition, which he conjectured would give a combinatorial interpretation of these identities. Still an undergraduate, Dyson did not prove these conjectures. They were resolved in 1948 by Atkin and Swinnerton-Dyer [At], although their celebrated paper [AS] appeared some years later.

Fortuitously, Dyson meanwhile had moved to the U.S. and published his conjectures as a short problem in the *American Mathematical Monthly* [D2]. Nathan Fine became interested in the problem and devoted three theorems in [F1] to enumeration of partitions with given rank (see below). Taken out of context, his results seemed completely mysterious, and would have remained so if not for publication of the book [F2] and Dyson's paper [D3]. We know now that Fine's results were based on the third-order mock theta function identities due to Watson [W], a technique used in [AS] as well.

Nobody seems to have noticed that Dyson's map can be used to give combinatorial proofs of Fine's results.

Dyson's paper [D3] (see also [D6]) was aimed at finding a simple proof of the formula for a generating function for partitions with given rank. This formula was used as a tool for practical calculations in [D1] and later was established in [AS]. Unaware of Fine's work, Dyson rediscovered one of Fine's then-unpublished equations, called it a "*new symmetry*," and proved it combinatorially. He then deduced the desired formula, and obtained a new proof of Euler's Pentagonal Theorem (see below). I refer to [D5] for Dyson's personal and historical account of these discoveries.

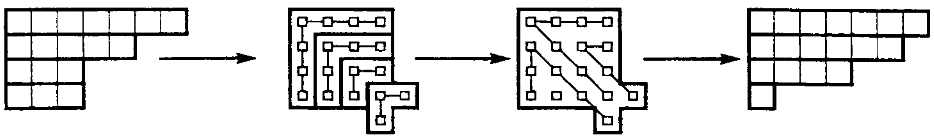

Figure 3. Sylvester's bijection φ: (7,5,3,3) → (7,6,4,1).

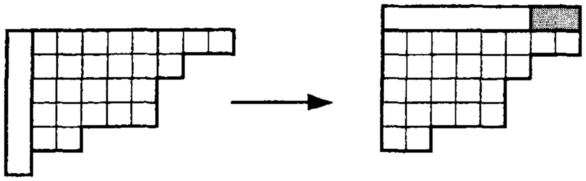

Figure 4. Dyson's map $\psi_r : \lambda \to \mu$, where $\lambda = (9,7,6,6,3,1) \in \mathcal{H}_{32,r+1}$, $\mu = (8,8,6,5,5,2) \in \mathcal{G}_{32+r,r-1}$, and $r = 2$.

Unfortunately, except for Andrews's paper [A5], nobody seems to have noticed that in fact Dyson's map, sometimes called *Dyson's adjoint* [BG], can be used to give combinatorial proofs of Fine's results. Even Andrews did not seem to realize that Dyson's map proves two other theorems of Fine as well. I return to that Andrews paper in the next section.

Define the *rank* of a partition λ as $r(\lambda) = a(\lambda) - \ell(\lambda)$. Denote by $\mathcal{P}_{n,r}$ the set of partitions of n with rank r, and let $p(n, r) = |\mathcal{P}_{n,r}|$. Similarly, denote by $\mathcal{H}_{n,r}$ ($\mathcal{G}_{n,r}$) the set of partitions of n with rank at most r (at least r). Let $h(n, r) = |\mathcal{H}_{n,r}|$, $g(n, r) = |\mathcal{G}_{n,r}|$. Clearly, $p(n, r) = h(n, r) - h(n, r-1)$, and (by comparing λ to λ') $g(n, r) = h(n, -r)$. Also, $h(n, r) + g(n, r+1) = \pi(n)$, where $\pi(n) = h(n, n) = \Sigma_r\, p(n, r)$ is the total number of partitions of n.

THEOREM 5 (Fine) *For all $n > 0$, we have $h(n, 1 + r) = h(n + r, 1 - r)$.*

Proof. I shall construct an explicit bijection $\psi_r : \mathcal{H}_{n,r+1} \to \mathcal{G}_{n+r,r-1}$, which implies the result. Start with the Young diagram $[\lambda]$ corresponding to a partition $\lambda \in \mathcal{H}_{n,r+1}$. Remove the first column with $\ell = \ell(\lambda)$ squares. Add the top row with $(\ell + r)$ squares. Let $[\mu]$ be the resulting Young diagram (see Figure 4.) Call the map $\psi_r : \lambda \to \mu$ *Dyson's map.*

By assumption of λ, we have $r(\lambda) = a(\lambda) - \ell \le r + 1$, so $a(\mu) = \ell + r \ge a(\lambda) - 1$. Thus μ is a partition indeed. And the same inequality shows that the inverse map is defined. Clearly, $|\mu| = \lambda - \ell + (\ell + r) = n + r$. Also, $r(\mu) = a(\mu) - \ell(\mu) = \ell(\lambda) + r - (\lambda_2' + 1) \ge r - 1$. Therefore, $\mu = \psi_r(\lambda) \in \mathcal{G}_{n+r,r-1}$, which completes the proof. □

I call the result of Theorem 5 the *Fine-Dyson relations.* The rest of the paper is built upon these relations and Dyson's map. First I prove the following four equations, which are listed in [F1] as one theorem as well.

THEOREM 6 (Fine) *We have:*

1) $p(n + 1, 0) + p(n, 0) + 2p(n - 1, 3) = \pi(n + 1) - \pi(n)$, *for* $n > 1$,
2) $p(n - 1, 0) - p(n, 1) + p(n - 2, 3) - p(n - 3, 4) = 0$, *for* $n > 3$,
3) $p(n - 1, 1) - p(n, 0) + p(n - 1, 2) - p(n - 2, 3) = 0$, *for* $n > 2$,
4) $p(n - 1, r) - p(n, r + 1) + p(n - r - 2, r + 3) - p(n - r - 3, r + 4) = 0$, *for* $n > r + 3$.

Proof. Taking $r = 0$ and $r = -1$ in equation 4), and using $p(m, r) = p(m, -r)$, we obtain 2) and 3), respectively. By Theorem 5, and from the formula $\pi(n) = h(n, 0) + g(n, 1) = h(n, 0) + h(n, -1)$, we deduce equation 1):

$$\begin{aligned}
&\pi(n + 1) - \pi(n) \\
&\quad = (h(n + 1, 0) + h(n + 1, -1)) - (h(n, 0) + h(n, -1)) \\
&\quad = (h(n + 1, 0) - h(n + 1, -1)) + (h(n, 0) - h(n, -1)) \\
&\qquad + 2\, h(n + 1, -1) - 2\, h(n, 0) \\
&\quad = p(n + 1, 0) + p(n, 0) + 2(h(n - 1, 3) - h(n - 1, 2)) \\
&\quad = p(n + 1, 0) + p(n, 0) + 2p(n - 1, 3).
\end{aligned}$$

Equation 4) follows in a similar manner:

$$\begin{aligned}
&p(n - r - 3, r + 4) - p(n - r - 2, r + 3) \\
&\quad = (h(n - r - 3, r + 4) - h(n - r - 3, r + 3)) \\
&\qquad - (h(n - r - 2, r + 3) - h(n - r - 2, r + 2)) \\
&\quad = h(n, -r - 2) - h(n - 1, -r - 1) - h(n, -r - 1) \\
&\qquad + h(n - 1, -r) \\
&\quad = -(h(n, -r - 1) - h(n, -r - 2)) \\
&\qquad + (h(n - 1, -r) - h(n - 1, -r - 1)) \\
&\quad = -p(n, -r - 1) + p(n - 1, -r) \\
&\quad = -p(n, r + 1) + p(n - 1, r).
\end{aligned}$$

This completes the proof. □

Since the equations in Theorem 6 follow immediately from the Fine-Dyson relations, one can obtain combinatorial proofs for these equations as well, by separating terms with positive and negative signs and then using Dyson's map to obtain identical sets of partitions on both sides. I present a variation on such a proof in case of another theorem from [F1].

THEOREM 7 (Fine) *For $r \ge n - 3$, we have $\pi(n) - \pi(n - 1) = p(n + r + 1, r)$.*

Proof. Denote by $\mathcal{F}_n$ the set of partitions $\lambda \vdash n$ with the smallest part $s(\lambda) \ge 2$. Observe that $|\mathcal{F}_n| = \pi(n) - \pi(n - 1)$. Indeed, one can always add a part (1) to every partition $\nu \vdash n - 1$ to obtain all partitions of n, except for those in $\mathcal{F}_n$.

Now, to a partition $\lambda \in \mathcal{F}_n$ apply Dyson's map $\psi_{r+1} : \lambda \to \mu$, corresponding to the rank $(r + 1)$. We have $\mu_1 = 1 + \ell(\lambda) + r \ge 2 + (n - 3) = n - 1$. On the other hand, $\mu_2 = \lambda_1 - 1 \le n - 1$ by construction. Therefore $\mu_1 \ge \mu_2$, and μ is a partition indeed. Because $s(\lambda) > 1$, we know that $\ell(\mu) = \ell(\lambda) + 1$. Thus $r(\mu) = (\ell(\lambda) + r + 1) - (\ell(\lambda) + 1) = r$, so $\mu \in \mathcal{P}_{n+r+1,r}$. Since the map is clearly reversible, we obtain the result. □

The Iterated Dyson's Map

As mentioned before, Andrews in [A5] proved combinatorially the following theorem from [F1]:

THEOREM 8 (Fine) *Let $\mathcal{D}_{n,r}$ be the set of partitions $\mu \in \mathcal{D}_n$ with rank $r(\mu) = r$. Let $\mathcal{U}_{n,2k+1}$ be the set of partitions $\lambda \in \mathcal{O}_n$, such that the largest part $a(\lambda) = 2k + 1$. Then:*

$$|\mathcal{U}_{n,2r+1}| = |\mathcal{D}_{n,2r+1}| + |\mathcal{D}_{n,2r}|.$$

One can view Theorem 8 as another refinement of Euler's theorem on partitions into odd and distinct parts. Andrews showed in [A5] that the theorem follows easily

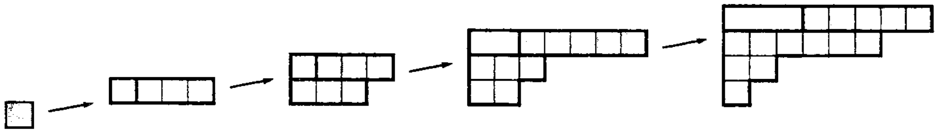

Figure 5. The iterated Dyson's map $\zeta : \lambda \to \mu$, where $\lambda = (5,5,3,3,1) \in \mathcal{U}_{17,5}$ and $\mu = (8,6,2,1) \in \mathcal{D}_{17,4}$.

from the properties of Dyson's map ψ_r. It is unfortunate that Andrews's proof was published in a little-known journal and was never studied further. I will now present a direct bijection between $\mathcal{O}_n$ and $\mathcal{D}_n$, which is different from Sylvester's and Glaisher's bijections [A3], and which proves Theorem 8. Naturally, this construction is motivated by [A5].

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathcal{O}_n$ be a partition into odd parts. Consider a sequence of partitions $\nu^1, \nu^2, \ldots, \nu^\ell$, such that $\nu^\ell = (\lambda_\ell)$, and ν^i is obtained by applying Dyson's map ψ_{λ_i} to ν^{i+1}. Now let $\mu = \nu^1$. Call the resulting map $\zeta : \lambda \to \mu$ the *iterated Dyson's map*. See Figure 5 for an example.

Dyson used his map to obtain a proof of Euler's Pentagonal Theorem.

THEOREM 9 *The iterated Dyson's map ζ defined above is a bijection between $\mathcal{O}_n$ and $\mathcal{D}_n$. Moreover, $\zeta(\mathcal{U}_{n,2r+1}) = \mathcal{D}_{n,2r} \cup \mathcal{D}_{n2r+1}$, for all $r \geq 0$.*

Clearly, Theorem 9 implies Theorem 8. It would be interesting to find further applications of the map ζ to other partition theorems.

Proof. First, note that $|\nu^i| = \lambda_i + \lambda_{i+1} + \cdots + \lambda_\ell$. Therefore $|\mu| = |\nu^1| = |\lambda| = n$, as required. Let us prove by induction that ν^i is a partition into distinct parts, such that $r(\nu^i)$ is either λ_i or $\lambda_i - 1$. The base of induction, when $i = \ell$ and $\nu^i = (\lambda_\ell)$, is obvious.

Suppose the claim holds for ν^{i+1}, i.e., $a(\nu^{i+1}) - \ell(\nu^{i+1})$ is either λ_{i+1} or $\lambda_{i+1} + 1$, depending on the parity. Since $a(\nu^i) = \ell(\nu^{i+1}) + \lambda_i$, we have

$$(\nu^i)_1 = a(\nu^i) \geq (a(\nu^{i+1}) - \lambda_{i+1}) + \lambda_i > a(\nu^{i+1}) - 1 = (\nu^i)_2,$$

and this inequality is maintained ($(\nu^i)_2 - (\nu^i)_3 = (\nu^{i+1})_1 - (\nu^{i+1})_2 > 0$, and so on); this implies that ν^i is indeed a partition into distinct parts. Now, observe that $\ell(\nu^i) = \ell(\nu^{i+1})$ or $\ell(\nu^{i+1}) - 1$. We have

$$r(\nu^i) = a(\nu^i) - \ell(\nu^i) = (\ell(\nu^{i+1}) + \lambda_i) - \ell(\nu^i) \in \{\lambda_i, \lambda_i - 1\},$$

which proves the induction step.

Note that we never used the fact that $\lambda \in \mathcal{O}_n$. This becomes important in the construction of the inverse map ξ^{-1}. Define the map ξ^{-1} by induction, starting with $\mu = \nu^1$ and applying the inverses of Dyson's maps ψ_r^{-1}. Clearly, the only freedom in the construction comes from the choice of r. But we need to have $r = a(\nu^i) - \ell(\nu^i)$ or $r = a(\nu^i) - \ell(\nu^i) - 1$, and r has to be odd; this makes the choice of r unique. Therefore the map ξ^{-1} is well defined, and ξ is a bijection. The second part of the theorem is immediate from the arguments above. This completes the proof. □

Dyson's Proof of Euler's Pentagonal Theorem

I already mentioned that Dyson used his map to obtain a simple proof of Euler's Pentagonal Theorem, Theorem 2 above. He writes, "This combinatorial derivation of Euler's formula is less direct but perhaps more illuminating, than the well-known combinatorial proof of Franklin." [D3] Twenty years later he adds, "This derivation is the only one I know that explains why the 3 appears in Euler's formula." [D5]

Here is how Dyson's proof goes. Let $P(t)$ and $G_r(t)$ be the generating functions for all partitions of n, and all partitions of n with rank $\geq r$:

$$G_r(t) = \sum_{n=1}^{\infty} g(n, r)t^n,$$

$$P(t) = 1 + \sum_{n=1}^{\infty} \pi(n)t^n = \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)}.$$

Write the relations $h(n, r) + g(n, r + 1) = \pi(n)$ and the Fine-Dyson relations $h(n, 1 + r) = h(n + r, 1 - r)$ in terms of $g(\cdot)$ alone:

$$g(n, r) + g(n, 1 - r) = \pi(n), \quad g(n, r) = g(n - r - 1, -2 - r).$$

In the language of generating functions, these relations imply the following two equations:

$$1 + G_r(t) + G_{1-r}(t) = P(t),$$
$$G_r(t) = t^{r+1}(1 + G_{-2-r}(t)).$$

Here 1 in both equations comes from taking into account the "empty" partition. Thus we have

$$G_r(t) = t^{r+1}\, P(t) - t^{r+1}\, G_{r+3}(t).$$

Iterating the above equation, we obtain:

$$\begin{aligned} G_r(t) &= t^{r+1}\, P(t) - t^{r+1}\, G_{r+3}(t) \\ &= t^{r+1}\, P(t) - t^{2r+5}\, P(t) + t^{2r+5}\, G_{r+6}(t) \\ &= t^{r+1}\, P(t) - t^{2r+5}\, P(t) + t^{3r+12}\, P(t) - t^{3r+12}\, G_{r+9}(t) \\ &\ \ \ldots \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} t^{\frac{m(3m-1)}{2} + rm}\, P(t). \end{aligned}$$

Substituting this into $P(t) - G_0(t) - G_1(t) = 1$, we deduce Euler's Pentagonal Theorem:

$$\prod_{i=1}^{\infty} \frac{1}{(1 - t^i)} \left(1 + \sum_{m=1}^{\infty} (-1)^m t^{\frac{m(3m-1)}{2}} + \sum_{m=1}^{\infty} (-1)^m t^{\frac{m(3m-1)}{2}}\right) = 1.$$

Dividing both sides by the product $P(t)$ and equating the coefficients gives us Theorem 2.

In fact, Euler [E] was interested in the recurrence relation for the number of partition $\pi(n)$. The above formula implies

$$\pi(n) = \pi(n-1) + \pi(n-2) - \pi(n-5) - \pi(n-7) + \pi(n-12) + \pi(n-15) - \dots$$

By analogy, Dyson [D5] obtained the following refinement of Euler's recurrence:

$$g(n, r) = \pi(n-r-1) - \pi(n-2r-5) + \pi(n-3r-12) - \dots$$

Naturally, one is tempted to convert the above simple analytic proof into a bijective proof of both recurrences. This turns out to be possible. Denote by $\mathcal{P}_n$ the set of all partitions of n. Write Dyson's recurrence as follows:

$$\begin{aligned} |\mathcal{H}_{n,-r}| &= |\mathcal{P}_{n-r-1}| - |\mathcal{P}_{n-2r-5}| + |\mathcal{P}_{n-3r-12}| - \dots \\ &= |\mathcal{G}_{n-r-1,-r-2} \cup \mathcal{H}_{n-r-1,-r-3}| \\ &\quad - |\mathcal{G}_{n-2r-5,-r-5} \cup \mathcal{H}_{n-2r-5,-r-6}| \\ &\quad + |\mathcal{G}_{n-3r-12,-r-8} \cup \mathcal{H}_{n-3r-12,-r-9}| - \dots \\ &= |\mathcal{G}_{n-r-1,-r-2}| + (|\mathcal{H}_{n-r-1,-r-3}| - |\mathcal{G}_{n-2r-5,-r-5}|) \\ &\quad - (|\mathcal{H}_{n-2r-5,-r-6}| - |\mathcal{G}_{n-3r-12,-r-8}|) + \dots \end{aligned}$$

Now Dyson's map ψ_{-r-1} gives a bijection between the left hand side and the first term on the right hand side of the equation. Similarly, maps ψ_{-r-4}, ψ_{-r-7}, etc., give bijections for the terms in the brackets. Thus we have a simple *bijective proof* of Dyson's recurrence. One can view the above bijection as a sign-reversing involution on the set of partitions $\lambda \in \mathcal{H}_{n,-r}$, or $\lambda \vdash n - rm - m(3m-1)/2$, where $m \geq 1$.

Similarly, after combining two involutions for $r = 0$ and 1, we easily obtain an involution γ proving Euler's recurrence:

$$\gamma : \bigcup_{m \text{ even}} \mathcal{P}_{n-m(3m-1)/2} \to \bigcup_{m \text{ odd}} \mathcal{P}_{n-m(3m-1)/2},$$

where m on both sides is allowed to take negative integer values, and the map γ (see Figure 6) is defined by the following rule:

for $\lambda \in \mathcal{P}_{n-m(3m-1)/2}$,

$$\gamma(\lambda) = \begin{cases} \psi_{-3m-1}(\lambda), & \text{if } r(\lambda) + 3m \leq 0, \\ \psi^{-1}_{-3m+2}(\lambda), & \text{if } r(\lambda) + 3m > 0. \end{cases}$$

Now comes a final surprise. Bijection γ is in fact well known! In this exact form it was discovered in 1985 by Bressoud and Zeilberger [BZ], for the sole purpose of finding a simple proof of Euler's recurrence. The authors, completely unaware of Dyson's proof, managed to rediscover a version of Dyson's map anyway. It seems, the Fine-Dyson relations and Dyson's map ψ_r are simply so fundamental they resurface despite the "missed opportunities" . . .

Final Remarks

There remains one last partition theorem of Fine [F1] without a simple combinatorial proof. Let $L(n)$ be the number of partitions $\lambda \vdash n$ with odd smallest part $s(\lambda)$. The theorem states that $L(n)$ is odd if and only if n is a square. Shouldn't one look for an involution proving this result? Any interested reader may draw inspiration from an involutive proof of the Rogers-Fine identity [A2].

The conditions in Fine's theorems 6 and 7 are slightly changed in this paper, either to correct or simplify the results (so as not to define $p(n, r)$ for $n \leq 0$). Dyson's map as defined here is the conjugate of the one in the literature. I find this version somewhat easier to work with.

The iterated Dyson's map ξ appears to be new. It is basically a recursive application of Andrews's recurrence relation for $|\mathcal{U}_{n,2k+1}|$ and $|\mathcal{D}_{n,r}|$ (see [A5]). Whether this bijection between partitions into odd and distinct numbers has other nice applications or not, the map ζ seems to give a natural proof of Theorem 8, just as Dyson's map gives a natural proof of Theorem 5. Unfortunately, the iterative construction of ξ is perhaps intrinsic. As Xavier Viennot once tole me, "Sometimes, a recursive bijection is the only one possible and one cannot do better."

Note that Dyson's proof of Euler's recurrence relation [D3] produces a bijection almost immediately once one employs Dyson's map. A different bijection, based on Franklin's involution, was obtained by means of the involution principle by Garsia and Milne [GM]. These two "automatic" approaches challenge Sylvester's paradigm that bi-

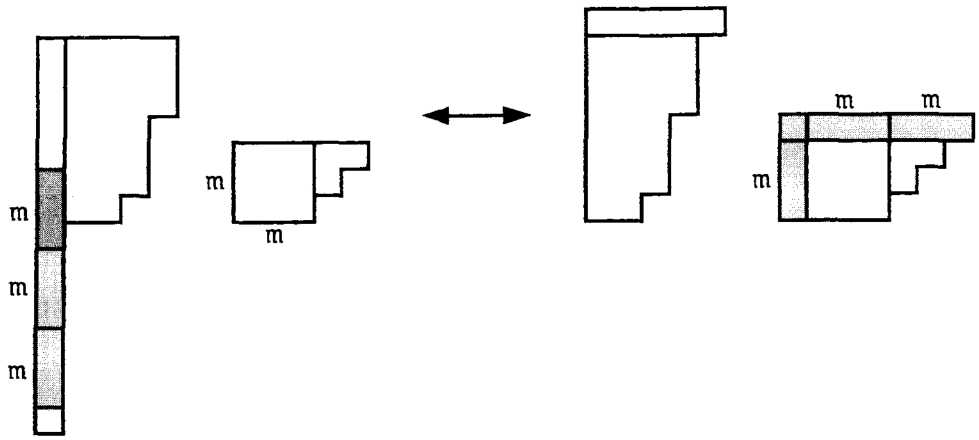

Figure 6. Bijection γ proving Euler's Pentagonal Theorem.

jections "should rather be regarded as something put into the two systems by the human intelligence than an absolute property inherent in the relation between the two [sets]" [S2].

In a recent paper [BG], Berkovich and Garvan defined a 2-modular version of Dyson's map. They used this new map to give a combinatorial proof of Gauss's famous identity. It would be interesting to convert this proof into a fully bijective proof of the identity and compare with Andrew's involutive proof [A2]. Similarly, one can consider an iterated version of this map and try to find new partition theorems this construction may prove.

This paper was motivated in part by the following quote: "[A5] *seems to be the only known application of Dyson's transformation*" [BG]. Let me add that had the preprint [BG] never been put on the internet, this paper might have never been written. That would have been another "missed opportunity" . . .

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