ASYMPTOTICS FOR THE NUMBER OF STANDARD TABLEAUX OF SKEW SHAPE AND FOR WEIGHTED LOZENGE TILINGS

ALEJANDRO H. MORALES, IGOR PAK, AND MARTIN TASSY

Abstract. We prove and generalize a conjecture in [MPP4] about the asymptotics of $\frac{1}{\sqrt{n!}} f^{\lambda/\mu}$, where $f^{\lambda/\mu}$ is the number of standard Young tableaux of skew shape λ/μ which have stable limit shape under the $1/\sqrt{n}$ scaling. The proof is based on the variational principle on the partition function of certain weighted lozenge tilings.

1. Introduction

In enumerative and algebraic combinatorics, *Young tableaux* are fundamental objects that have been studied for over a century with a remarkable variety of both results and applications to other fields. The asymptotic study of the number of standard Young tableaux is an interesting area in it own right, motivated by both probabilistic combinatorics (*longest increasing subsequences*) and representation theory. This paper is a surprising new advance in this direction, representing a progress which until recently could not be obtained by existing tools.

1.1. **Main results.** Let us begin by telling the story behind this paper. Denote by $f^{\lambda/\mu} = \mathrm{SYT}(\lambda/\mu)$ the number of standard Young tableaux of skew shape λ/μ. There is *Feit's determinant formula* for $f^{\lambda/\mu}$, which can also be derived from the Jacobi–Trudy identity for skew shapes. In some cases there are multiplicative formulas for $f^{\lambda/\mu}$, e.g. the *hook-length formula* (HLF) when $\mu = \varnothing$, see also [MPP3]. However, in general it is difficult to use Feit's formula to obtain even the first order of asymptotics, since there is no easy way to diagonalize the corresponding matrices.

It was shown in [Pak2] by elementary means, that when $|\lambda/\mu| = N$ and $\lambda_1, \ell(\lambda) \leq s\sqrt{N}$, we have:

$$c_1^N \leq \frac{\left(f^{\lambda/\mu}\right)^2}{N!} \leq c_2^N,$$

where $c_1, c_2 > 0$ are universal constants which depend only on s. Improving upon these estimates is of interest in both combinatorics and applications (cf. [MPP3, MPP4]).

In [MPP4], much sharper bounds on c_1, c_2 were given, when the diagrams λ and μ have a limit shape ψ/ϕ under $1/\sqrt{N}$ scaling in both directions (see below). Based on observations in special cases, we conjectured that there is always a limit

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{\left(f^{\lambda/\mu}\right)^2}{N!}$$

in this setting. The main result of this paper is a proof of this conjecture.

Date: April 30, 2018.

Theorem 1.1. *Let* $\{\lambda^{(N)}\}$ *and* $\{\mu^{(N)}\}$ *be two partition sequences with strong stable (limit) shapes* ψ *and* ϕ, *respectively* (*see* §*2.4 for precise definitions*). *Let* $\nu^{(N)} := \lambda^{(N)}/\mu^{(N)}$, *such that* $|\nu^{(N)}| = N + o(N/\log N)$. *Then*

$$\frac{1}{N}\left(\log f^{\nu^{(N)}} - \frac{1}{2} N \log N\right) \longrightarrow c(\psi/\phi) \quad \textit{as} \quad N \to \infty,$$

for some fixed constant $c(\psi/\phi)$.

The constant $c(\psi/\phi)$ is given in Corollary 4.6. The proof of the theorem is even more interesting perhaps than one would expect. In [Nar], Naruse developed a novel approach to counting $f^{\lambda/\mu}$, via what is now known as the *Naruse hook-length formula* (NHLF):

$$f^{\lambda/\mu} = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in \lambda \setminus D} \frac{1}{\mathsf{h}_\lambda(u)}, \tag{1.1}$$

where $\mathcal{E}(\lambda/\mu) \subseteq \binom{[\lambda]}{|\mu|}$ is a collection of certain subsets of the Young diagram $[\lambda]$, and $\mathsf{h}_\lambda(u)$ is the hook-length at $u \in \lambda$. The (usual) hook-length formula is a special case $\mu = \varnothing$. Let us mention that $\mathcal{E}(\lambda/\mu)$ can be viewed as the set of certain particle configurations, giving it additional structure [MPP3].

Although $\mathcal{E}(\lambda/\mu)$ can have exponential size, the NHLF can be useful in getting the asymptotic bounds [MPP4]. It has been reproved and studied further in [MPP1, MPP2, Kon, NO], including the q-analogues and generalizations to trees and shifted shapes. See §2.3 for the precise statements.

The next logical step was made in [MPP3], where a bijection between $\mathcal{E}(\lambda/\mu)$ and lozenge tilings of a certain region was constructed. Thus, the number of standard Young tableaux $f^{\lambda/\mu}$ can be viewed as a statistical sum of weighted lozenge tilings. In a special case of *thick hooks* this connection is especially interesting, as the corresponding weighted lozenge tilings were previously studied in [BGR] (see the example below).

Now, there is a large literature on random lozenge tilings of the hexagon and its relatives in connection with the *arctic circle* phenomenon, see [CEP, CKP, Ken]. In this paper we adapt the *variational principle* approach in these papers to obtain the arctic circle behavior for the weighted tilings as well. Putting all these pieces together implies Theorem 1.1.

Let us emphasize that the approach in this paper can be used to obtain certain probabilistic information on random SYTs of large shapes, e.g. in [MPP3, §8] we show how to compute asymptotics of various path probabilities. However, in the absence of a direct bijective proof of NHLF, our approach cannot be easily adapted to obtain limit shapes of SYTs as Sun has done recently [Sun] (see also §6.5).

1.2. **Thick hooks.** Let $\lambda = (a+c)^{b+c}$, $\mu = a^b$, $N = |\lambda/\mu| = c(a+b+c)$, where $a, b, c \geq 0$. This shape is called the *thick hook* in [MPP4]. The HLF applied to the 180 degree rotation of λ/μ gives:

$$f^{\lambda/\mu} = N! \, \frac{\Phi(a)\,\Phi(b)\,\Phi(c)^2\,\Phi(a+b+c)^2}{\Phi(a+b)\,\Phi(a+c)\,\Phi(b+c)\,\Phi(a+b+2c)}\,.$$

Here the *superfactorial* $\Phi(n) = 1!\cdot 2! \cdots (n-1)!$ is the integer value of the *Barnes G-function*, see e.g. [AsR].

On the other hand, $\mathcal{E}(\lambda/\mu)$ in this case in bijection with the set of lozenge tilings of the hexagon $\mathbb{H}(a,b,c) = \langle a \times b \times c \times a \times b \times c \rangle$, and the weight is simply a product of a linear function on horizontal lozenges (see below). The number of lozenge tilings in this cases is famously counted by the *MacMahon box formula* for the number $\mathsf{P}(a,b,c)$ of *solid partitions* which fit into a $[a \times b \times c]$ box:

$$\big|\mathcal{E}(\lambda/\mu)\big| \,=\, \mathsf{P}(a,b,c) \,=\, \frac{\Phi(a)\,\Phi(b)\,\Phi(c)\,\Phi(a+b+c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)}\,,$$

see e.g. [Sta2, §7.21]. It was noticed by Rains (see [MPP3, §9.5]), that in this example our weights are special cases of multiparameter weights studied in [BGR] in connection with closed formulas for *q-Racah polynomials*, cf. §6.2.

Now, Theorem 1.1 in this case does not give anything new, of course, as existence of the limit when $c \to \infty$, $a/c \to \alpha$ and $b/c \to \beta$, follows from either the Vershik–Kerov–Logan–Shepp *hook integral* of the strongly stable shapes [MPP4, §6.2] (see also [Rom]), or from the asymptotics of the superfactorial:

$$\log \Phi(n) \,=\, \frac{1}{2}\,n^2 \log n \,-\, \frac{3}{4}\,n^2 \,+\, 2n \log n \,+\, O(n).$$

This gives the exact value $c(\psi/\phi)$ as an elementary function of (α, β).

1.3. **Thick ribbons.** Let $\nu_k := \delta_{2k}/\delta_k$, where $\delta_k = (k-1, k-2, \ldots, 2, 1)$. This skew shape is a strong stable shape. The main theorem implies that there is a limit

$$\frac{1}{N}\left(\log f^{\nu_k} \,-\, \frac{1}{2}\,N \log N\right) \,\to\, C \quad \text{as } \ k \to \infty,$$

where $N = |\nu_k| = k(3k-1)/2$. This proves a conjecture in [MPP4, §13.7]. In that paper it was shown that $-0.3237 \le C \le -0.0621$. Both lower and upper bounds are further improved in [MPP5], but the exact value of C has no known closed formula. This paper describes C as solution of a certain very involved variational problem (see Corollary 4.6 and §6.5).

1.4. **Structure of the paper.** We start with Section 2 which reviews the notation and known results on tilings, standard Young tableaux and limit shapes. In Section 3 we state our main technical result (Theorem 3.3) on the variational principle for weighted lozenge tilings, whose proof is postponed until Section 5. In the technical Section 4 we deduce Theorem 1.1 from the variational principle. We conclude with final remarks and open problems in Section 6.

2. Background and notation

2.1. **Tilings and height functions.** Let R be a connected region in the triangular lattice. One can view a lozenge tiling of R as a stepped surface in $\mathbb{R}^3$ where the first two coordinates are the coordinates of the points in the lattice and the third coordinates is the height function $h(\cdot)$ of a lozenge tiling defined in the following way:

- For every edge (x,y) in R, $h(y)-h(x) = 1$ if (x,y) is a vertical edge and $h(y)-h(x) = 0$ otherwise.

In fact, there is a one to one correspondence between tilings of a given region and functions which verify this property defined up to a constant. Using this bijection, we will denote by $\mathtt{t}_h$ the tiling associated to a given height function h and we will do all the subsequent reasoning using height functions rather than tilings.

We extend the definition of height functions to any region of the lattice as follows: for general sets S, we say that a function $h : S \to \mathrm{Z}$ is a height function if its restriction on each simply connected component of S is a height function.

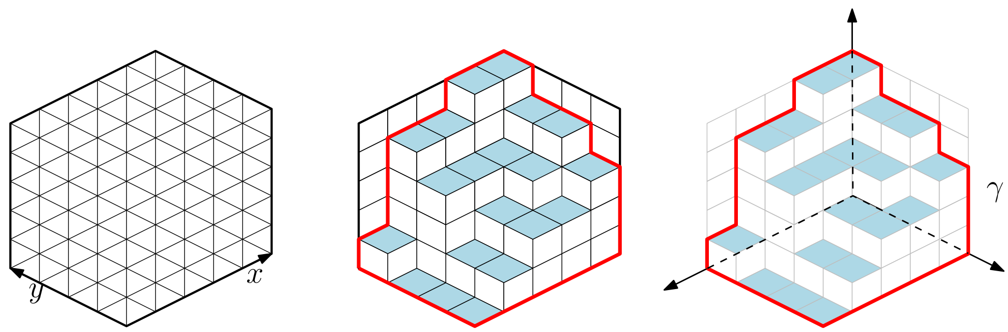


FIGURE 1. A region R of the triangular lattice. A lozenge tiling of that region and the associated admissible stepped curve (ASC).

FIGURE 2. Left: height function of the maximal tiling centered at x with height $g(x)$. Right: the local move on lozenges.

Let R be a lozenge tileable region. We say that the three dimensional curve obtained by traveling along ∂R and recording the height of each point is an admissible stepped curve (ASC).

Lemma 2.1. *Let R be a connected region in the triangular grid and let g be a height function on a subset S of R, such that for all $x = (x_1, x_2), y = (y_1, y_2) \in S$:*

$$g(y) - g(x) \leq \max\{y_1 - x_1, y_2 - x_2\}. \tag{2.1}$$

Then g can be extended into a height function on the whole region R.

The lemma is a variation on [PST, Thm. 4.1] (see also [Thu]). It can be viewed as a Lipschitz extendability property on height functions (cf. [CPT]). We include a quick proof for completeness.

Proof. Note that $h_x(y) = g(x) + \max\{y_1 - x_1, y_2 - x_2\}$ is the height function of the maximal tiling centered at x and with height $g(x)$ at x (see Figure 2). Define $h(y) := \min_{x\in S} h_x(y)$. Since the minimum of two height functions is still a height function, we conclude that h is itself a height function. Moreover, the inequality (2.1)) implies that for all pairs $x, y \in S$: $g(y) \leq h_x(y)$. We conclude that $h(y) = g(y)$, which implies the result. □

Finally, we need the following standard proposition which will be useful later in this article.

Proposition 2.2 (see [Thu])**.** *Every two lozenge tilings of a simply connected region R have equal number of lozenges of each type.*

In other words, the number of lozenges of each type depends only on R and not on the tiling. This follows, e.g. since every two tilings of R are connected by local moves which do not change the number of lozenges of each type (see Figure 2).

2.2. **Skew shapes and tableaux.** Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ denote integer partitions of length $\ell(\lambda) = r$ and $\ell(\mu) = s$. The size of the partition is denoted by $|\lambda|$. We denote by λ' the *conjugate partition*, and by $[\lambda]$ the corresponding *Young diagram* (in English notation). The *hook length* $\mathsf{h}_\lambda(x, y)$ of a cell $(x, y) \in \lambda$ is defined as $\mathsf{h}_\lambda(x, y) := \lambda_x - x + \lambda_y - y + 1$. It counts the number of cells directly to the right and directly below (x, y) in $[\lambda]$.

A *skew shape* λ/μ is defined as the difference of two shapes. Let $N = |\lambda/\mu|$. We always assume that the skew shape is connected. A *standard Young tableau* (SYT) of shape λ/μ is a bijective function $T : [\lambda/\mu] \to \{1, \ldots, N\}$, increasing in rows and columns. The number of such tableaux is denoted by $f^{\lambda/\mu}$. This counts the number of linear extensions of the poset defined on $[\lambda/\mu]$, with cells increasing downward and to the right.

2.3. **Naruse's hook-length formula.** As mentioned in the introduction, the Naruse hook-length formula (1.1) gives a positive formula for $f^{\lambda/\mu}$. It was restated in [MPP3] in terms of lozenge tilings as follows.

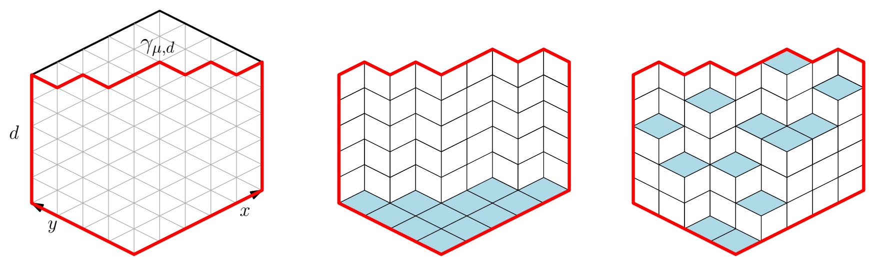


FIGURE 3. ASC and two lozenge tilings corresponding to excited diagrams in Naruse's formula.

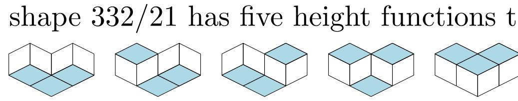
Let λ/μ be a skew shape with N cells. Let $\gamma_{\mu,d}$ be the ASC in the plane with upper side given by μ and bounded below by four sides of the hexagon of vertical height $d = \ell(\lambda) - \ell(\mu)$ (see Figure 3). Let $H_{\lambda/\mu}$ be the set of height functions h that extend $\gamma_{\mu,d}$ such that the corresponding lozenge tiling t_h has no horizontal lozenges with coordinates $(x, x-k)$ for $x - k > \lambda_x$. The weight of a horizontal lozenge of t_h at position (x, y) is the hook length $\mathsf{h}_\lambda(x, y) := \lambda_x - x + \lambda'_y - y + 1$. The weight of a tiling t_h is the product of the weights of its horizontal lozenges and we denote it by $\mathrm{hooks}_\lambda(\mathsf{t}_h)$,

$$\mathrm{hooks}_\lambda(\mathsf{t}_h) := \prod_{\lozenge \in \mathsf{t}_h} \mathsf{h}_\lambda(x_\lozenge, y_\lozenge).$$

Theorem 2.3 (Naruse [Nar]; lozenge tiling version [MPP3, §7])**.**

$$f^{\lambda/\mu} = \frac{N!}{\prod_{(x,y)\in\lambda} \mathsf{h}_\lambda(x, y)} \sum_{h \in H_{\lambda/\mu}} \mathrm{hooks}_\lambda(\mathsf{t}_h). \tag{2.2}$$

Example 2.4. The skew shape 332/21 has five height functions that extend $\gamma_{21,1}$:



Formula (2.2) yields in this case

$$f^{332/21} = \frac{5!}{5 \cdot 4^2 \cdot 3 \cdot 2^2} \big(5 \cdot 4 \cdot 4 + 5 \cdot 4 \cdot 1 + 5 \cdot 4 \cdot 1 + 5 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 1\big) = 16.$$

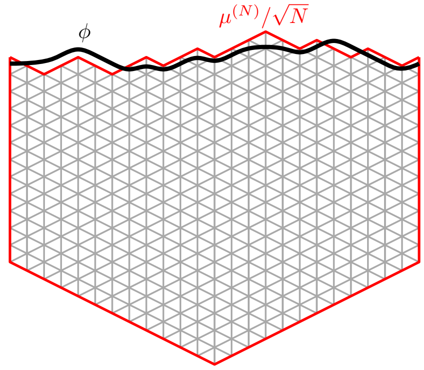


FIGURE 4. The sequence of shapes $\mu^{(N)}$ has a strongly stable shape ϕ with $|\mu^{(N)}| = \text{area}(\phi)\, N + o(N/\log N)$.

2.4. **Stable shapes.** Let $\psi : [0,a] \to [0,b]$ be a non-increasing continuous function. Assume a sequence of partitions $\{\lambda^{(N)}\}$ satisfies the following property

$$(\sqrt{N} - L)\psi < [\lambda^{(N)}] < (\sqrt{N} + L)\psi, \text{ for some } L > 0,$$

where $[\lambda]$ denotes the function giving the boundary of the Young diagram of λ. In this setting, we say that $\{\lambda^{(N)}\}$ has a strong stable shape ψ and denote it by $\lambda^{(N)} \to \psi$. Note that $\ell(\lambda^{(N)})$, $\ell(\lambda^{(N)\prime}) = O(\sqrt{N})$. Such shapes are called *balanced* (see e.g. [FeS]).

Let $\psi, \phi : [0,a] \to [0,b]$ be non-increasing functions, and suppose that $area(\psi/\phi) = 1$. Let $\{v_N = \lambda^{(N)}/\mu^{(N)}\}$ be a sequence of skew shapes with the strongly stable shape ψ/ϕ, i.e. $\lambda^{(N)} \to \psi$, $\mu^{(N)} \to \phi$ and satisfy the condition

$$|\mu^{(N)}| \,=\, \text{area}(\phi)\, N \,+\, o(N/\log N)\,. \tag{2.3}$$

Denote by $\mathcal{C} = \mathcal{C}(\psi/\phi) \subset \mathbb{R}_+^2$ the region between the curves. One can view $\mathcal{C}$ as the stable shape of the skew diagrams.

Finally, define the *hook function* $\hbar : \mathcal{C} \to \mathbb{R}_+$ to be the limit of the scaled function of the hooks:

$$\hbar(x,y) \,:=\, \lim_{N\to\infty} \frac{1}{\sqrt{N}}\, \mathsf{h}_{\lambda^{(N)}}\big(\lfloor x\sqrt{N}\rfloor, \lfloor y\sqrt{N}\rfloor\big)\,. \tag{2.4}$$

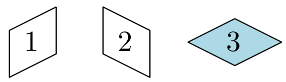
3. Variational principle for weighted lozenge tilings

Lozenge tilings is a dimer model and the existence of a variational principle which governs the limiting behavior of dimers under the uniform measure is a well known result. Our goal in this section will be to extend it to the case where we add weights to each tilings that depend on the position and the type of the lozenge tiles.

3.1. **Weighted tilings and smooth weights.** Let $D \subset \mathbb{R}^2$ be a connected domain in the plane, and let $\{w^{(i)} : D \to \mathbb{R}\}_{i\le 3}$ be three real valued functions corresponding to the weight of each type of lozenge. For a region $R \subset D$, define the *weight* of a height function h on R associated to the weight functions $w = (w^{(1)}, w^{(2)}, w^{(3)})$ as

$$\text{wt}(h) \,:=\, \prod_{\lozenge\in \mathsf{t}_h} \exp(w^{(i_\lozenge)}(x_\lozenge, y_\lozenge)), \tag{3.1}$$

where $(x_\Diamond, y_\Diamond)$ are the coordinates of the center of the tile $\Diamond$ and $i_\Diamond \in \{1,2,3\}$ is the type of the lozenge tile:



Given a weight function w, the partition function associated to an ASC γ is defined as:

$$Z(\gamma, w) := \sum_{h \in H_\gamma} \mathrm{wt}(h),$$

where H_γ is the set of height functions which extend γ. Let N_γ be the size of H_γ and let $\mathcal{L}^{(i)}(\gamma)$ be the (common) number of type i lozenges in each height function that extends γ.

Definition 3.1. *Let D be a domain in $\mathbb{R}^2$. A sequence of weight functions $\{w_n\}_{n\in\mathbb{N}}$ converges to a piecewise smooth function $\rho : D \to \mathbb{R}^3$ if it has the following property:*

$$(*) \qquad \lim_{n\to\infty} \sup_{(x_1,x_2)\in D} \|w_n(nx_1, nx_2) - \rho(x_1,x_2)\|_\infty = 0.$$

3.2. **The variational principle.** Our goal in this section is to establish a variational principle for weighted tilings. We recall the unweighted version of the variational principle from [Ken, Thm. 9]. Let $\mathrm{Lip}_{[0,1]}$ be the set of 1-Lipschitz functions $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfy

$$0 \le \partial_{x_1} f, \quad \partial_{x_2} f, \quad 1 - \partial_{x_1} f - \partial_{x_2} f \le 1$$

everywhere except on a set of Lebesgue measure 0. Let

$$\sigma(s,t) := \frac{1}{\pi}\Big(\Lambda(\pi s) + \Lambda(\pi t) + \Lambda\big(\pi(1-s-t)\big)\Big), \tag{3.2}$$

where $\Lambda(\cdot)$ is the *Lobachevsky function*, see e.g. [TM].

Theorem 3.2 ([Ken])**.** *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of ASC. Suppose that $\frac{1}{n}\gamma_n$ converges to a closed curve γ in $\mathbb{R}^3$ in the ℓ_∞ norm as $n \to \infty$. Then:*

$$\lim_{n\to\infty} \frac{1}{n^2} \log N_{\gamma_n} \to \Phi(g_{\max}),$$

where $g_{\max} : U \to R$ is the only extension of γ in $\mathrm{Lip}_{[0,1]}$ that maximizes the following integral:

$$\Phi(g) := \iint_U \sigma\big(\nabla g(x_1,x_2)\big)\, dx_1\, dx_2,$$

and U is the region enclosed by the projection of γ. Moreover, for all $\epsilon > 0$ the height function of a random tiling chosen from the weighted measure associated to w_n on height functions with boundary γ_n, stays within ϵ of $g_{\max}$ with probability $\to 1$ as $n \to \infty$.

The proof of this result is sketched in [Ken] and is the analogue of an earlier result for dominoes [CKP]. The argument in the latter paper extends to our setting of lozenges.

We are now ready to state the variational principle for the weighted case. The proof is postponed to Section 5.

Theorem 3.3 (Weighted variational principle)**.** *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of ASC, and let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of weight functions converging to a function ρ. Suppose that $\frac{1}{n}\gamma_n$ converges to a closed curved γ in $\mathbb{R}^3$ in the ℓ_∞ norm as $n \to \infty$. Then we have:*

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z(H_{\gamma_n}, w_n) = \Psi(f_{\max}).$$

Here $f_{\max} : U \to \mathbb{R}$ is the only extension of γ in $\mathrm{Lip}_{[0,1]}$ which maximizes the following integral:

$$\Psi(f) := \iint_U \Big(\sigma(\nabla f) + L(x_1, x_2, \nabla f)\Big)\, dx_1\, dx_2, \tag{3.3}$$

where U is the region enclosed by the projection of γ, and

$$L(x_1, x_2, \nabla f) := \rho(x_1, x_2) \cdot (\partial_{x_1} f, \partial_{x_2} f, 1 - \partial_{x_1} f - \partial_{x_2} f). \tag{3.4}$$

Moreover, for all $\epsilon > 0$, the height function of a random tiling chosen from the weighted measure associated to w_n on height functions with boundary γ_n, stays within ϵ of $f_{\max}$ with probability tending to 1.

4. From lozenge tilings to standard Young tableaux

In this section we apply the weighted variational principle to prove the main result on asymptotics of the number of skew SYT of skew shapes with strongly stable shapes.

Recall that $\{\nu_N = \lambda^{(N)}/\mu^{(N)}\}$ is a sequence of skew shapes with the strongly stable shape ψ/ϕ as defined in Section 2.4.

4.1. **The weight function of hook lengths.** In order to apply the weighted variational principle we need weight functions that converge in the sense of Definition 3.1. In order to obtain a partition function that matches Naruse's formula (2.2), the natural choice of weight function on $\mathcal{C}(\psi/\phi)$ is the following

$$w_N(x, y) := \big(0, 0, \log(\mathsf{h}_{\lambda^{(N)}}(x, y)/\sqrt{N})\big).$$

Denote by $\mathrm{wt}_N(h)$ the corresponding weight on height functions. Then

$$\mathrm{wt}(h) = (\sqrt{N})^{-|\mu^{(N)}|} \cdot \mathrm{hooks}_{\lambda^{(N)}}(\mathsf{t}_h).$$

However for this choice of weight function, $\log h_{\lambda^{(N)}}(x, y)$ can be very small for points (x, y) near the border of the shape $\lambda^{(N)}$; see Figure 5. In this regime, Property $(*)$ might not hold. To fix this, we change the weight function to cap these small values as follows. For $\epsilon > 0$ and (x, y) in $\mathcal{C}(\psi/\phi)$, let

$$w_N^{\epsilon}(x, y) := \Big(0,\ 0,\ \max\big\{\log\big(\mathsf{h}_{\lambda^{(N)}}(x, y)/\sqrt{N}\big), \log \epsilon\big\}\Big).$$

Denote by $\mathrm{wt}_N^{\epsilon}(h)$ the corresponding weights on a height function h. Similarly, denote by Z_N and Z_N^{ϵ} the corresponding partition functions associated to weights w_N and w_N^{ϵ} respectively.

4.2. **From lozenge tilings to counting tableaux.** We first show that the weighted variational principle, Theorem 3.3, applies to Z_N^{ϵ}. This implies that

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\epsilon} = c(\epsilon),$$

for some constant $c(\epsilon)$ depending on ϵ and the shapes ψ and ϕ (Lemma 4.1). We then show that $\log Z_N^{\epsilon}$ converges to $\log Z_N$ as $\epsilon \to 0$ (Lemma 4.2). Finally, we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = c,$$

for some constant c depending on ψ and ϕ (Corollary 4.4). In Section 4.3, we use this last result to prove Theorem 1.1.

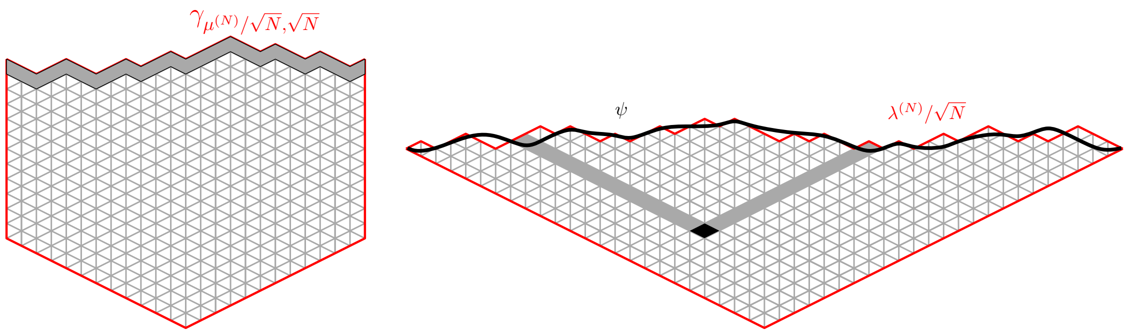


FIGURE 5. Left: For points (x, y) near the top border of the region the values of $\log h_\lambda(x, y)$ are small and can affect convergence of the weight function. Right: The hook measured in $h_{\lambda^{(N)}}(x, y)$.

Lemma 4.1. *We have:*

$$\lim_{N\to\infty} \frac{1}{N} Z_N^\epsilon \;=\; \sup_{f\in \mathrm{Lip}_{[0,1]}} \Psi_\epsilon(f),$$

where $\Psi_\epsilon(\cdot)$ *is the integral defined in* (3.3) *for the limiting weight function*

$$\rho_\epsilon(x, y) \;:=\; \Big(0,\ 0,\ \max\big\{\log \hbar(x, y), \log \epsilon\big\}\Big).$$

Proof. First, we verify that the weight function $w_N^\epsilon(x, y)$ converges to $\rho_\epsilon(x, y)$, in the sense of Definition 3.1. We verify property $(*)$. By convergence of the sequence of shapes, for N large enough, either both $\mathsf{h}_{\lambda^{(N)}}(x, y)/\sqrt{N}$ and $\hbar(x, y)$ defined in (2.4) are smaller than or equal ϵ or both are greater or equal to ϵ. In the first case, we have $w_N^\epsilon(x, y) = \rho_\epsilon(x, y) = (0, 0, \log \epsilon)$, and property $(*)$ vacuously holds.

In the second case we have that for all $(x, y) \in D$:

$$\begin{aligned}\Big|w_N^\epsilon(x\sqrt{N}, y\sqrt{N}) - \rho_\epsilon(x, y)\Big| \;&=\; \left|\log \frac{1}{\sqrt{N}} \mathsf{h}_{\lambda^{(N)}}\big(\lfloor x\sqrt{N}\rfloor, \lfloor y\sqrt{N}\rfloor\big) - \log \hbar(x, y)\right| \\ &\le\; k_\epsilon \left|\frac{1}{\sqrt{N}} \mathsf{h}_{\lambda^{(N)}}\big(\lfloor x\sqrt{N}\rfloor, \lfloor y\sqrt{N}\rfloor\big) - \hbar(x, y)\right|,\end{aligned}$$

where the inequality follows from the k-Lipschitz property of the log, for some constant k_ϵ. From the definition of hook lengths (see Figure 5), we also have:

$$\left|\frac{1}{\sqrt{N}} \mathsf{h}_{\lambda^{(N)}}\big(\lfloor x\sqrt{N}\rfloor, \lfloor y\sqrt{N}\rfloor\big) - \hbar(x, y)\right| \le \sqrt{2}\,\cdot\, \big\|\lambda^{(N)}/\sqrt{N} - \psi\big\|_\infty .$$

Thus, by convergence of the sequence of shapes, we have:

$$\lim_{N\to\infty} \Big|w_N^\epsilon(\sqrt{N}x, \sqrt{N}y) - \rho_\epsilon(x, y)\Big| \;\le\; \lim_{N\to\infty} k_\epsilon\sqrt{2}\cdot \big\|\lambda^{(N)}/\sqrt{N} - \psi\big\|_\infty \;=\; 0.$$

This proves property $(*)$.

By construction of the sequence of partitions $\{\mu^{(N)}\}$, we have that the corresponding sequence $\{\gamma_{\mu^{(N)},\sqrt{N}}\}$ of ASC satisfies that $\frac{1}{\sqrt{N}}\gamma_{\mu^{(N)},\sqrt{N}}$ converges to ϕ. Thus the weighted variational principle, Theorem 3.3, applies giving

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^\epsilon = \Psi_\epsilon(f_{\max}), \tag{4.1}$$

as desired. □

Lemma 4.2. *Let $\epsilon > 0$, there exists a function $F(\epsilon)$ satisfying $\lim_{\epsilon\to 0} F(\epsilon) = 0$ such that $\log Z_N^\epsilon = \log Z_N + F(\epsilon)N$.*

Proof. By the mediant inequality we have:

$$\frac{Z_N^\epsilon}{Z_N} \le \max_h \frac{\mathrm{wt}_N^\epsilon(h)}{\mathrm{wt}_N(h)}\,. \tag{4.2}$$

Outside of a border strip of $\mu^{(N)}$ of height $\lfloor \epsilon\sqrt{N}\rfloor$ the weights will not change. The hooks on the remaining lozenges in the strip are lower bounded by their depth. So the RHS in (4.2) can be bounded as follows,

$$\max_h \frac{\mathrm{wt}_N^\epsilon(h)}{\mathrm{wt}_N(h)} \le \frac{(e^{\log\epsilon})^{\epsilon N}}{\prod_{k=1}^{\lfloor\epsilon\sqrt{N}\rfloor}(e^{\log k/(\epsilon\sqrt{N})})^{\epsilon\sqrt{N}}} = \frac{(e^{\log\epsilon})^{\epsilon N}}{\exp\left(\sum_{k=1}^{\lfloor\epsilon\sqrt{N}\rfloor}\epsilon\sqrt{N}\log k/(\epsilon\sqrt{N})\right)}\,. \tag{4.3}$$

We can rewrite the denominator on the RHS above as

$$\exp\left(\sum_{k=1}^{\lfloor\epsilon\sqrt{N}\rfloor}\epsilon\sqrt{N}\log\frac{k}{\epsilon\sqrt{N}}\right) = \exp\left(\frac{\epsilon N}{\epsilon\sqrt{N}}\sum_{k=1}^{\lfloor\epsilon\sqrt{N}\rfloor}\log\frac{k}{\epsilon\sqrt{N}}\right) = e^{\epsilon N\int_0^\epsilon \log x dx}. \tag{4.4}$$

Finally, we denote $\epsilon\int_0^\epsilon \log x\,dx$ by the function $F(\epsilon)$. This function satisfies $\lim_{\epsilon\to 0}F(\epsilon) = 0$. Combining the bounds (4.2) and (4.3) with the simplification (4.4) gives

$$\frac{Z_N^\epsilon}{Z_N} \le e^{F(\epsilon)N},$$

where $F(\epsilon)$ satisfies the desired properties. ☐

Lemma 4.3. *Let $\Psi(\cdot)$ be the integral defined in (3.3) for the weight function*

$$\rho(x,y) := (0,\,0\,,\log\hbar(x,y))$$

and $f\in \mathrm{Lip}_{[0,1]}$. Then

$$\lim_{\epsilon\to 0}\sup_{f\in\mathrm{Lip}_{[0,1]}}\big|\Psi_\epsilon(f) - \Psi(f)\big| = 0.$$

In particular,

$$\lim_{\epsilon\to 0}\sup_{f\in\mathrm{Lip}_{[0,1]}}\Psi_\epsilon(f) = \sup_{f\in\mathrm{Lip}_{[0,1]}}\Psi(f)\,.$$

Proof. Using the Cauchy–Schwarz inequality, for $f\in\mathrm{Lip}_{[0,1]}$ we have:

$$\begin{aligned}\big|\Psi_\epsilon(f) - \Psi(f)\big| &= \iint_U (\rho_\epsilon(x_1,x_2)-\rho(x_1,x_2))\cdot(\partial_{x_1}f,\partial_{x_2}f,1-\partial_{x_1}f-\partial_{x_2}f)\,dx_1\,dx_2\\ &\le \iint_U\big|\rho_\epsilon(x_1,x_2)-\rho(x_1,x_2)\big|^{1/2}dx_1\,dx_2\iint_U\big|(\partial_{x_1}f,\partial_{x_2}f,1-\partial_{x_1}f-\partial_{x_2}f)\big|^{1/2}dx_1\,dx_2\\ &\le \mathrm{area}(U)\iint_U\big|\rho_\epsilon(x_1,x_2)-\rho(x_1,x_2)\big|^{1/2}dx_1\,dx_2\,,\end{aligned}$$

where in the last inequality we used the fact that the partial derivatives of f are bounded by 1. The last integral in the previous equation can we rewritten as:

$$\begin{aligned}\iint_U|\rho_\epsilon(x_1,x_2)-\rho(x_1,x_2)|^{1/2}dx_1\,dx_2 &= \iint_{\{\rho\le\log\epsilon\}}|\rho(x_1,x_2)-\log\epsilon|^{1/2}dx_1\,dx_2\\ &\le \iint_{\{\rho\le\log\epsilon\}}|\rho(x_1,x_2)|^{1/2}+|\log\epsilon|^{1/2}dx_1\,dx_2\,.\end{aligned}$$

The last integral converges to 0 when ϵ goes to 0 if the function $\rho = \log \mathsf{h}_\lambda$ is integrable on the domain U. Using a similar observation as in Lemma 4.1 we see that $h_\lambda(x,y) \le \sqrt{2}|\phi(y) - y|$. Since $\log \sqrt{2}|\phi(y)-y|$ is integrable for all x-section of U, we obtain than ρ is dominated by an integrable function on U and is itself integrable which finishes our proof. □

Corollary 4.4. $\lim_{N\to\infty} \frac{1}{N} Z_N = \sup_{f\in \mathrm{Lip}_{[0,1]}} \Psi(f)$.

Proof. By Lemma 4.2 we have that

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} \frac{1}{N}\log Z_N^\epsilon = \lim_{N\to\infty}\frac{1}{N}\log Z_N.$$

Applying Lemma 4.1 and Lemma 4.3 above yields the desired result. □

Note that $\lim_{N\to\infty} \frac{1}{N}\log Z_N^\epsilon = \sup_f \Psi_\epsilon(f)$ by the variational principle.

4.3. **The number of standard Young tableaux.** We are now ready to prove Theorem 1.1. We require the following technical result.

Lemma 4.5. *We have:*

$$\frac{1}{N}\left[\log\left(\sum_{h\in H_{\lambda^{(N)}/\mu^{(N)}}} \mathrm{hooks}_{\lambda^{(N)}}(\mathsf{t}_h)\right) - \frac{\mathrm{area}(\phi)}{2} N\log N\right] \to c,$$

where $c := \Psi(f_{\max})$ is a constant which depends only on ψ and ϕ.

Proof. Recall that for the weight function $w_N(x,y)$ and a height function h in $H_{\lambda^{(N)}/\mu^{(N)}}$ we have that

$$\begin{aligned}\mathrm{hooks}_{\lambda^{(N)}}(\mathsf{t}_h) &= \prod_{\Diamond\in \mathsf{t}_h} \mathsf{h}_{\lambda^{(N)}}(x_\Diamond, y_\Diamond) = (\sqrt{N})^{|\mu^{(N)}|} \times \prod_{\Diamond\in\mathsf{t}_h} e^{w^{i_\Diamond}(x_\Diamond y_\Diamond)} \\ &= (\sqrt{N})^{|\mu^{(N)}|}\times \mathrm{wt}(h), \end{aligned} \tag{4.5}$$

where $\mathrm{wt}(h)$ is a defined in (3.1). Then the log of the partition function of all height functions in $H_{\lambda^{(N)}/\mu^{(N)}}$ equals

$$\log \sum_{h\in H_{\lambda^{(N)}/\mu^{(N)}}} \mathrm{hooks}_{\lambda^{(N)}}(\mathsf{t}_h) \;=\; \log(\sqrt{N})^{|\mu^{(N)}|} \,+\, \log Z_N, \tag{4.6}$$

where $Z_N = \sum_{h\in H_{\lambda^{(N)}/\mu^{(N)}}} \mathrm{wt}(h)$. We treat each of the two summands in the RHS above separately.

By condition (2.3) on the area of ϕ in the definition of the stable shape we have that

$$\log(\sqrt{N})^{|\mu^{(N)}|} \;=\; \frac{1}{2}\left|\mu^{(N)}\right|\log N \;=\; \frac{\mathrm{area}(\phi)}{2} N\log N \,+\, o(N). \tag{4.7}$$

Next, by Corollary 4.4 we have

$$\lim_{N\to\infty}\frac{1}{N} Z_N \;=\; c, \tag{4.8}$$

where $c := \Psi(f_{\max})$ is a constant that only depends on ψ and ϕ.

Finally, we take the limit as $N\to\infty$ in (4.6) and use both (4.7) and (4.8) to obtain the desired result. □

Proof of Theorem 1.1. We take logs in (2.2) to obtain

$$\log f^{\nu^{(N)}} = \log|\nu^{(N)}|! - \left(\sum_{(x,y)\in\lambda^{(N)}} \log \mathsf{h}_{\lambda^{(N)}}(x,y)\right) + \log\left(\sum_{h\in H_{\lambda^{(N)}/\mu^{(N)}}} \mathrm{hooks}_{\lambda}(\mathsf{t}_h)\right) \tag{4.9}$$

Observe that $|\nu^{(N)}| = N + O(\sqrt{N})$ as $N\to\infty$. Then byy Stirling's formula we have

$$\log|\nu^{(N)}|! = N\log N - N + O(\sqrt{N}\log N). \tag{4.10}$$

Next, we use the definition and compactness of the stable shape $\mathcal{C}(\psi)$

$$\log \sum_{(x,y)\in\lambda^{(N)}} \mathsf{h}_{\lambda^{(N)}}(x,y) \;=\; N\iint_{\mathcal{C}(\psi)} \log\bigl(\sqrt{N}\hbar(x,y)\bigr)\,dx\,dy \;+\; o(N),$$

where the leading N outside the integral comes from a change of variables $x\to\sqrt{N}x$, $y\to\sqrt{N}y$ and the $\sqrt{N}$ inside the integral comes from rewriting $\mathsf{h}_{\lambda^{(N)}}(\cdot,\cdot)$ in terms of $\hbar(x,y)$ defined in (2.4). The error term $o(N)$ comes from approximating the sum with the scaled integral (cf. [MPP4, Thm. 6.3]).

By linearity of integration with respect to the integrand $\frac{1}{2}\log N + \log\hbar(x,y)$ we obtain

$$\log \sum_{(x,y)\in\lambda^{(N)}} \mathsf{h}_{\lambda^{(N)}}(x,y) \;=\; \frac{\mathrm{area}(\psi)}{2} N\log N \;+\; k(\psi)N \;+\; o(N), \tag{4.11}$$

where $k(\psi) = \iint_{\mathcal{C}(\psi)} \hbar(x,y)\,dx\,dy$. Lastly, applying to each term in (4.9) the bounds from (4.10), (4.11) and Lemma 4.5 respectively we obtain

$$\log f^{\nu^{(N)}} \;=\; \left(1 - \frac{\mathrm{area}(\psi/\phi)}{2}\right) N\log N \;+\; c(\psi/\phi)N \;+\; o(N),$$

where $c(\psi/\phi) := c + k(\psi)$ is the sum of the constant c from Lemma 4.5 and $k(\psi)$. Finally, since $\mathrm{area}(\psi/\phi) = 1$, the result follows. □

We end this section by extracting from the proof above the explicit expression for the constant of Theorem 1.1.

Corollary 4.6. *The constant $c(\psi/\phi)$ of Theorem 1.1 is given by*

$$c(\psi/\phi) \;:=\; k(\psi) \;+\; \Psi(f_{\max}),$$

where

$$k(\psi) \;=\; \iint_{\mathcal{C}(\psi)} \hbar(x,y)\,dx\,dy,$$

$$\Psi(f_{\max}) \;=\; \max_{f\in \mathrm{Lip}_{[0,1]}} \iint_U \Bigl(\sigma(\nabla f) \;+\; (1-\partial_x f - \partial_y f)\log\hbar(x,y)\Bigr)\,dx\,dy,$$

$\sigma(\cdot)$ is defined by (3.2), and U is the region enclosed by the projection of the curve bounded by ϕ.

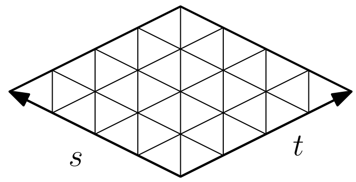


FIGURE 6. The slope of a periodic tiling.

5. Proof of the weighted variational principle

Our strategy to prove this theorem consists in three parts. In the first part we give a lemma (Lemma 5.1) that shows that fundamental domains with similar plane-like boundary conditions have the same number of tilings and that all those tilings contain a similar number of lozenges of each type. Both numbers depend on the slope of the domain. In the second part we give a lemma (Lemma 5.3) that shows that the weighted contribution of lozenges with similar plane-like boundary conditions is also the same. Finally, in the third part we use the two previous lemmas to prove the weighted variational principle.

5.1. **Tilings of similar plane-like regions (unweighted).** Let (s,t) be a pair of numbers such that $\{0 \le s, t, 1-s-t \le 1\}$, let $\epsilon > 0$ and let D_m be the $m \times m$ diamond of the hexagonal grid whose left corner is the origin.

Let $\overline{H}^{\epsilon}_{D_m}(s,t)$ be the set of admissible boundary height functions $\bar{h} : \partial D_m \to \mathrm{Z}$, such that:

- the left corner of the diamond has height 0
- for all $x = (x_1, x_2) \in \partial D_m$ we have

$$|\bar{h}(x_1, x_2) - (sx_1 + tx_2)| \le \epsilon m.$$

Lemma 5.1. *Let (s,t) be such that $\{0 \le s, t, 1-s-t \le 1\}$, let $\epsilon > 0$ and let $D_m \subset \mathbb{Z}^2$ and $\overline{H}^{\epsilon}_{D_m}(s,t)$ be as defined above . Then for each $\bar{h} \in \overline{H}^{\epsilon}_{D_m}(s,t)$ we have that*

$$\lim_{m\to\infty} \frac{1}{m^2} \log N(\bar{h}) = \lim_{m\to\infty} \frac{1}{m^2} \log \sum_{\bar{h}\in\overline{H}^{\epsilon}_{D_m}(s,t)} N(\bar{h}) = \sigma(s,t) + O(\epsilon \log(1/\epsilon)), \tag{5.1}$$

and

$$\lim_{m\to\infty} \frac{1}{m^2} \Big(\log \mathcal{L}^{(1)}(\bar{h}), \log \mathcal{L}^{(2)}(\bar{h}), \log \mathcal{L}^{(3)}(\bar{h})\Big) = (s, t, 1-s-t) + O(\epsilon)\mathbf{1}. \tag{5.2}$$

Proof. Let $\mathcal{P}_m(s,t)$ be the set of tilings of D_m with periodic boundary conditions with slope (s,t) and $N_m(s,t)$ be the number of tilings in $\mathcal{P}_m(s,t)$. Note that $\mathcal{P}_m(s,t)$ is also the set of tilings of a torus with slope (s,t). By [Ken, Thm. 8] we have that:

$$\frac{1}{m^2} \log N_m(s,t) \;=\; \sigma(s,t) + o(1),$$

and that each of those tilings has exactly $\{m^2 s, m^2 t, m^2(1-s-t)\}$ lozenges of each type. Additionally, if we choose a height function uniformly amongst all height functions in $\mathcal{P}_m(s,t)$ then we have the following concentration results:

$$\mathbb{P}\Big(|h(x_1,x_2) - (sx_1 + tx_2)| \ge \epsilon\Big) \;\le\; e^{4\epsilon m}. \tag{5.3}$$

This can be shown by applying the same martingale argument as in [CEP, Prop. 22]. Although the argument in this paper is made for simply connected regions, it extends for tilings of a torus with given slopes.

Denote by $\mathcal{P}^\epsilon_m(s,t)$ the set of periodic configurations on a torus of size m whose height function stays within ϵm of a linear plane of slope (s,t) that is:

$$\mathcal{P}^\epsilon_m(s,t) \,:=\, \left\{ h \in \mathcal{P}_m(s,t) \;:\; \max_{x\in D_m} \bigl\{ |h(x_1,x_2) - (sx_1+tx_2)| \bigr\} \geq \epsilon m \right\}.$$

Let $N^\epsilon_m(s,t)$ be the size of $\mathcal{P}^\epsilon_m(s,t)$. As a direct consequence of the inequality (5.3), we have:

$$\frac{1}{m^2}\log\bigl(N_m(s,t)(1-e^{-c\epsilon m})\bigr) \leq \frac{1}{m^2}\log N^\epsilon_m(s,t) \leq \frac{1}{m^2}\log N_m(s,t)$$

Therefore,

$$\lim_{m\to\infty}\frac{1}{m^2}\log N^\epsilon_m(s,t) \,=\, \sigma(s,t).$$

We must now distinguish between the case where $\epsilon \leq \frac{1}{2}(1-\max\{s,t,1-s-t\})$ and the case $\epsilon > \frac{1}{2}\max\{s,t,1-s-t\}$.

Case 1: Suppose $\epsilon \leq \frac{1}{2}(1-\max\{s,t,1-s-t\})$. Consider $\bar{h} \in \overline{H}^\epsilon_{D_m}(s,t)$ and $h_- \in \mathcal{P}^\epsilon_{m(1-3\epsilon)}(s,t)$. For all $x=(x_1,x_2)\in\partial D_{m(1-3\epsilon)}$ and $y=(y_1,y_2)\in\partial D_m$ we have

$$\begin{aligned}
&\bar{h}(y) - h_-(x) \,\leq \\
&\quad\leq \quad \Bigl[\bar{h}(y) - (sy_1+ty_2)\Bigr] \,+\, \Bigl[(sy_1+ty_2) - (sx_1+tx_2)\Bigr] \,+\, \Bigl[(sx_1+tx_2) - h_-(x)\Bigr] \\
&\quad\leq \quad \epsilon m \,+\, \max\{s,t,1-s-t\}\cdot\max\{y_1-x_1,y_2-x_2\} \,+\, \epsilon m \\
&\quad\leq \quad \bigl(1-\max\{s,t,1-s-t\}\bigr)\|x-y\|_1 \,+\, \max\{s,t,1-s-t\}\cdot\max\{y_1-x_1,y_2-x_2\} \\
&\quad\leq \quad \max\{y_1-x_1,y_2-x_2\}.
\end{aligned}$$

where $\|\cdot\|_1$ denotes the 1-norm.

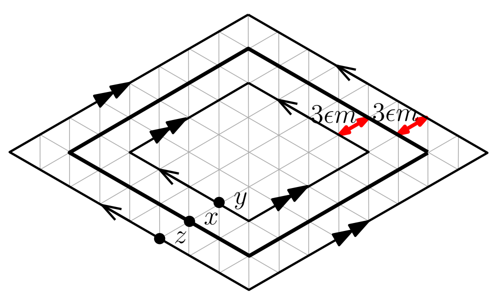


FIGURE 7. Illustration of the proof of Lemma 5.1. The number of tilings with boundary conditions in $\bar{h} \in \partial D_m$ is at least the number of tilings with periodic boundary conditions in $\partial D_{m(1-3\epsilon)}$ and at most the number of tilings with periodic boundary conditions in $\partial D_{m(1+\epsilon)}$.

Using Lemma 2.1, we deduce that there exist a height function h on D_m such that $h=\bar{h}$ on ∂D_m and $h = h_-$ on $\partial D_{m(1-3\epsilon)}$. As a consequence, we obtain that $N(\bar{h}) \geq N^\epsilon_{m(1-3\epsilon)}(s,t)$. For the same reasons, for $h_+ \in \mathcal{P}^\epsilon_{m(1+3\epsilon)}(s,t)$, for all $x\in\partial D_m$ and $z\in\partial D_{m(1+3\epsilon)}$ we have:

$$|\bar{h}(x) - h_+(z)| \,\leq\, \max\{z_1-x_1, z_2-x_2\}.$$

Thus, every boundary height functions in $\mathcal{P}^{\epsilon}_{m(1+3\epsilon)}(s,t)$ can be extended to $\bar{h}$ on ∂D_m. This implies:

$$N^{\epsilon}_{m(1-3\epsilon)}(s,t) \,\leq\, N(\bar{h}) \,\leq\, N^{\epsilon}_{m(1+3\epsilon)}(s,t),$$

which can be rewritten as

$$\frac{1}{m^2}\log N^{\epsilon}_{m(1-3\epsilon)}(s,t) \,\leq\, \frac{1}{m^2}\log N(\bar{h}) \,\leq\, \frac{1}{m^2}\log N^{\epsilon}_{m(1+3\epsilon)}(s,t).$$

Since $1/(m^2(1-3\epsilon)) = 1/m^2 + O(\epsilon)$, we deduce that

$$\lim_{m\to\infty}\frac{1}{m^2}\log N(\bar{h}) \,=\, \sigma(s,t) \,+\, O(\epsilon).$$

Finally, we can bound the number of boundary conditions in $H^{\epsilon}_{D_m}(s,t)$ by the number of different types of lozenges to the power of the length of ∂D_m. Since there are at most $3^{4m} = e^{o(m^2)}$ different boundary height functions in $\overline{H}^{\epsilon}_{D_m}(s,t)$, then this allow us to deduce (5.1), i.e.,

$$\lim_{m\to\infty}\frac{1}{m^2}\log\sum_{\bar{h}\in\overline{H}^{\epsilon}_{D_m}(s,t)} N(\bar{h}) = \sigma(s,t) + O(\epsilon).$$

For the second part of the statement, we notice that when attaching two tilings as described above (see Figure 7), we are adding at most $\epsilon^2 m^2$ tilings of each type. Hence we obtain that for all $i\in\{1,2,3\}$:

$$\mathcal{L}^{(i)}(h_-) - \epsilon^2 m^2 \leq \mathcal{L}^{(i)}(\bar{h}) \leq \mathcal{L}^{(i)}(h_+) + \epsilon^2 m^2.$$

Dividing by m^2 and taking the logarithm, we obtain (5.2).

Case 2: Suppose $\epsilon \geq \frac{1}{2}(1-\max\{s,t,1-s-t\})$. Let $\bar{h}\in\overline{H}^{\epsilon}_{D_m}(s,t)$. Without loss of generality we can assume that $\max\{s,t,1-s-t\} = 1-s-t$ so that $\epsilon \geq (s+t)/2$. The height difference between the top vertex and the bottom vertex of each vertical section of D_m is at most $4\epsilon m$. Hence, each of those vertical section contains at most $\lfloor 4\epsilon m\rfloor$ vertical edges. This means that the total number of non-horizontal lozenges in each tiling of a height function that extends $\bar{h}$ is smaller than $\lfloor 4\epsilon m^2\rfloor$ and implies directly (5.2). Notice that we can determine a tiling by specifying what is the position of the non-horizontal lozenges and their types. Hence the total number of tilings $N(\bar{h})$ is bounded by $\binom{m^2}{\lfloor 4\epsilon m^2\rfloor}2^{\lfloor 4\epsilon m^2\rfloor}$. By using Stirling's formula, we obtain

$$N(\bar{h}) \leq \binom{m^2}{\lfloor 4\epsilon m^2\rfloor}2^{\lfloor 4\epsilon m^2\rfloor} \,=\, e^{m^2 O(\epsilon\log(1/\epsilon))}.$$

Therefore, the total number of configurations with boundary $\bar{h}$ satisfies

$$\frac{1}{m^2}N(\bar{h}) \,=\, O\big(\epsilon\log(1/\epsilon)\big) \,+\, o(1).$$

Since $\sigma(0,0)=0$, this implies (5.1) and concludes our proof. □

Lemma 5.1 holds when we replace lozenges by equilateral triangles. This will be useful for the remainder of the proof as explained in Section 5.2.

Corollary 5.2. *Let T_m be an equilateral triangle of size m and $\overline{H}^{\epsilon}_{T_m}(s,t)$ be as defined above. Then for each $\bar{h}\in\overline{H}^{\epsilon}_{T_m}(s,t)$ we have that*

$$\lim_{m\to\infty}\frac{4}{\sqrt{3}m^2}\log N(\bar{h}) \,=\, \lim_{m\to\infty}\frac{4}{\sqrt{3}m^2}\log\sum_{\bar{h}\in\overline{H}^{\epsilon}_{D_m}(s,t)} N(\bar{h}) \,=\, \sigma(s,t) \,+\, O\big(\epsilon\log(1/\epsilon)\big), \tag{5.4}$$

and

$$\lim_{m\to\infty} \frac{4}{\sqrt{3}m^2}\left(\log \mathcal{L}^{(1)}(\bar{h}), \log \mathcal{L}^{(2)}(\bar{h}), \log \mathcal{L}^{(3)}(\bar{h})\right) \,=\, (s,t,1-s-t) \,+\, O(\epsilon)\mathbf{1}. \tag{5.5}$$

Proof. Let T_m be a triangle of size m and $\bar{h}$ be a boundary height function which stays within ϵm of the plane with slope (s,t). For each $h \in \bar{h}$, if we reflect h along one side we obtain a height function of a lozenge D_m which also stays within ϵm of the plane with slope (s,t). Hence we can bound $N(\bar{h})^2$ by the number of way to extend a boundary in $\overline{H}^{\epsilon}_{D_m}(s,t)$ and we obtain:

$$\frac{2}{m^2}\log N(\bar{h}) \,=\, \frac{1}{m^2}\log\left(N(\bar{h})^2\right) \,\le\, \frac{1}{m^2}\log\left[\sum_{\bar{h}\in\overline{H}^{\epsilon}_{D_m}(s,t)} N(\bar{h})\right] \,\le\, \sigma(s,t) \,+\, O\big(\epsilon\log(1/\epsilon)\big).$$

Now consider a triangle T_{m^2} of size m^2, $\bar{h}$ be a boundary height function which stays within ϵm of the plane with slope (s,t). We can fill partially T_{m^2} with $m - o(1)$ lozenges of size m each having the same periodic boundary height function with slope (s,t). Using a similar argument as the one in the previous lemma for attaching configurations, we can attach $\bar{h}$ to the height function on those lozenges and we obtain:

$$\sigma(s,t) \,+\, O\big(\epsilon\log(1/\epsilon)\big) \,\le\, \frac{1}{2m^2}\log N(\bar{h}),$$

as desired. $\square$

5.2. **Tilings of similar plane-like regions (weighted).** For the remainder of this proof we will be working with triangles since later in this proof we will need to approximate surfaces with piecewise-linear functions. Such approximations are done in a standard way using triangles (see for example Lemma 2.2 in [CKP]).

Since the weight of each individual lozenge tile depends on its position in the lattice, we now evaluate the weight contribution of a large triangle as a function of its position.

Lemma 5.3. *Let $x = (x_1, x_2) \in R^2$ and $\ell \in \mathbb{R}$ be such that ρ is smooth on $B(x,\ell)$. Let $T(x,\ell n)$ be the triangle of size ℓn centered at the point $x^n := (\lfloor nx_1\rfloor, \lfloor nx_2\rfloor)$ and let $\bar{h} \in \overline{H}^{\epsilon}_{T(x,\ell n)}(s,t)$. For a converging sequence of weights $\{w_n\}_{n\in\mathbb{N}}$ we have :*

$$\begin{aligned}\lim_{n\to\infty}\frac{4}{\sqrt{3}(\ell n)^2}\log Z(H_{\bar{h}}, w_\ell) \,&=\, \lim_{n\to\infty}\frac{4}{\sqrt{3}(\ell n)^2}\log\left[\sum_{\bar{h}\in\overline{H}^{\epsilon}_{T(x,\ell n)}(s,t)} Z\big(H_{\bar{h}}, w_\ell\big)\right]\\ &=\, \sigma(s,t) \,+\, L(x_1,x_2,s,t) \,+\, O\big(\epsilon\log(1/\epsilon)\big) \,+\, O(\ell),\end{aligned}\tag{5.6}$$

where $Z(H_{\bar{h}}, w_\ell)$ is the total weight of all configurations with boundary $\bar{h}$.

Proof. The sequence of weights $\{w_n\}_{n\in\mathbb{N}}$ is convergent, by Condition (ii) of Definition 3.1. Thus, for all $ny \in T(x,\ell n)$, and for each type of lozenge tile $i \in \{1, 2.3\}$, we have:

$$|w_n^i(ny) - \rho^i(x)| \le |w_n^i(ny) - \rho^i(y)| + |\rho^i(y) - \rho^i(x)| = o(1) + O(\ell).$$

Here we used the smoothness of ρ on $B(x,\ell)$ to bound $|\rho^i(y)-\rho^i(x)| = O(\ell)$. This means that for all height function $h \in H^{\epsilon}_{T(x,l)}(s,t)$ with boundary $\bar{h}$, we must have:

$$\begin{aligned}\mathrm{wt}(h) &= \prod_{\lozenge\in h} e^{w^{i_\lozenge}(x_\lozenge)} = \prod_{\lozenge\in h} e^{(\rho^{i_\lozenge}(x)+O(\ell)+o(1))}\\ &= \sum_{j=1}^{3} \exp\left[\frac{\sqrt{3}(\ell n)^2}{4}\big(x_j + o(1) + O(\epsilon)\big)\right]\cdot \exp\Big[\rho^1(x) + o(1) + O(\ell)\Big]\\ &= \exp\left[\frac{\sqrt{3}(\ell n)^2}{4}\Big(L(x_1,x_2,s,t) + o(1) + O(\ell) + O(\epsilon)\Big)\right],\end{aligned}$$

where $x_3 = 1-x_1-x_2$. Then the contribution of all configurations with boundary $\bar{h}$ is given by:

$$\begin{aligned}Z(H_{\bar h}, w_n) &= \sum_{h\in H^{\epsilon}_{T(x,\ell)}(s,t)} \mathrm{wt}(h)\\ &= \exp\left[\frac{\sqrt{3}(\ell n)^2}{4}\Big(L(x_1,x_2,s,t) + o(1) + O(\ell) + O(\epsilon)\Big)\right] N(\bar h).\end{aligned}$$

Applying Corollary 5.2 to the equation above, we obtain:

$$\begin{aligned}Z(H_{\bar h}, w_n) &= \exp\left[\frac{\sqrt{3}(\ell n)^2}{4}\Big(L(x_1,x_2,s,t) + o(1) + O(\epsilon)\Big)\right] \times\\ &\qquad \times \exp\left[\frac{\sqrt{3}(\ell n)^2}{4}\Big(\sigma(s,t) + o(1) + O\big(\epsilon\log(1/\epsilon)\big)\Big)\right]\\ &= \exp\left[\frac{\sqrt{3}(\ell n)^2}{4}\Big(\sigma(s,t) + L(x_1,x_2,s,t) + o(1) + O(\ell) + O\big(\epsilon\log(1/\epsilon)\big)\Big)\right].\end{aligned}$$

Then (5.6) follows by taking the logarithm to the equation above. Since the number of boundary height functions for a given triangle is bounded by $3^{3\ell n} = e^{o(\ell^2 n^2)}$, we also obtain:

$$\lim_{\ell\to\infty} \frac{4}{\sqrt{3}(\ell n)^2} \sum_{\bar h\in \overline{H}^{\epsilon}_{T(x,\ell)}(s,t)} Z(H_{\bar h}, w_\ell) = \sigma(s,t) + L(x_1,x_2,s,t) + O\big(\epsilon\log(1/\epsilon)\big) + O(\ell).$$

This finishes the proof. □

5.3. **Proof of Theorem 3.3.** We now prove the weighted variational principle. At this point our strategy is exactly the same as Theorem 4.3 in [CKP] or Theorem 2.9 in [MT]. We recall the following two lemmas from [CKP] which will be useful in our proof.

Lemma 5.4 ([CKP] Lemma 2.2)**.** *For $\ell > 0$, consider a mesh made up of equilateral triangles of side length ℓ (which we call an ℓ-mesh). Let $f \in \mathrm{Lip}_{[0,1]}$ be such that $f = \gamma$ on U, and let $\epsilon > 0$. If ℓ is sufficiently small then on at least $(1-\epsilon)$ fractions of the triangles in the ℓ-mesh that intersect U we have the following two properties:*

(1) The piecewise linear approximation $\widetilde{f}$ agrees with f to within $\ell\epsilon$
(2) For at least a $(1-\epsilon)$ fraction (in measure) of the points x of the triangle, the tilt $\nabla f(x)$ exists and is within ϵ of $\widetilde{f}(x)$.

Lemma 5.5 ([CKP] Lemma 2.3)**.** *Suppose that T is an equilateral triangle of length ℓ, and the height function f satisfies $|f_{\delta T} - \widetilde{f}| \le \epsilon\ell$ on δT, where $\widetilde{f}$ is the piecewise linear approximation from Lemma 5.4, then*

$$\Psi(f) \,=\, \Psi(\widetilde{f}) \,+\, o\big(\mathrm{area}(T)\big).$$

Remark 5.6. Note that Lemma 2.3 in [CKP] is stated for $\Phi(\cdot)$, however since $L(\cdots)$ in the definition of $\Psi(\cdot)$ is linear, the result still holds for $\Psi(\cdot)$.

Next, we approximate the partition function of all height functions which stays close to a given function f using Ψ and show that the error term goes to zero.

Lemma 5.7. *Let $f \in \mathrm{Lip}_{[0,1]}$ be such that $f = \gamma$ on U and let $\delta > 0$. If we denote by $Z(H_f^\delta, w_n)$ the total weight of height functions which stay within δn of f, then:*

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z(H_f^\delta, w_n) \,=\, \Psi(f) \,+\, o(1). \tag{5.7}$$

Proof. Let $\epsilon < 1$ and consider a sequence of grids $\{G_n\}_{n\in\mathbb{N}}$ which partition the triangular lattice into equilateral triangles of size ℓn. Denote by $\widetilde{G}_n$ the ℓ-mesh obtained by rescaling G_n and $\widetilde{f}$ the linear approximation of f on $\widetilde{G}_n$. See Figure 8.

We start by approximating $\Psi(f)$ by the terms on the RHS of (5.6) of Lemma 5.3. According to Lemma 5.4, for ℓ small enough we have:

$$\sup_{x\in T} |f(x) - \widetilde{f}(x)| \le \ell\epsilon$$

on all but a portion at most ϵ of the triangles in U. Next, we rewrite $\Psi(\widetilde{f})$ as

$$\Psi(\widetilde{f}) \,=\, \sum_{T\in U} \frac{4\ell^2}{\sqrt{3}} \Big(\sigma(\nabla\widetilde{f}) + L(x_1^T, x_2^T, \widetilde{f})\Big) \,+\, o\big(\mathrm{area}(U)\big), \tag{5.8}$$

where the error term $o\big(\mathrm{area}(U)\big)$ comes from bounding the following integral

$$\sum_{T\in U} \iint_T \big(\rho(x_1,x_2) - \rho(x_1^T, x_2^T)\big) \cdot (\partial_{x_1} f, \partial_{x_2} f, 1 - \partial_{x_1} f - \partial_{x_2} f)\, dx_1 dx_2,$$

using the uniform continuity of ρ on each component of U where it is smooth.

Combining (5.8) with Lemma 5.5 we have:

$$\left| \Psi(f) - \sum_{T\in U} \frac{4\ell^2}{\sqrt{3}} \Big(\sigma(\nabla\widetilde{f}) + L(x_1^T, x_2^T, \widetilde{f})\Big) \right| \,=\, o(1).$$

With this approximation, we are now ready to prove (5.7). Choose $\delta < \ell\epsilon$ and $\{h_n\}$ to be any height function with boundary γ_n. Define

$$a_n(x) \,:=\, \min\big\{h_n(x), \lfloor nf(x/n) + \delta\rfloor\big\} \quad \text{and} \quad g_n(x) \,:=\, \max\big\{a_n(x), \lfloor nf(x/n) - \delta\rfloor\big\}.$$

Then $|g_n/n - f| \le \delta/2$ and $g_n \in H_f^\delta$, by construction.

We can ignore the contribution of triangles that are not fully included in U which is $O(\delta) = O(\epsilon)$. The quantity $\log Z(H_f^\delta, w_n)$ is bounded from below by the weight of all height functions that agree with g_n on the boundary of all triangles in G_n completely contained in U after rescaling. This gives:

$$\log \prod_{T\in U} Z(H_{\bar{g}_{\partial T}}, w_n) \,=\, \sum_{T\in U} \log Z(H_{\bar{g}_{\partial T}}, w_n) \,\le\, \log Z(H_f^\delta, w_n),$$

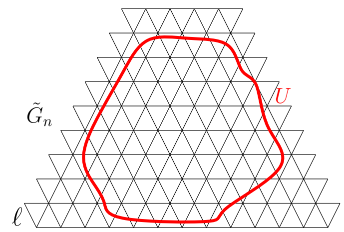


FIGURE 8. The grid $\widetilde{G}_n$ of equilateral triangles of size ℓ that partitions U used in the proof of Theorem 3.3.

where the product is taken over all triangles fully contained in U. Now include the $O(\epsilon)$ in the bound. Then $\log Z(H^\delta_f, w_n)$ is bounded from above by the the total free product of all height functions which stays within δ of $\widetilde{f}$ on each one of those triangles. In other words,

$$\log Z(H^\delta_f, w_n) \,\leq\, \log \prod_{T\in U} Z(H^\delta_{\widetilde{f}}, w_n) \,=\, \sum_{T\in U} \log Z(H^\delta_{\widetilde{f}}, w_n) \,+\, O(\epsilon).$$

Using Lemma 5.3 to approximate each $\log Z(H_{\bar{h}_{\partial T}}, w_n)$ and $\log Z(H^\delta_{\widetilde{f}}, w_n)$ in the above inequalities, we obtain:

$$\begin{aligned}
&\frac{1}{n^2}\log Z(H^\epsilon_f, w_n) \,= \\
&\quad= \frac{1}{n^2}\sum_{T\in U}\left[\frac{\sqrt{3}}{4}\ell^2 n^2 \Big(\sigma(\widetilde{f}) + L(x_1^T, x_2^T, \widetilde{f}) + o(1) + O(\ell) + O(\epsilon\log 1/\epsilon)\Big)\right] + o(1) \\
&\quad= \sum_{T\in U}\left[\frac{\sqrt{3}}{4}\ell^2 \Big(\sigma(\widetilde{f}) + L(x_1^T, x_2^T, \widetilde{f}) + o(1) + O(\ell) + O(\epsilon\log 1/\epsilon)\Big)\right] + o(1) \\
&\quad= \Psi(\widetilde{f}) + O(\ell) \,+\, O(\epsilon\log 1/\epsilon) \,+\, o(1).
\end{aligned}$$

Since both ℓ and ϵ can be chosen as small as needed when $\delta\to 0$, we have:

$$\frac{1}{n^2}\log Z(H^\epsilon_f, w_n) \,=\, \Psi(\widetilde{f}) \,+\, O(\ell) \,+\, O(\epsilon\log 1/\epsilon) \,+\, o(1) \,=\, \Psi(f) \,+\, o(1),$$

as desired. □

The function σ is strictly convex and L is linear, thus the function $\sigma + L$ is itself strictly convex. This implies that there exist a unique function $f_{\max}$ in $\mathrm{Lip}_{[0,1]}$ that maximize Ψ. By the previous lemma, we obtain:

$$\lim_{n\to\infty}\frac{1}{n^2}\log Z(H_{\gamma_n}, w_n) \,\geq\, \Psi(f_{\max}).$$

Moreover, the set of functions $f\in\mathrm{Lip}_{[0,1]}$ such that $f=\gamma$ on U is compact for the ℓ_∞ norm. Hence, for every fixed δ, there exist a finite covering of $\mathrm{Lip}_{[0,1]}$ with balls of radius δ. If we denote by $C(\delta)$ the number of balls in this coverings, this implies that for all $\delta>0$:

$$\lim_{n\to\infty}\frac{1}{n^2}\log Z(H_{\gamma_n}, w_n) \,\leq\, \Psi(f_{\max}) + \lim_{n\to\infty}\frac{1}{n^2}\log C(\delta) \,+ o(1) \,=\, \Psi(f_{\max}) + o(1).$$

Letting δ go to 0, this finishes the proof of Theorem 3.3.

6. Final remarks and open problems

6.1. There are other positive formulas for $f^{\lambda/\mu}$ using the Littlewood–Richardson coefficients and the *Okounkov–Olshanski formula*, see [MPP1, §9] for the discussion and references. It would be interesting to see if variational principle applies in either case.

6.2. In case of the thick hooks (see §1.2), the variational principle result (Theorem 3.3) is already interesting and is now well understood. It corresponds to a degenerate case of more general weights introduced in [BGR] and further studied in [Bet, DK] (see also [MPP3]), where both the frozen region and the probability density are computed.

It is worth comparing frozen regions in the uniform and weighed cases, see Figure 9. The uniform frozen region is famously a circle, while the weighted frozen region is an algebraic curve with only mirror symmetry. Let us mention that explicit product formulas for *q-Racah polynomials* allows a direct sampling from these weighted tilings in this case, see [Bet, §7.5] and [BGR, §9]. This approach does not generalize to other skew shapes.

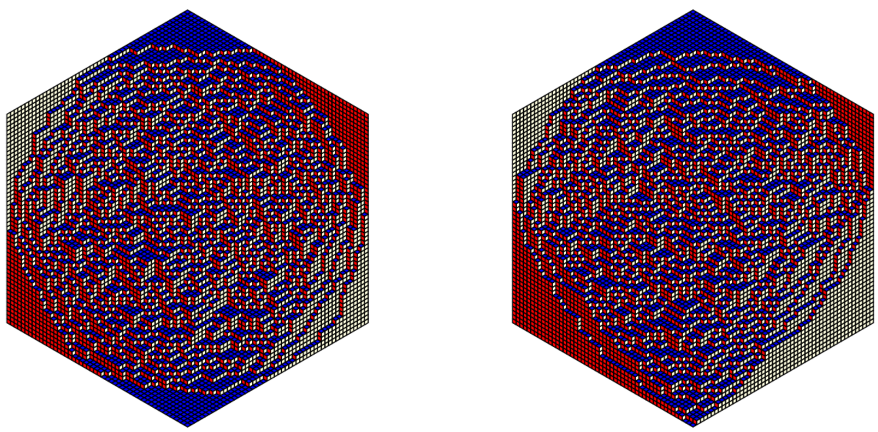


Figure 9. Uniform and weighted random lozenge tilings of the hexagon $\mathbb{H}(50,50,50)$ from [MPP3, Fig. 2].

6.3. It would be interesting to compute the frozen region explicitly for the weighted lozenge tilings in some important special cases, such as thick ribbons described in §1.3. From the variational principle we cannot even tell if these regions are bounded by algebraic curves.

6.4. Beside stable limits shapes, there are other asymptotic regimes when the problem of computing $f^{\lambda/\mu}$ is of interest, see [DF, MPP4, Sta1]. Except for the case when $|\mu| = O(1)$, obtaining better bounds is an interesting and difficult challenge.

6.5. In an important recent development, Sun showed the limit curves for random standard Young tableaux with stable limit shape [Sun], also by modifying the variational principle. This suggests that in principle one can apply the strategy sketched in [Pak1, §3.5] to conclude that there is no natural bijective proof of the Naruse hook-length formula NHLF (1.1). We are currently very far from even formulating this as a conjecture.

Let us mention that [Kon] gives a bijective proof of a recurrence involved in the proof of the NHLF. Unfortunately, there seem to be no way to use this bijection for uniform sampling of random standard Young tableaux of skew shape.

Acknowledgements. The authors would like to thank Nishant Chandgotia, Jehanne Dousse, Valentin Féray, Vadim Gorin, Rick Kenyon, Georg Menz, Leo Petrov, Dan Romik, Richard Stanley and Damir Yeliussizov for helpful discussions. We are especially thankful to Greta Panova for the ongoing extended collaboration in the area; notably, the idea of this paper is stemming from the lozenge tiling results in [MPP3]. The second author was partially supported by the NSF.

References

[AdR] R. Adin and Y. Roichman, Standard Young tableaux, in *Handbook of enumerative combinatorics*, CRC Press, Boca Raton, FL, 2015, 895–974.

[AHRV] O. Angel, A. E. Holroyd, D. Romik and B. Virág, Random sorting networks, *Adv. Math.* **215** (2007), 839–868.

[AsR] R. A. Askey and R. Roy, Barnes G-function, in *NIST Handbook of Mathematical Functions* (F. W. J. Olver et al., eds.), Cambridge Univ. Press, 2012.

[Bet] D. Betea, *Elliptic Combinatorics and Markov Processes*, Ph.D. thesis, Caltech, 2012.

[BGR] A. Borodin, V. Gorin and E. M. Rains, q-distributions on boxed plane partitions, *Selecta Math.* **16** (2010), 731–789.

[CPT] N. Chandgotia and I. Pak and M. Tassy, Kirszbraun–type theorems for graphs; `arXiv:1710.11007`.

[CEP] H. Cohn, N. Elkies and J. Propp, Local statistics for random domino tilings of the Aztec diamond, *Duke Math. J.* **85** (1996), 117–166.

[CKP] H. Cohn, R. Kenyon, and J. Propp, A variational principle for domino tilings. *J. AMS* **14** (2001), 297–346.

[Dau] D. Dauvergne, The Archimedean limit of random sorting networks; `arXiv:1802.08934`.

[DK] E. Dimitrov and A. Knizel, Log-gases on a quadratic lattice via discrete loop equations and q-boxed plane partitions, 75 pp.; `arXiv:1710.01709`.

[DF] J. Dousse and V. Féray. Asymptotics for skew standard Young tableaux via bounds for characters; `arXiv:1710.05652`.

[Feit] W. Feit, The degree formula for the skew-representations of the symmetric group, *Proc. AMS* **4** (1953), 740–744.

[FeS] V. Féray and P. Śniady, Asymptotics of characters of symmetric groups related to Stanley character formula, *Ann. of Math.* **173** (2011), 887–906.

[Ken] R. Kenyon, Lectures on dimers, in *Statistical mechanics*, AMS, Providence, RI, 2009, 191–230.

[Kon] M. Konvalinka, A bijective proof of the hook-length formula for skew shapes; `arXiv:1703.08414`.

[MPP1] A. H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes I. q-analogues and bijections, *J. Combin. Theory, Ser. A* **154** (2018), 350–405.

[MPP2] A. H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications, *SIAM Jour. Discrete Math.* **31** (2017), 1953–1989.

[MPP3] A. H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes III. Multivariate and product formulas; `arXiv:1707.00931`.

[MPP4] A. H. Morales, I. Pak and G. Panova, Asymptotics of the number of standard Young tableaux of skew shape, *Eur. J. Combin* **70** (2018), 26–49.

[MPP5] A. H. Morales, I. Pak and G. Panova, Asymptotics of the number of standard Young tableaux of skew shape II. The case of thick ribbons, in preparation.

[Nar] H. Naruse, Schubert calculus and hook formula, talk slides at *73rd Sém. Lothar. Combin.*, Strobl, Austria, 2014; available at `http://tinyurl.com/z6pagzu`.

[NO] H. Naruse and S. Okada, Skew hook formula for d-complete posets; `arXiv:1802.09748`.

[Pak1] I. Pak, Complexity problems in enumerative combinatorics, to appear in *Proc. ICM Rio de Janeiro*; `arXiv:1803.06636`.

[Pak2] I. Pak, On the number of tableaux of large diagrams, in preparation.

[PST] I. Pak, A. Sheffer, and M. Tassy, Fast domino tileability, *Discrete Comput Geom*, **56** (2016), 377–394.

[MT] G. Menz and M. Tassy, A variational principle for a non-integrable model; `arXiv:1610.08103`.

[PR] B. Pittel and D. Romik, Limit shapes for random square Young tableaux, *Adv. Appl. Math.* **38** (2007), 164–209.

[Rom] D. Romik, *The surprising mathematics of longest increasing subsequences*, Cambridge Univ. Press, New York, 2015.

[Sta1] R. P. Stanley, On the enumeration of skew Young tableaux, *Adv. Appl. Math.* **30** (2003), 283–294.

[Sta2] R. P. Stanley, *Enumerative Combinatorics*, vol. 1 and 2, Cambridge Univ. Press, 2012 and 1999.

[Sun] W. Sun, Dimer model, bead and standard Young tableaux: finite cases and limit shapes; available at https://www.archives-ouvertes.fr/hal-01631875v1

[Thu] W. P. Thurston, Groups, tilings and finite state automata, in *Lecture Notes*, AMS Summer Meetings, Bolder, CO, 1989.

[TM] W. P. Thurston and J. W. Milnor, *The geometry and topology of three-manifolds.* Princeton Univ. Princeton, NJ, 1979.

Department of Mathematics and Statistics, UMass, Amherst, MA
E-mail address: `amorales@math.umass.edu`

Department of Mathematics, University of California, Los Angeles, CA
E-mail address: `pak@math.ucla.edu`

Department of Mathematics, Dartmouth College, Hanover, NH
E-mail address: `mtassy@math.dartmouth.edu`