PRESBURGER ARITHMETIC WITH ALGEBRAIC SCALAR MULTIPLICATIONS

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ABSTRACT. We study complexity of integer sentences in $S_\alpha = (\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \alpha x)$, which is known to be decidable for quadratic $\alpha$, and undecidable for non-quadratic irrationals. When $\alpha$ is quadratic and the sentence has $r$ alternating quantifier blocks, we prove both lower and upper bounds as towers of height $(r - 3)$ and $r$, respectively. We also show that for $\alpha$ non-quadratic, already $r = 4$ alternating quantifier blocks suffice for undecidability.

1. Introduction

1.1. Foreword. It is generally well known that the areas of mathematics sometimes meet at unexpected places, leading to problems where tools from different areas can be used. Yet it is always exciting to witness this happening, as it both validates the common goals and justifies diverse techniques in their pursuit.

This paper arose at the meeting of two areas: mathematical logic (specifically, decidability of theories) and discrete geometry (integer points in polytopes). In recent years, there has been much effort by the authors, pursuing somewhat different but related goals, to understand when variations on Presburger Arithmetic (PA) become intractable. The meaning of “intractable” is traditionally different in the two areas, of course. In logic it is qualitative and stands for decidable, while in computational geometry and integer optimization it is quantitative and stands for computationally hard (as in NP-hard, PSPACE-hard, etc.)

In this paper we bridge the gap between the areas by asking questions of common interest. In fact, we get very close to completely filling the gap (see §8.1). As the reader shall see, we use similar number theoretic tools (Ostrowski representation of integers), as well as computational ideas.

Part of the problem in presenting the results is addressing both audiences, so we structure the paper a little differently. We first state the results and then give concise yet lengthy backgrounds in both area separately, emphasizing advances from both directions.

1.2. Main results. Let $\alpha$ be a fixed irrational number. The reader can always assume that $\alpha$ is algebraic, although some results in the paper hold in full generality.

Let $S_\alpha = (\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \alpha x)$. This is a first order theory over the reals, with a predicate for the integers, which allows addition and scalar multiplication by $\alpha$. This is an extension of Presburger Arithmetic. It is still decidable when $\alpha$ is quadratic [H2], but undecidable otherwise [HTy] (see below).
An integer sentence in $S_{\alpha}$, is a sentence whose quantified variables are constrained to integer values. Such sentences have the form:

\[(1.1) \quad S = Q_1x_1 \in \mathbb{Z}^{n_1} \ldots Q_rx_r \in \mathbb{Z}^{n_r} \Phi(x_1, \ldots, x_r),\]

where $Q_1, \ldots, Q_r \in \{\forall, \exists\}$ are $r$ alternating quantifiers, and $\Phi$ is a Boolean combination of linear inequalities in $x_1, \ldots, x_r$ with coefficients and constant terms in $\mathbb{Z}[\alpha]$. As the number $r$ of alternating quantifier blocks and the dimensions $(n_1, \ldots, n_r)$ increase, such sentences become harder to decide, and determining exactly how hard is an important problem in computational complexity.

Sentences $S$ in (1.1) have a nice geometric interpretation in many special cases. When the Boolean formula $\Phi$ is a conjunction of linear equations and inequalities, it gives a convex polyhedron $P$ defined over $\mathbb{Q}[\alpha]$. For $r = 1$ and $Q_1 = \exists$, the sentence $S$ asks for existence of an integer point in $P$:

\[(1.2) \quad \exists x \in P \cap \mathbb{Z}^n.\]

In a special case of $r = 2$, $Q_1 = \forall$ and $Q_2 = \exists$, the sentence $S$ asks whether projections of integer points in a convex polyhedron $P \subseteq \mathbb{R}^{k+m}$ cover all integer points in another polyhedron $R \subseteq \mathbb{R}^k$:

\[(1.3) \quad \forall x \in R \cap \mathbb{Z}^k \exists y \in \mathbb{Z}^m : x + y \in P.\]

Here both $P$ and $R$ are defined over $\mathbb{Q}[\alpha]$. Further variations on the theme and increasing number of quantifiers allow most general formulas with integer valuations of the polytope algebra (see e.g. [Bar]).

We think of $\alpha \in \overline{\mathbb{Q}}$ as being given by its defining $\mathbb{Z}[x]$ polynomial of degree $d$, with a rational interval to single out a unique root. We say that $\alpha \in \overline{\mathbb{Q}}$ is quadratic if $d = 2$. Similarly, the elements $\gamma \in \mathbb{Z}[\alpha]$ are represented in the form $\gamma = c_0 + c_1\alpha + \ldots + c_{d-1}\alpha^{d-1}$, where $c_0, \ldots, c_{d-1} \in \mathbb{Z}$. For example, $\alpha = \sqrt{2}$ is quadratic, is given by $\{\alpha^2 - 2 = 0, \alpha > 0\}$, so that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.

For $\gamma \in \mathbb{Z}[\alpha]$, the encoding length $\ell(\gamma)$ is the total bit length of $c_i$'s defined above. Similarly, the encoding length $\ell(S)$ is defined to be the total bit length of all symbols in $S$, with integer coefficients and constants represented in binary. In the following results, the constants $K, C$ vary from one context to another.

**Theorem 1.1.** Let $\alpha \in \overline{\mathbb{Q}}$ be a quadratic irrational number, and let $r \geq 1$. An integer sentence $S$ in $S_{\alpha}$ with $r$ alternating quantifier blocks can be decided in time at most

$$K2^{c\ell(S)^{1+2^{1-\ell(S)}}}$$

(tower of height $r$),

where the constants $K, C > 0$ depend only on $\alpha$.

In the opposite direction, we have the following lower bound:

**Theorem 1.2.** Let $\alpha \in \overline{\mathbb{Q}}$ be a quadratic irrational number, and let $r \geq 4$. Then, deciding integer sentences in $S_{\alpha}$ with $r$ alternating quantifier blocks and at most $cr$ variables and inequalities requires space at least:

$$K2^{c\ell(S)^{1+2^{1-\ell(S)}}}$$

(tower of height $r - 3$),

where the constants $c, K, C > 0$ only depend on $\alpha$. 

These results should be compared with the triply exponential upper bound and doubly exponential lower bounds for PA (discussed below). The borderline case of \( r = 3 \), the problem is especially interesting. We give the following lower bound, which only need a few variables:

**Theorem 1.3.** Let \( \alpha \in \mathbb{Q} \) be a quadratic irrational number. Then, deciding \( \exists^6 \forall^4 \exists^{11} \) integer sentences in \( S_\alpha \) with at most \( K \) inequalities is \( \text{PSPACE}-\text{hard} \), where the constant \( K \) depends only on \( \alpha \). Furthermore, for \( \alpha = \sqrt{2} \), one can take \( K = 10^6 \).

This should be compared with Grädel’s theorem on \( \Sigma^2_2 \)-completeness for \( \exists \forall \exists \) integer sentences in PA [Grä] (also discussed below).

On the other hand, for non-quadratic irrational numbers, we have:

**Theorem 1.4.** Let \( \alpha \in \mathbb{Q} \) be a non-quadratic irrational number. Then \( \exists^K \forall^K \exists^K \forall^K \) integer sentences in \( S_\alpha \) are undecidable, where \( K = 20000 \).

### 1.3. Complexity background.

Presburger Arithmetic PA is the decidable first order theory of \( (\mathbb{Z}, <, +) \), first introduced by Presburger in [Pre] and extensively studied by Skolem and others. A quantifier elimination algorithm for PA was given by Cooper [Coo] to effectively solve the decision problem of PA. General PA sentences have no bounds on the numbers of quantifiers, variables and Boolean operations. Oppen [Opp] showed that such sentences can be decided in at most triply exponential time (see also [RL]). In the opposite direction, a nondeterministic doubly exponential lower bound was obtained by Fischer and Rabin [FR] (see also [W1]) for deciding general PA sentences. As one restricts the number of alternations, the complexity of PA drops down by roughly one exponent (see [Für, Sca, RL]), but still remains exponential.

For a bounded number of variables, two important cases are known to be polynomial time decidable, namely the analogues of (1.2) and (1.3) with rational polyhedra \( P \) and \( R \). These are classical results by Lenstra [Len] and Kannan [Kan], respectively. Scarpellini [Sca] showed that all \( \exists^n \)-sentences are still polynomial time decidable for every \( n \) fixed. However, for two alternating quantifiers, Schöning proved in [Sch] that deciding \( \exists y \forall x : \Phi(x, y) \) is \( \text{NP} \)-complete. Here \( \Phi \) any Boolean combination of linear inequalities in two variables, instead of those in the particular form (1.3). This improved on an earlier result by Grädel in [Grä], who also showed that PA sentences with \( m + 1 \) alternating quantifier blocks and \( m + 5 \) variables are complete for the \( m \)-th level in the Polynomial Hierarchy \( \text{PH} \). In these results, the number of inequalities (atoms) in \( \Phi \) is still part of the input, i.e., allowed to vary.

Much of the recent work concerns the most restricted PA sentences:

\[
Q_1 z_1 \ldots Q_{r+1} z_{r+1} Q_{r+2} z_{r+2} : \Phi(z_1, \ldots, z_m, z_{m+1})
\]

for which the number of alternations \( (r + 2) \), number of variables and number of inequalities in \( \Phi \) are all fixed. Thus, the input of (1.4) is essentially a bounded list of integer coefficients and constants in \( \Phi \), encoded in binary. For \( r = 0 \), such sentences are polynomial time decidable by [Woo]. For \( r = 1 \), Nguyen and Pak [NP] showed that deciding \( \exists y \forall z : \Phi(x, y, z) \) is \( \text{NP} \)-complete. More generally, they showed that such sentences with \( r + 2 \) alternations, \( O(r) \) variables and inequalities are complete for the \( r \)-th level in \( \text{PH} \). Thus, limiting the “format” of a PA formula does not reduce the complexity by a lot. This is our main motivation for the lower bounds in theorems (1.3) and (1.2) for \( S_\alpha \).

We emphasize that the sudden jump from polynomial hierarchy in PA to super-exponential complexity in \( S_\alpha \) is due to the power of irrational quadratics. Specifically, any irrational
quadratic $\alpha$ has an infinite periodic continued fraction. From here, we can work with Ostrowski representations of integers in base $\alpha$, and code string relations such as shifts, suffix/prefix and subset, which were not all possible in PA. Such operations are rich enough to encode arbitrary automata computation, and in fact Turing Machine computation in bounded space.

Finally, in [KP], Khachiyan and Porkolab prove that for a bounded number of variables, one can decide in polynomial time if a convex semialgebraic set contains an integer point (see Theorem 8.3). In particular, for linear equations and inequalities, this implies the Integer Programming with algebraic coefficients:

**Theorem 1.5** ([KP]). Let $K = \mathbb{Q}$ be the field of algebraic numbers. For every fixed $n$, sentences of the form $\exists y \in \mathbb{Z}^n : Ay \leq \overline{b}$ with $A \in K^{m \times n}, \overline{b} \in K^m$ can be decided in polynomial time.

Note that the system $Ay \leq \overline{b}$ in the theorem can involve arbitrary algebraic irrationals. This is a rare positive result on irrational polyhedra. In fact, for a non-quadratic $\alpha$, this gives the only positive result on $S_\alpha$ that we know of (cf. §8.2).

1.4. Decidability background. It has long been known that the theory of $(\mathbb{R}, <, +, \mathbb{Z})$, equivalently the theory of $S_\alpha$ for rational $\alpha$, is decidable (arguably due to Skolem [Sko] and later rediscovered independently by Weispfenning [W2] and Miller [Mil]). However, the decidability of the theory of $S_\alpha$ for irrational $\alpha$ was determined only recently.

Hieronymi and Tychonievich showed in [HTy] that if an expansion of $(\mathbb{R}, <)$ can define a discrete set $D \subseteq \mathbb{R}_{\geq 0}$ and also satisfies a certain reasonable denseness condition, then it can actually define every subset of $D^n$ for every $n$. As an application, they proved the following result:

**Theorem 1.6** ([HTy]). For any $\alpha, \beta, \gamma \in \mathbb{R}$ that are $\mathbb{Q}$-linearly independent, the structure $(\mathbb{R}, <, +, \alpha\mathbb{Z}, \beta\mathbb{Z}, \gamma\mathbb{Z})$ defines multiplication, and thus its theory is undecidable.

Since $1, \alpha, \alpha^2$ are $\mathbb{Q}$-linearly independent for a non-quadratic $\alpha$, the theory of $S_\alpha$ is undecidable for such $\alpha$. Indeed, a careful analysis of their work shows that this result can be further specialized to give undecidability of integer sentences in $S_\alpha$:

**Corollary 1.7** ([HTy]). For any non-quadratic $\alpha$, integer sentences (1.1) of $S_\alpha$ are undecidable.

The main contrast between Theorem 1.5 and 1.7 is that the former only considers $\exists$-sentences. Neither Corollary 1.7 nor an upper bound on $r$ in (1.1) needed for undecidability was stated explicitly in [HTy], but both can be obtained by careful analysis of the proof. In Theorem 1.4, we not only give a proof of Corollary 1.7 but also explicitly quantify this result by showing that 4 alternating quantifier blocks are enough for undecidability. While our argument is based on the ideas in [HTy], substantial extra work is necessary to reduce the number of alternations to 4 from the upper bound implicit in the proof of Theorem 1.6.

When $\alpha$ is quadratic, Hieronymi proved the following surprising result:

**Theorem 1.8** ([H1, H2]). For $\alpha$ quadratic, integer sentences (1.1) of $S_\alpha$ are decidable. More generally, the structure $S_\alpha$ defines a model of Monadic Second Order Logic (MSO), and vice versa.
By this result for $\alpha$ quadratic, to decide integer sentences (1.1), one can translate them into corresponding sentences in MSO and then decide the latter. Thus, upper and lower complexity bounds for decision in MSO can theoretically be transferred to $S_{\alpha}$. However, an efficient direct translation between $S_{\alpha}$ and MSO was not described in [H1, H2]. Ideally, one would like to translate a sentence from $S_{\alpha}$ to MSO, and vice versa, with as few extra alternations as possible. In theorems 1.1 and 1.2, we explicitly quantify this translation.

1.5. Proofs outline. The most powerful feature of $S_{\alpha}$ is that we can talk about Ostrowski representation of integers, which will be used as the main encoding tool. We first obtain the upper bound in Theorem 1.1 by directly translating (1.1) into the language of automata using Ostrowski encoding. Next, we show the lower bound for 3 alternating quantifiers (Theorem 1.3) by a general argument on the Halting Problem with polynomial space constraint, again using Ostrowski encoding.

We generalize the above argument to get lower bound for any $r \geq 3$ alternating quantifier blocks (Theorem 1.2). This is done by first translating sentences from the weak Second Order Monadic logic (WMSO) to $S_{\alpha}$ sentences with only one extra alternation, and then invoke a known tower lower bound for WMSO. Overall, the paper make transitions between $S_{\alpha}$, finite automata and WMSO, all of which are different incarnations of the same logic theory.

Finally in the proof of Theorem 1.4 we can again use the expressibility of Ostrowski representation to reduce the upper bound of the number of alternating quantifier blocks needed for undecidability in $S_{\alpha}$ for non-quadratic $\alpha$. The use of Ostrowski representations allows us to replace more general arguments from [HTY] by explicit computations, and thereby reduce the quantifier-complexity of certain integer sentences in $S_{\alpha}$.

2. Notations

Ostrowski representation and continued fractions play a crucial role throughout the paper, and are first introduced in Section 3. We use the following notation:

- $\text{Ost}(X)$ denotes the set of convergents $q_n$ with non-zero coefficients in the Ostrowski representation of $X \in \mathbb{N}$.
- We write $v \in \text{Ost}(X)$ if $v$ is a convergent with a non-zero coefficient in the Ostrowski representation of $X$.

3. Preliminaries

3.1. Continued fractions and Ostrowski representation. Let $\alpha = [a_0; a_1, a_2, \ldots]$ be any irrational, with $a_i \in \mathbb{Z}_+$. The convergents of $\alpha$ follow the recurrence relation:

\begin{align}
(p_{-1}, q_{-1}) &= (1, 0); \quad (p_0, q_0) = (a_0, 1);
(p_n, q_n) &= (a_n p_{n-1} + p_{n-2}, a_n q_{n-1} + q_{n-2}) \quad \text{for } n \geq 1.
\end{align}

This can be written as:

\begin{align}
\begin{pmatrix} p_n \\ q_n \end{pmatrix} &= \left( \begin{array}{cc} a_n & 1 \\ 0 & 1 \end{array} \right) \Gamma_0 \cdots \Gamma_n \\
\end{align}

where $\Gamma_i = \left( \begin{array}{cc} a_i & 1 \\ 1 & 0 \end{array} \right)$. Let $\beta_n = \alpha q_n - p_n$. They have the properties:

\begin{align}
\beta_n > 0 \text{ if } 2 \mid n, \quad \beta_n < 0 \text{ if } 2 \nmid n.
\end{align}
\( (3.4) \quad \beta_0 > -\beta_1 > \beta_2 > -\beta_3 > \ldots \)

\( (3.5) \quad -\beta_n = a_{n+2}\beta_{n+1} + a_{n+3}\beta_{n+2} + a_{n+4}\beta_{n+3} + \ldots \quad \forall n \in \mathbb{N}. \)

These can be easily proved using \((3.1)\). We refer to [RS] for the basics of continued fractions.

**Fact 3.1.** Each \( X \in \mathbb{N} \) has a unique \( \alpha \)-Ostrowski representation:

\( (3.6) \quad X = \sum_{n=0}^{N} b_{n+1}q_n. \)

where \( 0 \leq b_1 < a_1, 0 \leq b_{n+1} \leq a_{n+1} \) and \( b_n = 0 \) whenever \( b_{n+1} = a_{n+1} \).

**Proof.** See [RS] Ch. II-§4].

From now on, when \( \alpha \) and \( X \) are clear from the context, we refer to \((3.6)\) simply as the Ostrowski representation of \( X \). We also denote the coefficient \( b_{n+1} \) by \( [q_n] \). Denote by \( \text{Ost}(X) \) the set of \( q_n \) with \([q_n] > 0\).

We set \( \zeta_\alpha := [a_1; a_2, \ldots] \), so that \( \zeta_\alpha = \frac{1}{\alpha - a_0} = \frac{1}{\alpha - [\alpha]} \). Let \( I_\alpha := [-\frac{1}{\zeta_\alpha}, 1 - \frac{1}{\zeta_\alpha}] \). Define \( f_\alpha : \mathbb{N} \to [0, 1] \) to be the function that maps \( X \) to \( \alpha X - U \), where \( U \) is the unique natural number such that \( \alpha X - U \in I_\alpha \). In other words:

\( (3.7) \quad f_\alpha(X) = \alpha X - U \iff -\frac{1}{\zeta_\alpha} \leq \alpha X - U < 1 - \frac{1}{\zeta_\alpha}. \)

Define \( g_\alpha(X) = U \), so that \( \alpha X = f_\alpha(X) + g_\alpha(X) \).

**Fact 3.2.** Let \( \beta_n = \alpha q_n - p_n \). We have:

\( (3.8) \quad f_\alpha(X) = \sum_{n=0} b_{n+1}\beta_n \quad \text{and} \quad g_\alpha(X) = \sum_{n=0} b_{n+1}p_n, \)

where the coefficients \( b_n \) are from \((3.6)\). Also \( f_\alpha(\mathbb{N}) = \{ f_\alpha(X) : X \in \mathbb{N} \} \) is a dense subset of the interval \([-\frac{1}{\zeta_\alpha}, 1 - \frac{1}{\zeta_\alpha}]\).

**Proof.** See [RS] Th. 1 on p. 25] and [RS] Th. 1 on p. 33].

3.2. **Periodic continued fractions.** An irrational \( \alpha \) is a quadratic if and only if it has a periodic continued fraction \( \alpha = [a_0; a_1, \ldots, a_m, b_0, b_1, \ldots, b_{k-1}] \). Let \( \beta = [b_0; b_1, \ldots, b_{k-1}] \).

It is clear that \( \beta = (\alpha + d)/(e\alpha + f) \) for some \( c, d, e, f \in \mathbb{Z} \). Therefore, sentences in the theory \((\mathbb{R}, <, +, \mathbb{Z}, x \to \alpha x)\) can be expressed in \((\mathbb{R}, <, +, \mathbb{Z}, x \to \beta x)\) and vice versa. Thus, for our complexity purposes, we can always assume that our quadratic irrational \( \alpha \) is purely periodic, i.e.,

\( (3.9) \quad \alpha = [a_0; a_1, \ldots, a_{k-1}] \)

with the minimum period \( a_0, \ldots, a_{k-1} \).

**Fact 3.3.** Let \( i \in \mathbb{N} \). There exist \( c_i, d_i \in \mathbb{Z} \) such that for every \( n \in \mathbb{N} \) with \( k | n \), we have:

\( (p_{n+i}, q_{n+i}) = c_i(p_n, q_n) + d_i(p_{n+1}, q_{n+1}). \)

The coefficients \( c_i, d_i \) can be computed in time \( \text{poly}(i) \).
Proof. By (3.2), we have:

\[
\begin{pmatrix}
 p_{n+i+1} & p_{n+i} \\
 q_{n+i+1} & q_{n+i}
\end{pmatrix} = 
\Gamma_0 \ldots \Gamma_{n+1} \Gamma_{n+2} \ldots \Gamma_{n+i+1} = 
\begin{pmatrix}
 p_{n+i+1} & p_n \\
 q_{n+i+1} & q_n
\end{pmatrix} \Gamma_{n+2} \ldots \Gamma_{n+i+1}
\]

Since \( \Gamma_k \Gamma_t = \Gamma_t \) for every \( t, k \in \mathbb{N} \) and \( k|N \), we have \( \Gamma_{n+2} \ldots \Gamma_{n+i+1} = \Gamma_2 \ldots \Gamma_{i+1} \). Let

\[
(3.10) \quad \Gamma_2 \ldots \Gamma_{i+1} = \left( \begin{array}{cc}
 d'_{i} & d_i \\
 c'_i & c_i
\end{array} \right)
\]

we have

\[
\begin{pmatrix}
 p_{n+i+1} & p_{n+i} \\
 q_{n+i+1} & q_{n+i}
\end{pmatrix} = 
\begin{pmatrix}
 p_{n+i+1} & p_n \\
 q_{n+i+1} & q_n
\end{pmatrix} \left( \begin{array}{cc}
 d'_{i} & d_i \\
 c'_i & c_i
\end{array} \right)
\]

So \( p_{n+i}, q_{n+i} = c_i (p_n, q_n) + d_i (p_{n+1}, q_{n+1}) \) and \( c_i, d_i \) only depend on \( i \). Note that \( c_i, d_i \) can be computed in time \( \text{poly}(i) \) by (3.10). \( \square \)

**Remark 3.4.** For \( i = 0 \), we have \( c_0 = 1, d_0 = 0 \). For \( i = 1 \), we have \( c_1 = 0, d_1 = 1 \). By (3.10), if we let \( \gamma_i(v, v') := c_i v + d_i v' \) then they follow the recurrence:

\[
(3.11) \quad \gamma_0(v, v') = v, \ \gamma_1(v, v') = v', \ \gamma_i(v, v') = a_i \gamma_{i-1}(v, v') + \gamma_{i-2}(v, v'),
\]

as similar to (3.1).

**Fact 3.5.** There are fixed \( \mu, \nu, \mu', \nu' \in \mathbb{Q} \) such that

\[
p_n = \mu q_n + \mu' q_{n+k}, \quad q_n = \nu p_n + \nu' p_{n+k}
\]

for every \( n \in \mathbb{N} \).

**Proof.** Again from (3.2), for every \( n \geq 0 \):

\[
\begin{pmatrix}
 p_n \\
 q_n
\end{pmatrix} = \Gamma_0 \Gamma_1 \ldots \Gamma_n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Since \( \Gamma_{i+k} = \Gamma_i \), we have:

\[
\begin{pmatrix}
 p_{n+k} \\
 q_{n+k}
\end{pmatrix} = (\Gamma_0 \ldots \Gamma_{k-1}) (\Gamma_k \ldots \Gamma_{n+k}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\Gamma_0 \ldots \Gamma_{k-1}) (\Gamma_0 \ldots \Gamma_n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= (\Gamma_0 \ldots \Gamma_{k-1}) \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{k-2} & p_{k-1} \\ q_{k-2} & q_{k-1} \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}.
\]

Note that \( p_{k-1}, q_{k-1}, p_{k-1}, q_{k-2} \) are constants. From here we easily get \( \mu, \nu, \mu' \) and \( \nu' \). \( \square \)

### 3.3. Logical formulas for working with Ostrowski representation.

Let \( \alpha \) be any irrational, not just quadratic. The convergents \( \{p_n/q_n\} \) can be characterized by the best approximation property. Namely, \( u/v \) with \( v > 1 \) is a convergent \( p_n/q_n \) if and only if

\[
(3.12) \quad \forall w, z \ (0 < z < v) \rightarrow |w - \alpha z| > |u - \alpha v|.
\]

From this, we have \((u, v) = (p_n, q_n)\) and \((u', v') = (p_{n+1}, q_{n+1})\) if and only if they satisfy

\[
C_{\forall} (u, v, u', v') := 1 < v < v' \quad \forall w, z \ (0 < z < v') \rightarrow
\]

\[
|w - \alpha z| \geq |u - \alpha v| > |u' - \alpha v'|.
\]

Note that \( C_{\forall} \) is a \( \forall \)-formula. More generally, consider the formula:

\[
C_{\forall} (u_0, v_0, \ldots, u_k, v_k) := 1 < v_0 < v_1 < \cdots < v_k \land
\]

\[
(3.14) \quad \forall w, z \ A \sum_{i=0}^{k} (0 < z < v_{i+1} \rightarrow |w - \alpha z| \geq |u_i - \alpha v_i| > |u_{i+1} - \alpha v_{i+1}|).
\]
By (3.3), we have have

To see this, note that

\( \forall \) and also

\( \text{Ost}(\forall) \)

(Similar to lemmas 4.6, 4.7 and 4.8 in [H2])

Proof. We have:

Fact 3.7.

Remark 3.6. Hereafter, we assume \( \text{C}_\forall(u, v, u', v') = \text{true} \), i.e., \((u, v) = (p_n, q_n)\) and \((u', v') = (p_{n+1}, q_{n+1})\) for some \( n \in \mathbb{N} \).

Define the following quantifier free relations:

\[
\text{After}(u, v, u', v', Z, Z') := (-\alpha v + u < \alpha Z - Z' < -\alpha v' + u')
\]

\[
(3.15)
\]

\[
\text{\(\widetilde{\text{After}}\)}(u, v, u', v', Z, Z') := (-\alpha v + u - \alpha v' + u' < \alpha Z - Z' < -\alpha v' + u')
\]

\[
(3.16)
\]

Fact 3.7. We have:

- \( \text{Ost}(Z) \subset \{q_{n+1}, q_{n+2}, \ldots \} \) if and only if \( \text{After}(u, v, u', v', Z, Z') \) holds for some \( Z' \).
- \( \text{Ost}(Z) \subset \{q_{n+1}, q_{n+2}, \ldots \} \) and \([q_{n+1}] < a_{n+2} \) if and only if \( \text{After}(u, v, u', v', Z, Z') \) holds for some \( Z' \).

Also \( Z' \) is uniquely determined by \( Z \) if \( \text{After} \) or \( \text{\(\widetilde{\text{After}}\)} \) holds.

Proof. (Similar to lemmas 4.6, 4.7 and 4.8 in [H2])

i) Assume \( n \) is odd. If \( \text{Ost}(Z) \subset \{q_{n+1}, q_{n+2}, \ldots \} \), then its Ostrowski representation is

\( Z = \sum_{k=n+1}^{N} b_{k+1}q_k \)

for some \( N \geq n + 1 \). From Fact 3.2, we have

\[
f_a(Z) = \sum_{k=n+1}^{N} b_{k+1} \beta_k.
\]

By (3.3), we have have \( \beta_k > 0 \) if \( k \) is odd and \( \beta_k < 0 \) if \( k \) is even. Combined with \( b_{k+1} \leq a_{k+1} \), we have:

\[
a_{n+3} \beta_{n+2} + a_{n+5} \beta_{n+4} + \ldots < f_a(Z) = \sum_{k=n+1}^{N} b_{k+1} \beta_k < a_{n+2} \beta_{n+1} + a_{n+4} \beta_{n+3} + \ldots
\]

By (3.3), this can be written as

\[
-\beta_{n+1} < f_a(Z) < -\beta_n.
\]

By (3.7), we have

\[
f_a(Z) = \alpha Z - Z',
\]

where \( Z' \in \mathbb{N} \) is unique such that \( aZ - Z' \in I_a \). Also note that

\[
\beta_n = \alpha v - u \quad \text{and} \quad \beta_{n+1} = \alpha v' - u'.
\]

So the above inequalities can be written as

\[
-\alpha v' + u' < \alpha Z - Z' < -\alpha v + u.
\]

When \( n \) is even, the inequalities reverse to

\[
-\alpha v + u < \alpha Z - Z' < -\alpha v' + u'.
\]

Thus \( \text{Ost}(Z) \subset \{q_{n+1}, q_{n+2}, \ldots \} \) implies \( \text{After}(u, v, u', v', Z, Z') \). The converse direction can be proved similarly, using (3.4) and (3.5).

ii) The only difference here is that \([q_{n+1}] = b_{n+2} \) can be at most \( a_{n+2} - 1 \). Details are left to the reader. \( \square \)

The relation \( v \in \text{Ost}(X) \), meaning that \( v = q_n \) appears in \( \text{Ost}(X) \), is \( \exists \)-definable:

\[
(3.17) \quad \exists Z_1, Z_2, Z_3 \ (v \leq Z_1 < v') \land \text{\(\widetilde{\text{After}}\)}(u, v, u', v', Z_2, Z_3) \land X = Z_1 + Z_2.
\]

and also \( \forall \)-definable:

\[
(3.18) \quad \forall Z_1, Z_2, Z_3 \left( (Z_1 < v) \land \text{\(\widetilde{\text{After}}\)}(u, v, u', v', Z_2, Z_3) \right) \rightarrow Z_1 + Z_2 \neq X.
\]

To see this, note that \( v \notin \text{Ost}(X) \) if and only if \( X = Z_1 + Z_2 \) for some \( Z_1, Z_2 \) with \( \text{Ost}(Z_1) \subset \{q_0, q_1, \ldots, q_{n-1}\} \) and \( \text{Ost}(Z_2) \subset \{q_{n+1}, q_{n+2}, \ldots \} \).

We will need one more quantifier-free formula:

\[
\text{Compatible}(u, v, u', v', X, Z, Z') := X < v' \land \text{\(\widetilde{\text{After}}\)}(u, v, u', v', Z, Z') \land
\]

\[
(3.19) \quad \left( X \geq v \rightarrow \text{\(\widetilde{\text{After}}\)}(u, v, u', v', Z, Z') \right).
\]
This is satisfied if and only if
- \( \text{Ost}(X) \subseteq \{q_0, \ldots, q_n\} \) (by \( X < v' \)),
- \( \text{Ost}(Z) \subseteq \{q_{n+1}, q_{n+2}, \ldots\} \) (by After),
- If \( q_n \in \text{Ost}(X) \), then \([q_{n+1}] \) in \( \text{Ost}(Z) \) is strictly less than \( a_{n+2} \) (by After).

In other words, Compatible is satisfied if and only if \( \text{Ost}(X) \) and \( \text{Ost}(Z) \) can be directly concatenated at the point \( v = q_n \) to form \( \text{Ost}(X + Z) \) (see 3.6).

4. QUADRATIC IRRATIONALS: UPPERBOUND

In this section we prove Theorem 1.1. It should be emphasized that the tower height in Theorem 1.1 only depends on the number of alternating quantifiers, but not on the number of variables in the sentence \( S \). First, we consider the case of a quantifier free formula.

**Proposition 4.1.** Let \( F(x) \) be a quantifier free (integer) formula in \( S_n \), i.e., a Boolean combination of linear inequalities in \( x \in \mathbb{Z}^n \) with coefficients/constants in \( \mathbb{Z}[\alpha] \). Then there is an automaton of size \( 2^{\delta \ell(F)} \) recognizing the set of solutions of \( F \). The constant \( \delta \) only depends on \( \alpha \).

**Proof.** Each variable \( x \) in \( F \) takes value over \( \mathbb{Z} \), but can be replaced by \( x_1 - x_2 \) for two variables \( x_1, x_2 \in \mathbb{N} \). So we can assume that all variables take values over \( \mathbb{N} \). Recall that coefficients/constants in \( \mathbb{Z}[\alpha] \) are given in the form \( c\alpha + d \) with \( c, d \in \mathbb{Z} \). So now each inequality in \( F \) can be reorganized into the form:

\[
\overline{\sigma}y + \alpha \overline{b}z \leq \overline{c}t + \alpha \overline{d}w.
\]

Here \( \overline{\sigma}, \overline{b}, \overline{c}, \overline{d} \) are tuples coefficients in \( \mathbb{N} \), and \( y, z, t, w \) are subtuples of \( x \). Now, for each homogeneous term \( \overline{\sigma}y \), we add in an additional variable \( u = \overline{\sigma}y \) and replace each appearance of \( \overline{\sigma}y \) in the inequalities by \( u \). By doing so, we introduce extra variables, but still keep the length \( \ell(F) \) linear. Now our formula splits into two parts. The first part consists of integer linear equalities:

\[
(*) \quad u = \overline{\sigma}y.
\]

The second part consists of inequalities of the form:

\[
(**) \quad u + \alpha v \leq w + \alpha z.
\]

We encode integer variables by their Ostrowski representations, and build an automaton that recognizes the solutions of \( F \). In other words, each \( x \in \mathbb{N} \) is encoded by the string \( \widehat{x} = b_1 b_2 \ldots \), where the \( b_n \)'s are from \( \{0, 1\} \). Here only a finite number of \( b_n \)'s are nonzero, so \( \widehat{x} \) is a finite string. Since \( a_n \)'s are periodic (3.9) and \( b_n \leq a_n \), we are working with a finite alphabet.

First, by the result in [11], integer addition in Ostrowski representation is recognizable by a finite automaton. In other words, the function \( \widehat{x} \mapsto \widehat{x} + \widehat{y} \) is regular. Now we rewrite each equality \( u = \overline{\sigma}y \) into single additions, using the doubling trick. For example, the equality \( u = 5y + 2z \) is equivalent to the following system:

\[
y_1 = y + y, \quad y_2 = y_1 + y, \quad y_3 = y_2 + y, \quad z_1 = z + z, \quad u = y_3 + z_1.
\]

Again, we are introducing additional variables while keeping \( \ell(F) \) linear. Each single addition \( x = y + z \) is recognizable by a finite automaton. Taking product of all such automata, one for each addition, we get a single automaton of size \( 2^{\gamma \ell(F)} \) that recognizes the first part \((*)\). Here \( \gamma \) is some constant dependent on \( \alpha \).
Now we build an automaton for each inequality (**), and later take their product automaton. Recall $f_\alpha$ and $g_\alpha$ from §3.2 and Fact 3.2. We have $\alpha x = f_\alpha(x) + g_\alpha(x)$ for every $x \in \mathbb{Z}$. Here $g_\alpha(x) \in \mathbb{Z}$ and $f_\alpha(x)$ always lies in the unit length interval $I_\alpha$. For $u, v, w, z \in \mathbb{N}$, we have $u + \alpha v < w + \alpha z$ if and only if:

$$u + g_\alpha(v) < w + g_\alpha(z), \text{ or } u + g_\alpha(v) = w + g_\alpha(z), f_\alpha(v) < g_\alpha(z)$$

So the proof is done if we can show that for input $u, v \in \mathbb{N}$:

i) The relation $u < v$ is recognizable by a finite automaton.

ii) The relation $f_\alpha(u) < f_\alpha(v)$ is recognizable by a finite automaton.

iii) The function $g_\alpha : \mathbb{Z} \to \mathbb{Z}$ is recognizable by a finite automaton.

Tasks i) and ii) are straightforward from basic properties of Ostrowski representation. We have $x < y$ if and only if $\hat{x}$ is lexicographically smaller than $\hat{y}$ when read from right to left. Also if $\hat{x} = b_1 b_2 \ldots$ and $\hat{y} = b_1' b_2' \ldots$ and $n$ is the smallest index where $b_n \neq b_n'$, then:

- $n$ odd : $b_n < b_n'$ if and only if $f_\alpha(x) < f_\alpha(y)$,
- $n$ even : $b_n < b_n'$ if and only if $f_\alpha(x) > f_\alpha(y)$.

(see [H2 Fact 2.13]). We have iii) left to show. \hfill \Box

**Lemma 4.2.** The function $g_\alpha : \mathbb{Z} \to \mathbb{Z}$ is recognizable by a finite automaton with Ostrowski encoding.

**Proof of Lemma 4.2.** We can assume that $\alpha$ is purely periodic, with minimum period $k$ (see Section 3.2). Also from Fact 3.3 there are fixed $\mu, \mu' \in \mathbb{Q}$ such that

$$p_n = \mu q_n + \mu' q_{n+k} \quad \text{for every } n \geq 0.$$

For $x \in \mathbb{N}$ with Ostrowski representation $x = \sum_{n=0}^{N} b_{n+1} q_n$ we define:

$$\text{Shift}(x) := \sum_{n=0}^{N} b_{n+1} q_{n+k}.$$

In other words, if $\hat{x} = b_1 b_2 \ldots$ then $\text{Shift}(x) = \underbrace{0^k b_1 b_2 \ldots}$ . So $x \mapsto \text{Shift}(x)$ is clearly recognizable by a finite automaton. By Fact 3.3

$$g_\alpha(x) = \sum_{n=0}^{N} b_{n+1} p_n = \sum_{n=0}^{N} b_{n+1}(\mu q_n + \mu' q_{n+k}) = \mu x + \mu' \text{Shift}(x).$$

Since $g_\alpha(x)$ is a linear combination of $x$ and $\text{Shift}(x)$ and linear equations are regular ([HTe]), we have an automaton for $g_\alpha : \mathbb{Z} \to \mathbb{Z}$.[4]

\hfill \Box

**Proof of Theorem 1.1.** Given the sentence (1.1), by negation, we can assume $Q_1 = \exists$. First, we build an automaton $A$ of size $2^{\delta \ell(F)}$ to recognize the quantifier free part $\Phi(x_1, \ldots, x_r)$.[4] Then we apply the power set construction (see e.g. [HUM §2.3.5]) to eliminate $Q_2 x_2 \ldots Q_r x_r$. This blows up the size of $A$ by at most $r-1$ exponentiations. Thus, the resulting automaton $A'$ has size at most a tower of height $r$ in $\delta \ell(F)$. Now we still have the outer quantifier $Q_1 = \exists$ remaining, i.e., we still need to decide if $A'$ has a solution. This is doable by a simple reachability argument, which runs in linear time relative to the size of $A'$.

---

[2]: The case of a sharp inequality can be handled similarly.
[3]: By clearing denominators in $\mu, \mu'$ and building automata for single additions.
[4]: Actually, we first need to make $Q_r = \exists$ so that additional variables in the proof of Lemma 4.4 can be inserted after $Q_r$. After that we make $Q_1 = \exists$. Apply negations whenever necessary.
5. Quadratic irrationals: PSPACE-hardness

In this section we prove Theorem 5.1. We will first show the lower bound for a general quadratic irrational \( \alpha \) (Theorem 5.1), and then specialize to \( \alpha = \sqrt{2} \) (Corollary 5.2). By a short sentence, we mean one with an integer sentence in \( \mathcal{S}_\alpha \) with a bounded number of variables, quantifiers and atoms (inequalities).

**Theorem 5.1.** Let \( \alpha \) be a fixed quadratic irrational and \( \mathcal{S}_\alpha = (\mathbb{R}, <, +, \mathbb{Z}, x \to \alpha x) \). Then deciding short \( \exists \forall \exists \) sentences in the theory \( \mathcal{S}_\alpha \) is PSPACE-hard.

The most important property for any quadratic irrational \( \alpha \) is the periodicity of its continued fraction. Before proving Theorem 5.1, we construct in §5.1 some explicit formulas in \( \mathcal{S}_\alpha \) to deal with the Ostrowski representation of an integer, in this case exploiting the periodicity of \( \alpha \). Then we recall the definitions of Turing machine computations in Subsection 5.2. The proof of Theorem 5.1 is in Subsection 5.3 which translates Turing machine computations into Ostrowski representations of integers. An explicit bound on the number of variables and inequalities for the constructed short sentences are given in §5.4 where we also treat the case \( \alpha = \sqrt{2} \).

5.1. Ostrowski representation for quadratic irrationals. We only need to consider a purely periodic \( \alpha \) with minimum period \( k \) (see Section 3.2). Let \( K = \text{lcm}(2, k) \).

We can define the set of convergents \( (p_n, q_n) \) for which \( K|n \). Recall \( \gamma_i \) from Remark 3.2 (also see Fact 3.3). Now define the formula:

\[
D^K_\gamma(u, v, u', v') = 1 < v < v' \land 0 < \alpha v - u \land \forall w, z
\]

\[
\bigwedge_{i=0}^{k+1} \left( 0 < z < \gamma_{i+1}(v, v') \rightarrow \right) \left| w - \alpha z \right| \geq |\gamma_{i}(u, u') - \alpha \gamma_{i}(v, v')| > |\gamma_{i+1}(u, u') - \alpha \gamma_{i+1}(v, v')|.
\]

We claim that \( D^K_\gamma \) is satisfied if and only if \( (u, v) = (p_{tk}, q_{tk}) \) and \( (u', v') = (p_{tk+1}, q_{tk+1}) \) for some \( t > 0 \). First, the condition \( \forall w, z [0 < z < \gamma_{i+1}(v, v') \rightarrow \ldots] \) implies that the pairs \( \gamma_i(u, u'), \gamma_i(v, v') \) \( 0 \leq i \leq k+1 \) are \( k+2 \) consecutive convergents (see §3.12 and §3.13). In other words, there is an \( n > 0 \) such that:

\[
(\gamma_i(u, u'), \gamma_i(v, v')) = (p_{n+i}, q_{n+i}), \quad 0 \leq i \leq k+1.
\]

Also by Remark 3.3, we have \( (\gamma_0(u, u'), \gamma_0(v, v')) = (u, v) \) and \( (\gamma_1(u, u'), \gamma_1(v, v')) = (u', v') \). So \( (u, v) = (p_n, q_n) \) and \( (u', v') = (p_{n+1}, q_{n+1}) \). Then by (3.11):

\[
(\gamma_2(u, u'), \gamma_2(v, v')) = (a_2u' + u, a_2v' + v) = (a_2p_{n+1} + p_n, a_2q_{n+1} + q_n).
\]

must be the next convergent \( (p_{n+2}, q_{n+2}) \). Combined with (3.1), we have

\[
p_{n+2} = a_{n+2}p_{n+1} + p_n = a_2p_{n+1} + p_n,
\]

which implies \( a_{n+2} = a_2 \). Similarly, we have \( a_{n+i} = a_i \) for all \( 2 \leq i \leq k+1 \). Since \( k \) is the minimum period of \( \alpha \), we must have \( k|n \). Also because \( 0 < \alpha v - u = aq_n - pn \), we have \( 2|n \) (see (3.3)). Therefore, \( D^K_\gamma(u, v, u', v') \) is true if and only if there is some \( t \geq 1 \) such that \( (u, v) = (p_{tk}, q_{tk}) \) and \( (u', v') = (p_{tk+1}, q_{tk+1}) \). In prenex normal form, \( D^K_\gamma \) is a \( \forall \exists \)-formula.

Next, we can also define the set of convergents \( q_n \) for which \( M|n \), where \( M \) is a large multiple of \( K \) to be specified later. To do this, we take a large enough prime \( P \) and define:

\[
D^K_M(u, v, u', v') = D^K_\gamma(u, v, u', v') \land v \equiv q_0 \pmod{P} \land v' \equiv q_1 \pmod{P}
\]
Let \( M > 0 \) be the least multiple of \( K \) such that \((q_M, q_{M+1}) \equiv (q_0, q_1) \pmod{P}\). Then \( D^M(u, v, u', v') = 1 \) if and only if there is a \( t \geq 1 \) such that \((u, v) = (p_{tM}, q_{tM})\). If \( P \) is large then \( M \) should also be large. Note that congruences can be expressed by \( \forall \) with one extra variable\(^3\). So \( D^M \) is a \( \forall^3 \)-formula in prenex normal form.

**Remark 5.2.** The multiple \( M = mK \) exists because we have:

\[
\begin{pmatrix}
p_{mK+1} \\
p_{mK}
\end{pmatrix}
\begin{pmatrix}
p_{mK+1} \\
p_{mK}
\end{pmatrix}
= \Gamma_0 \ldots \Gamma_{mK+1} = \Gamma_0 \Gamma_1 (\Gamma_2 \ldots \Gamma_{K-1} \Gamma_0 \Gamma_1)^m
\]

and the matrix \( \Gamma_2 \ldots \Gamma_{K-1} \Gamma_0 \Gamma_1 \) is invertible mod \( P \). So there is a smallest \( m > 0 \) such that:

\[
\begin{pmatrix}
p_{mK+1} \\
p_{mK}
\end{pmatrix}
\begin{pmatrix}
p_{mK+1} \\
p_{mK}
\end{pmatrix}
\equiv \Gamma_0 \Gamma_1 = \begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} \pmod{P}.
\]

Also by the recurrence (3.1), we have \((p_{mK+1}, q_{mK+i}) \equiv (p_i, q_i) \pmod{P}\) for every \( i \).

Recall from (3.6) that every \( T \in \mathbb{N} \) has a unique Ostrowski representation:

\[
T = \sum_{n=0}^{N} b_{n+1} q_n,
\]

with \( 0 \leq b_1 < a_1 \), \( 0 \leq b_{n+1} \leq a_{n+1} \) and \( b_n = 0 \) if \( b_{n+1} = a_{n+1} \). We denoted \([q_n] := b_{n+1}\). For the rest of the proof, we only consider numbers \( T \) that satisfy:

\[
[q_n] \quad \text{if} \quad 2 \nmid n,
\]

\[
[q_n] = 0, 1 \quad \text{if} \quad 2|n.
\]

This is guaranteed by the following formula:

\[
\text{ZeroOne}_{\forall\exists}(T) = \forall u, v, u', v' \ C_{\forall}(u, v, u', v') \rightarrow \exists Z_1, Z_2, Z_3
\]

\[
\begin{pmatrix} 0 > \alpha v - u \rightarrow [Z_1 < v \land \text{After}(u, v, u', v', Z_2) \land T = Z_1 \lor Z_2] \end{pmatrix} \land
\begin{pmatrix} 0 < \alpha v - u \rightarrow [Z_1 < 2v \land \text{Compatible}(u, v, u', v', Z_1, Z_2, Z_3) \land T = Z_1 \lor Z_2] \end{pmatrix}
\]

Here \text{After} and \text{Compatible} were defined earlier. Note that \text{ZeroOne}_{\forall\exists} is a \( \forall^4\exists^3 \)-formula.

For two natural numbers \( T \) and \( X \), the formula:

\[
\text{Pref}_{\forall\exists}(X, T) = \forall u, v, u', v' \ (C_{\forall}(u, v, u', v') \land v \leq X \land X < v') \rightarrow
\exists Z, Z' \ \text{Compatible}(u, v, u', v', X, Z, Z') \land T = X + Z.
\]

is true exactly when \( \text{Ost}(X) \) forms a prefix of \( \text{Ost}(T) \) if viewed as 0/1 strings. Note that \text{Pref}_{\forall\exists} is a \( \forall^4\exists^2 \)-formula in prenex normal form.

### 5.2. Turing machines

Consider any PSPACE-complete language \( \mathcal{L} \subset \{0,1\}^* \) and a 1-tape Turing Machine \( \mathcal{M} \) that can decide it. This means that given an input \( x \in \{0,1\}^* \) on its tape \( \mathcal{T} \), \( \mathcal{M} \) will run in space \( \text{poly}(|x|) \) and output 1 if \( x \in \mathcal{L} \) and 0 otherwise. More precisely, we have \( \mathcal{T} = x0\ldots \) at the beginning, and \( \mathcal{T} = 10\ldots \) or \( \mathcal{T} = 00\ldots \) at the end. WLOG, we can also assume \( \mathcal{M} \) has a unique halting state \( H \).

In [NW], a small universal 1-tape Turing machine \( \mathcal{U} = (Q, \Sigma, \sigma_1, \delta, q_1, q_2) \), with \( |Q| = 8 \) states and \( |\Sigma| = 4 \) tape symbols\(^4\). Using \( \mathcal{U} \), we can simulate \( \mathcal{M} \) in polynomial time and

---

\(^3\)We have \( x_1 \equiv x_2 \pmod{P} \) if and only if \( \forall w \ x_1 - x_2 - Pw = 0 \lor |x_1 - x_2 - Pw| \geq P \).

\(^4\)\( Q \) - states, \( \Sigma \) - tape symbols, \( \sigma_1 \in \Sigma \) - blank symbol, \( \delta : Q \times \Sigma \rightarrow Q \times \Sigma \times \{L, R\} \) - transitions, \( q_1 \in Q \) - start state, \( q_2 \in Q \) - unique halt state.
space. More precisely, suppose $\mathcal{M}$ is a PSPACE-complete TM as describe above and $x$ is an input to $\mathcal{M}$. Then we can encode $\mathcal{M}$ and $x$ in polynomial time as a string $\langle \mathcal{M}x \rangle \in \Sigma^*$. Upon input $\langle \mathcal{M}x \rangle$, $U$ simulates $\mathcal{M}$ on $x$, and halts with one of the two possible configurations:

$$U(\langle \mathcal{M}x \rangle) = \text{“yes”} \quad \text{if} \quad \mathcal{M}(x) = 1, \quad U(\langle \mathcal{M}x \rangle) = \text{“no”} \quad \text{if} \quad \mathcal{M}(x) = 0.$$  

(5.6) Here “yes” and “no” are the final state-tape configurations of $U$, which correspond to $\mathcal{M}$’s final configurations ($H, 10\ldots$) and ($H, 00\ldots$), respectively. By the encoding in [NW], these final “yes”/“no” configurations of $U$ have lengths $O(|\mathcal{M}|)$, which are constant when we fix $\mathcal{M}$. Furthermore, the computation $U(\langle \mathcal{M}x \rangle)$ takes time/space polynomial in the time/space of the computation $\mathcal{M}(x)$. Since $\mathcal{M}(x)$ runs in space $\text{poly}(|x|)$, so does $U$ upon input $\langle \mathcal{M}x \rangle$.

Consider the simulation $U(\langle \mathcal{M}x \rangle)$. Denote by $T_i \in \Sigma^\lambda$ the contents of $U$’s tape on step $i$-th. Here $\lambda = \text{poly}(|x|)$ is a polynomial bound on the tape length. Also denote by $s_i \in \mathbb{Q}$ the state of $U$ on step $i$-th. The $i$-th head position of $U$ is some number $1 \leq \pi_i \leq \lambda - 1$.

Altogether, for step $i$, we can encode the tape content $T_i$, the state $s_i$ and the tape head position $\pi_i$ by the string:

$$T_i = [\times, \times][\times, T_i(1)] \ldots [\times, T_i(\pi_i - 1)] [\times, T_i(\pi_i)] [\times, T_i(\pi_i + 1)] \ldots [\times, T_i(\lambda - 1)].$$  

Here $\times$ is a special marker symbol and $T_i(j) \in \Sigma$ is the $j$-th symbol of $T_i$. The marker block $[\times, \times]$ is at the beginning of each $T'_i$, which is distinct from the other $\lambda - 1$ blocks in $T'_i$. Note that $T'_i$ has in total $\lambda$ blocks. Now we concatenate $T'_i$ over all steps $1 \leq i \leq \rho$, where $\rho$ is the terminating step of the simulation. Let

$$T = T'_1 \ldots T'_\rho.$$  

We call $T$ the transcript of $U$ on input $\langle \mathcal{M}x \rangle$, denoted by $T = U(\langle \mathcal{M}x \rangle)$. The last segment in $T'_\rho$ contains the “yes” configuration if and only if $\mathcal{M}(x) = 1$. In total, $T$ has $\lambda \rho$ blocks.

Denote by $\mathcal{B} = \{|\times, \times|\} \cup \{|\times| \times \Sigma\} \cup (Q \times \Sigma)$ the set of all possible blocks in $T$, with $|\mathcal{B}| = 37$. Let $B_t \in \mathcal{B}$ be the $t$-th block in $T$. By the transition rules of $U$, the block $B_{t+\lambda}$ should only depend on $B_{t-1}$, $B_t$ and $B_{t+1}$. Thus, there is a function $f : B^3 \rightarrow \mathcal{B}$ such that:

$$B_{t+\lambda} = f(B_{t-1}, B_t, B_{t+1}) \quad \text{for every} \quad 0 \leq t < \lambda(\rho - 1).$$

Note that for the separator block $[\times, \times]$, we should have $f(B, [\times, \times], B') = 0$ for all $B, B'$.

5.3. Proof of Theorem 5.1. Recall the formulas $\mathbf{D}_x^\eta$, $\mathbf{D}_y^\eta$, $\mathbf{ZeroOne}_{\gamma, \eta}$, $\mathbf{Pref}_{\gamma, \eta}$ from Section 5.1. We encode the transcript $T$ by a number $T \in \mathbb{N}$ satisfying (5.3). To be precise, first we view $\mathcal{B}$ as a set of $37$ distinct strings in $\{0, 1\}^6$, each containing at least one $1$. Then we pick a large enough prime $P$ in $\mathbf{D}_x^\eta$ so that $M > 10$. Recall the notation $[q_n]$ in (5.3). If $B_t \in \mathcal{B}$ is the $t$-th block in $T$, then we should have:

$$[q_{tM}][q_{tM+2}] \ldots [q_{tM+10}] = B_t \quad \text{and} \quad [q_{tM+12}] \ldots [q_{(t+1)M-2}] = 0 \ldots 0.$$  

(5.8) For the rest of the proof, we view $\text{Ost}(T)$ as a binary string, and use $B_t$ to denote its $t$-th block.

Let $(u,v) = (p_{tM}, q_{tM})$ and $(u',v') = (p_{tM+1}, q_{tM+1})$ for some $t \geq 1$. For every triple $B, B', B'' \in \mathcal{B}$, we will construct a formula $\mathbf{Read}_{\gamma, B, B', B''}^\eta(u,v,u',v',T)$ to check if the three blocks $B_{t-1}, B_t, B_{t+1}$ in $T$ match with $B, B', B''$ in the sense of (5.8). We will also construct

---

Footnote:

It actually takes linear space and and quadratic time.
a formula $\text{Next}^{B,B',B'}_{\exists}(u,v,u',v',T)$ to check if the block $B_{t+\lambda}$ in $T$ agrees with the transition function $f$, i.e., $B_{t+\lambda} = f(B,B'B'')$. For the rest of the proof, the meaning of $c_i,d_i,a,b$ will change depending on the context.

- Constructing $\text{Next}^{B,B',B'}_{\exists}$: Let $r_1 = \lambda M$ and $r_2 = (\lambda + 1)M$. Then the block $B_{t+\lambda}$ correspond to those $[q_{tM+i}]$ with $r_1 \leq i < r_2$. By Fact 5.3, we can write each convergent $(p_{tM+i},q_{tM+i})$ with $r_1 - 1 \leq i \leq r_2$ as a linear combination $c_i(u,v) + d_i(u',v')$. Here the coefficients $c_i,d_i \in Z$ are independent of $t$, but do depend on $\lambda$. They can be computed explicitly in time poly($\lambda$). Let $B = f(B,B',B'')$. Then we sum up all $q_{tM+r_1+2j}$ for every $0 \leq j < 6$ such that the $j$-th bit in $\tilde{B}$ is ‘1’. This sum can be expressed as $av + bv'$ for some $a,b \in Z$ computable in time poly($\lambda$). Note that $c_i,d_i$ and $a,b$ depend on $\lambda$ and also the triple $B,B',B''$. Then $B_{t+\lambda} = \tilde{B}$ if and only if we can uniquely write $T = W_1 + (av + bv') + W_2$, where $W_1 < q_{tM+r_1-1}$ and $\text{Ost}(W_2) \subset \{q_n : n \geq tM + r_2\}$. Let $Z_1 = W_1 + (av + bv')$ and $Z_2 = W_2$. They satisfy:

i) $0 \leq Z_1 - (av + bv') < q_{tM+r_1-1}$

ii) $\text{Ost}(Z_2) \subset \{q_n : n \geq tM + r_2\}$.

Then the formula we want is:

$$\text{Next}^{B,B',B''}_{\exists}(u,v,u',v',T) := \exists Z_1,Z_2,Z_3 \ i) \land \text{ii}) \land T = Z_1 + Z_2.$$  

Here i) is written directly as linear inequalities in $u,v,u'$ and $Z_1$. By 5.15, we can express ii) as $\text{After}(pt_{tM+r_2-1},q_{tM+r_2-1},pt_{tM+r_2},q_{tM+r_2},Z_2,Z_3)$, which is again linear inequalities in $u,v,u',v'$ and $Z_2,Z_3$.

- Constructing $\text{Read}^{B,B',B'}_{\exists}$: Note that the blocks $B_{t-1}B_tB_{t+1}$ in $T$ correspond to $[q_n]$ with $(t-1)M \leq n < (t+2)M$. So we just need to express $(pt_{tM+i},q_{tM+i})$ for $-M - 1 \leq i \leq 2M$ as linear combinations $c_i(u,v) + d_i(u',v')$. Then we sum up all $q_{tM+i}$ that should correspond to the ‘1’ bits in $B,B',B''$, which is again some linear combination $av + bv'$. This time the coefficients $c_i,d_i,a,b$ do not depend on $\lambda$ and can be computed in constant time. Now we have $B_{t-1}B_tB_{t+1} = B'B'B''$ if and only if we can uniquely write $T = Z_1 + Z_2$, where $Z_1$ and $Z_2$ satisfy two conditions i’-ii’) similar to i-ii) above. The formula we want is:

$$\text{Read}^{B,B',B''}_{\exists}(u,v,u',v',T) := \exists Z_1,Z_2,Z_3 \ i') \land \text{ii'}) \land T = Z_1 + Z_2.$$  

Again i’-iii’) can be expressed as linear inequalities in $u,v,u',v'$ and $Z_1,Z_2,Z_3$.

So a single transition of $T$ from $B,B',B''$ to $f(B,B',B'')$ can be written as:

$$\text{Tran}^{B,B',B''}_{\exists}(u,v,u',v',T) := \text{Read}^{B,B',B''}_{\exists}(u,v,u',v',T) \land \text{Next}^{B,B',B''}_{\exists}(u,v,u',v',T).$$  

Note that $\text{Tran}_{\exists}$ is an $\exists^6$-formula. To ensure that $T$ obeys the transition rule $f : B^3 \to B$ every where, we simply require:

$$\forall u,v,u',v' \left( D^{M}_{\exists}(u,v,u',v') \land cv + dv' \leq T \right) \to \bigvee_{B,B',B'' \in B} \text{Tran}^{B,B',B''}_{\exists}(u,v,u',v',T).$$  

Here we write $q_{(t+\lambda)M} = cv + dv'$, with $c,d$ computable in poly($\lambda$) time. $D^{M}_{\exists}(u,v,u',v')$ means $v = q_{tM}$ is the beginning of some block $B_t$, and $q_{(t+\lambda)M} = cv + dv'$ is the beginning of the block $B_{t+\lambda}$, should it not exceed $T$.

We need one last formula to say that $T$ ends in the “yes” configuration (see 5.6). Recall that “yes” has fixed length. Assume “yes” starts at $v = q_{tM}$. Then just like before, we
can sum up all \(q_{iM+i}\) that correspond to ‘1’ bits in “yes”. This sum can be written as 
\[av + bv',\]
with \(a, b \in \mathbb{Z}\) explicit constants independent of \(\lambda\). Also observe that \(q_{iM-1} = q_{iM+i} - a_1q_{iM} = v' - a_1v\). So the formula:
\[
E_{\exists\forall}(T) = \exists u,v,u',v', Z \ D^M_{\forall}(u,v,u',v') \land Z < v' - a_1v \land T = Z + av + bv'
\]
is true if and only if \(T\) ends in “yes”. Note that \(E_{\exists\forall}\) is a \(\exists^5\forall^3\)-formula.

Finally, given \(x \in \{0, 1\}^\ell\), we can easily construct in time \(\text{poly}(\ell)\) the content of the first segment \(T'_1\) in \(T\) (see (5.7)). Again, \(T'_1\) is the starting configuration of the simulation \(U(\langle Mx \rangle)\), which is basically just \(\langle Mx \rangle\). Then we compute in time \(\text{poly}(\ell)\) the \(X \in \mathbb{N}\) whose Ostrowski representation corresponds to \(T'_1\). We also compute the tape length bound \(\lambda = \text{poly}(\ell)\) to be used in \(\text{Tran}_{\exists\forall}\). Now construct the sentence:
\[
(5.14) \quad \exists T \ ZeroOne_{\exists\forall}(T) \land \text{Pref}_{\exists\forall}(T, X) \land E_{\exists\forall}(T) \land \\
\left[\forall u,v,u',v' \ (D^M_{\forall}(u,v,u',v') \land cv + dv' \leq T) \rightarrow \bigvee_{B,B',B'' \in B} \text{Tran}_{\exists\forall}^{B,B',B''}(u,v,u',v',T)\right].
\]

Here \(ZeroOne_{\exists\forall}(T)\) ensures condition (5.3), \(\text{Pref}_{\exists\forall}(X, T)\) ensures that \(X\) is a prefix of \(T\), \(E_{\exists\forall}(T)\) says that \(T\) ends in “yes”, and the rest says that \(T\) follows the transition rules (see (5.12)). So (5.14) is an \(\exists\forall\)-sentence with total length \(\text{poly}(\ell)\), which is satisfied if and only if \(X \in L\). This proves Theorem 5.1

5.4. Analysis of the construction. We bound the number of variables in (5.14). Consider the last term \([\forall u,v \ldots]\). First, there are \(\forall^4\) variables \(u,v,u',v'\). Each \(\text{Tran}_{\exists\forall}^{B,B',B''}\) is an \(\exists^6\)-formula, which also commutes with the big disjunction. Also \(\neg D^M_{\forall}\) is an \(\exists^3\)-formula, which can be merged with the \(\exists^6\) part. Overall, the last term is of the form \(\forall^4\exists^6\).

Next, recall that \(\text{ZeroOne}_{\exists\forall}\) and \(\text{Pref}_{\exists\forall}\) in Section 5.1 are of the forms \(\forall^4\exists^3\) and \(\forall^4\exists^2\) and respectively. Since we are taking their conjunctions with the last term \(\forall^4\exists^6\), their outer \(\forall^4\) variables can be merged. However, their \(\exists\) variables need to be concatenated. Overall, we have \(\forall^4\exists^{11}\) for \(\text{ZeroOne}_{\exists\forall}\), \(\text{Pref}_{\exists\forall}\) and the last term. The term \(E_{\exists\forall}\) is \(\exists^5\forall^3\). Merging its \(\forall^3\) variables with the other three terms, we have \(\exists^5\forall^4\exists^{11}\). Lastly, we add in \(\exists T\) and get a \(\exists^6\forall^4\exists^{11}\) sentence.

The number of inequalities in all constructed formulas is bounded in the table below. Overall, the number of inequalities in (5.14) is at most:

\[34 + 26 + 14 + 10(k + 2) + 12 + 10(k + 2) + 16|B| = 810534 + 20(k + 2).\]

**Corollary 5.3.** For \(\alpha = \sqrt{2}\) deciding \(\exists^6\forall^4\exists^{11}\) sentences with at most \(10^6\) inequalities in \(S_\alpha\) is PSPACE-hard.

**Proof.** Note that \(\sqrt{2} + 1 = [2; 2, \ldots]\) has minimum period \(k = 1\). \(\square\)

---

\(^8\)We need to rewrite every implication “\(a \rightarrow b\)” as “\(\neg a \lor b\)”.
|\(x = y\) | 2 |
|\(|x| \geq |y|\) | 4 |
|\(|x| > |y|\) | 4 |
|\(\text{After, After}\) | 4 |
|\(\text{Compatible}\) | 10 |
|\(C_{\forall}\) | 12 |
|\(\text{ZeroOne}_{\forall \exists}\) | 34 |
|\(\text{Pref}_{\forall \exists}\) | 26 |
|\(\text{Read}_{\exists}\) | 8 |
|\(\text{Next}_{\exists}\) | 8 |
|\(\text{Tran}_{\exists}\) | 16 |
|\(D^K_{\forall}\) | \(3 + 10(k + 2)\) |
|\(D^M_{\forall}\) | \(11 + 10(k + 2)\) |
|\(E_{\exists \forall}\) | \(14 + 10(k + 2)\) |

6. Quadratic irrationals: General lower bound

In this section we prove Theorem 1.2. Recall Monadic Second Order logic MSO = (\(\mathbb{N}, \mathcal{P}(\mathbb{N}), s_\mathbb{N}, \in\)), where \(\mathcal{P}(\mathbb{N})\) is the (monadic) predicate for subsets of \(\mathbb{N}\), and \(s_\mathbb{N}\) is the successor function \(n \to n + 1\). Its weak variant is WMSO = (\(\mathbb{N}, \mathcal{P}_{\text{fin}}(\mathbb{N}), s_\mathbb{N}, \in\)), which only quantifies over finite subsets of \(\mathbb{N}\). We refer to \([GTW]\) for the equivalence between WMSO and the theory of automata equipped with quantifiers. First, we prove a similar lower bound for WMSO:

**Theorem 6.1.** Deciding a sentence \(S\) in WMSO with \(r + 2\) alternating quantifiers and \(O(r)\) variables requires space at least:

\[
\rho 2^{\eta^{\ell(S)}}.
\]

Here the tower has height \(r\), and \(\rho, \eta\) are absolute constants.

The proof is similar to that of Theorem 5.1. Recall that in \([\text{Next}_{\exists}^{B,B'B'}]\) if \(v = q_{tM}\) and \(v' = q_{tM+1}\) then the shifted convergent \(q_{(t+\lambda)M}\) can be written as \(cw + dv'\), with \(c, d \in \mathbb{Z}\) having lengths \(\text{poly}(\lambda)\). The resulting sentence \([5,14]\) has length \(\text{poly}(\lambda)\), and is PSPACE-complete to decide. To prove a tower lower bound, we need to construct a shift map

\(S_r : q_{tM} \mapsto q_{(t + g(\lambda))M}\),

so that \(g(\lambda)\) is a tower of height \((r - 2)\) in \(\lambda\). Here the formula \(S_r\) is allowed to have length \(\text{poly}(\lambda)\) and at most \(r - 2\) alternating quantifiers. The following construction is classical. It was first used in \([\text{Mey}\]) to prove that WMSO has non-elementary decision complexity, and was later improved on in \([\text{Sto}\]). An expository version is given in \([\text{GTW}, \text{Ch. } 13]\). For completeness, we reproduce it below in the setting of WMSO with some improvements on the number of alternating quantifiers. Afterwards, we translate it back to \(S_\alpha\).

We think of each subset \(X \in \mathcal{P}_{\text{fin}}(\mathbb{N})\) as a binary string of finite length. The relation \(n \in X\) simply means that the \(n\)-th bit in \(X\) is 1. Let

\[g_0(\lambda) = \lambda \quad \text{and} \quad g_{r+1}(\lambda) = g_r(\lambda) 2^{g_r(\lambda)}, \ r \geq 0.\]
The idea of the construction is as follows. We will iteratively define formulas $F_r(x, y, A, C)$ such that $F_{r+1}$ is true if and only if:

\[
\begin{align*}
y &= x + g_{r+1}(\lambda), \\
A &= 0^x[100\ldots0[100\ldots0[100\ldots0[\ldots100\ldots0]], \\
C &= 0^x[000\ldots0[100\ldots0[010\ldots0[\ldots1[111\ldots1]00^*.
\end{align*}
\]

Here $A, C$ have $2^{g_r(\lambda)}$ blocks, each of length $g_r(\lambda)$. The blocks in $C$ represent the integers $0, 1, \ldots, 2^{g_r(\lambda)} - 1$ in binary. The blocks in $A$ mark the beginning of the blocks in $C$. The first '1' in $A$ is at position $x$ and the last '1' in $A$ is at position $y$. In total, the difference $y - x$ is $g_r(\lambda)2^{g_r(\lambda)} = g_{r+1}(\lambda)$. First, we can define the basic quantifier-free case:

\[F_0(x, y, A, C) := \text{Singleton}(x) \land \text{Singleton}(y) \land y = x + \lambda\]

For this case $A$ and $C$ do not matter. Now recall the carry rule for addition by 1 in binary. If $X = x_0x_1\ldots$ in binary then $Y = X + 1 = y_0y_1\ldots$ satisfies:

\[
\begin{align*}
x_0 &= \neg y_0 \quad \text{(the least significant digit always switches)} \\
x_i &= 1, y_i = 0 \rightarrow x_{i+1} &= \neg y_{i+1} \quad \text{(carry rule)} \\
x_{i+1} &= y_{i+1} \quad \text{otherwise.}
\end{align*}
\]

In the context of WMSO, these rules can be summarized as:

\[
\begin{align*}
(6.1) & \quad 0 \in X \Leftrightarrow 0 \notin Y; \\
(6.2) & \quad i \in X \land i \notin Y \Leftrightarrow (i + 1 \in X \Leftrightarrow i + 1 \notin Y).
\end{align*}
\]

Observe that if we apply these rules on blocks of length $g_r(\lambda)$, starting with $0\ldots0$, then we get:

\[00\ldots0[01\ldots0[\ldots1[11\ldots1]00\ldots0[10\ldots0]\ldots\]

So the blocks do cycle back to $0\ldots0$ eventually. This needs to be taken care of in the definition of $F_{r+1}$, because we want each block of $C$ to be unique. We define:

\[
F_{r+1}(x, y, A, C) := \text{Singleton}(x) \land \text{Singleton}(y) \land x < y \land \\
& \forall z, w, t, D, E \left[ F_r(z, w, D, E) \land \text{Singleton}(t) \right] \rightarrow \\
& \left[ z = x \lor z = y \rightarrow z \in A, z \notin C; \\
& z < x \land y < z \rightarrow z \notin A, z \notin C; \\
& x = z, z < t < w \rightarrow t \notin A, t \notin C; \\
& x \leq z < w \leq y \rightarrow (z \in A \leftrightarrow w \in A); \\
& z \in A, w < y \rightarrow (z \in C \leftrightarrow w \notin C) \\
& x \leq z < w < y, z + 1 \notin A \rightarrow (z \in C, w \notin C \leftrightarrow (z + 1 \in C \leftrightarrow w + 1 \notin C)); \\
& x \leq z < w < y, z + 1 \in A \rightarrow (z \in C \rightarrow w \in C); \\
& w = y, z \leq t < w \rightarrow t \in C \right].
\]

For readability, we use $\land$, interchangeably to denote conjunctions. Lines (6.4) and (6.5) set up the first block in $A$ and $C$, and say that that $A$ and $C$ are all empty outside the range $[x, y]$. Line (6.6) expresses the increment rule (6.1) for every 2 consecutive blocks in $C$. Here $w$ and $z$ represent 2 corresponding digits in 2 consecutive blocks. Line (6.7) expresses the carry rule (6.2). Line (6.8) ensures that the blocks in $C$ do not cycle back to

---

9 Position indexing starts at 0.
10 Here $x + \lambda$ represents $\lambda$ iterations of the successor function $s_N$. 
0 \ldots 0$, because their last digits cannot decrease from 1 down to 0. Line (6.9) ensures that the last block is $1 \ldots 1$.

By induction, it is easy to see that $F_r$ has $r$ alternating quantifiers, starting with $\forall$. It is also clear that $F_{r+1}$ has 5 more variables $(x, y, A, C, t)$ than $F_r$. Therefore, $F_r$ has at most $5(r+1)$ variables. We can also bound their lengths:

$$\ell(F_0) = O(\lambda) \quad \text{and} \quad \ell(F_{r+1}) = \ell(F_r) + O(1) = O(\lambda + r).$$

Here $\ell(F_0) = O(\lambda)$ instead of $O(1)$ because we needed to iterate the successor function $s_\mathbb{N}$ $\lambda$ times to represent $y = x + \lambda$.

**Proof of Theorem 6.7** Consider the following decidable problem: Given a Turing machine $M$ and an input string $X$, does $M$ halt on $X$ within space $g_r(|M| + |X|)$. By a basic diagonalization argument, this problem requires space at least $g_r(|M| + |X|)$ to decide. By the same construction as in Theorem 5.14 we can write down a sentence $S$ with length $O(|M| + |X|)$ so that $S$ is true if and only if $M$ halts on $x$ within space $g_r(|M| + |X|)$. Here $\lambda = \Omega(|M| + |X|)$. The last part $[\forall u, v, u', v' \ldots]$ in (5.14) should be replaced by:

$$\forall x, y, A, C, F_r(x, y, A, C) \rightarrow \text{transition rules \ldots}$$

Here $x$ and $y$ are bits in the transcript $T = U((\langle M \rangle X))$, with $y = x + g_r(\lambda)$.

The resulting sentence $S$ has the form $\exists \ldots \forall \ldots \neg F_r \lor \ldots$. Since $F_r$ has $r$ alternating quantifiers, $S$ has $r + 2$ alternating quantifiers. The length $\ell(S)$ is roughly the input length $|M| + |X|$ plus $\ell(F_r)$, which is also $O(|M| + |X|)$.

**Proof of Theorem 5.2** We can easily translate the WMSO formula $F_r(x, y, A, C)$ with $r$ alternating quantifiers to a $S_\alpha$ formula $S_r$ with $(r+1)$-alternating quantifier. To do this, we replace each singleton variable in (6.3), say $x$, by a separate quadruple $(u_x, v_x, u'_x, v'_x)$, where $(u_x, v_x) = (p_{xM}, q_{xM})$ and $(u'_x, v'_x) = (p_{xM+1}, q_{xM+1})$. The Singleton$(x)$ predicate is replaced by $\mathbf{D}_y^{\mathbf{M}} u_x, v_x, u'_x, v'_x$, and similarly for other singleton variables. Each set variable, e.g. $A, C$, is replaced by an integer variable. The relation $\epsilon$ is now $\exists / \forall$-definable in $S_\alpha$ (see (3.17) and (3.18)). Recall from Fact 3.3 that if $v = q_{tM}$ and $v' = q_{t+1}M$ then $q_{(t+1)M} = cv + dv'$ for some constants $c, d \in \mathbb{Z}$ independent of $t$. We replace every $x + 1$ term in (6.3) is replaced by $c v_x + d v'_x$. Also $q_{(t+1)M} = c v + d v'$ for some $c, d \in \mathbb{Z}$ with log($c$), log($d$) = $O(\lambda)$. So the relation $y = x + \lambda$ in $F_0$ is replaced by $v_y = c v_x + d v'_x$. Observe that $S_0$ has just $O(1)$ atoms (inequalities), instead of $O(\lambda)$ atoms like $F_0$. By induction, $S_r$ has $O(r)$ inequalities and variables. The total length $\ell(S_r)$ (including symbols and integer coefficients) is still $O(r + \lambda)$.

Because of the $\mathbf{D}_y^{\mathbf{M}}$ predicate, $S_0$ now has 1 alternating quantifier. For $r > 0$, we can merge the $\forall$ quantifiers in $\mathbf{D}_y^{\mathbf{M}}$ predicates with the $\forall z, w, t, \ldots$ quantifiers in (6.3) (of course replaced by quadruples). Because $\epsilon$ is $\exists / \forall$-definable in $S_\alpha$, the body of the sentence $S_{r+1}$, consisting of Boolean combinations in $\epsilon / \exists \epsilon$, can be written using only $\forall$ quantifiers. These extra $\forall$ quantifiers can again be merged into the $\forall z, w, t$ part. This means $S_{r+1}$ has only one more alternating quantifier than $S_r$. So $S_r(u_x, v_x, u'_x, v'_x, u_y, v_y, A, C)$ is a $(r+1)$-alternating quantifier formula.

Now we are back to encoding Turing machine computations. In (5.14), we replace the last part $[\forall u, v, u', v' \ldots]$ by:

$$\forall u_x, v_x, u'_x, v'_x, u_y, v_y, A, C \quad (S_r(u_x, v_x, u'_x, v'_x, u_y, v_y, A, C) = \text{true} \land v_y \leq r) \rightarrow \text{transition rules \ldots}$$

$^{11}U$ is the universal TM used to emulate $M(X)$. 
In these transition rules, \( \text{Read}^{B,B',B''}_3 \) is kept as before with \( u_x, v_x, u'_x, v'_x \), but \( \text{Next}^{B,B',B''}_3 \) can be rewritten using the shifted convergents \( u_y, v_y, u'_y, v'_y \). Altogether, this expresses the transition rule for each jump \( y = x + g_r(\lambda) \). The resulting sentence \( S \) has the form \( \exists \ldots \forall \ldots ^{-}S_r \lor \ldots \). Since \( S_r \) has \( r + 1 \) alternating quantifiers, \( S \) has \( r + 3 \) alternating quantifiers. The number of variables and inequalities used is just \( O(r) \). \( \square \)

7. **Non-quadratic irrationals: Undecidability**

7.1. **Further tools.** In this section we are working with two different irrationals \( \alpha \) and \( \beta \). We denote by \( p_n/q_n \) and \( p'_n/q'_n \) the \( n \)-th convergent of \( \alpha \) and \( \beta \), respectively. Let \( \text{Ost}_\alpha := \{ q_n : n \in \mathbb{N} \} \) and \( \text{Ost}_\beta := \{ q'_n : n \in \mathbb{N} \} \). For \( X \in \mathbb{N} \), denote by \( \text{Ost}_\alpha(X) \) the set of \( q_n \) with non-zero coefficients in the \( \alpha \)-Ostrowski representation of \( X \). Then \( \text{Ost}_\beta(X) \) is defined accordingly for the \( \beta \)-Ostrowski representation of \( X \). All earlier notations can be easily adapted to \( \alpha \) and \( \beta \) separately. For brevity, we define the remaining functions and notations just for \( \alpha \). The corresponding versions for \( \beta \) are defined accordingly, with obvious relabeling.

For \( X \in \mathbb{N} \) and \( d \in \text{Ost}_\alpha \), if \( \sum_{n=0}^{\infty} b_{n+1}q_n \) is the \( \alpha \)-Ostrowski representation of \( X \), then we define

\[
X|_d^\alpha := \sum_{n \in \mathbb{N}, q_n \leq d} b_{n+1}q_n.
\]

**Fact 7.1.** Let \( X \in \mathbb{N} \). Then there is an interval \( I \) around \( f_\alpha(X) \) and \( d \in \text{Ost}_\alpha \) such that for all \( Y \in \mathbb{N} \)

\[
\text{Proof.} \text{ Let } \sum_{n=0}^{m} b_{n+1}q_n \text{ be the } \alpha \text{-Ostrowski representation of } X. \text{ Without loss of generality, we may assume that } \alpha q_m - p_m > 0. \text{ Then set }
\]

\[
Z_2 = X + q_{m+2} \text{ and } Z_1 = X + q_{m+3}.
\]

Since \( \alpha q_{m+2} - p_{m+2} > 0 \) and \( \alpha q_{m+3} - p_{m+3} < 0 \), we get from Fact 8.2 that

\[
f_\alpha(Z_1) < f_\alpha(X) < f_\alpha(Z_2).
\]

Now it follows easily from [H2, Fact 2.13] and Fact 8.2 that for all \( Y \in \mathbb{N} \)

\[
f_\alpha(Z_1) < f_\alpha(Y) < f_\alpha(Z_2) \implies Y|_{q_n}^\alpha = X,
\]

as desired. \( \square \)

**Fact 7.2.** Let \( X \in \mathbb{N} \) and let \( J \) be an open interval around \( f_\alpha(X) \). Then there is \( d \in \text{Ost}_\alpha \) such that for all \( Y \in \mathbb{N} \)

\[
Y|_d^\alpha = X \implies f_\alpha(Y) \in J.
\]

**Proof.** Let \( \sum_{n=0}^{m} b_{n+1}q_n \) be the \( \alpha \)-Ostrowski representation of \( X \). Let \( n \in \mathbb{N} \) be such that

- \( n > m + 1 \),
- \( \alpha q_n - p_n > 0 \) and
- \( (f_\alpha(X) + (\alpha q_{n+1} - p_{n+1}), f_\alpha(X) + (\alpha q_n - p_n)) \subseteq J \).
Let \( Y \in \mathbb{N} \) be such that \( Y^{\alpha}_{\beta_{n+2}} = X \). It is left to show that \( f_\alpha(Y) \in J \). By Fact 3.2 and [H2, Fact 2.13] we get that
\[
f_\alpha(X) + (\alpha q_{n+1} - p_{n+1}) = f_\alpha(X + q_{n+1}) < f_\alpha(Y) < f_\alpha(X + q_n) = f_\alpha(X) + (\alpha q_n - p_n).
\]
Thus \( f_\alpha(Y) \in J \). \( \square \)

7.2. Uniform definition of all finite subsets of \( \mathbb{N}^2 \). Let \( \alpha, \beta \) be two positive irrational numbers such that \( 1, \alpha, \beta \) are \( \mathbb{Q} \)-linearly independent. The goal of this section is to produce a predicate \( \text{Member} \subseteq \mathbb{N}^4 \) such that for every set \( S \subseteq \mathbb{N}^2 \) there is \( X \in \mathbb{N}^4 \) such that for all \( (s, t) \in \mathbb{N}^2 \),
\[
(s, t) \in S \iff \text{Member}(X, s, t).
\]
The \( \mathbb{Q} \)-linear independence of \( 1, \alpha, \beta \) is necessary as the existence of such an relation implies the undecidability of the theory. The failure of our argument in the case of \( \alpha, \beta \) can be traced back to the fact that the following lemma fails when \( 1, \alpha, \beta \) are \( \mathbb{Q} \)-linearly dependent.

Hereafter, we let \( X = (X_1, X_2), Y = (Y_1, Y_2) \) and \( Z = (Z_1, Z_2) \).

**Lemma 7.3.** Let \( X, Y \in \mathbb{N}^2 \). Then
\[
|f_\alpha(X_1) - f_\beta(X_2)| = |f_\alpha(Y_1) - f_\beta(Y_2)| \implies X = Y.
\]

**Proof.** Then there are \( U_1, U_2, V_1, V_2 \in \mathbb{N} \) such that
\[
|\alpha X_1 - U_1 + \beta X_2 - U_2| = |\alpha Y_1 - V_1 + \beta Y_2 - V_2|.
\]
By \( \mathbb{Q} \)-linear independence of \( 1, \alpha, \beta \), we get that \( X_1 = Y_1 \) and \( X_2 = Y_2 \). \( \square \)

**Definition 7.4.** Define \( g : \mathbb{N}^4 \to \mathbb{R} \) to be the function that maps \((X, Y)\) to
\[
|f_\alpha(X_2) - f_\beta(X_1)| - |f_\alpha(Y_2) - f_\beta(Y_1)|.
\]

**Definition 7.5.** Define \( \text{Best} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}^2 \times \mathbb{N} \) to be the set containing all \((d, e, X, Y_1)\) for which there is a \( Y_2 \in \mathbb{N} \) such that
- \( Y_1 \leq d, Y_2 \leq e, \)
- \( g(X, Y) < g(X, Z) \) for all \( Z \in \mathbb{N}_{\leq d} \times \mathbb{N}_{\leq e} \) with \( Z \neq Y \).

Observe that for given \((d, e, X) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}^2 \) there is at most one \( Y_1 \in \mathbb{N}_{\leq d} \) such that \( \text{Best}(d, e, X, Y_1) \) holds. We will later see in Lemma 7.7 that for given \( d \in \mathbb{N} \) we can take \( e \in \mathbb{N} \) large enough such that for all \( X_1 \in \mathbb{N} \) and \( Y_1 \leq d \) the set
\[
\{ X_2 \in \mathbb{N} : \text{Best}(d, e, X_1, X_2, Y_1) \}
\]
is cofinal in \( \mathbb{N} \).

**Lemma 7.6.** \( \text{Best} \) is \( \exists \forall \)-definable.

**Proof.** Observe that \( \text{Best}(d, e, X, Y_1) \) holds if and only if
\[
\exists X_2, U_1, U_2, V_1, V_2 \ \forall Z_1, Z_2, W_1, W_2 \ \ Y_1 \leq d \land Y_2 \leq e \land
f_\alpha(X_1) = \alpha X_1 - U_1 \land f_\alpha(X_2) = \alpha X_2 - U_2 \land f_\alpha(Y_1) = \alpha Y_1 - V_1 \land f_\beta(Y_2) = \beta Y_2 - V_2 \land
\left[ (Z_1 \leq d \land Z_2 \leq e \land f_\alpha(Z_1) = \alpha Z_1 - W_1 \land f_\beta(Z_2) = \beta Z_2 - W_2 \land (Z_1, Z_2) \neq (Y_1, Y_2) \right] \rightarrow \left[ |(\alpha X_2 - U_2) - (\alpha X_1 - U_1) - (\beta Y_2 - V_2) - (\alpha Y_1 - V_1)| < |(\alpha X_2 - U_2) - (\alpha X_1 - U_1) - (\beta Z_2 - W_2) - (\alpha Z_1 - W_1)| \right].
\]
Lemma 7.7. Let \( d \in \text{Ost}_\alpha, e_0 \in \text{Ost}_\beta, \overline{X} \in \mathbb{N}^2 \) and \( s \in \mathbb{N} \) be such that
\[
(1) \quad f_\alpha(X_1), f_\alpha(X_2) \in I_\beta,
(2) \quad f_\alpha(X_1) < f_\alpha(X_2),
(3) \quad s \leq d.
\]
Then there is \( e \in \text{Ost}_\beta \) and an open interval \( J \subseteq (f_\alpha(X_1), f_\alpha(X_2)) \) such that \( e \geq e_0 \) and for all \( Z \in \mathbb{N} \)
\[
f_\alpha(Z) \in J \implies \text{Best}(d, e, X_1, Z, s).
\]

Proof. Let \( e \in \text{Ost}_\beta \) be large enough such that for every \( w_1 \in \mathbb{N}_{\leq d} \) there is \( w_2 \in \mathbb{N}_{\leq e} \) such that
\[
f_\alpha(w_1) \in I_\beta \implies |f_\alpha(w_1) - f_\beta(w_2)| < f_\alpha(X_2) - f_\alpha(X_1).
\]
The existence of such an \( e \) follows from the finiteness of \( \mathbb{N}_{\leq d} \) and the density of \( f_\beta(\mathbb{N}) \) in \( I_\beta \). Let \( w \in \mathbb{N}_{\leq e} \) be such that
\[
|f_\alpha(s) - f_\beta(w)| < f_\alpha(X_2) - f_\alpha(X_1).
\]
By Lemma 7.3 we can find an \( \varepsilon > 0 \) such that for all \( (w_1, w_2) \in \mathbb{N}_{\leq d} \times \mathbb{N}_{\leq e} \) with \( (w_1, w_2) \neq (s, w) \)
\[
||f_\alpha(w_1) - f_\beta(w_2)| - |f_\alpha(s) - f_\beta(w)|| > \varepsilon.
\]
Set
\[
\delta := f_\alpha(X_1) + |f_\alpha(s) - f_\beta(w)|.
\]
Set \( J := (\delta - \frac{\varepsilon}{2}, \delta + \frac{\varepsilon}{2}) \). Let \( Z \in \mathbb{N} \) be such that \( f_\alpha(Z) \in J \). It is left to show that \( \text{Best}(d, e, X_1, Z, s) \) holds. We have that for all \( (w_1, w_2) \in \mathbb{N}_{\leq d} \times \mathbb{N}_{\leq e} \) with \( (w_1, w_2) \neq (s, w) \)
\[
g(X_1, Z, w_1, w_2) = |f_\alpha(Z) - f_\alpha(X_1) - |f_\alpha(w_1) - f_\beta(w_2)||
= |f_\alpha(Z) - \delta + |f_\alpha(s) - f_\beta(w)| - |f_\alpha(w_1) - f_\beta(w_2)||
\geq |f_\alpha(Z) - \delta| - ||f_\alpha(s) - f_\beta(w)| - |f_\alpha(w_1) - f_\beta(w_2)|| > \frac{\varepsilon}{2}.
\]
Moreover,
\[
g(X_1, Z, s, w) = |f_\alpha(Z) - f_\alpha(X_1) - |f_\alpha(s) - f_\beta(w)|| \leq |f_\alpha(Z) - \delta| \leq \frac{\varepsilon}{2}.
\]
Thus \( \text{Best}(d, e, X_1, Z, s) \) holds, as desired. \( \square \)

Lemma 7.8. Let \( d \in \text{Ost}_\alpha, s \in \mathbb{N}, \overline{X} \in \mathbb{N}^2 \) be such that
\[
(1) \quad f_\alpha(X_1), f_\alpha(X_2) \in I_\beta,
(2) \quad f_\alpha(X_1) < f_\alpha(X_2),
(3) \quad s \leq d.
\]
Then there are \( e_1 \in \text{Ost}_\beta, e_2 \in \text{Ost}_\alpha, Y \in \mathbb{N} \) such that
\[
(i) \quad f_\alpha(X_1) < f_\alpha(Y) < f_\alpha(X_2),
(ii) \quad d < e_1 < e_2
(iii) \quad \text{for all } Z \in \mathbb{N}
\]
\[
Z^{\frac{\alpha}{e_2}} = Y \implies \text{Best}(d, e_1, X_1, Z, s).
\]
Proof. By Lemma 7.7 there is an open interval $J \subseteq (f_\alpha(X_1), f_\alpha(X_2))$ and $e_1 \in \text{Ost}_\beta$ such that $e_1 > d$ and for all $Z \in \mathbb{N}$

$$f_\alpha(Z) \in J \Rightarrow \text{Best}(d, e_1, X_1, Z, s).$$

Take $Y \in \mathbb{N}$ such that $f_\alpha(Y) \in J$. By Fact 7.2 we can find $e_2 \in \text{Ost}_\alpha$ arbitrarily large such that $f_\alpha(Z) \in J$ for all $Z \in \mathbb{N}$ with $Z^{\alpha}_{i_{e_2}} = Y$. The statement of the Lemma follows. $\square$

**Definition 7.9.** Define **Admissible** $\subseteq \text{Ost}_\alpha^4 \times \text{Ost}_\beta^2 \times \mathbb{N}^6$ to be the set of all tuples

$$(d_1, d_2, d_3, d_4, e_1, e_2, X_1, X_2, X_3, X_4, s, t) \in \text{Ost}_\alpha^4 \times \text{Ost}_\beta^2 \times \mathbb{N}^6$$

such that

- $d_1, d_2, d_3$ are consecutive elements of $\text{Ost}_\alpha(X_1)$,
- $d_4 \in \text{Ost}_\alpha(X_3)$ and $d_1 \leq d_4 < d_2$,
- $e_1, e_2 \in \text{Ost}_\beta(X_2)$ and $d_1 \leq e_1 < d_2 \leq e_2 < d_3$,
- $\text{Best}(d_1, e_1, X_4^{\alpha}_{d_1}, X_4, s)$,
- $\text{Best}(d_2, e_2, X_4^{\alpha}_{d_2}, X_4, t)$

Define **Member** $\subseteq \mathbb{N}^6$ to be the set of all tuples $(X_1, X_2, X_3, X_4, s, t) \in \mathbb{N}^6$ such that there exist $d_1, d_2, d_3, d_4 \in \text{Ost}_\alpha, e_1, e_2 \in \text{Ost}_\beta$ with

**Admissible** $(d_1, d_2, d_3, d_4, e_1, e_2, X_1, X_2, X_3, X_4, s, t)$.

**Theorem 7.10.** Let $S \subseteq \mathbb{N}^2$ be finite. Then there are $X_1, X_2, X_3, X_4 \in \mathbb{N}$ such that for all $s, t \in \mathbb{N}$

$$(s, t) \in S \iff \text{Member}(X_1, X_2, X_3, X_4, s, t).$$

Proof. Let $S \subseteq \mathbb{N}^2$ be finite. Let $c_1, \ldots, c_{2n} \in \mathbb{N}$ be such that

$$S = \{(c_1, c_2), \ldots, (c_{2n-1}, c_{2n})\}.$$

Recall that the convergents of $\alpha$ and $\beta$ are $\{p_n/q_n\}$ and $\{p'_n/q'_n\}$, respectively. We will construct two strictly increasing sequences $(k_i)_{i=0,\ldots,2n}$ and $(l_i)_{i=1,\ldots,2n}$ of non-consecutive natural numbers and another sequence $(W_i)_{i=0,\ldots,2n}$ of natural numbers such that for all $i = 0, \ldots, 2n$

1. $W_j = W_{j_{k_i}}^\alpha$ for all $j \leq i$, and $f_\alpha(W_i) \in I_\beta$,
2. $q_{k_i} > \max\{c_1, \ldots, c_{2n}\}$,

and furthermore if $i \geq 1$, then

3. $q_{k_{i-1}} < q_{l_i} < q_{k_i}$,
4. for all $Z \in \mathbb{N}$

$$Z^{\alpha}_{q_{k_i}} = W_i \Rightarrow \text{Best}(q_{k_{i-1}}, q_{l_i}, W_{i-1}, Z, c_i).$$

We construct these sequences recursively. For $i = 0$, pick $k_0 \in \mathbb{N}$ such that

$$q_{k_0} > \max\{c_1, \ldots, c_{2n}\}.$$

Pick $W_0 \in \mathbb{N}$ such that $W_0 = W_0^{\alpha}_{q_{k_0}}$ and $f_\alpha(W_0) \in I_\beta$. Now suppose that $i > 0$ and that we already constructed $k_0, k_1, \ldots, k_{i-1}, l_1, \ldots, l_{i-1}$ and $W_1, \ldots, W_{i-1}$ such that the above conditions (1)-(4) hold for $j = 1, \ldots, i - 1$. We now have to find $k_i, l_i$ and $W_i$ that (1)-(4) also hold for $i$. We do so by applying Lemma 7.8. By Fact 7.1 we can take $T \in \mathbb{N}$ such that

(a) $f_\alpha(T) > f_\alpha(W_{i-1}), T^{\alpha}_{q_{k_{i-1}}} = W_{i-1}, f_\alpha(T) \in I_\beta$ and
(b) for all $Z \in \mathbb{N}$, $(f_\alpha(W_{i-1}) < f_\alpha(Z) < f_\alpha(T) \Rightarrow Z^{\alpha}_{q_{k_{i-1}}} = W_{i-1}).$
We now apply Lemma \[\text{7.3}\] with \(X_1 := W_{i-1}, X_2 := T, d := q_{k_i-1}\) and \(s := c_{i-1}\). We obtain \(e_1 \in \text{Ost}_\beta, e_2 \in \text{Ost}_\alpha\) and \(Y \in \mathbb{N}\) such that \(d < e_1 < e_2, f_\alpha(W_{i-1}) < f_\alpha(Y) < f_\alpha(T)\) and for all \(Z \in \mathbb{N}\)

\[
Z^{\alpha}_{e_2} = Y \implies \text{Best}(q_{k_i-1}, e_1, W_{i-1}, Z, c_{i-1}).
\]

If necessary, we increase \(e_2\) such that \(Y^{\alpha}_{e_2} = Y\). Choose \(k_i\) such that \(q_{k_i} = e_2\), choose \(l_i\) such that \(q_{l_i}' = e_1\). Set \(W_i := Y\). It is immediate that (2)-(4) hold for \(i = 1, \ldots, n\). For (1), observe that since \(f_\alpha(W_{i-1}) < f_\alpha(Y) < f_\alpha(T)\), we deduce from (b) that

\[
W_i^{\alpha}_{q_{k_i-1}} = Y^{\alpha}_{q_{k_i-1}} = W_{i-1}.
\]

Since (1) holds for \(j = 1, \ldots, i-1\), we get that for \(j < i-1\)

\[
W_i^{\alpha}_{q_{k_i}} = W_i^{\alpha}_{q_{k_i}} = W_j.
\]

Thus (1) holds for \(i\).

We have constructed \((k_i)_{i=0, \ldots, 2n}, (l_i)_{i=1, \ldots, 2n}\) and \((W_i)_{i=0, \ldots, 2n}\) satisfying (1)-(4) for each \(i = 0, 1, \ldots, 2n\). We now define \((Z_1, Z_2, Z_3, Z_4) \in \mathbb{N}^4\) by

- \(Z_1 := \sum_{i=0}^{2n} q_{k_i}\)
- \(Z_2 := \sum_{i=1}^{2n} q_{l_i}'\)
- \(Z_3 := \sum_{i=0}^{n} q_{k_{2i}}\)
- \(Z_4 := W_{2n}\).

Observe that we require the sequences \((k_i)_{i=0, \ldots, 2n}\) and \((l_i)_{i=1, \ldots, 2n}\) to be increasing sequences of non-consecutive natural numbers. Therefore the above description of \(Z_1, Z_2\) and \(Z_3\) immediately gives us the \(\alpha\)-Ostrowski representations of \(Z_1\) and \(Z_3\) and the \(\beta\)-Ostrowski representation of \(Z_2\). In particular,

\[
\text{Ost}_\alpha(Z_1) = \{q_{k_i} : i = 0, \ldots, n\}, \quad \text{Ost}_\beta(Z_2) = \{q_{l_i}' : i = 1, \ldots, n\},
\]

\[
\text{Ost}_\alpha(Z_3) = \{q_{k_i} : i = 0, \ldots, n, i \text{ even}\}.
\]

It is now left to prove that for all \(s, t \in \mathbb{N}\)

\[(s, t) \in S \iff \text{Member}(Z_1, Z_2, Z_3, Z_4, s, t)\).
\]

“\(\Rightarrow\)”: Let \((s, t) \in S\). Let \(i \in \{1, \ldots, 2n\}\) be such that \((s, t) = (c_i, c_{i+1})\). Observe that \(i\) is odd. We show that

\[
\text{Admissible}(q_{k_i-1}, q_{k_i}, q_{k_{i+1}}, q_{k_{i-1}}, q_{l_i}, q_{l_{i+1}}, Z_1, Z_2, Z_3, Z_4, c_i, c_{i+1})
\]

holds. By \(\text{7.2}\) and the fact that \(i - 1\) is even, we have that

\[
q_{k_{i-1}}, q_{k_{i}}, q_{k_{i+1}} \in \text{Ost}_\alpha(Z_1), q_{l_i}', q_{l_{i+1}} \in \text{Ost}_\beta(Z_2), q_{k_{i-1}} \in \text{Ost}_\alpha(Z_3).
\]

Trivially, \(q_{k_{i-1}} \leq q_{k_{i-1}} < q_{k_{i}}\). By (3) \(q_{k_{i-1}} < q_{l_i}' < q_{k_{i}} < q_{l_{i+1}} < q_{k_{i+1}}\). Now observe that by (1)

\[
Z_4^{\alpha}_{q_{k_{i-1}}} = W_{2n}^{\alpha}_{q_{k_{i-1}}} = W_{i-1},
\]

\[
Z_4^{\alpha}_{q_{k_{i}}} = W_{2n}^{\alpha}_{q_{k_{i}}} = W_{i},
\]

\[
Z_4^{\alpha}_{q_{k_{i+1}}} = W_{2n}^{\alpha}_{q_{k_{i+1}}} = W_{i+1}.
\]

Thus by (4)

\[
\text{Best}(q_{k_{i-1}}, q_{l_i}', Z_4^{\alpha}_{q_{k_{i-1}}}, Z_4, c_i) \land \text{Best}(q_{k_{i}}, q_{l_{i+1}}, Z_4^{\alpha}_{q_{k_{i}}}, Z_4, c_{i+1}).
\]
Thus (7.3) holds.

"⇐": Suppose that Member\((Z_1, Z_2, Z_3, Z_4, s,t)\) holds. Let \(d_1, d_2, d_3, d_4 \in \text{Ost}_\alpha, e_1, e_2 \in \text{Ost}_\beta\) be such that

\[
\text{Admissible}(d_1, d_2, d_3, d_4, e_1, e_2, Z_1, Z_2, Z_3, Z_4, s,t)
\]

holds. Then \(d_1, d_2, d_3\) are consecutive elements of \(\text{Ost}_\alpha(Z_1)\). Thus there is \(i \in \{1, \ldots, 2n-1\}\) such that

\[
d_1 := q_{k_i-1}, \ d_2 := q_{k_i}, \ d_3 := q_{k_i+1}.
\]

Since \(d_4 \in \text{Ost}_\alpha(Z_3)\) and \(d_1 \leq d_4 < d_2\), it follows that \(d_4 = d_1 = q_{k_i-1}\) and that \(i\) is odd. Since \(e_1, e_2 \in \text{Ost}_\beta(Z_2)\) and

\[
d_1 = q_{k_i-1} \leq e_1 < d_2 = q_{k_i} \leq e_2 \leq d_3 = q_{k_i+1},
\]

we get from (3) that \(e_1 = q'_{t_i}\) and \(e_2 = q'_{t_i+1}\). Thus by (7.3)

\[
\text{Best}(q_{k_i-1}, q'_{t_i}, Z_4|_{q_{k_i-1}}, Z_4, s) \land \text{Best}(q_{k_i}, q'_{t_i+1}, Z_4|_{q_{k_i}}, Z_4, t).
\]

By (4) we get that \(s = c_i\) and \(t = c_i+1\). Since \(i\) is odd, \((s,t) = (c_i, c_i+1) \in S\).

\[
\text{□}
\]

7.3. \(\exists v\)-Definability of Admissible and Member. For Admissible (Definition 7.9), we replace each variable \(d_i\), which earlier represented some convergent \(q_n \in \text{Ost}_\alpha\), by a 6-tuple \(\overline{d}_i = (u_i^-, v_i^-, u_i, u_i^+, v_i^+):\)

\[
(7.5) \quad (u_i^-, v_i^-, u_i, u_i^+, v_i^+) = (p_{n-1}, q_{n-1}, p_n, q_n, p_{n+1}, q_{n+1}) \text{ for some } n.
\]

We require \(C_{\exists v,\alpha}(u_i^-, v_i^-, u_i, u_i^+, v_i^+) = true\) to guarantee (7.5). Here \(v_i\) takes the earlier role of \(d_i\). Similarly, we replace each \(e_i\) in Admissible by a 6-tuple \(\overline{e}_i\) and also require that \(C_{\exists v,\beta}(\overline{e}_i) = true\). Here \(C_{\exists v,\alpha}\) and \(C_{\exists v,\beta}\) are from (3.14), with the extra subscript \(\alpha\) or \(\beta\) indicating which irrational is being considered. These \(C_{\exists v,\alpha}\) and \(C_{\exists v,\beta}\) conditions can be combined into a \(\exists^2\)-part. Altogether, the new Admissible has 42 variables.

Recall that Best is \(\exists^5\exists^4\)-definable (Lemma 7.3). The relation \(Y = X_{\overline{d}}^\alpha\) from (7.1) is \(\exists^2\)-definable:

\[
Y = X_{\overline{d}}^\alpha := Y < v^+ \land Z
\]

Compatible\((u,v,u^+,v^+,Y,Z,Z')\) \(\rightarrow W + Z \neq X\).

Here Compatible is from (3.19).

The relation \(\overline{d} \in \text{Ost}_\alpha(X)\), meaning \(v\) appears in \(\text{Ost}_\alpha(X)\), is \(\exists^3\)-definable (see (3.17)). The same holds for \(\overline{e} \in \text{Ost}_\beta(X)\) (just replace \(\alpha\) by \(\beta\)).

The relation

\[
\text{Consec}_{\exists}(\overline{d}_1, \overline{d}_2, X) := v_1 < v_2 \land \overline{d}_1 \in \text{Ost}_\alpha(X) \land \overline{d}_2 \in \text{Ost}_\alpha(X) \land \exists Y_1, Y_2 \ Y_1 = X_{\overline{d}_1}^\alpha \land Y_2 = X_{\overline{d}_2}^\alpha \land \text{After}(u_1^-, v_1^-, u_i, v_i, u_2 - Y_1),
\]

means \(v_1 < v_2\) appear consecutively in \(\text{Ost}_\alpha(X)\). This is \(\exists^{12}\)-definable.

It is now easy to see that Admissible \(\exists v\)-definable, and so is Member. A direct count reveals that Admissible is at most \(\exists^{50}\)\(\forall^{10}\), and Member is at most \(\exists^{100}\)\(\forall^{10}\).
7.4. Undecidability.

Theorem 7.11. The $\exists \forall \forall$-fragment is undecidable.

Proof. Here we follow an argument given in the proof of Thomas [Tho Th. 16.5]. Consider
$U = (Q, \Sigma, \sigma_1, \delta, q_1, q_2)$ a universal 1-tape Turing machine with 8 states and 4 symbols,
as given in [NW]. Here $Q = \{q_1, \ldots, q_8\}$ are the states, $\Sigma = \{\sigma_1, \ldots, \sigma_4\}$ are the tape
symbols, $\sigma_1$ is the blank symbol, $q_1$ is the start state and $q_2$ is the unique halt state.
Also, $\delta: [8] \times [4] \rightarrow [8] \times [4] \times \{\pm 1\}$ is the transition function. In other words, we have
$\delta(i, j) = (i', j', d)$ if upon state $q_i$ and symbol $\sigma_j$, the machine changes to state $q_{i'}$,
writes symbol $\sigma_{j'}$ and moves left ($d = -1$) or right ($d = 1$). Given an input $x \in \Sigma^*$, we will now
produce an $\exists \forall \forall$-sentence $\varphi_x$ such that $\varphi_x$ holds if and only if $U(x)$ halts.

We will now use sets $A_1, \ldots, A_8 \subseteq \mathbb{N}^2$ and $B_1, \ldots, B_4 \subseteq \mathbb{N}^2$ to code the computation on
$U(x)$. The $A_i$’s code the current state of the Turing machine. That is, for $(s, t) \in \mathbb{N}^2$,
we have $(s, t) \in A_i$ if and only if at step $s$-th of the computation, $U$ is in state $q_i$ and its
head over the $t$-th cell of the tape. The $B_j$’s code which symbols are written on the tape
at a given step of the computation. We have $(s, t) \in B_j$ if and only if at step $s$-th of the
computation, the symbol $\sigma_j$ is written on $t$-th cell of the tape. The computation $U(x)$ then
halts if and only if there are $A_1, \ldots, A_8 \subseteq \mathbb{N}^2$ and $B_1, \ldots, B_4 \subseteq \mathbb{N}^2$ such that:

a) $A_i$’s are pairwise disjoint; $B_j$’s are pairwise disjoint.

b) $(0, 0) \in A_1$, i.e., the computation starts in the initial state.

c) There exists some $(u, v) \in A_2$, i.e., the computation eventually halts.

d) For each $s \in \mathbb{N}$, there is at most one $t \in \mathbb{N}$ such that $(s, t) \in \bigcup_i A_i$, i.e., at each step
   of the computation, $U$ can only be in exactly one state.

e) If $x = x_0 \ldots x_n \in \Sigma^*$, then for every $0 \leq t \leq n$, we have $x_t = \sigma_j \iff (0, t) \in B_j$, 
i.e., the first rows of the $B_j$’s code the input string $x$.

f) Whenever $(s, t) \in B_j$,

f1) if $(s, t) \notin A_i$ for all $i \in \{0, \ldots, 8\}$, then $(s + 1, t) \in B_j$. That is, if the current head
   position is not at $t$, then the $t$-th symbol does not change.

f2) if $(s, t) \in A_i$ for some $i \in \{0, \ldots, 8\}$ and $\delta(i, j) = (\delta_i, \delta_j, \delta_3) \in [8] \times [4] \times \{\pm 1\}$, then
   $(s + 1, t) \in B_j$ and $(s + 1, t + \delta_3) \in A_j$. That is, if the head position is at $t$,
   and the state is $i$, then a transition rule is applied.

We use the predicate Member to code membership $(s, t) \in A_i, B_j$. By Theorem 7.10
there should exist tuples $X_i = (X_{i1}, \ldots, X_{i4})$, $Y_j = (Y_{j1}, \ldots, Y_{j4}) \in \mathbb{N}^4$ that represent $A_i$ and $B_j$.
In other words, we have


t \in A_i \iff \text{Member}(X_i, s, t) \quad \text{and} \quad \text{Member}(Y_j, s, t).

For the input condition e), there exist $Z_j = (Z_{j1}, \ldots, Z_{j4}) \in \mathbb{N}^4$ that represent $A_i$
and $B_j$. In other words, we have


t \in B_j \iff \text{Member}(Z_j, 0, t) \forall 0 \leq t \leq n.

Note that $Z_j$ can be explicitly constructed from the input $x$ (see Theorem 7.10’s proof).
Now the sentence $\varphi_x$ that encodes halting of $U(x)$ is:

$\varphi_x \coloneqq \exists X_1, \ldots, X_8, Y_1, \ldots, Y_4 \in \mathbb{N}^4, u, v \in \mathbb{N} \forall s, t, t' \in \mathbb{N}$

$\bigwedge_{i \neq i'} \neg(\text{Member}(X_i, s, t) \land \text{Member}(X_{i'}, s, t))$

$\land \bigwedge_{j \neq j'} \neg(\text{Member}(Y_j, s, t) \land \text{Member}(Y_{j'}, s, t))$
\[ \land \text{Member}(X_1, 0, 0) \land \text{Member}(X_2, u, v) \]
\[ \land \left( \bigvee_i \text{Member}(X_i, s, t) \land \left( \bigvee_i \text{Member}(X_i, s, t') \rightarrow t = t' \right) \right) \]
\[ \land \bigwedge_j \left( \text{Member}(Z_j, 0, t) \rightarrow \text{Member}(Y_j, 0, t) \right) \]
\[ \land \bigwedge_j \left( \text{Member}(Y_j, s, t) \rightarrow \left[ \bigwedge_i \neg \text{Member}(X_i, s, t) \land \text{Member}(Y_j, s + 1, t) \right] \right) \]
\[ \lor \bigvee_i \left[ \text{Member}(X_i, s, t) \land \text{Member}(Y_{d^2_{ij}}, s + 1, t) \land \text{Member}(X_{d^3_{ij}}, s + 1, t + \delta_{ij}^3) \right] \].

Since \text{Member} is \exists \forall-definable, the sentence \( \varphi_x \) is \exists \forall \exists \forall. Whether \( U(x) \) halts or not is undecidable, so is \( \varphi_x \). A direct count shows that \text{Member} appears at most 200 times in \( \varphi_x \). From the last estimate in Section 7.3, we see that \( \varphi_x \) is at most a \( \exists^k \forall^k \exists^k \forall^k \) sentence, where \( k = 20000 \). This completes the proof. \( \square \)

8. Final remarks and open problems

8.1. Comparing theorems 1.5 and 1.3, we see a big complexity jump by going from 1 to 3 alternating quantifier blocks, even the field is quadratic. The interesting open questions are the complexity of deciding (1.1) when \( r = 2, 3 \) with \( \alpha \) non-quadratic. We make the following conjecture:

**Conjecture 8.1.** For \( \alpha \) non-quadratic and \( r = 3 \), integer sentences (1.1) are undecidable.

Similarly, when \( \alpha \) is quadratic we make the following conjecture:

**Conjecture 8.2.** For \( \alpha \) quadratic and \( r = 2 \), deciding integer sentences (1.1) with a fixed number of variables and inequalities is NP-hard.

We note that for \( \alpha = \sqrt{5} \), \( \exists \forall \)-sentences in \( S_{\alpha} \) can already express non-trivial questions, such as the following: Given \( a, b \in \mathbb{Z} \), decide whether there is a Fibonacci number \( F_n \) congruent to \( a \) modulo \( b \)? Note that the sequence \( \{ F_n \mod b \} \) is periodic with period \( O(b) \), called the Pisano period. These periods were introduced by Lagrange and heavily studied in number theory (see e.g. [Sil, §29]), but the question above is likely computationally hard.

8.2. The main theorem by Khachiyan and Porkolab in [KP] is the following general Integer Programming result over \( \mathbb{Q} \).

**Theorem 8.3 (KP).** Consider a first order formula \( F(y) \) over the reals of the form:
\[ y \in \mathbb{R}^k : P(y, x_1, \ldots, x_m), \]
where \( P(y, x_1, \ldots, x_w) \) is a Boolean combination of equalities/inequalities of the form
\[ g_i(y, x_1, \ldots, x_w) \square_i 0 \]
with \( \square_i \in \{ >, <, = \} \) and \( g_i \in \mathbb{Z}[y, x_1, \ldots, x_w] \). Let \( k, m, n_1, \ldots, n_m \) be fixed, and suppose the set
\[ S_F := \{ y \in \mathbb{R}^n : F(y) = \text{true} \} \]
is convex. Then we can either decide in polynomial time that \( S_F \cap \mathbb{Z}^k = \emptyset \), or produce in polynomial time some \( y \in S_F \cap \mathbb{Z}^k \).
This immediately implies Theorem 1.5. Here there is no restriction on the number of \( g_i \)'s and their degrees. The coefficients of \( g_i \)'s are encoded in binary.

Note that convexity is crucially important in the theorem. In [MA], it is shown that given \( a, b, c \in \mathbb{Z} \), deciding \( \exists y \in \mathbb{N}^2 : ay_1^2 + by_2 + c = 0 \) is \( \text{NP} \)-complete. Here the semialgebraic set

\[
\{ y \in \mathbb{R}^2 : 0 \leq ay_1^2 + by_2 + c < 1 \}
\]

is not necessarily convex.

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