STRICT UNIMODALITY OF q-BINOMIAL COEFFICIENTS

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ABSTRACT. We prove strict unimodality of the q-binomial coefficients $\binom{n}{k}_q$ as polynomials in q. The proof is based on the combinatorics of certain Young tableaux and the semigroup property of Kronecker coefficients of S_n representations.

INTRODUCTION

A sequence $(a_1, a_2, \ldots, a_n)$ is called *unimodal*, if for some k we have

$$a_1 \,\le\, a_2 \,\le\, \ldots \,\le\, a_k \,\ge\, a_{k+1} \,\ge\, \ldots \,\ge\, a_n.$$

The *q-binomial* (Gaussian) *coefficients* are defined as:

$$\binom{m+\ell}{m}_q \;=\; \frac{(q^{m+1}-1)\,\cdots\,(q^{m+\ell}-1)}{(q-1)\,\cdots\,(q^{\ell}-1)} \;=\; \sum_{n=0}^{\ell m} p_n(\ell,m)\, q^n.$$

Sylvester's theorem establishes unimodality of the sequence

$$p_0(\ell,m),\, p_1(\ell,m),\, \ldots\,,\, p_{\ell m}(\ell,m).$$

This celebrated result was first conjectured by Cayley in 1856, and proved by Sylvester using Invariant Theory, in a pioneer 1878 paper [17]. In the past decades, a number of new proofs and generalizations were discovered both by algebraic and combinatorial tools, see Section 3. In the previous paper [12], we found a new proof of Sylvester's theorem using combinatorics of Kronecker and Littlewood–Richardson coefficients. Here we use the recently established *semigroup property* of Kronecker coefficients to prove *strict unimodality* of q-binomial coefficients:

Theorem 1. *For all $\ell, m \ge 8$, we have the following strict inequalities:*

$$(\circ) \qquad p_1(\ell,m) \,<\, \ldots \,<\, p_{\lfloor \ell m/2 \rfloor}(\ell,m) \,=\, p_{\lceil \ell m/2 \rceil}(\ell,m) \,>\, \ldots \,>\, p_{\ell m-1}(\ell,m).$$

These and the remaining cases are covered in Theorem 6. Note that neither combinatorial nor algebraic tools imply $(\circ)$ directly, as Sylvester's theorem is notoriously hard to prove and extend. In our previous paper [12], we proved strict unimodality of the diagonal coefficients $\binom{2m}{m}_q$ by combining technical algebraic tools from [13] and Almkvist's analytic unimodality results.

The following result lies in the heart of the proof of the theorem.

Lemma 2 (Additivity Lemma)**.** *Suppose inequalities $(\circ)$ as in the theorem hold for pairs (ℓ, m_1) and (ℓ, m_2). Suppose also that at least one of integers $\{\ell, m_1, m_2\}$ is even, and at least one ≥ 3. Then $(\circ)$ holds for $(\ell, m_1 + m_2)$.*

Although stated combinatorially, the only proof we know is algebraic. We first establish the lemma and then combine it with computational results to derive the theorem. We conclude with historical remarks, brief overview of the literature and open problems.

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1. Kronecker coefficients

We adopt the standard notation in combinatorics of partitions and representation theory of S_n (see e.g. [8, 16]). We use $g(\lambda,\mu,\nu)$ to denote the *Kronecker coefficients*:

$$\chi^\lambda \otimes \chi^\mu \, = \, \sum_{\nu\vdash n} g(\lambda,\mu,\nu)\, \chi^\nu\,, \quad \text{where} \ \ \lambda,\mu\vdash n.$$

The following technical result was never stated before, but is implicit in [12] (see also [18, §4]).

Lemma 3. *Let $n=\ell m$, $\tau_k=(n-k,k)$, where $0\le k\le n/2$ and set $p_{-1}(\ell,m)=0$. Then*

$$g(m^\ell,m^\ell,\tau_k) \, = \, p_k(\ell,m) \, - \, p_{k-1}(\ell,m).$$

Proof. Let $\lambda\vdash n$, $\pi\vdash k$ and $\theta\vdash n-k$, and let "$*$" denote the Kronecker product of symmetric functions. Littlewood's formula states that

$$s_\lambda * (s_\pi\, s_\theta) \, = \, \sum_{\alpha\vdash k,\, \beta\vdash n-k} c^\lambda_{\alpha\beta}\, (s_\alpha * s_\pi)\, (s_\beta * s_\theta)\,,$$

where $c^\lambda_{\mu\nu}$ denote the Littlewood–Richardson coefficients. Clearly, $s_\nu * s_a = s_\nu$, for all $\nu\vdash a$. We obtain:

$$s_\lambda * (s_k\, s_{n-k}) \, = \, \sum_{\alpha\vdash k,\, \beta\vdash n-k} c^\lambda_{\alpha\beta}\, s_\alpha\, s_\beta \, = \, \sum_{\alpha\vdash k,\, \beta\vdash n-k,\, \nu\vdash n} c^\lambda_{\alpha\beta}\, c^\mu_{\alpha\beta}\, s_\mu\,.$$

By the Jacobi–Trudi formula, we have:

$$s_{\tau_k} \, = \, s_{(n-k,k)} \, = \, s_k\, s_{n-k} \, - \, s_{k-1}\, s_{n-k+1}\,.$$

This gives:

$$s_\lambda * s_{\tau_k} \, = \, s_\lambda * (s_k\, s_{n-k}) \, - \, s_\lambda * (s_{k-1}\, s_{n-k+1}) \, = \, \sum_{\mu\vdash n} a_k(\lambda,\mu)\, s_\mu \, - \, \sum_{\mu\vdash n} a_{k-1}(\lambda,\mu)\, s_\mu\,,$$

where

$$a_k(\lambda,\mu) \, = \, \sum_{\alpha\vdash k,\, \beta\vdash n-k} c^\lambda_{\alpha\beta}\, c^\mu_{\alpha\beta}\,.$$

Taking the coefficient at s_μ in the expansion of $s_\lambda * s_{\tau_k}$ in terms of Schur functions, we get:

$$(\circledast) \qquad g(\lambda,\mu,\tau_k) \, = \, a_k(\lambda,\mu) \, - \, a_{k-1}(\lambda,\mu).$$

Let $\lambda=\mu=(m^\ell)$. Recall that $c^{(m^\ell)}_{\alpha\beta}=1$ if α and β are complementary partitions within the rectangle (m^ℓ); and $c^{(m^\ell)}_{\alpha\beta}=0$ otherwise (see e.g. [10]). Therefore,

$$a_k(m^\ell,m^\ell) \, = \, \sum_{\alpha\vdash k,\, \alpha\subset(m^\ell)} 1^2 \, = \, p_k(\ell,m).$$

Substituting this into ($\circledast$), gives the result. ∎

Theorem 4 (Semigroup property)**.** *Suppose $\lambda,\mu,\nu,\alpha,\beta,\gamma$ are partitions of n, such that $g(\lambda,\mu,\nu)>0$ and $g(\alpha,\beta,\gamma)>0$. Then $g(\lambda+\alpha,\mu+\beta,\nu+\gamma)>0$.*

Remark 5. This result was conjectured by Klyachko in 2004, and recently proved in [3]. It is the analogue of the semigroup property of Littlewood–Richardson coefficients proved by Brion and Knop in 1989 (see [21] for the history and the related results). Unfortunately, the Knutson–Tao *saturation theorem* does not generalize to Kronecker coefficients (see e.g. [5, §2.5]). Let us mention the following useful extension by Manivel [9]: in conditions of the theorem, we have

$$g(\lambda+\alpha,\mu+\beta,\nu+\gamma) \, \ge \, \max\bigl\{ g(\lambda,\mu,\nu),\, g(\alpha,\beta,\gamma)\bigr\}\,.$$

2. The proofs

Proof of Lemma 2. Let $\lambda = \mu = (m_1^\ell)$, $\alpha = \beta = (m_2^\ell)$, $\nu = (\ell m_1 - r, r)$, $\gamma = (\ell m_2 - s, s)$. By the strict unimodality assumption for (ℓ, m_1) and (ℓ, m_2) and Lemma 3, we have

$$g(m_1^\ell, m_1^\ell, \nu) > 0, \qquad g(m_2^\ell, m_2^\ell, \gamma) > 0,$$

for all $r, s \geq 0, \neq 1$. Apply Theorem 4 to the fixed partitions above. Now, for all $k = r + s$, we then have

$$g\big((m_1+m_2)^\ell, (m_1+m_2)^\ell, \tau_k\big) = g(m_1^\ell + m_2^\ell, m_1^\ell + m_2^\ell, \nu + \gamma) > 0,$$

where $n = (m_1 + m_2)\ell$ and $\tau_k = (n-k, k)$ as before. For $k \leq 3$ we can choose $(r, s) = (0, k)$ or $(k, 0)$, as at it is implicit that $\ell, m_1, m_2 \geq 2$ and one of them is ≥ 3. For $3 < k \leq \lfloor n/2 \rfloor - 1$ we have that $k \leq \lfloor \ell m_1/2 \rfloor + \lfloor \ell m_2/2 \rfloor$, so there are values $r, s \geq 2$, $r \leq \lfloor \ell m_1/2 \rfloor$ and $s \leq \lfloor \ell m_2/2 \rfloor$, such that $k = r + s$. Finally, when $k = \lfloor n/2 \rfloor$, by the parity conditions we have that at least one of ℓm_1, ℓm_2 is even, so we can choose $(r, s) = (\ell m_1/2, \lfloor \ell m_2/2 \rfloor)$ or $(\lfloor \ell m_1/2 \rfloor, \ell m_2/2)$. $\square$

Theorem 6. *Let $m, \ell \geq 2$. Strict unimodality* $(\circ)$ *as in Theorem 1 holds for pairs (ℓ, m), $\ell \leq m$, if and only if $\ell = m = 2$, or $\ell, m \geq 5$ with the exception of the following values:*

$$\{(5,6),\ (5,10),\ (5,14),\ (6,6),\ (6,7),\ (6,9),\ (6,11),\ (6,13),\ (7,10)\}.$$

Proof. A direct calculation gives strict unimodality for each $\ell \in \{8, \ldots, 15\}$, and $8 \leq m < 16$. For each fixed $\ell \in \{8, \ldots, 15\}$ and $m \geq 16$, we have that $m = 8a + b$ for $a \geq 1$ and $8 \leq b < 16$. Applying the additivity lemma successively with $\ell, m_1 = 8k + b, m_2 = 8$ for $k = 0, 1, \ldots, a-1$, shows that $(\circ)$ holds for all $\ell \in \{8, \ldots, 15\}$ and $m \geq 16$.

Fixing any $m \geq 8$ and applying the additivity lemma in the direction of ℓ the same way by expressing $\ell = 8a' + b'$, shows that $(\circ)$ holds for all $m, \ell \geq 8$.

A direct calculation also gives strict unimodality for all values of $\ell \in \{5, 6, 7\}$ and $5 \leq m \leq 20$ with the exception of the listed cases, where the middle three coefficients of the expansion of $\binom{\ell+m}{m}_q$ are equal. Now we apply the additivity lemma for each value of $\ell = 5, 6, 7$ and $m = 10a + b$ where $10 \leq b \leq 19$ and induct over a with the values $m_1 = 10(a-1) + b$ and $m_2 = 10$. The cases $\ell > m$ follow from the symmetry.

Now, case $\ell = 2$ is straightforward, since $p_{2i}(2, m) = p_{2i+1}(2, m)$ for all $i < n/4$. On the other hand, cases $\ell = 3, 4$ have been studied in [6, 19] using an explicit symmetric chain decomposition. Since all chain lengths there are ≥ 3, we obtain equalities for the middle coefficients. $\square$

3. Final remarks

3.1. Let us quote a passage from [17] describing how Sylvester viewed his work:

> "I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power."

The grandeur notwithstanding, it does reveal Sylvester's excitement over his discovery.

3.2. Proving unimodality is often difficult and involves a remarkable diversity of applicable tools, ranging from analytic to bijective, from topological to algebraic, and from Lie theory to probability. We refer to [1, 2, 15] for a broad overview of the subject.

3.3. The Additivity Lemma gives an example of a 2-dim *Klarner system*, which always have a finite basis (see [14]).

3.4. The equation (KOH) in [20], based on O'Hara's combinatorial approach to unimodality of q-binomial coefficients [11], gives a useful recurrence relation (cf. [4, 7]). It would be interesting to see if (KOH) can be used to prove Theorem 1.

3.5. An important generalization of Sylvester's theorem is the unimodality of $s_\lambda(1, q, \ldots, q^m)$ as a polynomial in q, see [8, p. 137]. We conjecture that if the *Durfee square* size of λ is large enough, then these coefficients are strictly unimodal. An analogue of (KOH) in this case is in [4].

3.6. In a different direction, we believe that for every $d \geq 1$ there exists $L(d)$, s.t. $p_k(\ell, m) - p_{k-1}(\ell, m) \geq d$ for all $L(d) < k \leq \ell m/2$, and m, ℓ large enough. Unfortunately, the tools in this paper are not directly applicable. However, for $\ell = m$, this follows from Prop. 11 in [15], and further extension of Thm. 5.2 in [12] on strict unimodality of the number of partitions into distinct odd parts. Then, combined with Manivel's extension (see Remark 5), and the finite basis theorem (see §3.3), this would prove the conjecture in a similar manner as the proof of Theorem 6. We plan to return to this problem in the future.

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