A Bijective Proof of the Hook-Length Formula and its Analogs

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1. The main result of the article is a bijective proof of the multiplicative formula for the dimension of an irreducible representation of the symmetric group, which is usually called the "hook-length formula." We also prove a formula for the Poincaré series for the multiplicity of the isotypic component in the symmetric algebra $S(V)$ $(V = \mathbb{C}^n)$ considered as a graded S_n-module. We use classical combinatorial interpretations (see [1, 2]) and establish the bijection in their terms.

2. A set of pairs $(i, j) \in \mathbb{Z}^2$ satisfying the conditions $1 \leq j \leq \lambda_i$, $\lambda_1 \geq \lambda_2 \geq \cdots \in \mathbb{N}$, is called a *Young diagram* $\lambda = (\lambda_1, \lambda_2, \ldots)$. We suppose that $\mathbb{Z}^2$ is situated in the plane $\mathbb{R}^2$ so that the first coordinate i increases from top to bottom, while the second coordinate j increases from left to right. A diagram λ is depicted by the set of 1×1-cells with centers at the points $(i, j) \in \lambda$. A function $A\colon \lambda \to \mathbb{N}$ represented by numbers written in the cells of a diagram λ that strictly increase from top to bottom along the columns and do not decrease from left to right along the rows is said to be a *tableau A of shape λ*. Denote by $|A|$ the sum of the numbers in a tableau A; $|\lambda| := \sum_{i=1}^{\infty} \lambda_i$. We call a tableau A *standard* if each number $1, \ldots, |\lambda|$ occurs in A just once. We denote by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ the diagram transposed to λ.

Recall that the equivalence classes $[T_\lambda]$ of irreducible representations of the symmetric group S_n are parametrized by diagrams λ, $|\lambda| = n$; $\dim T_\lambda =: K_\lambda$ is the number of standard tableaux of shape λ. Consider the space $\mathcal{H} = \bigoplus \mathcal{H}^i$ of *harmonic polynomials* in n variables (i.e., those anihilated by $\mathbb{C}[\partial/\partial X_1, \ldots, \partial/\partial X_n]^{S_n}$) as an S_n-module, and put $\mathbf{K}_\lambda(t) := \sum_i \dim \operatorname{Hom}(T_\lambda, \mathcal{H}^i) t^i$. Then the regular representation of S_n is realized in $\mathcal{H}$ (see [3]), $\mathbf{K}_\lambda(t)$ being the generating function for the number of standard tableaux of shape λ with a given charge (see below). Similarly, $\mathbf{P}_\lambda(t) := \sum_k \dim \operatorname{Hom}(T_\lambda, S^k(V)) t^k$ is the generating function for the number of tableaux A of shape λ with a given sum $|A|$ (see [1]).

Theorem 1.

$$\mathbf{K}_\lambda(t) = \frac{\prod_{i=1}^{n}(1 + t + \cdots + t^{i-1})}{\prod_{(i,j)\in\lambda}(1 + t + \cdots + t^{h(i,j)-1})/t^{i-1}},$$

where $h(i, j) = \lambda_i + \lambda'_j - i - j + 1$.

Corollary 1 (hook-length formula).

$$\mathbf{K}_\lambda(1) = K_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i, j)}.$$

Theorem 2.

$$\mathbf{P}_\lambda(t) = \prod_{(i,j)\in\lambda} \frac{t^{i-1}}{1 - t^{h(i,j)}}.$$

Remark. The polynomials $\mathbf{K}_\lambda(t)$ play an important role in representation theory of the group $\mathrm{GL}(n)$ over the complex or a finite field (see [1]); they are particular cases of the Kazhdan–Lustzig polynomials. The generating function $\mathbf{P}_\lambda$ is also well-known in combinatorics (see [1, 4]), as well as in representation theory (see [5–7]); its superanalog is also known (see [8]).

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3. To prove Theorem 1, we establish a bijection between the following sets (λ is fixed, $|\lambda| = n$):
I is the set of pairs of the form

(a standard tableau of shape λ;

a function $f : \lambda \to \mathbb{Z}$ such that $(\lambda_i - j) \geq f(i, j) \geq -(\lambda'_j - i)$, $\forall (i, j) \in \lambda$),

II is the set of bijective fillings of the diagram λ by the numbers $\{1, 2, \dots, n\}$.

Fix a filling $A \in \mathrm{II}$ and set $f \equiv 0$. We shall order the numbers in A, beginning from the right column and moving to the left, and from bottom to top inside the columns. Suppose that the filling is already ordered up to the cell (i_0, j_0), i.e., the numbers increase along rows and columns for all (i, j) such that either $j > j_0$ or $j = j_0$ and $i > i_0$. We shall execute the following procedure:

Set $i = i_0$, $j = j_0$, $a = A(i_0, j_0)$.

1) Put

$$b = A(i, j+1) \quad \text{for } j < \lambda_i, \quad \text{and} \quad b = n+1 \quad \text{for } j = \lambda_i;$$
$$c = A(i+1, j) \quad \text{for } i < \lambda'_j, \quad \text{and} \quad c = n+1 \quad \text{for } i = \lambda'_j.$$

2) If $a > b$ and $c > b$, then $A(i, j+1) := a$, $A(i, j) := b$; $f(i, j_0) := f(i, j_0) + 1$, $j := j + 1$; go to 1).
3) If $a > c$ and $b > c$, then $A(i+1, j) := a$, $A(i, j) := c$; exchange the values $f(i, j_0)$ and $f(i+1, j_0)$; $f(i, j_0) := f(i, j_0) - 1$, $i := i + 1$; go to 1).
4) If $c > a$ and $b > a$, the procedure is over.

Finally we order the filling (an element of the set II) and obtain a standard tableau A and a function f (i.e., an element of the set I).

Proposition 1. *The constructed map* $\mathrm{II} \to \mathrm{I}$ *is a bijection.*

Let $\omega(A)$ be the permutation of numbers $1, \dots, |\lambda|$ obtained by reading a tableau A along the rows from right to left and from top to bottom. Put $\operatorname{ind}(1) := 0$; $\operatorname{ind}(i+1) := \operatorname{ind}(i)$ if $\omega^{-1}(i+1) > \omega^{-1}(i)$, and $\operatorname{ind}(i+1) := \operatorname{ind}(i) + 1$ if $\omega^{-1}(i+1) < \omega^{-1}(i)$. The number $c(A) := \sum_{i=1}^{|\lambda|} \operatorname{ind}(i)$ is said to be the *charge* of the standard tableau A. We define a grading on the set I as $c(A) + \sum_{(i,j)\in\lambda} f(i, j)$, and transfer it to the set II by means of the bijection.

Proposition 2. *The grading thus obtained is equivalent to the standard grading on the set* S_n *defined by the number of inversions* (*see* [9]) (*equivalence means the equality of numbers of elements with a given grading*).

From Propositions 1 and 2 we derive the equation

$$\mathbf{K}_\lambda(t) \prod_{(i,j)\in\lambda} \frac{1 + t + \dots + t^{h(i,j)-1}}{t^{\lambda'_j - i}} = \prod_{i=1}^{n} (1 + t + \dots + t^{i-1}),$$

which immediately implies Theorem 1.

To prove Theorem 2, we establish a bijection between the two following infinite graded sets:
III is the set of pairs of the form

(a standard tableau A of shape λ; a partition $\mu = (\mu_1 \geq \dots \geq \mu_n)$, $n = |\lambda|$),

where the grading is defined as $c(A) + |\mu|$,
IV is the set of tableaux B of shape λ with the grading $|B|$.

We construct an explicit map $\mathrm{III} \to \mathrm{IV}$. Let $I(i, j) = \operatorname{ind}(A(i, j))$, $D(i, j) = \mu_{n+1-A(i,j)}$, $(i, j) \in \lambda$. It is easy to see that I is a tableau of shape λ, $|I| = c(A)$, and D is a filling which is nondecreasing along rows and columns, $|D| = |\mu|$. Now we put $B(i, j) := I(i, j) + D(i, j)$. It is the desired tableau of shape λ, and $|B| = c(A) + |\mu|$, i.e., our map preserves the grading.

Proposition 3. *The constructed map* III $\to$ IV *is a bijection.*

Therefore, from Proposition 3 and Theorem 1 we obtain

$$\mathbf{P}_\lambda(t) = \frac{\sum_A t^{c(A)}}{\prod_{i=1}^n (1-t^i)} = \frac{\prod_{i=1}^n (1-t^i) \prod_{(i,j)\in\lambda} t^{i-1}(1-t^{h(i,j)})^{-1}}{\prod_{i=1}^n (1-t^i)} = \prod_{(i,j)\in\lambda} \frac{t^{i-1}}{1-t^{h(i,j)}},$$

thus establishing Theorem 2.

Remark. The latter bijection is a combinatorial analog of the Kostant decomposition (see [3]), $S(V) = \mathcal{A} \otimes \mathcal{H}$, where $\mathcal{A}$ is the algebra of invariants with respect to the Weyl group (in our case, the algebra of symmetric polynomials).

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Invariant Semilinear Elliptic Equations on Manifolds of Constant Negative Curvature

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0. At present, numerous results on solvability of semilinear elliptic equations of the form $-\Delta u = f(u)$ in $\mathbb{R}^n$ are known (see [1, 2] and detailed references therein). Similar equations are of interest on arbitrary complete Riemannian manifolds. However, the papers referred to lean heavily on specific features of the flat case (for example, on the presence of homotheties). It seems that, besides $\mathbb{R}^n$, the most simple case is that of the hyperbolic space $\mathbb{H}_n$ (with curvature -1). We consider its standard model, namely, the unit ball in $\mathbb{R}^n$ supplied with the Poincaré metric. Consider the equation

$$-\Delta u + mu = f(u) \tag{1}$$

in $\mathbb{H}_n$, where Δ is the Laplace–Beltrami operator. We are interested in decaying solutions, more precisely, in those belonging to the invariant Sobolev space $H^1(\mathbb{H}_n)$. This space is defined as the completion of $C_0^\infty(\mathbb{H}_n)$ with respect to the norm given by

$$\|u\|^2 = \int |\nabla u|^2 dV + \int u^2 dV, \tag{2}$$

where dV is the invariant volume form in $\mathbb{H}_n$, and the gradient ∇ and $|\cdot|$ correspond to the Poincaré metric [3]. This space coincides with the space of functions from $L^2(\mathbb{H}_n)$ having finite norms defined by (2). Note also that $H^1(\mathbb{H}_2) = H_0^1(\Omega_2)$, where $\Omega_2 \subset \mathbb{R}^2$ is the Euclidean unit disk. To Eq. (1) we assign the action functional

$$S(u) = \frac{1}{2}\int |\nabla u|^2 dV - \int \left(F(u) - \frac{mu^2}{2}\right) dV, \tag{3}$$

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