5. The author is grateful to W. N. Everitt for drawing his attention to the close connection of Theorem 2 with the results obtained in [6] and to M. Sh. Birman for useful comments concerning this theorem.

References

1. M. A. Naimark, Linear Differential Operators [in Russian], Nauka, Moscow (1969).
2. W. N. Everitt, I. W. Knowles, and T. T. Read, Proc. Roy. Soc. Edinburgh. Ser. A, **103**, 215–228 (1986).
3. I. M. Rapoport, On Certain Asymptotic Methods in Theory of Differential Equations [in Russian], Izd. AN UkrSSR, Kiev (1954).
4. Yu. B. Orochko, Mat. Zametki, **48**, 86–94 (1990).
5. Yu. B. Orochko, Ukr. Mat. Zh., **44**, 948–956 (1992).
6. W. N. Everitt and A. Zettl, Proc. London. Math. Soc., **64**, 524–544 (1992).

Translated by Yu. B. Orochko

Resolutions for S_n-Modules, Associated with Rim Hooks, and Combinatorial Applications

I. M. Pak and A. E. Postnikov

UDC 512.547+511.217

1. In [1] a method for constructing resolutions that "materialize" classical formulas for S_n-modules is given. In this paper we present a new resolution. In a special case it "materializes" the well-known combinatorial fact that the value of the inversion polynomial $f_n(t)$ for trees with $n+1$ vertices at $t=-1$ is equal to the number of up-down permutations (see [2, 3]).

Let $A_q(n) = A_q := \mathbb{C}\langle x_1, x_2, \dots, x_n\rangle/(x_ix_j - qx_jx_i, i<j)$ be the algebra of functions on the quantum space equipped with the natural gradation $A_q = A_q^0 \oplus A_q^1 \oplus A_q^2 \oplus \dots$. The braid group $\mathrm{Br}(n)$, defined by the generators $s_1, s_2, \dots, s_{n-1}$ and the relations $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, $s_is_j = s_js_i$ for $|i-j|\ge 2$, acts on A_q in the following way:

$$s_i(x_1^{b_1}\dots x_i^{b_i}x_{i+1}^{b_{i+1}}\dots x_n^{b_n}) := x_1^{b_1}\dots x_{i+1}^{b_i}x_i^{b_{i+1}}\dots x_n^{b_n}.$$

It is clear that for $q=\pm 1$ the relation $s_i^2 = \mathrm{id}$ also holds, i.e., $A_{\pm 1}$ is a graded S_n-module. From now on we will consider the case $q=-1$.

Let $a=(a_1,\dots,a_n)$, where $0\le a_1\le\dots\le a_n$ and $|a| := a_1+\dots+a_n$. We consider the submodule $\tau_a \subset A_{-1}$ generated by the monomials $x_{\sigma(1)}^{b_1}\dots x_{\sigma(n)}^{b_n}$, where $0\le b_i\le a_i$, $i=1,\dots,n$, and $\sigma\in S_n$. The module τ_a is also graded: $\tau_a = \tau_a^0\oplus\tau_a^1\oplus\dots\oplus\tau_a^{|a|}$.

Now we construct a skew Young diagram $\theta(a)\subset\mathbb{Z}_+^2$, $\theta(a) = \{\theta_1,\dots,\theta_n\}$, such that $\theta_1 = (1,n)$ and if $\theta_{l-1} = (i,j)$, then $\theta_l = (i, j-1)$ for a_l even and $\theta_l = (i+1, j)$ for a_l odd, $l = 2,3,\dots,n$. Obviously, the diagram $\theta = \theta(a)$ thus obtained is a rim hook (see [4, 5]).

Recall that to any skew Young diagram γ, $|\gamma| = n$, there corresponds a representation π_γ of the symmetric group S_n. If γ is an ordinary Young diagram, then the representation π_γ is irreducible (see [4, 5]).

Theorem. *Let $\theta=\theta(a)$. There exists a natural resolution*

$$0\to\tau_a^0\to\tau_a^1\to\dots\to\tau_a^{|a|}\to\pi_\theta\to 0 \quad \textit{if } a_1 \textit{ is even},$$
$$0\to\tau_a^0\to\tau_a^1\to\dots\to\tau_a^{|a|}\to 0 \quad \textit{if } a_1 \textit{ is odd}.$$

Since the Euler–Poincaré characteristic is equal to zero, we get the following formula for characters

$$\chi^{|a|} - \chi^{|a|-1} + \dots + (-1)^{|a|}\chi^0 = \begin{cases} \chi_\theta & \text{if } a_1 \text{ is even},\\ 0 & \text{if } a_1 \text{ is odd},\end{cases} \tag{1}$$

Moscow State University. Moscow Independent University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 28, No. 2, pp. 72–75, April–June, 1994. Original article submitted April 19, 1993.

0016–2663/94/2802–0132 $12.50

where χ^i is the character of the representation τ_a^i and χ_θ is the character of π_θ.

The authors also found a purely combinatorial proof of formula (1) based on the construction of an involution on the tableaux of a special kind. The proof of the theorem and the construction of the involution will be published later.

In the remaining part of the paper we will consider some examples and consequences of the theorem and continue the investigation of the graded S_n-module A_{-1}.

2. Suppose that S_n acts on $V = \mathbb{C}^n$ by the permutation of coordinates. Consider the Weil algebra $E(V) = S(V)\otimes\Lambda(V) = \bigoplus S^k(V)\otimes\Lambda^l(V)$ as a graded S_n-module, where $S^k(V)\otimes\Lambda^l(V)$ is the component of degree $2k+l$. It can be shown that A_{-1} is isomorphic to $E(V)$ as a graded S_n-module. Using the results of [6] concerning the decomposition of the bigraded S_n-module $E(V)$, we immediately obtain the following formula:

$$\sum_{k=0}^{\infty} \dim \operatorname{Hom}(\pi_\lambda, A_{-1}^k)t^k = \prod_{(i,j)\in\lambda} \frac{t^{2i}+t^{2j+1}}{1-t^{2h(i,j)}}, \tag{2}$$

where $\lambda \subset \mathbb{Z}_+^2$ is a Young diagram, $|\lambda| = n$, and $h(i,j)$ is the length of the hook at the box (i,j) (see [4, 5]).

3. Let $a = (0, 1, \dots, n-1)$ and $f_n(t) := \sum_{k=0}^{\binom{n}{2}} \dim \tau_a^k t^{\binom{n}{2}-k}$. It is known that $f_n(1) = \dim\tau_a = (n+1)^{n-1}$ is the number of labeled trees with $n+1$ vertices, $f_n(1+t) = t^{-n}\sum c_{nk}t^k$, where c_{nk} is the number of connected graphs with $n+1$ vertices and k edges, and $f_n(t)$ is the inversion polynomial for trees with $n+1$ vertices (see [2, 3]).

We will show that the theorem implies $f_n(-1) = \mathrm{ud}_n$, where ud_n is the number of up-down permutations, i.e.,

$$\mathrm{ud}_n := |\{\sigma \in S_n \mid \sigma(1) < \sigma(2) > \sigma(3) < \dots\}|.$$

Indeed, in this case the diagram $\theta = \theta(a)$ has a "staircase" form: $\theta = (n, n, n-1, n-2, \dots) \setminus (n-1, n-2, n-3, \dots)$. If we consider (1) as an identity for dimensions, we immediately obtain $f_n(-1) = \dim\theta = \mathrm{ud}_n$. Note that $\sum_{n=0}^{\infty} \mathrm{ud}_n t^n/n! = \tan t + \sec t$ (see [2, 3]).

4. Let $a = (0, k, 2k, \dots, (n-1)k)$. This case is a natural generalization of Sec. 3 to k-dimensional trees (see [7]). In this case $\dim\tau_a = (kn+1)^{n-1}$, $\dim\theta(a) = \mathrm{ud}_n$, for odd k, and $\dim\theta(a) = 1$ for even k.

5. Let $a = (a_1, \dots, a_n)$, $a_1 = \dots = a_n = k$, $\tau_{nk} := \tau_a$, and $g_{nk}(t) := \sum \dim\tau_a^i t^i$. It can be shown that $g_{nk}(t) = (1+t+\dots+t^k)^n$. For S_n-invariants the following formula holds:

$$\sum_{i,n} \dim(\tau_{nk}^i)^{S_n} t^i z^n = \frac{(1+zt)(1+zt^3)\dots}{(1-z)(1-zt^2)\dots}, \tag{3}$$

where the right-hand side contains k factors. Since $\varinjlim \tau_{nk}^i = A_{-1}^i(n)$ as $k \to \infty$, it follows from (3) that

$$\sum_{i,n} \dim(A_{-1}^i(n))^{S_n} t^i z^n = \prod_{i=0}^{\infty} \frac{1+zt^{2i+1}}{1-zt^{2i}}. \tag{4}$$

On the other hand, from (2) for $\lambda = (n)$ we get

$$\sum_i \dim(A_{-1}^i(n))^{S_n} t^i = \frac{(1+t)(1+t^3)\dots(1+t^{2n-1})}{(1-t^2)(1-t^4)\dots(1-t^{2n})}. \tag{5}$$

Taking the sum of the right-hand sides of (5) over all n and comparing this sum with the right-hand side of (4), we obtain the well-known Euler identity (see [2])

$$\sum_{n=0}^{\infty} z^n \prod_{i=0}^{n} \frac{1+t^{2i-1}}{1-t^{2i}} = \prod_{i=0}^{\infty} \frac{1+zt^{2i+1}}{1-zt^{2i}}.$$

In this connection we note that S_n-invariants of the algebra $A_{-1}(n)$ are generated by the polynomials $e_i(x_1^2, \dots, x_n^2)$ and $p_{2i-1}(x_1, \dots, x_n)$, $i = 1, \dots, n$, where e_i are elementary and p_l are power symmetric polynomials (e.g., see [4]).

References

1. A. V. Zelevinsky, Funkts. Anal. Prilozhen., **21**, No. 2, 74–75 (1987).
2. I. P. Goulden and D. M. Jackson, Combinatorial Enumeration, John Wiley & Sons, New York (1983).
3. G. Kreweras, Period. Math. Hung., **11**, No. 4, 309–320 (1980).
4. I. G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford (1979).
5. G. James and A. Kerber, The Representation Theory of the Symmetric Group, Addison-Vesley, P.G. (1981).
6. A. A. Kirillov and I. M. Pak, Funkts. Anal. Prilozhen., **24**, No. 3, 9–13 (1990).
7. J. W. Moon, J. Combin. Theory, **6**, 196–199 (1969).

Translated by A. E. Postnikov

The Skew Field of Rational Functions on $GL_q(n, K)$

A. N. Panov

UDC 512.554.3+512.813.4

For every field K, the ring of regular functions on the quantum matrix algebra $M_q(n, K)$ is defined by generators and imposed relations [1, 2]. This ring is a K-algebra and is generated, as a K-algebra, by the elements $\{a_{ij} \mid 1 \leq i, j \leq n\}$ and $K[q, q^{-1}]$. The variable q commutes with a_{ij}, and the matrix elements are related by the conditions $a_{ij}a_{ik} = q^{-1}a_{ik}a_{ij}$ for $j < k$, $a_{ki}a_{mi} = q^{-1}a_{mi}a_{ki}$ for $k < m$, $a_{ij}a_{km} = a_{km}a_{ij}$ for $i < k$, $j > m$, and $a_{ij}a_{km} - a_{km}a_{ij} = (q^{-1} - q)a_{im}a_{kj}$ for $i < k$, $j < m$. We denote the ring of regular functions on the quantum matrix algebra $M_q(n, K)$ by $K[M_q(n, K)]$ or $\mathfrak{F}_q$. For every $\varepsilon \in K$, $\varepsilon \neq 0$, the algebra $\mathfrak{F}_q$ contains the ideal $\mathfrak{F}_q(q - \varepsilon)$, and we can define the specialization $\mathfrak{F}_\varepsilon = \mathfrak{F}_q/\mathfrak{F}_q(q - \varepsilon)$. Hereafter we will use the common notation $\mathfrak{F}_q$ for both rings $\mathfrak{F}_q$ and $\mathfrak{F}_\varepsilon$, stating explicitly whenever we use it whether q is a variable or an element of the field K.

For a permutation σ we denote by $l(\sigma)$ the number of inversions in the rearrangement $\sigma(1), \dots, \sigma(n)$. The quantum determinant $\det_q$ is an element of the algebra $\mathfrak{F}_q$ that is the sum $\sum(-q)^{-l(\sigma)}a_{1\sigma(1)} \dots a_{n\sigma(n)}$ over all permutations $\sigma \in S_n$. If q is generic (i.e., q is either a variable or an element of K that is not a root of unity), then the quantum determinant generates the center of the ring $\mathfrak{F}_q$ [1].

Proposition 1 [2, 3, 5]. *The ring $\mathfrak{F}_q$ is a Noetherian integral domain.*

The ring $K[GL_q(n)]$ of regular functions on the quantum group $GL_q(n, K)$ is defined as a localization of the ring $\mathfrak{F}_q$ by the multiplicative subset generated by $\det_q$. The ring $K[GL_q(n)]$ is also a Noetherian integral domain. A Noetherian integral domain has a skew field of fractions [4, Theorem 3.6.12]. The skew fields of fractions of the rings $K[GL_q(n)]$ and $\mathfrak{F}_q$ coincide.

Definition 1. The skew field of fractions $\operatorname{Fract} K[GL_q(n)]$ will be called the *skew field of rational functions on the quantum group* $GL_q(n, K)$ and denoted by $K(GL_q(n))$ or F_q.

The main purpose of this article is to describe the skew field of fractions F_q in simpler terms. To this end, we consider algebras of twisted polynomials and their skew fields of fractions.

Definition 2. Let $S = (s(i, j))$ be a skew-symmetric $\mathbb{Z}$-matrix of size $t \times t$. The *algebra of twisted polynomials* is a K-algebra $A_q(S)$ generated by $x_1, \dots, x_t$ and $K[q, q^{-1}]$ with the relations $x_ix_j = q^{s(i,j)}x_jx_i$ and $qx_i = x_iq$.

Proposition 2. *The ring $A_q(S)$ is a Noetherian integral domain.*

Samara State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 28, No. 2, pp. 75–77, April–June, 1994. Original article submitted April 2, 1993.

0016–2663/94/2802–0134 $12.50