Constructing Uniquely Realizable Graphs

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Abstract

In the *Graph Realization Problem* (GRP), one is given a graph G, a set of non-negative edge-weights, and an integer d. The goal is to find, if possible, a realization of G in the Euclidian space $\mathbb{R}^d$, such that the distance between any two vertices is the assigned edge weight. The problem has many applications in mathematics and computer science, but is NP-hard when the dimension d is fixed. Characterizing tractable instances of GRP is a classical problem, first studied by Menger in 1931. We construct two new infinite families of GRP instances which can be solved in polynomial time. Both constructions are based on the blow-up of fixed small graphs with large expanders. Our main tool is the *Connelly's condition* in *Rigidity Theory*, combined with an explicit construction and algebraic calculations of the *rigidity* (*stress*) *matrix*. As an application of our results, we describe a general framework to construct uniquely k-colorable graphs. These graphs have the extra property of being uniquely vector k-colorable. We give a deterministic explicit construction of such a family using Cayley expander graphs.

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1 Introduction

The *Graph Realization Problem* (GRP) is one of the most well studied problems in distance geometry and has received attention in many disciplines. In that problem, one is given a graph $G = (V, E)$ on n vertices, a set of non–negative edge weights $\{w_{ij} : (i, j) \in E\}$, and a positive integer d. The goal is to compute a realization of G in the Euclidean space $\mathbb{R}^d$, i.e. a mapping $\mathbf{p} : V \to \mathbb{R}^d$ such that $\|\mathbf{p}(i) - \mathbf{p}(j)\| = w_{ij}$ ($\|\mathbf{x}\|$ stands for the Euclidian length of the vector $\mathbf{x}$), or determine if such realization does not exist.

The Graph Realization Problem and its variants arise from applications in different areas both in mathematics and computer science. In *molecular conformation* (see, e.g., [9, 26]), solving the GRP in dimension three allows construction of the 3-dimensional structure of the molecule. In *wireless sensor network localization* (see, e.g., [6, 16]), where one is interested in inferring the locations of sensor nodes in a sensor network. And in *computer vision*, where image reconstruction is performed from selected pairwise distances of labeled sources [8, 25].

In geometry, the GRP is widely studied in the context of the *theory of rigid structures*. One thinks of a graph as having metal bars instead of edges (the length of each bar is the weight of the edge), and the vertices are points in $\mathbb{R}^d$. The bars are connected by joints and therefore movement is possible (but the bars cannot change length). Loosely speaking, a framework is called globally *rigid* if there is only one possible realization in $\mathbb{R}^d$. A triangle is for example a rigid framework in $\mathbb{R}^2$, while a square is not. Two triangles sharing a common edge is also not a globally rigid structure in $\mathbb{R}^2$, even if they are continuously and infinitesimally rigid (see [10] and Figure 1). Note also, that some frameworks can be rigid in $\mathbb{R}^d$ but not in $\mathbb{R}^{d'}$ (take e.g. a closed chain of bars of lengths 2, 2, 2, 3 and 3, which is rigid in $\mathbb{R}$ but not in $\mathbb{R}^2$). For a comprehensive survey on rigidity theory we refer the reader to [10].

It is often required that the solution to the GRP is *unique* (that is the case in the examples we mentioned). This leads us to a related *Unique Realization Problem* (URP): given a realization of a graph G in $\mathbb{R}^d$, is there another realization in the same dimension? (We consider realizations equivalent under rotations, reflections or translations).

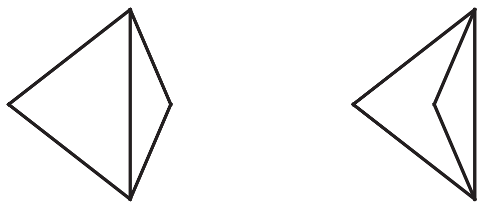

Figure 1: Two realizations of the same two triangle framework.

Solving either the GRP or the URP when $d \geq 1$ is fixed is NP-hard [36]; of course, in many applications the interesting case is $d = 2$ or $d = 3$. On the positive side, for $d = 1, 2$, one can solve a restricted (generic lengths) version of the URP in polynomial time [28], while GRP and unrestricted URP may still be NP-hard in this setting [6, 17]. Unfortunately, The GRP cannot be formulated as a semidefinite program (SDP), due to the non-convex dimension constraint. If we disregard the dimension constraint for a moment, then GRP can be formulated as an SDP, and if a solution exists then an approximate solution can be computed efficiently in dimension at most n, see [23] (here n is the number of vertices of the graph), and in some cases dimension $o(n)$ suffices (see [1, 7]). For an in-depth survey about the SDP approach to the problem we refer the reader to [37].

1.1 Our Contribution

We address the following natural problem. Since the GRP and URP are NP-hard in general, can we identify large families of instances for which the realization problem can be efficiently solved? In 1931, Menger [34] resolved the problem in the special case of the complete graph K_n on n vertices (giving a necessary and sufficient condition for the existence of a solution to the GRP, and showing that this solution can be computed efficiently). In a different (non-algorithmic) language, Connelly [12] showed that the family of *Cauchy polygons* has a unique realization in $\mathbb{R}^d$ for all $d \geq 2$, developing tools which were later used to find several other ad hoc examples (see [10, 13, 38]). With a notable exception of the work of So [37] who studied the GRP of graphs based on certain k-trees, there has been little progress in this direction, either algorithmic or non-algorithmic. In this paper, we continue this line of research by describing new infinite families of tractable GRP instances.

Our construction is based on the following idea. Suppose that a realization $\mathbf{p}$ of a graph G is unique not just in $\mathbb{R}^d$, but in any dimension up to n. In that case, the only solution to the SDP of the GRP is the realization $\mathbf{p}$. This fact lends itself to an efficient approximate solution (up to an arbitrary precision) of the problem. This sort of uniqueness is captured by the notion of *universal rigidity* which was studied extensively in [11, 12, 13]. Our main idea is to construct a *blow-up* of G (for a carefully chosen graph G) with a sufficiently good expander, which gives a large universally rigid framework. We do this explicitly in two special cases: for G the complete graph on k vertices, and G the Cauchy polygon C_8 (see below). We use tools developed earlier to give sufficient conditions for the blow-ups to be universally rigid, which by the above observation become tractable GRP instances (in approximation). The main tool that we use is Connelly's sufficient condition for a realization to be universally rigid. For completeness and reader's convenience, we give a concise proof of this result (Theorem 4), following closely the sketch and ideas in [11, 12].

1.2 Uniquely k-colorable and vector k-colorable graphs

In the k-colorability problem, given a graph G one is asked to compute (if possible) an assignment of colors to the vertices (using k colors) s.t. two adjacent vertices receive different colors. The least number k such that G is k-colorable is called the chromatic number of G and is commonly denoted by $\chi(G)$. A *vector k-coloring* of an n-vertex graph $G = (V, E)$ is an assignment of unit vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \in \mathbb{R}^n$ to the vertices of G such that $\langle \mathbf{x}_i, \mathbf{x}_j \rangle \leq -1/(k-1)$ for every $(v_i, v_j) \in E$. The least positive *real* number k such that G admits a vector k-coloring is called the vector chromatic number of G, usually denoted by $\chi_v(G)$ [30]. It is not hard to see that $\chi_v(G) \leq \chi(G)$ (by identifying each color class with one vertex of a regular $(k-1)$-dimensional simplex centered at the origin), and indeed $\chi_v(G)$ can be much smaller than $\chi(G)$ [18]. A graph G is *uniquely* k-colorable (resp. vector k-colorable) if G admits only one proper k-coloring, up to color permutations (resp. congruency).

Theorem 3 below gives a recipe how to "cook" uniquely k-colorable graphs. The resulting graphs have the extra property of being uniquely vector k-colorable as well. The latter is not obvious at all, as k need not even be an integer. We could not find any mention in the literature about uniquely vector k-colorable graphs, nor the relation to unique k-colorability. Hopefully this paper can instigate research in this fascinating unexplored direction. The fact that our graphs have this extra property is the product of our proof method for unique k-colorability. The proof uses a reduction from k-colorability to the GRP. Specifically, the graph is embedded in a $(k-1)$-simplex, and unique k-colorability is reduced to rigidity of the $(k-1)$-simplex embedding. Since the inner product of every two simplex vertices is $-1/(k-1)$, this gives the result about unique vector k-colorability. Note that in the definition of vector coloring, the inner product is required to be at least $-1/(k-1)$, not necessarily equal. Therefore unique vector k-colorability reduces to rigidity with cables as well as bars, thus taking full advantage of the rigidity framework.

At the end of Section 2 we use our recipe to obtain an explicit construction of a family of uniquely k-colorable expander graphs. Our graphs are also vertex-transitive, and the construction uses good Cayley expanders (see e.g. [33, 29]).

Finally let us mention that our results about k-colorability are related to the work of Alon and Kahale on planted k-colorable graphs [3].

1.3 Definitions and Preliminaries

Before we press on with a formal description of our results, we establish a few key notions and useful facts.

A d-dimensional *tensegrity* framework $\mathbf{p}$ is a configuration of n points in $\mathbb{R}^d$, in which every two points are connected using a strut, or a cable, or a bar, or not connected at all. (A strut is a structural component which resists longitudinal compression, a cable resists longitudinal extension, and a bar resits both). We say that a configuration $\mathbf{q}$ satisfies the tensegrity constraints of $\mathbf{p}$ if for every strut connecting point u and point v we have $\|\mathbf{q}(u) - \mathbf{q}(v)\| \geq \|\mathbf{p}(u) - \mathbf{p}(v)\|$, for every cable $\|\mathbf{q}(u) - \mathbf{q}(v)\| \leq \|\mathbf{p}(u) - \mathbf{p}(v)\|$ and for every bar $\|\mathbf{q}(u) - \mathbf{q}(v)\| = \|\mathbf{p}(u) - \mathbf{p}(v)\|$. We say that $\mathbf{p}$ is *universally rigid* if every configuration $\mathbf{q}$ (in any dimension) that satisfies the tensegrity constraints is congruent to $\mathbf{p}$ (in particular, all constraints are satisfied with equality).

The *tensor product* of a graph G with a graph H is the graph $G \otimes H$. Its vertex set is $V(G \otimes H) = V(G) \times V(H)$, and its edge set satisfies $((u, x), (v, y)) \in E(G \otimes H)$ iff $(u, v) \in E(G)$ and $(x, y) \in E(H)$. The tensor product is a well known notion in graph theory, also called the graph weak product, or the Kronecker product [21]. The adjacency matrix $A(G \otimes H)$ is easily seen to be the matrix tensor (or Kronecker) product of the adjacency matrices of G and H. A well known fact is that if $\lambda_1, ..., \lambda_s$ are the eigenvalues of $A(G)$ and $\mu_1, ..., \mu_t$ are those of $A(H)$ (listed according to multiplicity), the eigenvalues of $A(G \otimes H)$ are $\lambda_i \mu_j$ for $i = 1, \ldots s$, $j = 1, \ldots t$.

For a graph G, let G^+ be the graph obtained from G by adding a new vertex v_0 and connecting it to all vertices of G.

2 Main results

Our main contribution is a "recipe" how to cook tractable, universally rigid, GRP instances. Informally, our recipe is to take a small universally rigid framework, and tensor it with a good expander graph. We demonstrate this construction using two examples: *The k-simplex configuration* (Theorem 1) and the *Cauchy polygon* on eight vertices (Theorem 2). Finally, we show how to use the k-simplex result to obtain a construction of uniquely k-cloroable graphs, which are also uniquely vector k-colorable.

The k-Simplex Configuration. A regular k-simplex is a k-dimensional regular polytope which is the convex hull of its $k+1$ vertices. We use K_k to denote the complete graph on k vertices $x_1, \dots, x_k$. For an arbitrary graph H, the *simplex configuration* of the graph product $K_k \otimes H$ assigns all vertices of the from $(x_i, *)$ with the i^{th} vertex of the regular $(k-1)$-simplex in $\mathbb{R}^{k-1}$. In other words, each color class of $K_k \otimes H$ is mapped to another vertex of the $(k-1)$-simplex. In case we have the additional vertex v_0 (connected to all other vertices), then it is assigned the barycenter of the simplex.

Theorem 1. *Let H be an r-regular graph on n vertices that satisfies $\lambda(H) < r/(k-1)$, and let $G = K_k \otimes H$. Then the simplex configuration of G^+ is universally rigid. Furthermore, the* unique *solution to the corresponding GRP problem can be computed in polynomial time.*

Let us emphasize the algorithmic task referred to in Theorem 1. The tensegrity framework of the graph G is indeed constructed as in Theorem 1, but the algorithm receives only a list of pairwise distances, in an arbitrary order, and is required to reconstruct the k-simplex configuration out of that. This task in general is NP-hard.

We remark that in this graph theoretical setting, the mathematical notions that are used to prove universal rigidity reduce to standard calculations involving quadratic forms that arise in the context of the Rayleigh quotient definition for the eigenvalues of a matrix (another example is the Hoffman bound [19] for the size of independent sets in a graph). Thus in some sense, the language of universal rigidity can be viewed as a generalization of those ideas in a geometric setting.

The Cauchy polygon configuration. Our second result has a true geometric flavor. Denote by C_8 the *Cauchy polygon* on eight vertices, $x_1, \ldots, x_8$, which form a regular 8-gon inscribed into a unit circle (see Figure 2). We refer to [12, 10, 35] for more about Cauchy polygons and their role in rigidity theory. Note that the short edges in C_8 have length $\sqrt{2-\sqrt{2}}$, and long edges have length $\sqrt{2}$. For a graph $G = C_8 \otimes H$, the C_8-configuration of G^+ assigns all vertices of the form $(x_i, *)$ with the i^{th} vertex of the Cauchy Polygon in $\mathbb{R}^2$, and v_0 with the origin (v_0 is the additional vertex that is connected to all vertices of the C_8).

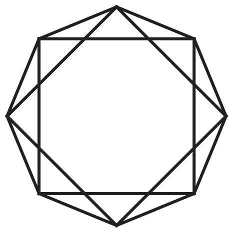

Figure 2: Cauchy polygon C_8.

Theorem 2. *Let H be an r-regular graph on n vertices that satisfies $\lambda(H) < \sqrt{2}r/3$, and also let $G = C_8 \otimes H$. The C_8-configuration of G^+ is universally rigid. Furthermore, the solution to the corresponding GRP problem can be computed in polynomial time.*

Let us note a connection between the realizations of $G = C_8 \otimes H$ and certain 3-circular 8-colorings of G (see [39] for definitions). We elaborate more in Section 8.2.

Uniquely k-colorable graphs. The next theorem is a corollary of Theorem 1. It describes a general construction of a family of uniquely k-colorable graphs, which have the extra property of being uniquely vector k-colorable as well.

Theorem 3. *Let H be an r-regular graph with $\lambda < r/(k-1)$. Then $G = K_k \otimes H$ is uniquely k-colorable and uniquely vector k-colorable. Furthermore, the k-colorability problem for G can be solved in polynomial time.*

Finally, let us show how to use Theorem 3 to obtain an explicit construction of such family. Let H be an r-regular Ramanujan expander graphs (such as described in [33]), satisfying $\lambda(H) \leq 2\sqrt{r-1}$. If $r \geq 4(k-1)^2$, then $G = K_k \otimes H$ is uniquely k-colorable. In particular, when $\{H_p\}$ are the 17-regular Ramanujan graphs on $n = (p^2 - 1)$ vertices (p is a prime), as in [33], we can take $k = 3$ and set $G_p = K_3 \otimes H_p$. The graphs $\{G_p\}$ are then 34-regular uniquely 3-colorable graphs on $3n$ vertices, and vector chromatic number 3. Furthermore, since the Ramanujan graph and K_k are both vertex transitive, so is G.

Paper's organization The remainder of the paper is organized as follows. In Section 3 we discuss in details Connlley's universal rigidity result. In Section 4 we discuss the algorithmic perspective of our result. Then in Sections 5, 6 and 7 we prove our main results. Conclusions and open problems are given in Section 8. For completeness, the proof of Connelly's sufficient condition for universal rigidity, Theorem 4, is given in Appendix B.

3 Universal Rigidity

The main tool we use in proving our results is Connelly's universal rigidity theorem. In this section we set the backdrop for the theorem, and give its exact statement.

For a graph G, let $\mathcal{P}(G)$ be the space of all configurations of the graph G, and let $\mathcal{P}^d(G)$ be the space of configurations that lie entirely in the $\mathbb{R}^d$ (or congruent to a configuration in $\mathbb{R}^d$).

An *equilibrium stress matrix* of a configuration $\mathbf{p}(G)$ is a matrix $\mathbf{\Omega}$ indexed by $V \times V$ satisfying:

1. $\mathbf{\Omega}$ is a symmetric matrix,
2. If $(u, v) \notin E$, and $u \neq v$ then $\mathbf{\Omega}(u, v) = 0$.
3. For every $u \in V$, $\sum_{v \in V} \mathbf{\Omega}(u, v) = 0$.
4. For every $u \in V$, $\sum_{v \in V} \mathbf{\Omega}(u, v)\mathbf{p}(v) = 0$.

The stress matrix $\mathbf{\Omega}$ is considered *proper* if it satisfies $\mathbf{\Omega}_{uv} \geq 0$ for every strut $\{u, v\}$ and $\mathbf{\Omega}_{uv} \leq 0$ if it is a cable.

We say that the edge directions of a configuration $\mathbf{p}$ in $\mathcal{P}^d(G)$ lie *on a conic at infinity* if there exists a non-zero symmetric $d \times d$ matrix $\mathbf{Q}$ such that for all edges (u, v) of G $[\mathbf{p}(u) - \mathbf{p}(v)]^t \mathbf{Q} [\mathbf{p}(u) - \mathbf{p}(v)] = 0$. A symmetric matrix $\mathbf{Q}$ is *positive semidefinite* (or PSD for short) if for every vector $\mathbf{x}$, $\mathbf{x}^t \mathbf{Q} \mathbf{x} \geq 0$.

The following is a sufficient condition for a configuration to be universally rigid. It was derived by Connelly in a series of papers, see [11] for example. The complete proof is given in full details in Appendix B.

Theorem 4. *Suppose that G is a graph with $d+2$ or more vertices and $\boldsymbol{p}$ is a configuration in $\mathcal{P}^d(G)$. Suppose that there is a proper PSD equilibrium stress matrix $\mathbf{\Omega}(\boldsymbol{p}, G)$ whose rank is $n - d - 1$. Also suppose that the edge directions of $\boldsymbol{p}$ do not lie on a conic at infinity. Then $\boldsymbol{p}$ is universally rigid.*

4 Semidefinite programming and the algorithmic perspective

In Theorems 1 and 2 we claim that one can *efficiently* find the unique solution to the specific GRP. In this section we elaborate on the algorithmic aspect of this problem. As we mentioned already, the GRP is naturally expressed as a semidefinite programm. What follows is a self-contained discussion, which appears in [37] and [22] as well.

Let $\mathbf{e}_i$ be the i^{th} standard basis vector of $\mathbb{R}^n$. By $\mathbf{A} \succeq 0$ we mean that $\mathbf{A}$ is a symmetric PSD matrix. For two $n \times n$ matrices $\mathbf{A}, \mathbf{B}$ define $\mathbf{A} \circ \mathbf{B} = \sum_{i,j=1}^{n} \mathbf{A}_{ij} \cdot \mathbf{B}_{ij}$. Let $\mathbf{X} = [\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_n]$ be a $k \times n$ matrix ($\mathbf{x}_i$ is the i^{th} column of $\mathbf{X}$). One can easily verify that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = (\mathbf{e}_i - \mathbf{e}_j)^t \mathbf{X}^t \mathbf{X} (\mathbf{e}_i - \mathbf{e}_j) = (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^t \circ (\mathbf{X}^t \mathbf{X}).$$

Define the matrix $\mathbf{L}_{ij} = (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T$. Given a graph $G = (V, E)$ and a set of weights $\{w_{ij} : (i,j) \in E\}$, the GRP is equivalent to the following feasibility SDP:

$$\begin{aligned} &\text{Find } \mathbf{Y} \text{ s.t.} \\ &\quad \mathbf{L}_{ij} \circ \mathbf{Y} = w_{ij}^2 \qquad \forall (i,j) \in E(G), \\ &\quad \mathbf{Y} \succeq 0. \end{aligned} \tag{4.1}$$

Let us explain why this formulation is equivalent to the GRP. For a symmetric matrix $\mathbf{Y}$, $\mathbf{Y}$ is PSD iff there exists a matrix $\mathbf{B}$ such that $\mathbf{Y} = \mathbf{B}^t \mathbf{B}$ (the Cholesky decomposition). Let $\mathbf{Y}$ be a solution to the above SDP, and write $\mathbf{Y} = \mathbf{X}^t \mathbf{X}$. The first constraint ensures that the columns of $\mathbf{X}$, treated as a configuration of the graph G, satisfy the distance constraints.

Unfortunately, this still does not quite suffice for the algorithmic part of our result, since the SDP can only be computed up to a finite precision in polynomial time (the running time is proportional to $\log 1/\varepsilon$, where ε is the desired precision). Universal rigidity does not exclude the existence of a configuration $\mathbf{q}$ which looks very different than $\mathbf{p}$ (maybe even in a different dimension), and satisfies the distance constraints up to a tiny error. So $\mathbf{q}$ might be an output of the SDP, when the precision is finite. Fortunately, there is a family of SDP solvers called *path-following* algorithms, see for example [2], which compute a series of solutions $(\mathbf{Y}_i, \mathbf{S}_i)$, $i = 1, 2, \ldots$, such that $\lim_{i \to \infty}(\mathbf{Y}_i, \mathbf{S}_i) = (\mathbf{Y}^*, \mathbf{S}^*)$, where $\mathbf{Y}^*, \mathbf{S}^*$ are optimal solutions to the primal and dual SDP programs. Since in our case, $\mathbf{Y}^*$ is the unique solution to the primal SDP, then using a path-following method guarantees that indeed the output is an approximation of the unique solution. More about the path-following method in our context is given in Appendix A.

Finally, recall that the solution to the SDP is only an approximation. Consider the concrete example of the k-simplex configuration of a graph G, which we defined in Section 2. Suppose that we fed the SDP solver with the constraints implied by G and indeed it came back with the k-simplex configuration, but now the points are accurate up to some small error. Since the distance between two points (in the original k-simplex configuration) is much larger than the error (which can be made arbitrarily small), we can group the vertices of the graph according to vertices whose assigned vectors are close to each other, say their inner product is positive (it should be $-1/(k-1)$ in the exact solution). This ensures that we indeed reconstruct the k-simplex configuration exactly.

Eigenvector Approach. Let us remark that in the case of Theorem 1, one can use a different algorithmic approach to efficiently extract the k-simplex configuration. Indeed, the $k-1$ eigenvectors corresponding to the least eigenvalue $-r$ of the adjacency matrix A of $K_k \otimes H$, carry information about the k-coloring. For example, any vector which is constant on every color class, and whose entries sum to zero is an eigenvector of A belonging to the eigenspace of $-r$. This argument is traced for example in [3], [31] or [12]. This might as well be the case for the Cauchy configuration in Theorem 2, although we haven't checked the details. A prominent advantage of this approach is the fact that it is computationally lighter than solving an SDP. On the other hand, one advantage of the SDP approach is captured in the following setting. Suppose that on top of the graph $K_k \otimes H$, one adds additional edges that respect the partitioning imposed by K_k. This operation may jumble the spectrum of the graph, and in general the aforementioned eigenvector approach to extract the realization using eigenvectors, may not work. The SDP approach however is resilient to such “noise”. This line of research was followed in [5] for example, under the title “the semi-random model”.

5 Proof of Theorem 1

Recall that K_k is the complete graph on k vertices $x_1, \ldots, x_k$, and H is some r-regular graph satisfying $\lambda < r/(k-1)$ ($\lambda = \max_{i \geq 2} |\lambda_i|$). Let $G = K_k \otimes H$, and the configuration $\mathbf{p}$ of G^+ assigns every vertex of the form $(x_i, *)$ with the i^{th} vertex of the k-simplex (we think of the k-simplex whose vertices lie on the unit ball in $\mathbb{R}^k$), and v_0 with the origin. We shall prove the theorem for the case $k = 3$, as the general k case is easily deduced in a similar fashion.

If $k = 3$, then the 3-simplex is just an equilateral triangle (the dimension is $d = 2$). Let P_1, P_2, P_3 be the three vertices of an equilateral triangle whose center of mass is at the origin O and $|OP_i| = 1$ for $i = 1, 2, 3$. The configuration $\mathbf{p}$ assigns P_i to all vertices of the form $(x_i, *)$, and O to the vertex v_0. In this configuration, $\overrightarrow{OP_i}$ is a unit vector, and $\overrightarrow{P_iP_j}$ is of length $\sqrt{3}$ for every $i \neq j$.

The heart of the proof lies in providing a PSD equilibrium stress matrix whose nullspace has dimension $d+1$ which is 3 in our case. Let $\mathbf{A}$ be the $n \times n$ adjacency matrix of H, and $\mathbf{\Gamma}$ be the adjacency matrix of K_3. We let $\mathbf{1}_{3n} \in \mathbb{R}^{3n}$ be the column all-one vector, $\mathbf{I}_{3n}$ is the identity $3n \times 3n$ matrix. Define

$$\mathbf{\Omega} = \left(\begin{array}{c|c} \mathbf{I}_{3n} + \frac{1}{r}\mathbf{\Gamma} \otimes \mathbf{A} & -3 \cdot \mathbf{1}_{3n} \\ \hline -3 \cdot \mathbf{1}_{3n}^t & 9n \end{array}\right).$$

The matrix $\mathbf{\Omega}$ is an $(3n+1) \times (3n+1)$ matrix and indeed in G^+ there are

$3n+1$ vertices. The theorem follows from the following Lemmas 5–7. Before proceeding with the proof, let us remark how the general-k setting looks like. As mentioned before, each color class of G is assigned with a different vertex of the regular $(k-1)$-simplex whose barycenter is the origin. The point v_0 is assigned the origin. The matrix $\mathbf{\Omega}$ changes in the natural way: replace 3 by k, and 9 by k^2. The proofs of Lemmas 5–7 are adjusted in the same straightforward way to accommodate the general k setting.

Lemma 5. *The edge directions of* $\boldsymbol{p}$ *do not lie on a conic in infinity.*

Lemma 5 follows immediately by noticing that the stressed directions in $\mathbf{p}$ contain least three distinct directions, and a conic on the projective line has at most two points.

Lemma 6. $\mathbf{\Omega}(G^+, \boldsymbol{p})$ *is an equilibrium stress matrix.*

Lemma 7. *The dimension of the null space of* $\mathbf{\Omega}$ *is 3, and* $\mathbf{\Omega}$ *is PSD.*

The proof of Lemma 6 is a straightforward verification procedure and is given in Section 7. We now give the proof of Lemma 7.

Proof. (Lemma 7) Our first observation is that $\mathbf{1}_{3n+1}$ is an eigenvector of $\mathbf{\Omega}$ corresponding to the eigenvalue 0 (this is true for every equilibrium stress matrix, by the third property in its definition). One can also verify that $\boldsymbol{\xi} = (1, 1, \ldots, 1, -3n) \in \mathbb{R}^{3n+1}$ is an eigenvector of $\mathbf{\Omega}$ corresponding to the eigenvalue $3+9n$.

Define the subspace $W = \text{span}\{\mathbf{1}_{3n+1}, \boldsymbol{\xi}\}$. A symmetric $m \times m$ matrix has a set of m orthogonal eigenvectors. Since $\dim(W) = 2$, one can find the remaining $(3n+1) - 2$ eigenvectors of $\mathbf{\Omega}$ in a subspace perpendicular to W. Consider $\mathbf{z} = (x_1, x_2, \ldots, x_{3n}, y)$ s.t. $\mathbf{z} \perp W$. In particular, $\mathbf{z} \perp \mathbf{1}_{3n+1}$, which implies

$$y = -\sum_{i=1}^{3n} x_i.$$

Also $\mathbf{z}\perp\boldsymbol{\xi}$, which implies

$$y = \frac{1}{3n}\sum_{i=1}^{3n} x_i.$$

The only way to satisfy both equations is by forcing $\sum_{i=1}^{3n} x_i = 0$, which gives $y = 0$. Therefore the vector $\mathbf{z} = (x_1, x_2, \ldots, x_{3n}, 0) = (\mathbf{x}, 0)$. Let $\mathbf{z}\perp W$ be an eigenvector of $\boldsymbol{\Omega}$ corresponding to the eigenvalue λ. Since the last entry of $\mathbf{z}$ is 0,

$$\boldsymbol{\Omega}\mathbf{z} = \lambda\mathbf{z} \Rightarrow \left(\mathbf{I}_{3n} + \frac{1}{r}\boldsymbol{\Gamma}\otimes\mathbf{A}\right)\mathbf{x} = \lambda\mathbf{x}.$$

The eigenvalues of $\mathbf{I}_{3n} + \frac{1}{r}\boldsymbol{\Gamma}\otimes\mathbf{A}$ are the eigenvalues of $\frac{1}{r}\boldsymbol{\Gamma}\otimes\mathbf{A}$ when adding 1 to every eigenvalue.

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{\Gamma}$ are easily seen to be $2, -1, -1$ (in the general k setting they are $(k-1)$ and -1 with multiplicity $k-1$). The eigenvalues of $\mathbf{A}$ are r with multiplicity 1 (this is true for every connected r-regular graph, and we know H is connected, since otherwise its second largest eigenvalue in absolute value would be r), and all others are $< r/2$ in absolute value. By the discussion in Section 1.3, the eigenvalues of $\boldsymbol{\Gamma}\otimes\mathbf{A}$ are $-r$ with multiplicity 2, and the smallest eigenvalue is no smaller than $2\cdot(-r/2) = -r$. Multiplying by $1/r$, we get -1 with multiplicity 2, and the rest have absolute value < 1. Returning to $\mathbf{I}_{3n} + \frac{1}{r}\boldsymbol{\Gamma}\otimes\mathbf{A}$, we have 0 with multiplicity 2, and the remaining eigenvalues are positive.

To conclude, 0 is an eigenvalue of $\boldsymbol{\Omega}$ with multiplicity 3 (this gives the required rank of the null space). The remaining eigenvalues are positive. It is easy to see that a matrix is PSD iff all its eigenvalues are non-negative. This gives the second part of the lemma. $\square$

The proof of Theorem 1 now easily follows. Lemmas 5–7, together with Theorem 4 imply that the simplex configuration for G^+ is universally rigid. The discussion in Section 4 implies the algorithmic part of the theorem.

5.1 Proof of Theorem 3

Fix an integer $k > 0$, and let H be a r-regular graph satisfying $\lambda < r/(k-1)$. We need to prove that $G = K_k \otimes H$ is uniquely k-colorable. Assume otherwise. Then we can construct two non-congruent realizations of G^+, by mapping color class i to the i^{th} vertex of the k-simplex. This however contradicts the universal rigidity established in Theorem 1.

For the vector chromatic number part, we need the slightly more general version of Theorem 4 which appears in Appendix B. In that case, the bars are replaced with struts (i.e, the edges can now stretch). A configuration $\mathbf{p}$ is universally rigid if every configuration with edge lengths at least the ones in $\mathbf{p}$ is congruent to $\mathbf{p}$. The conditions in Theorem 1 imply also this version of universal rigidity. Observe that if $\mathbf{x}_i, \mathbf{x}_j$ are two unit vectors, then $\arccos(\langle \mathbf{x}_i, \mathbf{x}_j \rangle)$ is just the angle between $\mathbf{x}_i$ and $\mathbf{x}_j$, which determines the length of the edge $(i,j) \in E$. Therefore, using the broader notion of rigidity, if G has two different vector k-colorings, then the two configurations that assign each color class with a different vector of the coloring, give two non-congruent realizations of G, contradicting the universal rigidity.

For the algorithmic part, observe that once we compute the realization of G^+, we can group the vertices according to the vectors that the SDP assigned them (take all vertices whose vectors are at distance, say, at most 0.01, to the same color class). Another way to obtain the coloring is to compute the eigenvectors corresponding to the $(k-1)$ smallest eigenvalues. Those eigenvectors encode the k-coloring (see [31]).

6 Proof of Theorem 2

We think of C_8, the Cauchy polygon on eight vertices, bounded in the unit circle in $\mathbb{R}^2$, centered around the origin. Let H be a r-regular graph satisfying $\lambda < \sqrt{2}r/3$ ($\lambda = \max_{i\geq 2}|\lambda_i|$). Let $G = C_8 \otimes H$, and the configuration $\mathbf{p}$ of G^+ assigns every vertex of the form $(x_i, *)$ with the i^{th} vertex of C_8, and v_0 with the origin.

Again, the heart of the proof lies in providing a PSD equilibrium stress matrix whose nullspace has dimension $d+1$ which is 3 in our case. Let $\mathbf{A}$ be the $n \times n$ adjacency matrix of H, and $\mathbf{\Gamma}$ be the following weighted adjacency matrix of C_8.

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & -1 & 0.5 & 0 & 0 & 0 & 0.5 & -1 \\ -1 & 0 & -1 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0.5 & -1 & 0 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0 & -1 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 & -1 & 0 & -1 \\ -1 & 0.5 & 0 & 0 & 0 & 0.5 & -1 & 0 \end{pmatrix}$$

We let $\mathbf{1}_{8n} \in \mathbb{R}^{8n}$ be the column all-one vector, $\mathbf{I}_{8n}$ is the identity $8n \times 8n$ matrix. Define

$$\mathbf{\Omega} = \left(\begin{array}{c|c} \sqrt{2}\mathbf{I}_{8n} + \frac{1}{r}\mathbf{\Gamma} \otimes \mathbf{A} & (1-\sqrt{2}) \cdot \mathbf{1}_{8n} \\ \hline (1-\sqrt{2}) \cdot \mathbf{1}_{8n}^t & 8n(\sqrt{2}-1) \end{array}\right).$$

The matrix $\mathbf{\Omega}$ is an $(8n+1) \times (8n+1)$ matrix and indeed in G^+ there are $8n+1$ vertices. The theorem follows from the following three lemmas:

Lemma 8. *The edge directions of* $\boldsymbol{p}$ *do not lie on a conic in infinity.*

Lemma 9. $\Omega(G^{+}, \boldsymbol{p})$ *is an equilibrium stress matrix.*

Lemma 10. *The dimension of the null space of* $\mathbf{\Omega}$ *is 3, and* $\mathbf{\Omega}$ *is PSD.*

The proofs of Lemmas 8 and 9 are a straightforward verification procedure and are given in Section 7. We now give the proof of Lemma 10.

Proof. (Lemma 10) Our first observation is that $\mathbf{1}_{8n+1}$ is an eigenvector of $\mathbf{\Omega}$ corresponding to the eigenvalue 0 (this is true for every equilibrium stress matrix, by the third property in its). One can also verify that $\boldsymbol{\xi} = (1, 1, \ldots, 1, -8n) \in \mathbb{R}^{8n+1}$ is an eigenvector of $\mathbf{\Omega}$ corresponding to the eigenvalue $(\sqrt{2} - 1)(8n + 1)$.

Define the subspace $W = \mathrm{span}\{\mathbf{1}_{8n+1}, \boldsymbol{\xi}\}$. A symmetric $m \times m$ matrix has a set of m orthogonal eigenvectors. Since $\dim(W) = 2$, one can find the remaining $(8n + 1) - 2$ eigenvectors of $\mathbf{\Omega}$ in a subspace perpendicular to W. Consider $\mathbf{z} = (x_1, x_2, \ldots, x_{8n}, y)$ s.t. $\mathbf{z} \perp W$. In particular, $\mathbf{z} \perp \mathbf{1}_{8n+1}$, which implies

$$y = -\sum_{i=1}^{8n} x_i.$$

Also $\mathbf{z} \perp \boldsymbol{\xi}$, which implies

$$y = \frac{1}{8n} \sum_{i=1}^{8n} x_i.$$

The only way to satisfy both equations is by forcing $\sum_{i=1}^{8n} x_i = 0$, which gives $y = 0$. Therefore the vector $\mathbf{z} = (x_1, x_2, \ldots, x_{8n}, 0) = (\mathbf{x}, 0)$. Let $\mathbf{z} \perp W$ be an eigenvector of $\mathbf{\Omega}$ corresponding to the eigenvalue λ. Since the last entry of $\mathbf{z}$ is 0,

$$\mathbf{\Omega z} = \lambda \mathbf{z} \Rightarrow \left(\sqrt{2} \cdot \mathbf{I}_{8n} + \frac{1}{r} \mathbf{\Gamma} \otimes \mathbf{A} \right) \mathbf{x} = \lambda \mathbf{x}.$$

The eigenvalues of $\sqrt{2} \cdot \mathbf{I}_{8n} + \frac{1}{r} \mathbf{\Gamma} \otimes \mathbf{A}$ are the eigenvalues of $\frac{1}{r} \mathbf{\Gamma} \otimes \mathbf{A}$ when adding $\sqrt{2}$ to every eigenvalue. The eigenvalues of $\mathbf{\Gamma}$ can be computed (using MATLAB for example), and they are $\{-\sqrt{2}, -\sqrt{2}, -1, -1, -1, \sqrt{2}, \sqrt{2}, 3\}$. The eigenvalues of $\mathbf{A}$ are r with multiplicity 1 (this is true for every connected r-regular graph, and we know H is connected, since otherwise its smallest eigenvalue would be $-d$, and we know that it is larger than $-\sqrt{2}r/3$), and all others are $< \sqrt{2}r/3$ in absolute value (by the conditions on the theorem). By the discussion in Section 1.3, the eigenvalues of $\mathbf{\Gamma} \otimes \mathbf{A}$ are $-\sqrt{2}r$ with multiplicity 2, and the smallest eigenvalue is larger than $3 \cdot (-\sqrt{2}r/3) = -\sqrt{2}r$. Multiplying by $1/r$, we get $-\sqrt{2}$ with multiplicity 2, and the rest have absolute value $< \sqrt{2}$. Returning to $\sqrt{2}\mathbf{I}_{8n} + \frac{1}{r}\mathbf{\Gamma} \otimes \mathbf{A}$, we have 0 with multiplicity 2, and the remaining eigenvalues are positive.

To conclude, 0 is an eigenvalue of $\mathbf{\Omega}$ with multiplicity 3 (this gives the required rank of the null space). The remaining eigenvalues are positive. It is easy to see that a matrix is PSD iff all its eigenvalues are non-negative. This gives the second part of the lemma. □

The proof of Theorem 2 now easily follows. Lemmas 8–10, together with Theorem 4 imply that Cauchy-polygon configuration for G^+ is universally rigid. The discussion in Section 4 implies the algorithmic part of the theorem.

7 Missing proofs from Sections 5 and 6

7.1 Proof of Lemma 6

Let us go over the required properties in the definition of a stress matrix (Section 3). $\mathbf{\Omega}$ is a symmetric matrix. There is no limitation on the diagonal entries, and it has non-zero entries only where there is an edge of G^+. The sum of every row i is indeed 0: For the first $3n$ rows, $a_{ii} = 1$, and the sum of the remaining entries is $1/r$ times the degree of a vertex i, which is $2r$. This is balanced by the -3 at the last column. For the last row, the sum is $3n \cdot (-3) + 9n = 0$. As for the last property, for every $v \in V(G)$, treating the configuration as vectors,

$$\sum_{w\in V}\mathbf{\Omega}(v,w)\mathbf{p}(w) =$$

$$\overrightarrow{OP_1} + \frac{1}{r}\sum_{w\in V_2}\mathbf{\Omega}(v,w)\overrightarrow{OP_2} + \frac{1}{r}\sum_{w\in V_3}\mathbf{\Omega}(v,w)\overrightarrow{OP_3} - 3\cdot\overrightarrow{0} = \sum_{i=1}^{3}\overrightarrow{OP_i} = 0.$$

For the last row, corresponding to $v = v_0$, we have

$$\sum_{w\in V}\mathbf{\Omega}(v_0,w)p(w) =$$

$$-3n\sum_{w\in V_1}\overrightarrow{OP_1} - 3n\sum_{w\in V_2}\overrightarrow{OP_2} - 3n\sum_{w\in V_3}\overrightarrow{OP_3} + 9n\cdot\overrightarrow{0} = -3n\sum_{i=1}^{3}\overrightarrow{OP_i} = 0.$$

7.2 Proof of Lemma 8

The four vertices of the Cauchy polygon are $P_1 = (1,0), P_2 = \frac{1}{2}(\sqrt{2},\sqrt{2}), P_3 = (0,1)$ and $P_{v_0} = (0,0)$ (omitting the 0-padding of the points). Let

$$\mathbf{Q} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a symmetric 2×2 matrix. Solving $[P_i - P_{v_0}]\mathbf{Q}[P_i - P_{v_0}]^t = 0$ for $i = 1,3$ gives $a = c = 0$. Solving $[P_2 - P_{v_0}]\mathbf{Q}[P_2 - P_{v_0}]^t = 0$ gives $b = 0$.

7.3 Proof of Lemma 9

Let us go over the required properties in the definition of a stress matrix (Section 3). The stress matrix $\mathbf{\Omega}$ is a symmetric matrix. There is no limitation on the diagonal entries, and besides it we have non-zero entries only where there is an edge of G^+. The sum of every row i is indeed 0: For the first $8n$ rows, $a_{ii} = \sqrt{2}$, and the sum of the remaining entries is $1/r$ times the degree times the weight, which is gives -1. This is balanced by the $1-\sqrt{2}$ at the last column.

For the last row, the sum is $8n \cdot (1-\sqrt{2}) + 8n \cdot (\sqrt{2}-1) = 0$.

Let us treat the configuration as vectors. Look at a vertex $v = (x_1, u)$, where x_1 corresponds to the point $P_1 = (1, 0)$ (since C_8 is symmetric for every vertex, we can just consider this case). The vertex v has $4d$ neighbors, d of each form $(x_2, *), (x_3, *), (x_7, *), (x_8, *)$. The vectors corresponding to its neighbors of the form $(x_3, *)$ and $(x_7, *)$ are antipodal, and therefore cancel each other (as long as both are assigned with the same weight, which is 0.5 in our case). The sum of every two vectors of the form $(x_2, *)$ and $(x_8, *)$ is $\frac{1}{r}\sqrt{2} \cdot \overrightarrow{OP_1}$. Therefore,

$$\sum_{w \in V} \mathbf{\Omega}(v, w) p(w) = -1 \cdot r \cdot \frac{1}{r}\sqrt{2} \cdot \overrightarrow{OP_1} + \sqrt{2} \cdot \overrightarrow{OP_1} = \overrightarrow{0}.$$

For the last row, corresponding to $v = v_0$, we have

$$\sum_{w \in V} \mathbf{\Omega}(v_0, w) p(w) = 8n(\sqrt{2}-1) \cdot \overrightarrow{0} + (1-\sqrt{2}) \sum_{i=1}^{8} \sum_{w \in H_{v_i}} \overrightarrow{OP_i} = \overrightarrow{0}.$$

8 Conclusions and open problems

8.1 In this paper we characterized two families of tractable GRP instances, using a blow-up of the regular k-simplex and the Cauchy polygon with a suitable r-regular graph. Both the regular k-simplex and the Cauchy polygon C_8 are universally rigid frameworks by themselves. Our results suggests perhaps the following more general method for generating universally rigid frameworks (and thus tractable GRP instances): take a universally rigid framework R and an r-regular graph H. Find a condition on $\lambda(H)$ (the second largest eigenvalue in absolute value) so that $R \otimes H$ is universally rigid. An interesting question for future research would be to prove that indeed the method works in this full generality, or come up with a counter example.

8.2 Let us note that every realization of $G = C_8 \otimes H$ into $\mathbb{R}^d$ corresponds to a 3-circular 8-coloring of G (see [39] for a comprehensive survey). As of yet, we could not prove the reverse claim, although we conjecture that it holds, and that G is a uniquely 3-circular 8-colorable graph. Explicit constructions of c-circular k-colorable graphs with good expansion is an open question in general (cf. [39]), and we propose our graphs $G = C_8 \otimes H$ as good candidates in the case $c = 3$ and $k = 8$. Finally, let us mention that in this spirit general frameworks are related to general graph homomorphism problems, where uniquely homomorphic graphs are of importance again (see [27]).

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A Self-Duality

One of the requirements for the path-following methods to apply is that both the primal and dual SDP have strictly feasible solutions (by strict we mean positive-definite). This will not be the case for the SDP we just described since the rank of $\mathbf{Y}$ is the dimension of the configuration $\mathbf{p}$ for the GRP. However, $\mathbf{Y}$ is positive-definite iff it has full rank, which in most cases will not be true. To overcome this problem, we use the self-dual method (see [14, 15]) which embeds both the primal and dual SDP in a new SDP, which is self-dual, and has strict feasible solutions. The solutions of the new SDP give the solution to the embedded programs iff the original primal and dual programs are gap free (that is, they both have the same maximal/minimal solution value). Let us show that indeed this is the case for us. Let $\mathbf{b}$ be the vector $\mathbf{b} = (w_{ij}^2)$ for $(i, j) \in E$. The dual to (4.1) is

$$\begin{aligned}
&\min_{\mathbf{x},\mathbf{S}} \ \mathbf{b}^t\mathbf{x} \\
&\sum_{(i,j)\in E} \mathbf{x}_{ij}\mathbf{L}_{ij} + \mathbf{S} = 0, \qquad \text{(A.1)} \\
&\mathbf{S} \succeq 0.
\end{aligned}$$

The solution $\mathbf{x} = 0$ and $\mathbf{S} = 0$ is a feasible solution to (A.1) whose value is 0. By the weak duality theorem, this is the optimal value of the dual SDP (since the primal has value 0).

We are going to embed both our primal and dual programs in the following self–dual SDP:

$$
\begin{aligned}
\min \quad & \theta\beta \\
& \mathbf{L}_{ij}\mathbf{Y} - \tau\mathbf{b}_{ij} + \theta\bar{\mathbf{b}}_{ij} = 0 \qquad (i,j) \in E, \\
& -\sum_{i=1}^{|E|} \mathbf{x}_{ij}\mathbf{L}_{ij} - \mathbf{Z} = 0, \\
& \mathbf{b}^t\mathbf{x} + \theta - \rho = 0, \\
& -\bar{\mathbf{b}}^t\mathbf{x} - \mathbf{I} \bullet \mathbf{Y} - \tau - \nu = -\beta, \\
& \mathbf{Y}, \mathbf{Z} \succeq 0, \qquad \mathbf{x} \in \mathbb{R}^{|E|}, \qquad \theta, \rho, \tau, \nu \geq 0.
\end{aligned}
$$

Where $\bar{\mathbf{b}}_{ij} = \mathbf{b}_{ij} - tr(\mathbf{L}_{ij})$. One can verify that this SDP is self dual, and since setting all parameters to 0 is a feasible solution, by the self-duality we get that this is indeed the optimal value of the SDP, and $\theta = 0$ in the optimum. Furthermore, [14] shows that $\tau > 0$ iff the original pair of primal and dual SDP were gap free. This is indeed the case for us, and we get that $\tau > 0$ in the optimal solution. Therefore the optimal solution to the self-dual satisfies

$$\mathbf{L}_{ij}\mathbf{Y} = \tau\mathbf{b}_{ij} = \tau w_{ij}^2.$$

Therefore, if the original SDP had a unique solution, then the self-dual has a unique $\mathbf{Y}$ that satisfies the first constraint (since if $\mathbf{Y}$ is part of a solution to the self-dual, $\tau^{-1}\mathbf{Y}$ is a solution to the primal).

The second requirement for applying a path-following method is that the matrices $\mathbf{L}_{ij}$ be linearly independent. Since there are no parallel edges in G, this is indeed the case.

B Proof of Theorem 4

For two configurations $\mathbf{p}, \mathbf{q}$ we use the notation $\mathbf{p} \asymp \mathbf{q}$ to denote the fact that $\|\mathbf{p}(u) - \mathbf{p}(v)\| = \|\mathbf{q}(u) - \mathbf{q}(v)\|$ for every bar $(u, v) \in E$, $\|\mathbf{p}(u) - \mathbf{p}(v)\| \geq \|\mathbf{q}(u) - \mathbf{q}(v)\|$ for a cable $(u, v) \in E$ and $\|\mathbf{p}(u) - \mathbf{p}(v)\| \leq \|\mathbf{q}(u) - \mathbf{q}(v)\|$ for a strut.

Lemma 11. *Let $\mathbf{\Omega}(\boldsymbol{p}, G)$ be a proper equilibrium stress matrix for the configuration $\boldsymbol{p}(G)$. Let $\boldsymbol{q}$ be another configuration for G s.t. $\boldsymbol{p} \asymp \boldsymbol{q}$. If $\mathbf{\Omega}$ is PSD, $\mathbf{\Omega}(\boldsymbol{q}, G)$ is an equilibrium stress as well. Furthermore, all tensegrity constraints hold with equality.*

Proof. The first three requirements that $\mathbf{\Omega}$ needs to satisfy as an equilibrium stress matrix do not depend on $\mathbf{p}$. Therefore, they hold for $\mathbf{q}$ as well. It remains to verify the last property, that is

$$\forall u \in V \sum_{w \in V} \mathbf{\Omega}_{uw} \mathbf{q}(w) = 0.$$

Let us define the matrix $\mathbf{\Psi}^{(uv)}$ to be the $n \times n$ matrix with -1 in the (u, v) and (v, u) entries, 1 in the (u, u) and (v, v) entries, and 0 otherwise. Define $\omega_{uv} = -\mathbf{\Omega}_{uv}$. We claim that

$$\mathbf{\Omega} = \sum_{(u,v) \in V \times V} \omega_{uv} \mathbf{\Psi}^{(uv)}.$$

This is clear for the off-diagonal entries. For the diagonal entries, observe that since $\mathbf{\Omega}$ is an equilibrium stress matrix, the sum of every row is 0, therefore the diagonal entry $\mathbf{\Omega}_{uu}$ must equal the negative of the sum of all entries of the u-row. This is equivalent to having the 1 entry in $\mathbf{\Psi}^{(uu)}$. For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ consider the quadratic form $\mathbf{x}^t \mathbf{\Omega} \mathbf{x}$. It is easy to see that $\mathbf{x}^t \mathbf{\Psi}^{(uv)} \mathbf{x} = (x_u - x_v)^2$, therefore it follows that

$$\mathbf{x}^t\boldsymbol{\Omega}\mathbf{x} = \mathbf{x}^t\left(\sum_{(u,v)\in V\times V}\omega_{uv}\boldsymbol{\Psi}^{(uv)}\right)\mathbf{x} = \sum_{(u,v)\in V\times V}\omega_{uv}\mathbf{x}^t\boldsymbol{\Psi}^{(uv)}\mathbf{x} = \sum_{(u,v)\in V\times V}\omega_{uv}(x_u - x_v)^2. \tag{B.1}$$

Think of the entries of the points in $\mathbf{p}$ as formal variables (that is, n^2 variables), and let us define the following function $E(\mathbf{p})$ (which is usually called the energy function in rigidity theory):

$$E(\mathbf{p}) = \sum_{(u,v)\in V\times V}\omega_{uv}\|\mathbf{p}(u) - \mathbf{p}(v)\|^2.$$

Let $\mathbf{p}_0$ be a configuration such that $\boldsymbol{\Omega}(\mathbf{p}_0, G)$ is an equilibrium stress matrix. The first observation that we make is that $\nabla E(\mathbf{p}_0) = 0$. To see this, let $\mathbf{Z}$ be the $d \times n$ matrix such that the u^{th} column of $\mathbf{Z}$ is $\mathbf{p}_0(u)$. Let $\mathbf{x}_u$ be the u^{th} row of $\mathbf{Z}$ (i.e., the j^{th} entry in $\mathbf{x}_u$ is the u^{th} entry of $\mathbf{p}_0(j)$ for $j = 1, \ldots, n$). Fix a vertex $u \in V$ and consider the variable x_{uv} – the v^{th} entry of $\mathbf{x}_u$.

$$\frac{\partial E}{\partial x_{uv}} = 2\sum_{w\in V}\omega_{vw}(x_{uv} - x_{wv}).$$

The last two properties of a stress matrix imply that if $\boldsymbol{\Omega}(\mathbf{p}_0, G)$ is an equilibrium stress, then

$$\forall u \in V \sum_{w\in V}\omega_{vw}(\mathbf{p}(u) - \mathbf{p}(w)) = 0.$$

Combining the last two equations we get that if $\boldsymbol{\Omega}(\mathbf{p}_0, G)$ is an equilibrium stress matrix, $\nabla E(\mathbf{p}_0) = 0$.

The next observation that we make is that $E(\mathbf{p}_0) = 0$. Define $\mathbf{y} = t\mathbf{p}_0$, then $E(\mathbf{y}) = E(t\mathbf{p}_0) = t^2 E(\mathbf{p}_0)$ (the last equality just follows from the quadratic form of E). Using the chain rule,

$$\mathrm{E}'(\mathbf{y}) = \nabla E(\mathbf{y}) \cdot \mathbf{y}' = \nabla E(t\mathbf{p}_0) \cdot \mathbf{p}_0.$$

On the other hand

$$\mathrm{E}'(\mathbf{y}) = (t^2 E(\mathbf{p}_0))' = 2tE(\mathbf{p}_0).$$

Combining the two and setting $t = 1$ (and recalling that $\nabla E(\mathbf{p}_0) = 0$), we get $E(\mathbf{p}_0) = 0$.

The last observation that we make is that $E(\mathbf{p}) \geq 0$ for every configuration $\mathbf{p}$. By Equation (B.1), $E(\mathbf{p})$ can be reexpressed as follows ($\mathbf{x}_i$ is the i^{th} row of $\mathcal{P}$):

$$E(\mathbf{p}) = \mathbf{x}_1 \mathbf{\Omega} \mathbf{x}_1^t + \mathbf{x}_2 \mathbf{\Omega} \mathbf{x}_2^t + \cdots + \mathbf{x}_n \mathbf{\Omega} \mathbf{x}_n^t.$$

Since $\mathbf{\Omega}$ is PSD, it holds that $\mathbf{x}_i \mathbf{\Omega} \mathbf{x}_i^t \geq 0$ for every i, hence $E(\mathbf{p}) \geq 0$.

Now we are ready to prove that $\mathbf{\Omega}(\mathbf{q}, G)$ is an equilibrium stress matrix as well, if $\mathbf{p} \asymp \mathbf{q}$. First observe that for every $(u, v) \in E$, $\omega_{uv}\|\mathbf{q}(u) - \mathbf{q}(v)\| \leq \omega_{uv}\|\mathbf{p}(u) - \mathbf{p}(v)\|$. This is obvious for bars (which hold with equality). For struts, $\|\mathbf{q}(u) - \mathbf{q}(v)\| \geq \|\mathbf{p}(u) - \mathbf{p}(v)\|$, and $\mathbf{\Omega}_{uv} \geq 0$ since $\mathbf{\Omega}$ is a proper stress matrix. Therefore, $\omega_{uv} \leq 0$, and the inequality holds. The same argument implies the inequality for cables. This observation, combined with $E(\mathbf{q}) \geq 0$ gives

$$0 \leq E(\mathbf{q}) \leq E(\mathbf{p}) = 0 \Rightarrow E(\mathbf{q}) = 0.$$

This also means that the struts and cables constraints must hold with equality. If $E(\mathbf{q}) = 0$, then $\mathbf{q}$ is a minimum point for E, and therefore $\nabla E(\mathbf{q}) = 0$. However this implies that

$$\forall u \in V \sum_{w \in V} \omega_{uw}(\mathbf{q}(u) - \mathbf{q}(w)) = 0.$$

Or, put differently,

$$\forall u \in V \sum_{w \in V} \omega_{uw}\mathbf{q}(u) = \sum_{w \in V} \omega_{uw}\mathbf{q}(w).$$

Combining this observation with the third property of a stress matrix,

$$\sum_{w \in V} \boldsymbol{\Omega}(u,w)\mathbf{q}(w) = -\sum_{w \in V} \omega_{uw}\mathbf{q}(w) = -\mathbf{q}(u)\sum_{w \in V} \omega_{uw} = \mathbf{q}(u)\sum_{w \in V} \boldsymbol{\Omega}(u,w) = 0.$$

So we conclude that

$$\forall u \in V \sum_{w \in V} \boldsymbol{\Omega}(u,w)\mathbf{q}(w) = 0.$$

□

Before we state the next lemma, we remind the reader that an affine transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ is a function of the form $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$; $\mathbf{T}$ is linear if $\mathbf{b} = 0$. For a configuration $\mathbf{p}$, $\mathbf{T}(\mathbf{p})$ stands for the *ordered* set $\{\mathbf{T}(\mathbf{p}(u)) : u \in V\}$.

Lemma 12. *If $\boldsymbol{\Omega}(\boldsymbol{p}, G)$ is a PSD equilibrium stress matrix of degree $n-d-1$, G has at least $d+2$ vertices, and $\boldsymbol{q} \asymp \boldsymbol{p}$ is another configuration, then there exists an affine transformation $\boldsymbol{T}$ such that $\boldsymbol{T}(\boldsymbol{p}) = \boldsymbol{q}$.*

Proof. The configuration $\mathbf{p}$ has dimension d, therefore there exists an isometry $\mathbf{T}_1 = \mathbf{A}_1\mathbf{x}+\mathbf{d}_1$ ($\mathbf{A}_1$ is an orthogonal matrix, therefore $\mathbf{T}_1$ preserves edge lengths) such that $\mathbf{T}_1(\mathbf{p})$ lies in $\mathbb{R}^d$ (we pad every vector in $\mathbb{R}^d$ with $n-d$ zeros for consistency of dimensions). Let $\mathbf{p}^* = \mathbf{T}_1(\mathbf{p})$. Define $\mathbf{b}_1 = \mathbf{p}^*(1)$ and set $\mathbf{p}' = \{\mathbf{p}^*(u) - \mathbf{b}_1 : u \in V\}$. Similarly define $\mathbf{q}' = \{\mathbf{q}(u) - \mathbf{b}_2 : u \in V\}$, where $\mathbf{b}_2 = \mathbf{q}(1)$. Clearly if $\mathbf{p} \asymp \mathbf{q}$ then $\mathbf{p}' \asymp \mathbf{q}'$. Therefore it suffices to prove that there exists a *linear* affine transformation $\mathbf{T}$ such that $\mathbf{T}(\mathbf{p}') = \mathbf{q}'$. If this is indeed the case, then $\mathbf{q}'(u) = \mathbf{A}\mathbf{p}'(u)$ for some matrix $\mathbf{A}$. Recalling that $\mathbf{q}'(u) = \mathbf{q}(u) - \mathbf{b}_2$, and $\mathbf{p}'(u) = \mathbf{p}^*(u) - \mathbf{b}_1 = \mathbf{A}_1\mathbf{p}(u) + \mathbf{d}_1 - \mathbf{b}_1$, we get $\mathbf{q}(u) - \mathbf{b}_2 = \mathbf{A}(\mathbf{A}_1\mathbf{p}(u) + \mathbf{d}_1 - \mathbf{b}_1)$. This implies $\mathbf{q}(u) = \mathbf{C}\mathbf{p}(u) + \mathbf{c}$ for $\mathbf{C} = \mathbf{A}\mathbf{A}_1$ and $\mathbf{c} = \mathbf{b}_2 + \mathbf{A}(\mathbf{d}_1 - \mathbf{b}_1)$, which is just a vector in $\mathbb{R}^n$. In other words, $\mathbf{q} = \mathbf{U}(\mathbf{p})$ for the affine transformation $\mathbf{U}(\mathbf{x}) = \mathbf{C}\mathbf{x} + \mathbf{c}$. It can be verified rather easily that if $\boldsymbol{\Omega}(\mathbf{p}, G)$ is an equilibrium stress matrix, so is $\boldsymbol{\Omega}(\mathbf{T}(\mathbf{p}), G)$ (for an arbitrary affine transformation $\mathbf{T}$). Hence from now on we shall consider only the configurations $\mathbf{p}'$ and $\mathbf{q}'$

Recall the definition of the matrix $\mathbf{Z}$ from above, the columns of $\mathbf{Z}$ are the vectors $\{\mathbf{p}'(u) : u \in V\}$. Observe that by our construction of $\mathbf{p}'$, $\mathbf{p}'(1) = 0$ and let us assume w.l.o.g that the vectors $\mathbf{p}(2), \ldots, \mathbf{p}(d+1)$ are linearly independent. Since also $\mathbf{q}'(1) = 0$, every linear transformation satisfies $\mathbf{T}(\mathbf{p}'(1)) = \mathbf{q}'(1)$. We can certainly define a linear affine transformation $\mathbf{T}$ such that $\mathbf{T}(\mathbf{p}'(i)) = \mathbf{q}'(i)$ for $i = 2, \ldots, d+1$. If this also holds for $i = d+2, \ldots, n$ then we are done. If not, then there is some index $i > d+1$ such that $\mathbf{T}(\mathbf{p}'_i) \neq \mathbf{q}'_i$. Therefore, $\mathbf{T}(\mathbf{p}'(i)) - \mathbf{q}'(i)$ is not the zero vector, so there exists some coordinate i_0 which is non-zero. Define the vector $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ to be: e_j is the i_0 coordinate of $\mathbf{T}(\mathbf{p}'(j)) - \mathbf{q}'(j)$. Let $\mathbf{Z}^*$ be the matrix $\mathbf{Z}$ in which row $d+2$ is replaced with $\mathbf{e}$. Observe that the first $d+1$ entries in row $d+2$ are 0 (just because $\mathbf{T}(\mathbf{p}'(i)) = \mathbf{q}'(i)$), and in the remaining $n-d-1$ entries there is at least one non-zero entry. It is not hard to see that this implies that the rank of $\mathbf{Z}^*$ is at least $d+1$. The fourth property in the definition of a stress matrix (Section 3) implies that if $\boldsymbol{\Omega}(\mathbf{p}', G)$ is an equilibrium stress matrix, then the rows of $\mathbf{Z}$ are in the kernel of $\boldsymbol{\Omega}$. Also if a linear affine transformation is applied to $\mathbf{p}'$, the rows of the new matrix $\mathbf{Z}$ will be in the kernel (using the linearity of matrix multiplication). Similarly to $\mathbf{Z}$, we can define the matrix $\mathbf{Y}$ whose columns are the vectors in $\mathbf{q}'$. Since $\boldsymbol{\Omega}(\mathbf{q}', G)$ is an equilibrium stress (by Lemma 11), the rows of $\mathbf{Y}$ are also in the kernel of $\boldsymbol{\Omega}$. These two facts imply that the vector $\mathbf{e}$ is going to be in the kernel of $\boldsymbol{\Omega}$ as well. To conclude, all the rows of $\mathbf{Z}^*$ belong to the kernel of $\boldsymbol{\Omega}$ and their rank is at least $d+1$. The vector $\mathbf{1}_n \in \mathbb{R}^n$ always belongs to the kernel of $\boldsymbol{\Omega}$ (third property of a stress matrix); since the first coordinate of every row in $\mathbf{Z}^*$ are 0, $\mathbf{1}_n$ does not belong to the span of the rows of $\mathbf{Z}^*$. To conclude, the kernel of $\boldsymbol{\Omega}$ contains the span of the rows of $\mathbf{Z}^*$, which has dimension at least $d+1$, and the vector $\mathbf{1}_n$. The rank of $\mathbf{\Omega}$ is then at most $n-(d+2)$, which contradicts our rank assumption. Hence, $\mathbf{T}(\mathbf{p}'(i)) = \mathbf{q}'(i)$ for every i, and we have shown that there is a linear transformation such that $\mathbf{T}(\mathbf{p}') = \mathbf{q}'$ as required. □

Lemma 13. *Let $\boldsymbol{\Omega}(\boldsymbol{p}, G)$ be a proper PSD equilibrium stress matrix for the configuration $\boldsymbol{p}(G)$. Let $\boldsymbol{q}$ be another configuration for G s.t. $\boldsymbol{p} \asymp \boldsymbol{q}$. If there is an affine transformation $\boldsymbol{T}$ such that $\boldsymbol{T}(\boldsymbol{p}) = \boldsymbol{q}$, then either the directions of $\boldsymbol{p}$ lie on a conic at infinity, or $\boldsymbol{q}$ is congruent to $\boldsymbol{p}$*

Proof. Lemma 11 implies that the tensegrity constraints hold with equality. Therefore since $\mathbf{p} \asymp \mathbf{q}$, we have that for every $(u,v) \in E(G)$,

$$
\begin{aligned}
0 &= \|\mathbf{q}(u) - \mathbf{q}(v)\|^2 - \|\mathbf{p}(u) - \mathbf{p}(v)\|^2 = \\
&= \|(\mathbf{A}\mathbf{p}(u) + \mathbf{b}) - (\mathbf{A}\mathbf{p}(v) + \mathbf{b})\|^2 - \|\mathbf{p}(u) - \mathbf{p}(v)\|^2 = \\
&= \|\mathbf{A}\mathbf{p}(u) - \mathbf{A}\mathbf{p}(v)\|^2 - \|\mathbf{p}(u) - \mathbf{p}(v)\|^2 = \|\mathbf{A}(\mathbf{p}(u) - \mathbf{p}(v))\|^2 - \|\mathbf{p}(u) - \mathbf{p}(v)\|^2.
\end{aligned}
$$

Using the fact that $\|\mathbf{A}\mathbf{x}\|^2 = (\mathbf{A}\mathbf{x})^t\mathbf{A}\mathbf{x} = \mathbf{x}^t\mathbf{A}^t\mathbf{A}\mathbf{x}$, the latter can be restated as

$$0 = [\mathbf{p}(u) - \mathbf{p}(v)]^t \, \mathbf{A}^t\mathbf{A} \, [\mathbf{p}(u) - \mathbf{p}(v)] - [\mathbf{p}(u) - \mathbf{p}(v)]^t \, \mathbf{I}_n \, [\mathbf{p}(u) - \mathbf{p}(v)] \, ,$$

which gives

$$[\mathbf{p}(u) - \mathbf{p}(v)]^t \left(\mathbf{A}^t\mathbf{A} - \mathbf{I}_n\right) [\mathbf{p}(u) - \mathbf{p}(v)] = 0.$$

Define $\mathbf{Q} = \mathbf{A}^t\mathbf{A} - \mathbf{I}_n$. If $\mathbf{Q} \neq 0$, then by the definition in Section 3, the directions of $\mathbf{p}$ indeed lie on a conic at infinity. If $\mathbf{Q} = 0$, this means that $\mathbf{A}^t\mathbf{A} = \mathbf{I}_n$, or in other words, $\mathbf{A}$ is an orthogonal matrix. Thus $\mathbf{T} = \mathbf{A}\mathbf{x} + \mathbf{b}$ is an isometry, and therefore $\mathbf{p}$ and $\mathbf{q}$ are congruent. □

Theorem 4 now follows easily from these lemmas. Let $\mathbf{p}$ be a configuration in $\mathbb{R}^d$ for the graph G, satisfying the conditions of Theorem 4. That is, $\mathbf{\Omega}(\mathbf{p}, G)$ is a proper PSD equilibrium stress matrix with rank $n-d-1$. Further, we assume that the directions of $\mathbf{p}$ do not lie on a conic at infinity. Let $\mathbf{q}$ be another configuration that satisfies the tensegrity constraints of $G(\mathbf{p})$. Lemma 12 asserts that there exists an affine transformation $\mathbf{T}$ such that $\mathbf{T}(\mathbf{p}) = \mathbf{q}$. Lemma 13 then gives that either the directions of $\mathbf{p}$ lie on a conic at infinity or $\mathbf{T}$ is a congruence. But the conditions of the theorem exclude the former, and we are left with $\mathbf{p}$ and $\mathbf{q}$ are congruent.