Constructing Uniquely Realizable Graphs

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Abstract

In the *Graph Realization Problem* (GRP), one is given a graph G, a set of non-negative edge-weights, and an integer d. The goal is to find, if possible, a realization of G in the Euclidian space $\mathbb{R}^d$, such that the distance between any two vertices is the assigned edge weight. The problem has many applications in mathematics and computer science, but is NP-hard when the dimension d is fixed. Characterizing tractable instances of GRP is a classical problem, first studied by Menger in 1931 in the case of a complete graph. We construct two new infinite families of GRP instances whose solution can be approximated up to an arbitrary precision in polynomial time. Both constructions are based on the blow-up of fixed small graphs with large expanders. Our main tool is the *Connelly's condition* in *Rigidity Theory*, combined with an explicit construction and algebraic calculations of the *rigidity* (*stress*) *matrix*.

An important application of our results is a deterministic construction of *uniquely k-colorable graphs* with arbitrarily large girth. Constructing such graphs is also a classical problem; a non-constructive proof of existence was given by Bollobás and Sauer [8] in 1976. We present an explicit, asymptotically optimal (logarithmic girth) construction of vertex transitive uniquely k-colorable graphs, based on Cayley expanders. Other applications are also discussed.

Introduction

The *Graph Realization Problem* (GRP) is one of the most well studied problems in distance geometry with a number of applications in different areas both in mathematics and computer science (see [57]). In that problem, one is given a graph $G = (V, E)$ on n vertices, a set of non–negative edge weights $\{w_{ij} : (i, j) \in E\}$, and a positive integer d. The goal is to compute a realization of G in the Euclidean space $\mathbb{R}^d$, i.e. a mapping $p : G \to \mathbb{R}^d$ such that $\|p(i) - p(j)\| = w_{ij}$ ($\|u\|$ stands for the Euclidian length of the vector u), or determine if such realization does not exist. It is often desirable that the solution to the GRP is *unique*. This leads us to a related *Unique Realization Problem* (URP): given a realization of a graph G in $\mathbb{R}^d$, is there another realization in the same dimension? Here we consider realizations equivalent under rotations, reflections or translations.

In geometry, the GRP and URP were widely studied in the context of the *theory of rigid structures* (see Section 3). One thinks of a graph as *bar and joint frameworks* in $\mathbb{R}^d$, with joints corresponding to vertices and bars corresponding to graph edges (the length of each bar is the weight of the edge). Loosely speaking, a framework is called *globally rigid* if there is only one possible realization in $\mathbb{R}^d$. A triangle is for example a globally rigid framework in $\mathbb{R}^2$, while a square is not (see [12, 58]). Different variants of the GRP also arise in practical settings (especially in two or three dimensions), where one wants to deduce the locations of different objects in space when supplied with the pairwise distances between the objects. Such applications may include *molecular conformation* [17, 36], *wireless sensor network localization* [4, 22], and *computer vision* [10, 35].

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Solving either the GRP or the URP when $d \geq 1$ is fixed is NP-hard [54]; of course, in many applications the interesting case is $d = 2$ or $d = 3$. On the positive side, for $d = 1, 2$, one can solve a restricted (generic lengths) version of the URP in polynomial time [38], while GRP and unrestricted URP may still be NP-hard in this setting [4, 25]. The GRP can be formulated as a semi-definite program (SDP), and if a solution exists then an approximate solution can be computed efficiently in dimension at most n, see [32] (here n is the number of vertices of the graph), and in some cases dimension $o(n)$ suffices (see [1, 5]). For an in-depth survey about the SDP approach to the problem we refer the reader to [57].

In this paper, we address the following natural problem. Since the GRP and URP are NP-hard in general, can we identify large families of instances for which the realization problem can be efficiently solved? We present two large families of such constructions. Before describing them, let us quickly review the history of the problem.

In 1931, Menger [49] resolved the problem in the special case of a complete graph K_n vertices (giving a necessary and sufficient condition for the existence of a solution to the GRP, and showing that this solution can be computed efficiently). An alternative proof was later proposed by Schoenberg [55], and independently by Young and Householder [59]. In a different direction, Connelly [14] showed that the family of *Cauchy polygons* has a unique realization in $\mathbb{R}^d$ for all $d \geq 2$, developing tools which were later used to find several other ad hoc examples (see [12, 15, 58]). With a notable exception of the work of So [57] who studied the GRP of graphs based on certain k-trees, there has been little progress in this direction, either algorithmic or non-algorithmic.

Our construction is based on the following idea. Suppose that a realization p of a graph G is unique not just in $\mathbb{R}^d$, but in any dimension up to n. That is, (a) the realization in $\mathbb{R}^d$ is unique and (b) there is no other realization of G in $\mathbb{R}^s$ for $s \neq d$. In that case, the only solution to the SDP of the GRP (in any dimension up to n) is the realization p. This fact lends itself to an efficient approximate solution (up to an arbitrary precision) of the problem. This sort of uniqueness is captured by the notion of *universal rigidity* which was studied extensively in [13, 14, 15]. Our main idea is to construct a *blow-up* of G (for a carefully chosen graph G) with a sufficiently good expander, which gives a large universally rigid framework. We do this explicitly in two special cases: for G the complete graph on k vertices, and G the Cauchy polygon C_8 (see below). We use tools developed earlier to give sufficient conditions for the blow-ups to be universally rigid, which by the above observation become tractable GRP instances (in approximation). The main tool that we use is Connelly's sufficient condition for a realization to be universally rigid. For completeness and reader's convenience, we give in the Appendix a concise proof of this result (Theorem 12), following closely the sketch and ideas in [13, 14].

Applications to uniquely k-colorable graphs

Our result has also important implications to the problem of k-colorability. Specifically, we present an explicit construction of a family of uniquely k-colorable graphs with an arbitrary large girth. The problem of finding graphs with large girth and chromatic numbers has a storied history. It was noticed by Tutte [21] in 1947 (writing under a pseudonym) that triangle-free graphs can have arbitrary large chromatic numbers (this result was later rediscovered by Zykov in 1949 and Mycielski in 1955). These results were later extended by Erdős [24] in 1959 to general girth (the *girth* of a graph is the length of its shortest cycle), with an explicit construction given by Lovász [46] in 1968. In a different direction, Harary, Hedetniemi and Robinson [34] found *uniquely* k-colorable graphs with no K_k, and finally Bollobás and Sauer [8], found a common generalization, showing existence of uniquely k-colorable graphs of large girth by using a probabilistic construction (see also [23]). Subsequent derandomization efforts by Chao and Chen [11], Greenwell and Lovász [31], and Müller [51] produced ad hoc families with various girth parameters.

We give an explicit construction of a large family of uniquely k-colorable graphs with arbitrarily large girth, thus derandomizing the Bollobás and Sauer theorem. Our construction has logarithmic girth, matching the classical Erdős bound (up to a constant). We believe it is the first such tight construction, and note that it can be viewed as a far-reaching extension of [31]. Our graphs are also vertex-transitive, and are built by using a blow-up of a complete graph K_k with certain good Cayley expanders (see e.g. [48, 40]). Not only do we give an explicit construction, we also show that these graphs can be k-colored in polynomial time when the labels are permuted and additional edges are added (respecting the planted k-coloring). This is related to k-colorability problem with a planted k-coloring (see [3]). Let us also mention recent breakthroughs [39, 20] in empirical testing of unique k-colorability, using a computer algebra approach.

The remainder of the paper is organized as follows. In Section 1 we state the main results, and in the following Section 2 we present a number of applications of these results to graph colorings and further colorability testing. In Section 3 we discuss in detail the relevant mathematical notions of rigidity theory, and the SDP for the GRP. Then, in sections 4 and 6, we presend the proofs. Conclusions, remarks, and open problems are given in Section 7. The proof of Connelly's sufficient condition for universal rigidity, Theorem 12, is given in Appendix A.

1 Main results

In this section we present formally our main results. Throughout the paper we consider only finite simple graphs (undirected, no loops or multiple edges). We start with a formal definition of two main complexity problems in this paper.

The Graph Realization Problem
Input: Graph $G = (V, E)$ given in an arbitrary description, a set of non-negative edge weights $\{w_e : e \in E\}$, and an integer d.
Output: A map $p : V \to \mathbb{R}^d$ satisfying for every edge $e = (i, j)$, $\|p(i) - p(j)\| = w_e$ or FAIL.

The k-colorability Problem
Input: A graph $G = (V, E)$ given in an arbitrary description, and an integer k.
Output: A k-coloring of G, or FAIL if none exists.

Recall that the *realization* in $\mathbb{R}^d$ of a graph $G = (V, E)$ with edge weights $\{w_{ij}\}$, is a mapping $p : G \to \mathbb{R}^d$, such that $\|p(i) - p(j)\| = w_{ij}$, where $\|u\|$ is the length of vector u. Throughout, we use the word *configuration* interchangeably for *realization*.

Definition 1. A realization $p \in \mathbb{R}^d$ of $G = (V, E)$ is *universally rigid*, if any other realization q of G (in any dimension) that satisfies for every edge $(i, j) \in E$, $\|p(i) - p(j)\| = \|q(i) - q(j)\|$ is congruent to p.

Definition 2. The *tensor product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \times G_2$, is the graph $G = (V, E)$ with $V = V_1 \times V_2$, and $(u, v) \in E$ for $u = (x, y), v = (x', y')$, if $(x, x') \in E_1$ and $(y, y') \in E_2$.

The *graph tensor product* is a well known notion in graph theory [41]. A strongly related notion of graph product is called the *strong graph product*, but unlike the tensor product, we consider vertices x

and x' adjacent in G_1 if $(x, x') \in E_1$ or if $x = x'$, and the same for y, y' in G_2 (see [41] for other graph products and their properties).

For a graph G, we denote by G^+ the graph obtained from G by adding a new vertex v_0 and connecting it to all vertices of G. The spectrum of a graph G is the set of eigenvalues of its adjacency matrix $A = A(G)$. Since the adjacency matrix is symmetric, it has n real eigenvalues, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We let $\lambda = \max_{i \geq 2} |\lambda_i|$.

Let H be an arbitrary graph, K_k the complete graph on k vertices, and $G = K_k \times H$. The **k-simplex-configuration** of the graph G^+ assigns all vertices $\{v_i\} \times V(H)$ at the i^{th} vertex of a regular $(k-1)$-simplex in $\mathbb{R}^{k-1}$ (for $i = 1, \ldots, k$), and v_0 at the origin O. For example, a 3-simplex-configuration is an equilateral triangle in the plane centered at O. We are now ready to present our *First Construction* of graphs where GRP can be solved.

Theorem 3. *Let H be an r-regular graph on n vertices that satisfies $\lambda(H) < r/(k-1)$, and $G = K_k \times H$. Then the k-simplex-configuration of G^+ is universally rigid. Furthermore, the (unique) solution to the corresponding GRP problem can be approximated in polynomial time up to an arbitrary precision.*

The edge lengths of $G = K_k \times H$ are all the same (by the properties of the regular simplex). Therefore this result can be restated in graph theoretic language. It is easy to see that G is k-colorable (color all vertices $\{v_i\} \times V(H)$ with color i). Next, observe that if G^+ is universally rigid, then G is uniquely k-colorable (if there were two different k-colorings of $K_k \times H$, then by mapping color class i to the i^{th} vertex of the $(k-1)$-simplex, we get a contradiction to the universal rigidity of G^+). Now observe that

$$\operatorname{girth}(G_1 \times G_2) \geq \max\{\operatorname{girth}(G_1), \operatorname{girth}(G_2)\} \quad \text{for every } G_1,\ G_2.$$

As a corollary of Theorem 3 we obtain an explicit deterministic construction of uniquely k-colorable graphs with large girth, thus derandomizing the result of Bollobás and Sauer [8], and extending the results of Greenwell and Lovász [31]. We describe this result in details in Subsection 2.2 below.

We also prove that these graphs have a unique *vector* k-coloring, and we compute exactly their Lovász-number and Shannon capacity. A *vector k-coloring* of a graph $G = (V, E)$ is an assignment of unit vectors $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$ to the vertices of G such that $\langle x_i, x_j \rangle \leq -1/(k-1)$ for every $(v_i, v_j) \in E$. The least k such that G admits a vector k-coloring is called the vector chromatic number of G, usually denoted by $\chi_v(G)$ [43]. It is not hard to see that $\chi_v(G) \leq \chi(G)$, and indeed $\chi_v(G)$ can be much smaller than $\chi(G)$ [27].

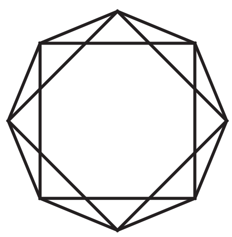

Figure 1: Cauchy polygon C_8.

Our *Second Construction* of a universally rigid framework uses a "true" geometric setting. Denote by C_8 the *Cauchy polygon* on eight vertices which form a regular 8-gon inscribed into a unit circle (see Figure 1). We refer to [14, 12, 53] for more about Cauchy polygons and their role in Rigidity

Theory. Note that the short edges in C_8 have length $\sqrt{2-\sqrt{2}}$, and long edges have length $\sqrt{2}$. For a graph $G = C_8 \times H$, the C_8-**configuration** of G^+ assigns all vertices $\{v_i\} \times V(H)$ with the i^{th} vertex of the Cauchy Polygon in $\mathbb{R}^2$ ($i = 1, \ldots, 8$), and v_0 with the origin.

Theorem 4. *Let H be an r-regular graph on n vertices that satisfies $\lambda(H) < \sqrt{2}r/3$, and also let $G = C_8 \times H$. The C_8-configuration of G^+ is universally rigid. Furthermore, the solution to the corresponding GRP problem can be approximated in polynomial time up to an arbitrary precision.*

Let us note a connection between realizations of $G = C_8 \times H$ and certain 3-circular 8-colorings of G (see [60] for definitions). We make this connection precise in Subsection 7.3.

2 Extensions and applications

2.1 Colorability testing

Let us further emphasize complexity theoretic implications by slightly extending Theorem 6. Observe that if G as in the theorem is a spanning subgraph of a graph G', then G' is either uniquely k-colorable, or not k-colorable at all. The idea of adding edges on top of some underlying well-defined structure, is also captured by the notion of *semi-random graphs*, where a random graph is augmented by adding "adversarial" edges. For more on semi-random graphs see survey [26].

Corollary 5. *Let H be an r-regular graph with $\lambda < r/(k-1)$. Suppose further that $G = K_k \times H$ is a spanning subgraph of G'. Then the k-colorability problem for G' can be solved in polynomial time.*

Proof. Suppose that G is a spanning subgraph of G'. If G' is k-colorable, then clearly it is uniquely so (otherwise, there are two colorings of G). The SDP gives in that case the same solution for G and G', and therefore one can find the coloring of G' in the same manner as for G (by solving the SDP for the GRP and grouping vertices into color classes according to the distances between the assigned vectors). On the other hand, if G' is not k-colorable, then the SDP is unfeasible (as there is not solution to the GRP). This can be observed form the additional parameters introduced in the self-dual SDP that we described in Subsection 3.4 below. ■

2.2 Explicit construction of uniquely k-colorable graphs

The following theorem is a corollary of Theorem 3:

Theorem 6. *Let H be an r-regular graph with girth g and $\lambda < r/(k-1)$. Then $G = K_k \times H$ is uniquely k-colorable, uniquely vector k-colorable, and has girth at least g. Furthermore, the k-colorability problem for G can be solved in polynomial time.*

Let us emphasize that although G is given explicitly, with a "planted" k-coloring, in the k-colorability problem for G the vertices can have different labels, permuted arbitrarily. Thus finding the (unique) "planted" k-coloring is difficult, NP-hard in general, but is polynomial in this case.

Corollary 7. *Let H be a r-regular vertex transitive graph on n vertices, with $\lambda = 2\sqrt{r-1}$ (explicit construction e.g. [48]), and let $G = K_k \times H$. Then, (a) the graph G is vertex transitive and has girth $\Omega(\log_r n)$, and (b) the graph G is uniquely k-colorable if $r \geq 4(k-1)^2$.*

The corollary describes a deterministic explicit construction of uniquely k-colorable graphs with arbitrarily large girth. The corollary follows from Theorem 6, as for the Ramanujan expander graphs in [48], we have (a) the girth of H (which is at least the girth of G) is $\Omega(\log_r n)$, (b) the graph H satisfies $\lambda(H) \leq 2\sqrt{r-1}$ (which is smaller than $r/(k-1)$ for the choice of r as in the corollary), and (c) the graph H is vertex transitive, and so is K_k, hence their tensor product is vertex transitive.

Example 8. We follow [48] to construct an example to satisfy the conditions of Corollary 7. Fix $p = 17$, let q be a prime with Legendre symbol $\left(\frac{q}{p}\right) = 1$, and let $\Gamma_q = \mathrm{PSL}(2, q)$ be a group of order $n = |\Gamma_q| = (q^3 - q)/2$. The Ramanujan graphs defined in [48] are certain r-regular Cayley graphs $H_q = Cayley(\Gamma_q, S_q)$, with $r = |S| = p + 1 = 18$, $\lambda = \sqrt{17}$, and $2 \log_p q \leq \mathrm{girth}(H_q) \leq 2 \log_p n + 3$. In summary, $\{H_q\}$ are 18-regular Cayley graphs on $n = \Theta(q^3)$ vertices with girth $\Theta(\log q)$.

We can now take $k = 3$ and set $G_q = K_3 \times H_q$. As in the corollary, $r = 18 > 4(k-1)^2 = 16$. Graphs $\{G_q\}$ are then 36-regular uniquely 3-colorable graphs on $3n$ vertices, with vector chromatic number 3, and girth $\Theta(\log n)$. These graphs are constructed explicitly and can be made into Cayley graphs by taking $\mathbb{Z}_3 \times \Gamma_q$ with generators $\{(\pm 1, s), s \in S_q\}$.

2.3 Lovász's theta function

Now consider the *Lovász number* (also called *Lovász's theta function*) of a graph G, denoted by $\vartheta(G)$, and the *Shannon capacity* of G, denoted by $\Theta(G)$. These two important graph parameters are intimately related to each other [47]; while the Lovász number is usually hard to calculate for large families of graphs, the Shannon capacity open for some very small graphs (see Subsection 7.4). Recall one of the several equivalent ways to define $\vartheta(G)$. A vector k-coloring is *strict* if $\langle x_i, x_j \rangle = -1/(k-1)$ for every edge of the graph. Using this notion, $\vartheta(\bar{G})$ ($\bar{G}$ is the complement of G) is the least $k \geq 1$ such that $\bar{G}$ admits a strict vector k-coloring. The Shannon capacity of a graph G is defined to be $\lim_{b \to \infty} (\alpha(G^b))^{1/b}$, where $\alpha(G)$ is the size of the maximal independent set in G, and G^b is the *strong graph product* of G with itself b times (see below Definition 2).

A nice feature of graphs $G = K_k \times H$ where H is a r-regular Ramanujan graph on n vertices (see Corollary 7), is that they satisfy also $\vartheta(G) = \Theta(G) = n$. One way to see this is to use the fact that for all vertex transitive graphs F, we have $\vartheta(F) \cdot \vartheta(\bar{F}) = |V(F)|$ (see [47]). Using the above definition, and observing that $\chi_v(G) \leq \vartheta(\bar{G}) \leq \chi(G)$, we get $\vartheta(\bar{G}) = k$, and therefore $\vartheta(G) = n$. Now recall that

$$(*) \qquad \alpha(F) \leq \Theta(F) \leq \vartheta(F)$$

for a general graph F (this was shown by Lovász [47]). Since we obviously have $\alpha(G) \geq n$, we conclude that $\alpha(G) = \Theta(G) = n$ as well.

In a different direction, we can show that $\vartheta(G) = n$ by combining $(*)$ and the following extension of *Hoffman bound* on the size of an independent set in a graph [28]:

$$\alpha(F) \leq \Theta(F) \leq \vartheta(F) \leq N \cdot \frac{-\lambda_N}{\lambda_1 - \lambda_N},$$

where $N = |V(F)|$. When $F = G = K_k \times H$ as in Corollary 7, we have $N = kn$, $\lambda_1 = (k-1)r$, and $\lambda_N = -r$ (we compute these eigenvalues in Section 4), so both the lower and upper bounds are equal to n. This implies that $\vartheta(G) = \Theta(G) = n$.

3 Preliminaries

Before we prove our main results, we present some tools that we shall use in the proof.

3.1 Universal Rigidity

As before, let $G = (V, E)$ be a simple graph, and let $\{w_e : e \in E\}$ be non-negative edge weights. Two *realizations* $p, q : V \to \mathbb{R}^d$ of a framework $\mathcal{F} = \big(V, E, \{w_e\}\big)$ are called *equivalent*, write $p \simeq q$, if p can be obtained from q as a composition of rotations, reflections, and orthogonal translations. A framework $\mathcal{F}$ is called *globally rigid* in $\mathbb{R}^d$ if it has a unique realization defined as above. A framework $\mathcal{F}$ is called *universally rigid* if there is a dimension $s \geq 1$, such that $\mathcal{F}$ is globally rigid in every dimension $d \geq s$, and $\mathcal{F}$ is *not* realizable in $\mathbb{R}^d$ for $d < s$. It is easy to see that such $s \leq n$ for every universally rigid $\mathcal{F}$.

Let us note that universally rigidity is a very strong (restrictive) notion of rigidity. Even though we will not use them, other notions include: *continuous rigidity* (every continuous family of realizations must be constant, up to equivalent), *infinitesimal rigidity* (there is no first order deformation of the framework), *second order rigidity*, etc. All these notions are different, although some imply the other. For examples, two triangles sharing a common edge is also not a globally rigid structure in $\mathbb{R}^2$, even if they are continuously and infinitesimally rigid (see [12] and Figure 2). Similarly, a framework can be globally rigid in $\mathbb{R}^d$ but not in $\mathbb{R}^{d'}$ (take e.g. a closed chain of bars of lengths 2, 2, 2, 3 and 3, which is rigid in $\mathbb{R}$ but not in $\mathbb{R}^2$). For comprehensive surveys on rigidity theory we refer the reader to [12, 58].

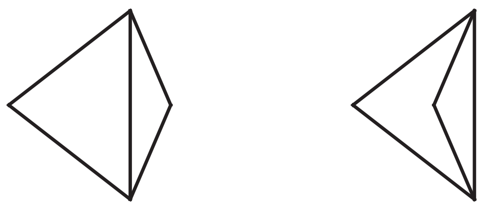

Figure 2: Two realizations of the same five bar framework.

In this section we state the main result in the area, Connelly's Theorem 12, giving sufficient conditions to universal rigidity. We let $\mathcal{P}(G)$ be the space of all configurations of the graph G, and let $\mathcal{P}^d(G)$ denote the space of configurations that lie entirely in the $\mathbb{R}^d$ (that is, are congruent to a configuration in $\mathbb{R}^d$).

Definition 9. *An* ***equilibrium stress matrix*** *of a configuration $p(G)$ is a matrix Ω indexed by $V \times V$ so that*

1. *Ω is a symmetric matrix,*
2. *If $(u, v) \notin E$, and $u \neq v$ then $\Omega(u, v) = 0$.*
3. *For every $u \in V$, $\sum_{v \in V} \Omega(u, v) = 0$.*
4. *For every $u \in V$, $\sum_{v \in V} \Omega(u, v)p(v) = 0$.*

Definition 10. *We say that the edge directions of a configuration p in $\mathcal{P}^d(G)$ lie* ***on a conic at infinity*** *if there exists a non-zero symmetric $d \times d$ matrix Q such that for all edges (u, v) of G*

$$[p(u) - p(v)]^t Q [p(u) - p(v)] = 0.$$

Definition 11. *We say that a symmetric matrix A is* ***positive semi-definite*** *(or PSD for short) if for every vector x, $x^t Q x \geq 0$.*

The following is a sufficient condition for a configuration to be universally rigid. It was derived by Connelly in a series of papers, see [13] for example. The complete proof is given in full details in Section A.

Theorem 12 (Connelly)**.** *Suppose that G is a graph with $d+2$ or more vertices and p is a configuration in $\mathcal{P}^d(G)$. Suppose there exists a PSD equilibrium stress matrix $\Omega(p,G)$ whose rank is $n-d-1$. Also suppose that the edge directions of p do not lie on a conic at infinity. Then p is universally rigid.*

We should mention that the original Connelly's result is more general, and works also for frameworks of bars, struts and cables. The proof in Section A is presented in full generality. Also, it was recently proved by Gortler and Thurston [30] that for generic universally rigid frameworks, the sufficient conditions in the theorem are also necessary.

3.2 The Kronecker Product

If $A = (a_{i,j})$ is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product, usually denoted by $A \otimes B$, is the $mp \times nq$ block matrix

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{pmatrix}$$

A well known fact is that if $\lambda_1, ..., \lambda_n$ are the eigenvalues of A and $\mu_1, ..., \mu_q$ are those of B (listed according to multiplicity), the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ for $i = 1, \ldots n$, $j = 1, \ldots q$.

Let G, H be two graphs with A, B their respective adjacency matrices. It is well known and rather easy to verify that $A \otimes B$ is the adjacency matrix of the graph $G \times H$.

3.3 A Semidefinite Program for the GRP

The Graph Realization Problem can be formulated as a Semidefinite Program. We follow [57, 30] in our description of the SDP. Let e_i be the i^{th} standard basis vector of $\mathbb{R}^n$. By $A \succeq 0$ we mean that A is a symmetric PSD matrix. For two $n \times n$ matrices A, B define

$$A \circ B = \sum_{i,j=1}^{n} A_{ij} \cdot B_{ij}.$$

Let $X = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_n]$ be a $k \times n$ matrix ($\mathbf{x}_i$ is the i^{th} column of X). One can easily verify that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = (e_i - e_j)^t X^t X (e_i - e_j) = (e_i - e_j)(e_i - e_j)^t \circ (X^t X).$$

Define the matrix $L_{ij} = (e_i - e_j)(e_i - e_j)^T$. Given a graph $G = (V, E)$ and a set of weights $\{w_{ij} : (i,j) \in E\}$, the GRP is solved by the following SDP:

$$\begin{aligned} &\max_{Y} 0 \\ &L_{ij} \circ Y = w_{ij}^2, \quad \text{for all } (i,j) \in E(G), \\ &Y \succeq 0. \end{aligned} \tag{3.1}$$

For a symmetric matrix Y it is a known result that Y is PSD iff there exists a matrix B such that $Y = B^t B$ (the Cholesky decomposition). Let Y be a solution to the above SDP, and write $Y = X^t X$, then the constraint ensures that the columns of X, treated as a configuration of the graph G, satisfy the distance constraints.

Unfortunately, this still does not quite suffice for the algorithmic part of our result, since the SDP can only be computed up to a finite precision in polynomial time (the running time is proportional to $\log 1/\varepsilon$, where ε is the desired precision). Universal rigidity does not exclude the existence of a configuration q which looks very different than p (maybe even in a different dimension), and satisfies the distance constraints up to a tiny error. So q might be an output of the SDP, when the precision is finite. Fortunately, there is a family of SDP solvers called *path-following* algorithms, see for example [2], which compute a series of solutions (Y_i, S_i), $i = 1, 2, \ldots$, such that $\lim_{i\to\infty}(Y_i, S_i) = (Y^*, S^*)$, where Y^*, S^* are optimal solutions to the primal and dual SDP programs. Since in our case, Y^* is the unique solution to the primal SDP, then using a path-following method guarantees that indeed the output is an approximation of the unique solution.

3.4 Self-Duality

One of the requirements for the path-following methods to apply is that both the primal and dual SDP have strictly feasible solutions (by strict we mean positive-definite). This will not be the case for the SDP we just described since the rank of Y is the dimension of the configuration p for the GRP. However, matrix Y is positive-definite if and only if it has full rank, which in most cases will not be true. To overcome this problem, we use the self-dual method (see [18, 19]) which embeds both the primal and dual SDP in a new SDP, which is self-dual, and has strict feasible solutions. The solutions of the new SDP give the solution to the embedded programs if and only if the original primal and dual programs are gap free (that is, they both have the same maximal/minimal solution value). We used this property earlier, in the proof of Corollary 5.

Let b be the vector $b = (w_{ij}^2)$ for $(i, j) \in E$. The dual to (3.1) is

$$\begin{aligned} &\min_{\mathbf{x},S} \; b^t\mathbf{x} \\ &\sum_{(i,j)\in E} \mathbf{x}_{ij} L_{ij} + S = 0, \\ &S \succeq 0. \end{aligned} \tag{3.2}$$

The solution $\mathbf{x} = 0$ and $S = 0$ is a feasible solution to (3.2) whose value is 0. By the weak duality theorem, this is the optimal value of the dual SDP (since the primal has value 0). We are going to embed both our primal and dual programs in the following self–dual SDP:

$$\begin{aligned} \min \quad & \theta\beta \\ & L_{ij}Y - \tau b_{ij} + \theta\bar{b}_{ij} = 0 \qquad (i,j) \in E, \\ & -\sum_{i=1}^{|E|} \mathbf{x}_{ij} L_{ij} - Z = 0, \\ & b^t\mathbf{x} + \theta - \rho = 0, \\ & -\bar{b}^t\mathbf{x} - I \circ Y - \tau - \nu = -\beta, \\ & Y, Z \succeq 0, \qquad \mathbf{x} \in \mathbb{R}^{|E|}, \qquad \theta, \rho, \tau, \nu \geq 0, \end{aligned}$$

where $\bar{b}_{ij} = b_{ij} - tr(L_{ij})$.

One can verify that this SDP is self dual, and since setting all parameters to 0 is a feasible solution, by the self-duality we get that this is indeed the optimal value of the SDP, and $\theta = 0$ in the optimum. Furthermore, it was shown in [18] that $\mu > 0$ if and only if the original pair of primal and dual SDP were feasible and gap free. This is indeed the case for us, and we get that $\tau > 0$ in the optimal solution. Therefore the optimal solution to the self-dual satisfies

$$L_{ij}Y = \tau b_{ij} = \tau w_{ij}^2.$$

Therefore, if the original SDP had a unique solution, then the self-dual has a unique Y that satisfies the first constraint. This follows since if Y is part of a solution to the self-dual, then $\tau^{-1} \cdot Y$ is a solution to (3.1).

The second requirement for applying a path-following method is that the matrices L_{ij} be linearly independent. Since there are no parallel edges in G, this is indeed the case.

4 Proof of Theorem 3

Recall that K_k is the complete graph on k vertices, and H is some r-regular graph satisfying $\lambda < r/(k-1)$ ($\lambda = \max_{i\geq 2} |\lambda_i|$). Let $G = K_k \times H$, and the configuration p of G^+ assigns every vertex in H_{v_i} with the i^{th} vertex of the $(k-1)$-simplex and v_0 with the origin; here we think of the $(k-1)$-simplex whose vertices lie on the unit ball in $\mathbb{R}^k$. We shall prove the theorem for the case $k = 3$, as the general k case is easily deduced in a similar fashion.

If $k = 3$, then the 2-simplex is an equilateral triangle (the dimension is $d = 2$). Let $V(K_3) = \{v_1, v_2, v_3\}$, and let P_1, P_2, P_3 be the three vertices of an equilateral triangle whose center of mass is at the origin O and $|OP_i| = 1$ for $i = 1, 2, 3$. The configuration $p(G^+)$ is then defined by assigning P_i to all vertices $\{v_i\} \times V(H)$, and O to the vertex v_0. In this configuration, $\overrightarrow{OP_i}$ is a unit vector, and $\overrightarrow{P_iP_j}$ is of length $\sqrt{3}$ for every $i \neq j$.

The heart of the proof lies in providing a PSD equilibrium stress matrix whose nullspace has dimension $d+1$ which is 3 in our case. Let $A = (a_{ij})$ be the $n \times n$ adjacency matrix of H, and Γ be the adjacency matrix of K_3. We let $\mathbf{1} \in \mathbb{R}^{3n}$ be the column all-one vector, I_{3n} is the identity $3n \times 3n$ matrix. Define

$$\Omega = \left(\begin{array}{c|c} I_{3n} + \frac{1}{r}\Gamma \otimes A & -3 \cdot \mathbf{1} \\ \hline -3 \cdot \mathbf{1}^t & 9n \end{array} \right).$$

The matrix Ω is an $(3n+1) \times (3n+1)$ matrix and indeed in G^+ there are $3n+1$ vertices. The theorem follows from the following three lemmas:

Lemma 13. *The edge directions of p do not lie on a conic at infinity.*

Proof. W.l.o.g. assume that $P_1 = (\sqrt{3}/2, 1/2), P_2 = (-\sqrt{3}/2, 1/2), P_3 = (0, -1)$ and $P_{v_0} = (0, 0)$ (we omit the 0-padding of the points). Let

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a symmetric 2×2 matrix. We will show that if $[P_i - P_{v_0}]Q[P_i - P_{v_0}]^t = 0$ for $i = 1, 2, 3$ then Q is the zero matrix, a contradiction.

For $i = 3$, the equation gives $c = 0$. For $i = 1, 2$ we get

$$\frac{3a}{4} + \frac{\sqrt{3}b}{2} = 0$$
$$\frac{3a}{4} - \frac{\sqrt{3}b}{2} = 0$$

This gives $a = 0$ and $b = 0$. Therefore Q has to be the zero matrix, a contradiction. ■

Lemma 14. *$\Omega(G^+, p)$ is an equilibrium stress matrix.*

Proof. Let us go over the required properties in Definition 9: Ω is a symmetric matrix. There is no limitation on the diagonal entries, and besides it we have non-zero entries only where there is an edge of G^+ (since $\Gamma \otimes A$ is the adjacency matrix of G). The sum of every row i is indeed 0: For the first $3n$ rows, $a_{ii} = 1$, and the sum of the remaining entries is $1/r$ times the degree of a vertex i, which is $2r$. This is balanced by the -3 at the last column. For the last row, the sum is $3n \cdot (-3) + 9n = 0$. Fix some $v \in V(G)$, and treat the configuration as vectors, w.l.o.g. assume $v = (v_1, x)$ for some $x \in V(H)$.

$$\sum_{w \in V} \Omega(v,w)p(w) = \overrightarrow{OP_1} + \frac{1}{r} \sum_{\substack{w=(v_2,z):\\(z,x)\in E(H)}} \Omega(v,w)\overrightarrow{OP_2} + \frac{1}{r} \sum_{\substack{w=(v_3,z):\\(z,x)\in E(H)}} \Omega(v,w)\overrightarrow{OP_3} - 3 \cdot \overrightarrow{0} = \sum_{i=1}^{3} \overrightarrow{OP_i} = 0.$$

For the last row, corresponding to $v = v_0$, we have

$$\sum_{w \in V} \Omega(v_0,w)p(w) = -3n \sum_{\substack{w=(v_1,x):\\x\in V(H)}} \overrightarrow{OP_1} - 3n \sum_{\substack{w=(v_2,x):\\x\in V(H)}} \overrightarrow{OP_2} - 3n \sum_{\substack{w=(v_3,x):\\x\in V(H)}} \overrightarrow{OP_3} + 9n \cdot \overrightarrow{0} = -3n \sum_{i=1}^{3} \overrightarrow{OP_i} = 0,$$

as desired. ■

Lemma 15. *The dimension of the null space of Ω is 3, and Ω is PSD.*

Proof. Our first observation is that $\mathbf{1}_{3n+1}$ is an eigenvector of Ω corresponding to the eigenvalue 0 (this is true for every equilibrium stress matrix, by the third property in Definition 9). One can also verify that $\xi = (1, 1, \ldots, 1, -3n) \in \mathbb{R}^{3n+1}$ is an eigenvector of Ω corresponding to the eigenvalue $3 + 9n$.

Define $W = \text{span}\{\mathbf{1}_{3n+1}, \xi\}$. A symmetric $m \times m$ matrix has a set of m orthogonal eigenvectors. Since $\dim(W) = 2$, one can find the remaining $(3n + 1) - 2$ eigenvectors of Ω in a subspace perpendicular to W. Consider $\mathbf{z} = (x_1, x_2, \ldots, x_{3n}, y)$ s.t. $\mathbf{z} \perp W$. In particular, $\mathbf{z} \perp \mathbf{1}_{3n+1}$, which implies

$$y = -\sum_{i=1}^{3n} x_i.$$

Also $\mathbf{z} \perp \xi$, which implies

$$y = \frac{1}{3n} \sum_{i=1}^{3n} x_i.$$

The only way to satisfy both equations is by forcing $\sum_{i=1}^{3n} x_i = 0$, which gives $y = 0$. Therefore the vector $\mathbf{z} = (x_1, x_2, \ldots, x_{3n}, 0) = (\mathbf{x}, 0)$. Let $\mathbf{z} \perp W$ be an eigenvector of Ω corresponding to the eigenvalue λ. Since the last entry of $\mathbf{z}$ is 0,

$$\Omega \mathbf{z} = \lambda \mathbf{z} \Rightarrow \left(I_{3n} + \frac{1}{r} \Gamma \otimes A \right) \mathbf{x} = \lambda \mathbf{x}.$$

The eigenvalues of $I_{3n} + \frac{1}{r}\Gamma \otimes A$ are the eigenvalues of $\frac{1}{r}\Gamma \otimes A$ when adding 1 to every eigenvalue.

$$\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of Γ are easily seen to be $2, -1, -1$. The eigenvalues of A are r with multiplicity 1 (this is true for every connected r-regular graph, and we know H is connected, since otherwise its smallest eigenvalue would be $-r$, and we know that it is larger than $-r/2$), and all others are $< r/2$ in absolute value. By the discussion in Subsection 3.2, the eigenvalues of $\Gamma \otimes A$ are $-r$ with multiplicity 2, and the smallest eigenvalue is smaller than $2 \cdot (-r/2) = -r$. Multiplying by $1/r$, we get -1 with multiplicity 2, and the rest have absolute value < 1. Going back to $I_{3n} + \frac{1}{r}\Gamma \otimes A$, we have 0 with multiplicity 2, and the remaining eigenvalues are positive.

To conclude, 0 is an eigenvalue of Ω with multiplicity 3 (this gives the required rank of the null space). The remaining eigenvalues are positive. It is easy to see that a matrix is PSD iff all its eigenvalues are non-negative. This gives the second part of the lemma. ■

The proof of Theorem 3 now easily follows. Lemmas 13–15, together with Theorem 12 imply that the the simplex configuration for G^+ ($G = K_k \times H$) is universally rigid. The discussion in Subsection 3.3 implies the algorithmic part of the theorem.

5 Proof of Theorem 6

Fix an integer $k > 0$, and let H be a r-regular graph satisfying $\lambda < r/(k-1)$ and $girth(H) \geq g$. We need to prove that $G = K_k \times H$ is uniquely k-colorable, and that the girth of G is at least g. For the first part, assume that G is not uniquely k-colorable. Then we can construct two non-congruent realizations of G^+, by mapping color class i to the i^{th} vertex of the $(k-1)$-simplex. This however contradicts the universal rigidity established in Theorem 3.

For the vector chromatic number part, we need the slightly more general version of Theorem 12 which appears in Appendix A. In that case, the bars are replaced with struts (i.e, the edges can now stretch). A configuration p is universally rigid if every configuration with edge lengths at least the ones in p is congruent to p. The conditions in Theorem 3 imply also this version of universal rigidity. Observe that if x_i, x_j are two unit vectors, then $\arccos(\langle x_i, x_j \rangle)$ is just the angle between x_i and x_j, which determines the length of the edge $(i, j) \in E$. Therefore, using the broader notion of rigidity, if G has two different vector k-colorings, the the two configurations that assign each color class with a different vector of the coloring, give two non-congruent realizations of G, contradicting the universal rigidity.

For the girth part, take a simple cycle C in G, $(x_1, y_1) - (x_2, y_2) - \cdots - (x_t, y_t) = (x_1, y_1)$. In this case, $y_1 - y_2 - \cdots - y_t = y_1$ is a cycle in H by the definition of the vertex set of G. Also, $x_j \neq x_{j+1}$ for every j, otherwise there is a self loop in H.

For the algorithmic part, observe that once we compute the realization of G^+, we can group the vertices according to the vectors that the SDP assigned them (take all vertices with close by vectors to the same color class). Another way to obtain the coloring is to compute the eigenvectors corresponding to the $(k-1)$ smallest eigenvalues. Those eigenvectors encode the k-coloring.

6 Proof of Theorem 4

We think of C_8, the Cauchy polygon on eight vertices, bounded in the unit circle in $\mathbb{R}^2$, centered around the origin. Let H be a r-regular graph satisfying $\lambda < \sqrt{2}r/3$ ($\lambda = \max_{i\geq 2}|\lambda_i|$). Let $G = C_8 \times H$, and the configuration p of G^+ assigns vertices $\{v_i\} \times V(H)$ with the i^{th} vertex of C_8 (the first vertex of C_8 is the point (1,0) and we number the remaining vertices in a counter-clockwise manner), and v_0 with the origin.

Again, the heart of the proof lies in providing a PSD equilibrium stress matrix whose nullspace has dimension $d+1$ which is 3 in our case. Let Let $A = (a_{ij})$ be the $n \times n$ adjacency matrix of H, and Γ be the following weighted adjacency matrix of C_8.

$$\Gamma = \begin{pmatrix} 0 & -1 & 0.5 & 0 & 0 & 0 & 0.5 & -1 \\ -1 & 0 & -1 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0.5 & -1 & 0 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0 & -1 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 & -1 & 0 & -1 \\ -1 & 0.5 & 0 & 0 & 0 & 0.5 & -1 & 0 \end{pmatrix}$$

We let $\mathbf{1} \in \mathbb{R}^{8n}$ be the column all-one vector, I_{8n} is the identity $8n \times 8n$ matrix. Define

$$\Omega = \left(\begin{array}{c|c} \sqrt{2}I_{8n} + \frac{1}{r}\Gamma \otimes A & (1-\sqrt{2}) \cdot \mathbf{1} \\ \hline (1-\sqrt{2}) \cdot \mathbf{1}^t & 8n(\sqrt{2}-1) \end{array}\right).$$

The matrix Ω is an $(8n+1) \times (8n+1)$ matrix and indeed in G^+ there are $8n+1$ vertices. The theorem follows from the following three lemmas:

Lemma 16. *The edge directions of p do not lie on a conic at infinity.*

Proof. We consider four vertices of the Cauchy polygon, $P_1 = (1,0), P_2 = (\sqrt{2}/2, \sqrt{2}/2), P_3 = (0,1)$ and $P_{v_0} = (0,0)$ (we omit the 0-padding of the points). Let

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a symmetric 2×2 matrix. Solving $[P_i - P_{v_0}]Q[P_i - P_{v_0}]^t = 0$ for $i = 1, 3$ gives $a = c = 0$. Similarly, solving $[P_2 - P_{v_0}]Q[P_2 - P_{v_0}]^t = 0$ gives $b = 0$. Thus, Q is a zero matrix, a contradiction. ■

Next we need to provide a PSD equilibrium stress matrix whose nullspace has dimension $d+1$ which is 3 in our case. The matrix Ω is an $(8n+1) \times (8n+1)$ matrix and indeed in G^+ there are $8n+1$ vertices.

Lemma 17. *$\Omega(G^+, p)$ is an equilibrium stress matrix.*

Proof. Let us go over the required properties in Definition 9. Ω is a symmetric matrix. There is no limitation on the diagonal entries, and besides it we have non-zero entries only where there is an edge of G^+. The sum of every row i is indeed 0: For the first $8n$ rows, $a_{ii} = \sqrt{2}$, and the sum of the remaining entries is $1/r$ times the degree times the weight, which is gives -1. This is balanced by the $1 - \sqrt{2}$ at the last column. For the last row, the sum is $8n \cdot (1 - \sqrt{2}) + 8n \cdot (\sqrt{2} - 1) = 0$.

Fix some $v \in V(G)$, and treat the configuration as vectors. Suppose that $v = (v_1, x)$, where v_1 corresponds to the point $P_1 = (1, 0)$ (since C_8 is symmetric for every vertex, we can just consider this case), and $x \in V(H)$. The vertex v has $4r$ neighbors, r of each type $\{v_i\} \times V(H)$ for $i = 2, 3, 7, 8$.

The vectors corresponding to vertices $\{v_3\} \times V(H)$ and $\{v_7\} \times V(H)$ are antipodal, and therefore cancel each other (as long as both are assigned with the same weight, which is 0.5 in our case). The sum of every two vectors, one in $\{v_2\} \times V(H)$ and the other in $\{v_8\} \times V(H)$, is $\frac{1}{r}\sqrt{2} \cdot \overrightarrow{OP_1}$.

$$\sum_{w \in V} \Omega(v, w) p(w) = -1 \cdot r \cdot \frac{1}{r}\sqrt{2} \cdot \overrightarrow{OP_1} + \sqrt{2} \cdot \overrightarrow{OP_1} = \overrightarrow{0}.$$

For the last row, corresponding to $v = v_0$, we have

$$\sum_{w \in V} \Omega(v_0, w) p(w) = 8n(\sqrt{2} - 1) \cdot \overrightarrow{0} + (1 - \sqrt{2}) \sum_{i=1}^{8} \sum_{\substack{w = (v_i, x): \\ x \in V(H)}} \overrightarrow{OP_i} = \overrightarrow{0},$$

as desired. ∎

Lemma 18. *The dimension of the null space of Ω is 3, and Ω is PSD.*

Proof. Our first observation is that $\mathbf{1}_{8n+1}$ is an eigenvector of Ω corresponding to the eigenvalue 0 (this is true for every equilibrium stress matrix, by the third property in Definition 9). One can also verify that $\xi = (1, 1, \ldots, 1, -8n) \in \mathbb{R}^{8n+1}$ is an eigenvector of Ω corresponding to the eigenvalue $(\sqrt{2} - 1)(8n + 1)$.

Define $W = \operatorname{span}\{\mathbf{1}_{8n+1}, \xi\}$. A symmetric $m \times m$ matrix has a set of m orthogonal eigenvectors. Since $\dim(W) = 2$, one can find the remaining $(8n + 1) - 2$ eigenvectors of Ω in a subspace perpendicular to W. Consider $\mathbf{z} = (x_1, x_2, \ldots, x_{8n}, y)$ s.t. $\mathbf{z} \perp W$. In particular, $\mathbf{z} \perp \mathbf{1}_{8n+1}$, which implies

$$y = -\sum_{i=1}^{8n} x_i.$$

Also $\mathbf{z} \perp \xi$, which implies

$$y = \frac{1}{8n} \sum_{i=1}^{8n} x_i.$$

The only way to satisfy both equations is by forcing $\sum_{i=1}^{8n} x_i = 0$, which gives $y = 0$. Therefore the vector $\mathbf{z} = (x_1, x_2, \ldots, x_{8n}, 0) = (\mathbf{x}, 0)$. Let $\mathbf{z} \perp W$ be an eigenvector of Ω corresponding to the eigenvalue λ. Since the last entry of $\mathbf{z}$ is 0,

$$\Omega \mathbf{z} = \lambda \mathbf{z} \Rightarrow \left(\sqrt{2} \cdot I_{8n} + \frac{1}{r}\Gamma \otimes A\right) \mathbf{x} = \lambda \mathbf{x}.$$

The eigenvalues of $\sqrt{2} \cdot I_{8n} + \frac{1}{r}\Gamma \otimes A$ are the eigenvalues of $\frac{1}{r}\Gamma \otimes A$ when adding $\sqrt{2}$ to every eigenvalue. The eigenvalues of Γ can be computed (using MATLAB for example), and they are $\{-\sqrt{2}, -\sqrt{2}, -1, -1, -1, \sqrt{2}, \sqrt{2}, 3\}$. The eigenvalues of A are r with multiplicity 1 (this is true for every connected r-regular graph, and we know H is connected, since otherwise its smallest eigenvalue would be $-r$, and we know that it is larger than $-\sqrt{2}r/3$), and all others are $< \sqrt{2}r/3$ in absolute value (by the conditions on the theorem). By the discussion in Subsection 3.2, the eigenvalues of $\Gamma \otimes A$ are $-\sqrt{2}r$ with multiplicity 2, and the smallest eigenvalue is larger than $3 \cdot (-\sqrt{2}r/3) = -\sqrt{2}r$. Multiplying by $1/r$, we get $-\sqrt{2}$ with multiplicity 2, and the rest have absolute value $< \sqrt{2}$. Going back to $\sqrt{2}I_{8n} + \frac{1}{r}\Gamma \otimes A$, we have 0 with multiplicity 2, and the remaining eigenvalues are positive.

To conclude, 0 is an eigenvalue of Ω with multiplicity 3 (this gives the required rank of the null space). The remaining eigenvalues are positive. It is easy to see that a matrix is PSD iff all its eigenvalues are non-negative. This gives the second part of the lemma. ■

The proof of Theorem 4 now easily follows. Lemmas 16–18, together with Theorem 12 imply that Cauchy-polygon configuration for G^+, $G = C_8 \times H$, is universally rigid. The discussion in Subsection 3.3 implies the algorithmic part of the theorem.

7 Final remarks and open problems

7.1 In this paper we characterized two families of tractable GRP instances, using a tensor product between the regular d-simplex or the Cauchy polygon with a suitable r-regular graph. Both the regular d-simplex and the Cauchy polygon C_8 are universally rigid frameworks by themselves. Our results suggests perhaps the following more general method for generating universally rigid frameworks (and thus tractable GRP instances): take a universally rigid framework R and an r-regular graph H. Find a condition on $\lambda(H)$ (the second largest eigenvalue in absolute value) so that $R \times H$ is universally rigid. An interesting question for future research would be to prove that indeed the method works in this full generality, or come up with a counter example.

7.2 Let us note that every realization of $G = C_8 \times H$ into $\mathbb{R}^d$ corresponds to a 3-circular 8-coloring of G (see [60] for a comprehensive survey). On the other hand, it is a priori not true that *every* 3-circular 8-coloring of G corresponds to a realization of G. Thus, although explicit constructions of c-circular k-colorable graphs with large girth is open in general (they are known to exist via probabilistic method, see [60]), our graphs $G = C_8 \times H$ are good candidates in the case $c = 3$ and $k = 8$. In fact, we conjecture that G are in fact uniquely 3-circular 8-colorable graphs. More generally, we conjecture that Cauchy polygons C_k give rise to uniquely 3-circular k-colorable graphs of large girth, for sufficiently large k. Finally, let us mention that in this spirit general frameworks are related to general graph homomorphism problems, where uniquely homomorphic graphs are of importance again (see [37]).

7.3 Since testing for infinitesimal (first order) rigidity can be done in polynomial time (see Subsection 3.1 for definitions), there is now a large library of such frameworks, including frameworks with a large groups of symmetries, cables and struts (see [13, 15, 53]). Many of these frameworks are probably also universally rigid, though this may require new constructions of equilibrium stress matrices. By taking their tensor product with expanding graphs, potentially one can construct new large interesting families of universally rigid graphs.

7.4 The Shannon capacity of a graph and its Lovász number are typically hard quantities to compute, and remain unknown in many cases. It took over twenty years to compute $\Theta(C_5) = \sqrt{5}$ (here C_5 is a cycle on five vertices). This was left as an open question by Shannon in 1956, and resolved by Lovász in 1979 by using the fact that C_5 is vertex transitive and $C_5 \simeq \bar{C}_5$. Computing $\Theta(C_\ell)$ remains an open problem for odd $\ell \geq 7$.

7.5 Let us mention that in the graph theoretical setting of Subsection 2.2, the mathematical notions that are used to prove universal rigidity reduce to standard calculations involving quadratic forms that arise in the context of the Rayleigh quotient definition for the eigenvalues of a matrix (see the proof of Theorem 12 in the appendix). In a similar manner, one can also obtain the Hoffman bound for the independence number a graph [28], an approach extended recently by Bilu to vector chromatic numbers [7]. Thus in a certain sense, the language of universal rigidity can be viewed as a generalization of these ideas in a geometric setting.

7.6 Motivated by the *Brooks' theorem* and earlier results on triangle-free graphs, the classical problem of Grünbaum [33] asks whether for a given g and k, there is a graph with maximal degree k, girth g and chromatic number k (see also [42]). While false for large k and $g \geq 5$ (see [44]), one can ask what is the smallest possible maximal degree δ one can have? Even though small cases remain open, this problem has been largely resolved by now, with $\delta = \Theta(k \log k)$ being the optimal (see [9, 44, 45]). With some notable exceptions, most results in the field are proved by a probabilistic method, so the graphs in Example 8 derandomize these results in some special cases. It would be interesting to see if other expander constructions (see [40]) allow decrease in the maximal degree, and give explicit constructions in the (modified) Grünbaum problem.

Now, in a similar manner one can ask what is the smallest possible maximal degree δ of a graph G one can have, so that G has a unique k-coloring? Is $\delta = O(k \log k)$ still valid? This problem seem unexplored. We refer to [42, 50] for more on graph colorings and further references.

7.7 Our results seem to fit well with known results and ideas in Rigidity Theory. First, the Ramanujan graphs were previously used in [56] to construct graphs with large girth, which are infinitesimally rigid in $\mathbb{R}^2$. Thus one can view our Example 8 as an advanced generalization of this result. Second, the product graphs $K_k \times H$ viewed as a cone over a simplex is a generalization of a classical operation to ensure universal rigidity of a framework in higher dimensions (see e.g. [6] for a cable-and-strut suspension over a simplex in $\mathbb{R}^d$). Finally, our tensor product operation $G \to G \times H$ is somewhat related to *coning*, recently studied in the context of global rigidity [16].

7.8 Cauchy polygons, as defined in [14] have two fewer edges, but still universally rigid. We use our definition of C_8 for simplicity; it is also critical in our calculations of the equilibrium stress matrix Ω. The idea behind Connelly's asymmetric definition is to generalize *Cauchy's arm lemma* used in the proof of the *Cauchy rigidity theorem* of convex polytopes in $\mathbb{R}^3$. We refer to [52] for definitions and the references.

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A Proof of Theorem 12

Let us extend the notion of universal rigidity of a configuration p by allowing struts and cables in addition to bars. Such a framework is also known as a *tensegrity* framework. Putting a strut instead of the edge (u, v), means that every configuration q must satisfy $\|q(u) - q(v)\| \geq \|p(u) - p(v)\|$, and a cable means $\|q(u) - q(v)\| \leq \|p(u) - p(v)\|$. Now, p is universally rigid if every configuration q that satisfies the tensegrity constraints (i.e., bars, struts and cables) is congruent to p (in particular, all constraints are satisfied with equality). The stress matrix Ω is considered *proper* if it satisfies $\Omega_{uv} \geq 0$ for a strut and $\Omega_{uv} \leq 0$ for a cable.

We shall prove the following theorem, which in particular implies Theorem 12.

Theorem 19. *Suppose that G is a graph with $d + 2$ or more vertices and p is a configuration in $\mathcal{P}^d(G)$. Suppose that there is a proper PSD equilibrium stress matrix $\Omega(p, G)$ whose rank is $n-d-1$. Also suppose that the edge directions of p do not lie on a conic at infinity. Then p is universally rigid.*

For two configurations p, q we use the notation $p \simeq q$ to denote the fact that $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for every bar $(u, v) \in E$, $\|p(u) - p(v)\| \geq \|q(u) - q(v)\|$ for a cable $(u, v) \in E$ and $\|p(u) - p(v)\| \leq \|q(u) - q(v)\|$ for a strut.

Lemma 20. *Let $\Omega(p, G)$ be a proper equilibrium stress matrix for the configuration $p(G)$. Let q be another configuration for G s.t. $p \simeq q$. If Ω is PSD, $\Omega(q, G)$ is an equilibrium stress as well. Furthermore, all tensegrity constraints hold with equality.*

Proof. The first three requirements that Ω needs to satisfy as an equilibrium stress matrix do not depend on p. Therefore, they hold for q as well. It remains to verify the last property, that is

$$\sum_{w \in V} \Omega(u, w) q(w) = 0, \quad \text{for all } u \in V.$$

Let us define the matrix $\Psi^{(uv)}$ to be the $n \times n$ matrix with -1 in the (u, v) and (v, u) entries, 1 in the (u, u) and (v, v) entries, and 0 otherwise. Define $\omega_{uv} = -\Omega(u, v)$. We claim that

$$\Omega = \sum_{(u,v) \in V \times V} \omega_{uv} \Psi^{(uv)}.$$

This is clear for the off-diagonal entries. For the diagonal entries, observe that since Ω is an equilibrium stress matrix, the sum of every row is 0, therefore the diagonal entry $\Omega(u, u)$ must equal the negative of the sum of all entries of the u-row. This is equivalent to having the 1 entry in $\Psi^{(uu)}$. For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ consider the quadratic form $\mathbf{x}^t \Omega \mathbf{x}$. It is easy to see that $\mathbf{x}^t \Psi^{(uv)} \mathbf{x} = (x_u - x_v)^2$, therefore it follows that

$$\mathbf{x}^t \Omega \mathbf{x} = \mathbf{x}^t \left(\sum_{(u,v) \in V \times V} \omega_{uv} \Psi^{(uv)} \right) \mathbf{x} = \sum_{(u,v) \in V \times V} \omega_{uv} \mathbf{x}^t \Psi^{(uv)} \mathbf{x} = \sum_{(u,v) \in V \times V} \omega_{uv} (x_u - x_v)^2. \tag{A.1}$$

Think of the entries of the points in p as formal variables (that is, n^2 variables), and let us define the following function $E(p)$ (which is usually called the energy function in rigidity theory):

$$E(p) = \sum_{(u,v) \in V \times V} \omega_{uv} |p(u) - p(v)|^2.$$

Let p_0 be a configuration such that $\Omega(p_0, G)$ is an equilibrium stress matrix. The first observation that we make is that $\nabla E(p_0) = 0$. To see this, let $\mathcal{P}$ be the $n \times n$ matrix such that the u^{th} column of $\mathcal{P}$ is $p_0(u)$. Let $\mathbf{x}_u$ be the u^{th} row of $\mathcal{P}$ (i.e., the j^{th} entry in $\mathbf{x}_u$ is the u^{th} entry of $p_0(j)$ for $j = 1, \ldots, n$). Fix a vertex $u \in V$ and consider the variable x_{uv} – the v^{th} entry of $\mathbf{x}_u$.

$$\frac{\partial E}{\partial x_{uv}} = 2 \sum_{w \in V} \omega_{vw}(x_{uv} - x_{wv}).$$

The last two properties in Definition 9 imply that if $\Omega(p_0, G)$ is an equilibrium stress, then

$$\sum_{w \in V} \omega_{vw}(p(u) - p(w)) = 0, \quad \text{for all } u \in V.$$

Combining the last two equations we get that if $\Omega(p_0, G)$ is an equilibrium stress matrix, $\nabla E(p_0) = 0$.

The next observation that me make is that $E(p_0) = 0$. Define $g = tp_0$, then $E(g) = E(tp_0) = t^2 E(p_0)$ (the last equality just follows from the quadratic form of E). Using the chain rule,

$$\mathrm{E}'(g) = \nabla E(g) \cdot g' = \nabla E(tp_0) \cdot p_0.$$

On the other hand

$$\mathrm{E}'(g) = (t^2 E(p_0))' = 2tE(p_0).$$

Combining the two and setting $t = 1$ (and recalling that $\nabla E(p_0) = 0$), we get $E(p_0) = 0$.

The last observation that we make is that $E(p) \geq 0$ for every configuration p. By Equation (A.1), $E(p)$ can be reexpressed as follows ($\mathbf{x}_i$ is the i^{th} row of $\mathcal{P}$):

$$E(p) = \mathbf{x}_1 \Omega \mathbf{x}_1^t + \mathbf{x}_2 \Omega \mathbf{x}_2^t + \cdots + \mathbf{x}_n \Omega \mathbf{x}_n^t.$$

Since Ω is PSD, it holds that $\mathbf{x}_i \Omega \mathbf{x}_i^t \geq 0$ for every i, hence $E(p) \geq 0$.

Now we are ready to prove that $\Omega(q, G)$ is an equilibrium stress matrix as well, if $p \simeq q$. First observe that for every $(u, v) \in E$, $\omega_{uv}\|q(u) - q(v)\| \leq \omega_{uv}\|p(u) - p(v)\|$. This is obvious for bars (which hold with equality). For struts, $\|q(u) - q(v)\| \geq \|p(u) - p(v)\|$, and $\Omega(u, v) \geq 0$ since Ω is a proper stress matrix. Therefore, $\omega_{uv} \leq 0$, and the inequality holds. The same argument implies the inequality for cables. This observation, combined with $E(q) \geq 0$ gives

$$0 \leq E(q) \leq E(p) = 0 \Rightarrow E(q) = 0.$$

This also means that the struts and cables constraints must hold with equality. If $E(q) = 0$, then q is a minimum point for E, and therefore $\nabla E(q) = 0$. However this implies that

$$\sum_{w \in V} \omega_{uw}(q(u) - q(w)) = 0, \quad \text{for all } u \in V.$$

In other words,

$$\sum_{w \in V} \omega_{uw} q(u) = \sum_{w \in V} \omega_{uw} q(w), \quad \text{for all } u \in V.$$

Combining this observation with the third property in Definition 9,

$$\sum_{w \in V} \Omega(u, w) q(w) = -\sum_{w \in V} \omega_{uw} q(w) = -q(u) \sum_{w \in V} \omega_{uw} = q(u) \sum_{w \in V} \Omega(u, w) = 0.$$

We conclude

$$\sum_{w \in V} \Omega(u,w)q(w) = 0, \quad \text{for all } u \in V.$$

■

Before we state the next lemma, we remind the reader that an affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a function of the form $T = Ax+b$, $A \in M_{n\times n}(\mathbb{R})$ and $b \in \mathbb{R}^n$; T is linear if $b = 0$. For a configuration p, $T(p)$ stands for the *ordered* set $\{T(p(u)) : u \in V\}$.

Lemma 21. *If $\Omega(p,G)$ is a PSD equilibrium stress matrix of degree $n-d-1$, G has at least $d+2$ vertices, and $q \simeq p$ is another configuration, then there exists an affine transformation T such that $T(p) = q$.*

Proof. The configuration p has dimension d, therefore there exists an isometry $T_1 = A_1x + d_1$ (A_1 is an orthogonal matrix, therefore T_1 preserves edge lengths) such that $T_1(p)$ lies in $\mathbb{R}^d$ (we pad every vector in $\mathbb{R}^d$ with $n-d$ zeros for consistency of dimensions). Let $p^* = T_1(p)$. Define $b_1 = p^*(1)$ and set $p' = \{p^*(u) - b_1 : u \in V\}$. Similarly define $q' = \{q(u) - b_2 : u \in V\}$, where $b_2 = q(1)$. Clearly if $p \simeq q$ then $p' \simeq q'$. Therefore it suffices to prove that there exists a *linear* affine transformation T such that $T(p') = q'$. If this is indeed the case, then $q'(u) = Ap'(u)$ for some matrix A. Recalling that $q'(u) = q(u) - b_2$, and $p'(u) = p^*(u) - b_1 = A_1p(u) + d_1 - b_1$, we get $q(u) - b_2 = A(A_1p(u) + d_1 - b_1)$. This implies $q(u) = Cp(u) + c$ for $C = AA_1$ and $c = b_2 + A(d_1 - b_1)$, which is just a vector in $\mathbb{R}^n$. In other words, $q = U(p)$ for the affine transformation $U = Cx + c$. It can be verified rather easily that if $\Omega(p,G)$ is an equilibrium stress matrix, so is $\Omega(T(p),G)$ (for an arbitrary affine transformation T). Hence from now on we shall consider only the configurations p' and q'

Recall the definition of the matrix $\mathcal{P}$ from above, the columns of $\mathcal{P}$ are the vectors $\{p'(u) : u \in V\}$. Observe that by our construction of p', $p'(1) = 0$ and let us assume w.l.o.g that the vectors $p(2), \ldots, p(d+1)$ are linearly independent. Since also $q'(1) = 0$, every linear transformation satisfies $T(p'(1)) = q'(1)$. We can certainly define a linear affine transformation T such that $T(p'(i)) = q'(i)$ for $i = 2, \ldots, d+1$. If this also holds for $i = d+2, \ldots, n$ then we are done. If not, then there is some index $i > d+1$ such that $T(p'_i) \neq q'_i$. Therefore, $T(p'(i)) - q'(i)$ is not the zero vector, so there exists some coordinate i_0 which is non-zero. Define the vector $r = (r_1, r_2, \ldots, r_n)$ to be: r_j is the i_0 coordinate of $T(p'(j)) - q'(j)$. Let $\mathcal{P}^*$ be the matrix $\mathcal{P}$ in which row $d+2$ is replaced with r. Observe that the first $d+1$ entries in row $d+2$ are 0 (just because $T(p'(i)) = q'(i)$), and in the remaining $n-d-1$ entries there is at least one non-zero entry. It is not hard to see that this implies that the rank of $\mathcal{P}^*$ is at least $d+1$. The fourth property in Definition 9 implies that if $\Omega(p',G)$ is an equilibrium stress matrix, then the rows of $\mathcal{P}$ are in the kernel of Ω. Also if a linear affine transformation is applied to p', the rows of the new matrix $\mathcal{P}$ will be in the kernel (using the linearity of matrix multiplication). Similarly to $\mathcal{P}$, we can define the matrix $\mathcal{Q}$ whose columns are the vectors in q'. Since $\Omega(q',G)$ is an equilibrium stress (by Lemma 20), the rows of $\mathcal{Q}$ are also in the kernel of Ω. These two facts imply that the vector r is going to be in the kernel of Ω as well. To conclude, all the rows of $\mathcal{P}^*$ belong to the kernel of Ω and their rank is at least $d+1$. The vector $\mathbf{1}_n \in \mathbb{R}^n$ always belongs to the kernel of Ω (third property in Definition 9); since the first coordinate of every row in $\mathcal{P}^*$ are 0, $\mathbf{1}_n$ does not belong to the span of the rows of $\mathcal{P}^*$. To conclude, the kernel of Ω contains the span of the rows of $\mathcal{P}^*$, which has dimension at least $d+1$, and the vector $\mathbf{1}_n$. The rank of Ω is then at most $n-(d+2)$, which contradicts our rank assumption. Hence, $T(p'(i)) = q'(i)$ for every i, and we have shown that there is a linear transformation such that $T(p') = q'$ as required. ■

Lemma 22. *Let $\Omega(p, G)$ be a proper PSD equilibrium stress matrix for the configuration $p(G)$. Let q be another configuration for G s.t. $p \simeq q$. If there is an affine transformation T such that $T(p) = q$, then either the directions of p lie on a conic at infinity, or q is congruent to p*

Proof. Lemma 20 implies that the tensegrity constraints hold with equality. Therefore since $p \simeq q$, we have that for every $(u, v) \in E(G)$,

$$\begin{aligned} 0 &= \|q(u) - q(v)\|^2 - \|p(u) - p(v)\|^2 = \|(Ap(u) + b) - (Ap(v) + b)\|^2 - \|p(u) - p(v)\|^2 = \\ &= \|Ap(u) - Ap(v)\|^2 - \|p(u) - p(v)\|^2 = \|A(p(u) - p(v))\|^2 - \|p(u) - p(v)\|^2. \end{aligned}$$

Using the fact that $\|Ax\|^2 = (Ax)^t Ax = x^t A^t Ax$, the latter can be restated as

$$0 = [p(u) - p(v)]^t A^t A [p(u) - p(v)] - [p(u) - p(v)]^t I_n [p(u) - p(v)],$$

which gives

$$[p(u) - p(v)]^t \left(A^t A - I_n\right) [p(u) - p(v)] = 0.$$

Define $Q = A^t A - I_n$. If $Q \neq 0$, then by Definition 10, the directions of p indeed lie on a conic at infinity. If $Q = 0$, this means that $A^t A = I_n$, or in other words, A is an orthogonal matrix. Thus $T = Ax + b$ is an isometry, and therefore p and q are congruent. ∎

Theorem 12 now follows easily from these lemmas. Let p be a configuration in $\mathbb{R}^d$ for the graph G, satisfying the conditions of Theorem 12. That is, $\Omega(p, G)$ is a proper PSD equilibrium stress matrix with rank $n - d - 1$. Further, we assume that the directions of p do not lie on a conic at infinity. Let q be another configuration that satisfies the tensegrity constraints of $G(p)$. Lemma 21 asserts that there exists an affine transformation T such that $T(p) = q$. Lemma 22 then gives that either the directions of p lie on a conic at infinity or T is a congruence. But the conditions of the theorem exclude the former, and we are left with p and q are congruent.