GEOMETRIC REALIZATIONS OF POLYHEDRAL COMPLEXES

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Abstract. We discuss two problems in combinatorial geometry. First, given a geometric polyhedral complex in $\mathbb{R}^3$ (a family of 3-polytopes attached face-to-face), can it always be realized over $\mathbb{Q}$? We give a negative answer to this question, by presenting an *irrational polyhedral complex* with 1278 convex polyhedra. We then present a *universality theorem* on the space of realizations of such complexes.

Second, given a pure d-dimensional topological polyhedral complex (embedded in $\mathbb{R}^d$), we ask when it can be realized geometrically (that is, rectilinearly embedded in $\mathbb{R}^d$). We present both positive and negative results in this direction.

1. Introduction

The notion of realizing topological configurations geometrically is illustrated elegantly in Fáry's theorem [F], which states that every planar graph can be drawn in the plane such that each edge is a straight line segment. In two influential papers [T1, T2], Tutte first showed necessary and sufficient conditions for realizing 2-connected planar graphs, with all faces (non-strictly) convex. He then showed that for 3-connected planar graphs one can make all faces *strictly convex*. Much of this paper was motivated by the possibility of extending Tutte's results to 3 and higher dimensions.

We begin with the classical *Steinitz theorem* (see e.g. [G, P, R, Z1]). It states that all 3-connected planar graphs can be realized as graphs of 3-polytopes. As a consequence of the proof, all polytopes in $\mathbb{R}^3$ can be realized over $\mathbb{Q}$ (i.e. realized with rational vertex coordinates) by applying small perturbations of the vertices which preserve combinatorial structure (the faces of the polytope). There are several directions into which this result has been shown to have negative analogues:

(1) In $\mathbb{R}^d$, $d \geq 4$, there exist irrational convex polytopes (see [R, RZ]),

(2) There exists an irrational 2-dim polyhedral complex *immersed* into $\mathbb{R}^3$ (see [Br, Z2]),

(3) There exists a 3-dim topological simplicial complex that is not geometrically realizable (see [Ca, HZ, K]).

In light of Steinitz's theorem, (1) and (2), it is natural to ask whether every 3-dim polyhedral complex can be realized over $\mathbb{Q}$. A 3-dim polyhedral complex is a natural generalization of a Schlegel diagram of a 4-polytope, so this question occupies an intermediate position between Steinitz's theorem and (1). Our first result answers this question in the negative (see below for definitions and notation).

Theorem 1.1 *In $\mathbb{R}^3$, there exists an irrational 3-dim geometric polyhedral complex consisting of 1278 convex polyhedra (one pentagonal pyramid and 1277 triangular prisms).*

In other words, we show that there exists an arrangement of finitely many convex polytopes, attached face-to-face, and which cannot be realized over $\mathbb{Q}$. In particular, in contrast with a single polytope, one cannot perturb (in unison) the vertices of the polytopes to make them all rational. Our construction also shows that Brehm's construction (2) presented in [Z2] can be replaced with a polyhedral complex that is (convexly) *embedded* into $\mathbb{R}^3$. We refer to Subsection 7.1 for discussion of a strongly related construction by Richter-Gebert.

The proof of Theorem 1.1 follows the same general approach as (1), going back to Perles's first original construction of an irrational polytope in $\mathbb{R}^8$ (see [G]). We start with an irrational point and line configuration in the plane, and then use polyhedral gadgets to constrain the realization space emulating the configuration. At the end, we explicitly construct an irrational arrangement of 1278 polyhedra.

Later in the paper, we use a similar approach to prove two variations on the universality theorem by Brehm [Br, Z2] and Richter-Gebert [R, §10]. Roughly speaking, we show that every algebraic equation can be encoded by a combinatorics of polyhedral complexes. Since these results are rather technical and their history is tumultuous, we postpone them until Section 5, and their discussion until Subsection 7.2.

Our second result is a variation on (3). There are of course various topological obstructions to embedding an abstract simplicial complex into $\mathbb{R}^d$. Furthermore, a *geometric* embedding is even harder to obtain, even if we assume that we start with a *topological polyhedral complex*, i.e. a complex that is already embedding into $\mathbb{R}^d$. The results in [HZ, K] (see also [AB, Ca, Wi]) rely on topological triangulations whose 1-skeletons contain a nontrivial knot with 5 or fewer edges (this creates an obstruction to a rectilinear embedding). By a much simplified variation on a construction from the proof of Theorem 1.1, we show that if one replaces "simplicial" with "polyhedral", one obtains very small examples of complexes which are not geometrically realizable, much smaller than those in [HZ, Wi].

Theorem 1.2 *There exists a topological* 3*-dim polyhedral complex* X *in* $\mathbb{R}^3$ *with* 8 *vertices and* 3 *polyhedra, that is not geometrically realizable.*

In fact, it is easy to show that this is a minimal such example, i.e. two polyhedra are not enough (see Remark 4.1. In a different direction, we may extend this polyhedral complex to a polyhedral subdivision of a ball:

Theorem 1.3 *There exists a topological* 3*-dim polyhedral complex* X' *in* $\mathbb{R}^3$ *consisting of* 9 *vertices and* 9 *polyhedra, such that* X' *is homeomorphic to a ball, and the complex* X *of Theorem 1.2 is a subcomplex of* X'*. In particular,* X' *is not geometrically realizable.*

Heuristically, both (3) and Theorem 1.2 say that one cannot possibly extend the Fáry and Tutte theorems into $\mathbb{R}^3$. To put both our results into one scheme, we have:

$$\text{topological polyhedral complex} \quad \not\Rightarrow_{\text{thm 1.2}} \quad \text{geometric polyhedral complex},$$

$$\text{geometric polyhedral complex} \quad \not\Rightarrow_{\text{thm 1.1}} \quad \text{rational polyhedral complex}.$$

Our final result in a positive result complementing Theorems 1.1 and 1.2. Our actual result in full generality is somewhat involved (see Section 6), so we state it here only for simplicial complexes.

We restrict ourselves to simplicial complexes which are homeomorphic to a ball, and which are *vertex decomposable* (see e.g. [BP, Wo]). This is a topological property that implies *shellability*. A d-dim simplicial ball X is vertex decomposable if either it is a single simplex, or recursively, it has a boundary vertex $v \in \partial X$ such that the the deletion $X \smallsetminus v$ is also a vertex decomposable d-ball. We say that X is *strongly* vertex decomposable if in addition, this vertex v is adjacent to exactly d boundary edges.

Theorem 1.4 *Let X be a topological d-dim simplicial complex in $\mathbb{R}^d$ that is homeomorphic to a ball and strongly vertex decomposable. Then there is a geometric simplicial complex Y in $\mathbb{R}^d$ such that Y is a realization of X.*

This result may seem restrictive, but for $d = 2$ it is equivalent Fáry's theorem. To see this, note first that it suffices to prove Fáry's theorem for triangulations (added edges can be removed later). But in the plane, every triangulation X is vertex decomposable [BP] (see also [FPP] for a short proof). But then, by definition, X is also strongly vertex decomposable, and thus Theorem 1.4 is just Fáry's theorem.

In the most interesting case of $d = 3$, we then extend our theorem to general polyhedral complexes with triangular interior faces and any given boundary realization (Theorem 6.1). This is a rare positive result in this direction. We use an inductive argument to construct the desired realization. We postpone the (technical) statement and the discussion of this result.

The rest of the paper is structured as follows. We begin with Definitions and Notations in Section 2. We then discuss irrational polyhedral complexes in Section 3, but move some figures to the Appendix. A short Section 4 contains proofs of theorems 1.2 and 1.3, based on the same ideas. In the next, lengthy Section 5 we present two universality theorems. Again, based on ideas in Section 3, it can be viewed as an advanced generalization of that construction; this is the only section where our exposition is not self-contained. We then turn to positive results in Section 6, proving Theorem 1.4 and a more technically involved Theorem 6.1. We conclude with historical remarks and further discussion in Section **??**.

2. Definitions and notation

Given $A \subseteq \mathbb{R}^n$, let $\mathrm{conv}(A)$ and $\mathrm{aff}(A)$ denote the convex and affine hulls of A in $\mathbb{R}^n$, respectively. Given a set $A \subseteq \mathbb{R}^n$, we write $\mathrm{int}(A)$ for the topological interior of A. If A is a manifold then we write ∂A for the manifold boundary of A. Let $B^d = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ denote the unit d-ball. If two topological spaces A, B are homeomorphic, we shall write $A \sim B$.

A *polytope* P is the convex hull of finitely many points $x_1, \ldots, x_k \in \mathbb{R}^n$. A polytope P is a *d-polytope* if $\mathrm{aff}(P)$ is a d-dimensional affine subspace of $\mathbb{R}^n$. We call a poset X a *geometric d-polyhedron in* $\mathbb{R}^n$ if X is the face poset of a d-polytope. By abuse of notation we will sometimes refer to a geometric polyhedron as a polytope.

For a geometric d-polyhedron X, an element $F \in X$ is a *face* of X. We shall call a 0-face of X a *vertex*, a 1-face an *edge*, a $(d-1)$-face a *facet*, and the d-face the *cell.* A polyhedron is called *simplicial* if each of its facets is a $(d-1)$-simplex.

A poset X, ordered by set inclusion, is a *topological d-polyhedron (in $\mathbb{R}^n$)* if there exists a geometric d-polyhedron Y in $\mathbb{R}^n$ and a poset isomorphism $\varphi : X \to Y$ such that for each $F \in X$, $F \subseteq \mathbb{R}^n$ and $F \sim \varphi(F)$. Note that every geometric polyhedron is a topological polyhedron.

A *topological d-polyhedral complex (in $\mathbb{R}^n$)* is a set $X = \bigcup_{i=1}^n X_i$, where each X_i is a topological d-polyhedron in $\mathbb{R}^n$, and such that if $A \in X_i$ and $B \in X_j$ then $A \cap B \in X_i \cap X_j$. We call X a *geometric d-polyhedral complex (in $\mathbb{R}^n$)* if each X_i is a geometric d-polyhedron. For brevity, we will write *polyhedron* instead of *topological polyhedron* and *polyhedral complex* instead of *topological polyhedral complex.*

Note that every polyhedron is a polyhedral complex. If $X = \bigcup_{i=1}^k X_i$ is a polyhedral complex in $\mathbb{R}^n$, we shall write $\mathcal{P}_X = \{X_1, \ldots, X_k\}$ and $|X| = \bigcup_{F \in X} F$. Note that $|X| \subseteq \mathbb{R}^n$.

If $F \in X$, we define $\mathcal{P}_X(F) = \{P \in \mathcal{P}_X \mid F \in P\}$. Let $\partial X = \{F \in X \mid F \subseteq \partial X\}$. Note that ∂X is also a polyhedral complex. Two polyhedral complexes are *isomorphic*, written $X \simeq Y$, if they are isomorphic as posets under inclusion.

A *(geometric) realization* of a polyhedral complex X is a geometric polyhedral complex Y such that $X \simeq Y$. We say that Y *(geometrically) realizes* X. If a polyhedral complex has a geometric realization, then we say that it is *(geometrically) realizable.*

3. An irrational polyhedral complex

In what follows we will work in real projective space $\mathbb{RP}^d$, and we extend the definitions of convex hull and affine hull appropriately. We regard $\mathbb{R}^d \subseteq \mathbb{RP}^d$ under the standard inclusion $(p_1, \ldots, p_d) \mapsto [p_1 : \ldots : p_d : 1]$. We say that distinct points $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{RP}^d$ are *collinear* if they are contained in the same line. We say that distinct projective lines $\ell_1, \ldots, \ell_n$ are *concurrent* if $\ell_1 \cap \cdots \cap \ell_n$ is non-empty. If $e_1, \ldots, e_n$ are edges of a polytope and the edge supporting lines $\mathrm{aff}(e_1), \ldots, \mathrm{aff}(e_n)$ are concurrent, we say that the edges $e_1, \ldots, e_n$ are *concurrent.*

An *(abstract) point and line configuration* $\mathcal{L} = ([n], E)$ consists of a finite set $[n] = \{1, \ldots, n\}$, together with a set of *(abstract) lines* $E = \{e_1, \ldots, e_k\}$, where each $e_i \subseteq [n]$. We require that each point is contained in at least 2 lines, and each line contains at least 3 points. A *realization* of $\mathcal{L}$ is a set of points $\Lambda = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subseteq \mathbb{RP}^d$ such that each collection $\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}\}$ of 3 or more points is collinear if and only if $\{i_1, \ldots i_k\} \subseteq e$ for some $e \in E$. A line $\ell \subset \mathbb{RP}^d$ is a *line of* Λ if $\ell \cap \Lambda = \{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}\}$ and $\{i_1, \ldots, i_k\} \in E$. A point and line configuration $\mathcal{L}$ is said to be *realizable* over a field F if there is a realization Λ of $\mathcal{L}$ such that each point of Λ has coordinates in F.

A realizable point and line configuration $\mathcal{L}$ is said to be *irrational* if it is not realizable over $\mathbb{Q}$. That is, for every realization Λ of $\mathcal{L}$ there is some point $\mathbf{p} \in \Lambda$ such that $\mathbf{p}$ has an irrational coordinate. The following 9-point configuration due to Perles is irrational.

Lemma 3.1 (Perles, [G]) *The point and line configuration depicted in Figure 1 is irrational.*

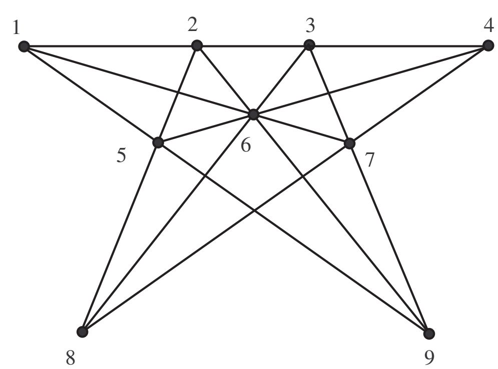

Figure 1. The 9-point Perles configuration

We say that a geometric polyhedral complex X *generates* a realization Λ of a point and line configuration $\mathcal{L}$, if each point of Λ is the intersection of affine hulls of faces of X. We say that a polyhedral complex X is *realizable* over a field F if there is a geometric realization X' of X such that each vertex of X' has coordinates in F. Note that *realizable over* $\mathbb{R}$

is equivalent to *realizable*. A geometric polyhedral complex X is called *irrational* if it is not realizable over $\mathbb{Q}$. In this section we will construct a geometric polyhedral complex X such that every realization of X generates the Perles configuration. This implies that X is irrational.

In what follows we let T denote any geometric realization of a triangular prism in $\mathbb{R}^3$. The edges of T not contained in the triangular facets of T are the *lateral* edges of T, and the facets containing these edges (i.e. the tetragonal facets) are the *lateral* facets.

Lemma 3.2 *In every geometric realization T of a triangular prism, the lateral edges of T are concurrent.*

Proof. Let ℓ_1, ℓ_2, ℓ_3 denote the supporting lines of the lateral edges of T. These lines are pairwise coplanar. Indeed, $P_{1,2} = \operatorname{aff}(\ell_1 \cup \ell_2)$, $P_{1,3} = \operatorname{aff}(\ell_1 \cup \ell_3)$ and $P_{2,3} = \operatorname{aff}(\ell_2 \cup \ell_3)$ are the supporting planes of the lateral facets of T. Since ℓ_1, ℓ_2 are coplanar projective lines, they intersect in a point $\mathbf{p}$. Thus $\{\mathbf{p}\} = \ell_1 \cap \ell_2 \subseteq P_{1,3} \cap P_{2,3} = \ell_3$. Therefore $\{\mathbf{p}\} = \ell_1 \cap \ell_2 \cap \ell_3$. $\square$

A *belt* is a polyhedral complex B consisting of triangular prisms $T_1, \ldots, T_m$ attached consecutively along their lateral facets (see Fig. 2). We introduce notation that will be used in Lemma 3.3 below. Let $B = \bigcup_{i=1}^m T_i$ be a belt. For $i = 1, \ldots, m$ and $j = 1, 2, 3$ let $\mathbf{z}_{(i,j)}$ and $\mathbf{z}_{(i,j')}$ denote the adjacent vertices of T_i contained in opposite triangular facets. For $i = 1, \ldots, m$ and $j = 1, 2$ let $F_{(i,j)}$ denote the facet containing the vertices $\mathbf{z}_{(i,1)}, \mathbf{z}_{(i,1')}, \mathbf{z}_{(i,2)}, \mathbf{z}_{(i,2')}$. Then each $F_{(i,k)}$ is a lateral facet of T_i. We assume that the prisms are attached so that $F_{(i,2)} = F_{(i+1,1)}$ for all $i = 1, \ldots, m-1$.

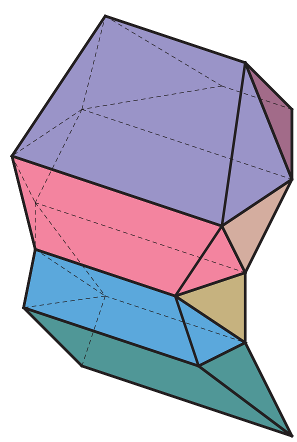

FIGURE 2. A belt of 7 prisms

When attached to other polytopes, a belt forces concurrency of the edges along which it is attached. This is ensured by the following lemma. It will be a crucial ingredient in the proofs that follow.

Lemma 3.3 *For every belt $B = \bigcup_{i=1}^m T_m$, the set of all lateral edges of all prisms T_i is a set of concurrent lines.*

Proof. Let $\ell_{(i,j)} = \operatorname{aff}(\mathbf{z}_{(i,j)}, \mathbf{z}_{(i,j')})$ denote the supporting line of the j^{th} lateral edge of T_i. For each $i = 1, \ldots, m$, by Lemma 3.2 there is some $\mathbf{p}_i \in \mathbb{R}^3$ such that $\{\mathbf{p}_i\} = \ell_{(i,1)} \cap \ell_{(i,2)} \cap \ell_{(i,3)}$.

Let $i \in \{1, \dots, m-1\}$. Then $\{\mathbf{p}_i\} = \ell_{(i,1)} \cap \ell_{(i,2)} \cap \ell_{(i,3)} = \ell_{(i,1)} \cap \ell_{(i,2)} = \ell_{(i+1,1)} \cap \ell_{(i+1,2)} = \ell_{(i+1,1)} \cap \ell_{(i+1,2)} \cap \ell_{(i+1,3)} = \{\mathbf{p}_{i+1}\}$. Thus

$$\{\mathbf{p}_1\} = \cdots = \{\mathbf{p}_m\} = \bigcap_{1 \le i \le m,\, 1 \le j \le 3} \ell_{(i,j)}.$$

□

Proof of Theorem 1.1. For $k = 1, \dots, 5$, let

$$\mathbf{x}_k = \left(\cos\left(\frac{2\pi k}{5}\right), \sin\left(\frac{2\pi k}{5}\right), 0\right) \quad \text{and} \quad \mathbf{x}'_k = \left(\cos\left(\frac{2\pi k}{5}\right), \sin\left(\frac{2\pi k}{5}\right), 2\right).$$

Note that the convex hull of the $\mathbf{x}_k$ and $\mathbf{x}'_k$ is a regular pentagonal prism, call it R. Let

$$\mathbf{a} = \left(0, 0, 1 - \frac{1}{\sqrt{5}}\right) \quad \text{and} \quad \mathbf{a}' = \left(0, 0, 1 + \frac{1}{\sqrt{5}}\right).$$

We subdivide R into 7 polytopes as follows. Let M denote the pentagonal pyramid with base vertices $\mathbf{x}_k$ and apex $\mathbf{a}$, and M' the pentagonal pyramid with base vertices $\mathbf{x}'_k$ and apex $\mathbf{a}'$. For $i = 1, \dots, 5$, let T_i denote the triangular prism whose two triangular faces F_i and F'_i consist of the vertices $\mathbf{a}, \mathbf{x}_i, \mathbf{x}_{i+1}$ and $\mathbf{a}', \mathbf{x}'_i, \mathbf{x}'_{i+1}$, respectively, where addition is modulo 5. Note that the prisms T_i all share the lateral edge $\mathrm{conv}(\mathbf{a}, \mathbf{a}')$. Now remove the "top" pentagonal pyramid M'. The result is a geometric polyhedral complex K consisting of 6 geometric polyhedra. We call K the *core* (see Fig. **??**). It forms the centerpiece of our construction.

For $i = 1, \dots, 5$, let $e_i = \mathrm{conv}(\mathbf{x}_i, \mathbf{x}_{i+1})$ denote the edges of the base of the pentagonal prism M, let $e'_i = \mathrm{conv}(\mathbf{x}'_i, \mathbf{a}')$ denote the edges on the top of K containing $\mathbf{a}'$, and let $\ell_i = \mathrm{aff}(e_i)$ and $\ell'_i = \mathrm{aff}(e'_i)$. Let $P_i = \mathrm{aff}(T_i \cap T_{i-1})$, and let $P_b = \mathrm{aff}(\mathbf{x}_1, \dots, \mathbf{x}_5)$ denote the supporting plane of the base of the pentagonal pyramid M.

We define 9 points $\mathbf{p}_1, \dots \mathbf{p}_9$ in P_b as follows. Let $\mathbf{p}_2 = \mathbf{x}_1$, $\mathbf{p}_3 = \mathbf{x}_5$, $\mathbf{p}_5 = \mathbf{x}_2$, $\mathbf{p}_7 = \mathbf{x}_4$. Define $\mathbf{p}_1, \mathbf{p}_4, \mathbf{p}_8, \mathbf{p}_9$ by

$$\{\mathbf{p}_1\} = \ell_2 \cap \ell_5, \quad \{\mathbf{p}_4\} = \ell_3 \cap \ell_5, \quad \{\mathbf{p}_8\} = \ell_1 \cap \ell_3, \quad \{\mathbf{p}_9\} = \ell_2 \cap \ell_4.$$

Finally, let $\ell_c = \mathrm{aff}(\mathbf{a}, \mathbf{a}')$, and define $\mathbf{p}_6$ by $\{\mathbf{p}_6\} = \ell_c \cap P_b$.

One can directly check that the points $\mathbf{p}_1, \dots, \mathbf{p}_9$ constitute a realization Λ of the Perles configuration (in fact the points are labeled so that $\mathbf{p}_i$ corresponds to vertex i in Figure 1). Indeed, all collinearities are satisfied (see Fig. **??**). For example, $\mathbf{p}_1, \dots, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ are all contained in the line ℓ_5 by definition, and $\mathbf{p}_3, \mathbf{p}_6, \mathbf{p}_8$ are collinear by a direct calculation. Furthermore, it is clear from the definitions that in any geometric realization of K, the collinearities $\{2,5,8\}$, $\{1,5,9\}$, $\{4,7,8\}$, $\{3,7,9\}$, $\{1,2,3,4\}$ are satisfied, since they correspond to the lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$, respectively. However, the collinearities $\{2,6,9\}$, $\{4,5,6\}$, $\{1,6,7\}$, $\{3,6,8\}$ may fail. To obtain a geometric polyhedral complex X such that these last four collinearities hold in any realization of X, we attach four belts to K, one for each collinearity.

In order for X to be a polyhedral complex, when we attach four belts to K we must ensure that they don't intersect. To achieve this, the belts we use are long thin arcs consisting of hundreds of triangular prisms. We have produced an explicit construction on the computer, of which we give an overview here. The full details and code can be found on the website listed in Appendix A.

We force the collinearity $\{2,6,9\}$ to hold in any realization by attaching a belt B_1 to K as follows. Draw two simple arcs φ_1, φ'_1, which intersect K in at most their endpoints, and such that $\varphi_1(0) = \varphi_1'(0) = \mathbf{x}_3$, $\varphi_1(1) = \mathbf{a}'$, and $\varphi_1'(1)$ is a point lying in the plane P_1, above the edge e_1' and close to but not directly above $\mathbf{a}'$. Place a large number N of points roughly equidistantly along each arc, and label these points $\mathbf{z}_1, \ldots, \mathbf{z}_N$ and $\mathbf{z}_1', \ldots, \mathbf{z}_N'$, respectively. We demand that $\mathbf{z}_1 = \mathbf{z}_1' = \mathbf{x}_3$, $\mathbf{z}_N = \mathbf{a}'$, and $\mathbf{z}_N' = \varphi_1'(1)$. These points determine a collection of triangles $\Delta_i = \operatorname{conv}(\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{z}_{i+1}')$, for $i = 1, \ldots, N-1$ and $\Delta_i' = \operatorname{conv}(\mathbf{z}_i, \mathbf{z}_i', \mathbf{z}_{i+1}')$ for $i = 2, \ldots, N-1$.

For each triangle Δ_i, we obtain a triangular prism with triangular facet Δ_i by letting the lateral edges be segments of the lines passing through $\mathbf{p}_9$ and one of $\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{z}_{i+1}'$. Similarly for the triangles Δ_i'. We demand in particular that the lateral edge containing $\mathbf{z}_1 = \mathbf{x}_3$ is the edge e_2, and that the lateral edge containing $\mathbf{z}_N$ is the edge e_1'. We are free to choose the length of the remaining lateral edges. By choosing this length to be very short for all lateral edges save for those containing points close to $\mathbf{z}_1$ and $\mathbf{z}_N$, we make it possible to attach other belts while avoiding intersections.

Concretely, let $m \in \mathbb{Z}_+$ and let $f_m : [0,1] \to [0,1]$ be defined by

$$f_m(t) = (1-t)^m + t^m.$$

Let $L(e)$ denote the length of the edge e of K, and let $g(t) = (1-t)L(e_2) + tL(e_1')$. Define a function h_m by $h_m(t) = f_m(t)g(t)$. Let E_i denote the lateral edge containing $\mathbf{z}_i$ and E_i' the lateral edge containing $\mathbf{z}_i'$. Then we choose the length of E_i to be $h_m(\frac{i-1}{N-1})$ and the length of E_i' to be $h_m(\frac{i-1}{N})$. Taking m large, we may ensure that the lengths of the lateral edges of each prism are very short except near the edges e_2 and e_1'. In our explicit construction we take $m = 80$.

The collection of the resulting triangular prisms forms a belt B_1'. Reflect B_1' across the plane P_1, and call the result B_1''. Then $B_1 = B_1' \cup B_1''$ is the desired belt. It intersects K in the three edges e_2, e_4, and e_1' (see Fig. 3). In our explicit construction we take $N = 80$, so that B_1' and B_1'' each consist of $2(80-1)+1 = 159$ prisms, for a total of $2(159) = 318$ prisms in the belt B_1.

We now show that in any realization of the geometric polyhedral complex $K \cup B_1$, the collinearity $\{2, 6, 9\}$ is satisfied. Let Z denote any geometric realization of $K \cup B$. Let $\overline{\mathbf{p}_i}$, $\overline{\ell_i}$, $\overline{\ell_i'}$ and $\overline{P_1}, \overline{P_b}$ denote the points, lines, and planes of Z corresponding to $\mathbf{p}_i$, ℓ_i, ℓ_i', and P_1, P_b. Since the belt B is attached to K along the three edges e_2, e_4, and e_1', the lines $\overline{\ell_2}$, $\overline{\ell_4}$, and $\overline{\ell_1'}$ must be concurrent by Lemma 3.3, and their point of intersection is $\{\overline{\mathbf{p}_9}\} = \overline{\ell_2} \cap \overline{\ell_4}$. So in particular, $\overline{\mathbf{p}_9} \in \overline{\ell_1'}$. Clearly $\overline{\mathbf{p}_9} \in \overline{P_b}$ and $\overline{\ell_1'} \subseteq \overline{P_1}$, so we have $\overline{\mathbf{p}_9} \in \overline{\ell_1'} \cap \overline{P_b} \subseteq \overline{P_1} \cap \overline{P_b}$. But $\overline{P_1} \cap \overline{P_b}$ is a line containing $\overline{\mathbf{p}_2}$ and $\overline{\mathbf{p}_6}$. Thus $\overline{\mathbf{p}_2}$, $\overline{\mathbf{p}_6}$, $\overline{\mathbf{p}_9}$ a are collinear.

We may force the remaining three collinearities to hold in any realization by attaching three more belts to K. The construction of these remaining belts is analogous to the construction of B_1. In particular, attaching a belt B_2 to the edges e_3, e_5, and e_2' forces the collinearity $\{4, 5, 6\}$, attaching a belt B_3 to the edges e_5, e_2, and e_4' forces the collinearity $\{1, 6, 7\}$, and attaching a belt B_4 to the edges e_1, e_3, and e_5' forces the collinearity $\{3, 6, 8\}$. Let X denote the resulting geometric polyhedral complex with all 4 belts attached (see Fig. 3). Choosing the arcs which define these belts to curve in the appropriate way, we may ensure that the belts do not intersect. In our explicit construction, each of these four belts consists of 318 triangular prisms. There are 5 triangular prisms in K, for a total of $4(318) + 5 = 1277$ triangular prisms in X. Together with the pentagonal pyramid M, we have a grand total of 1278 polytopes comprising X. For more pictures and further details see Appendix A.

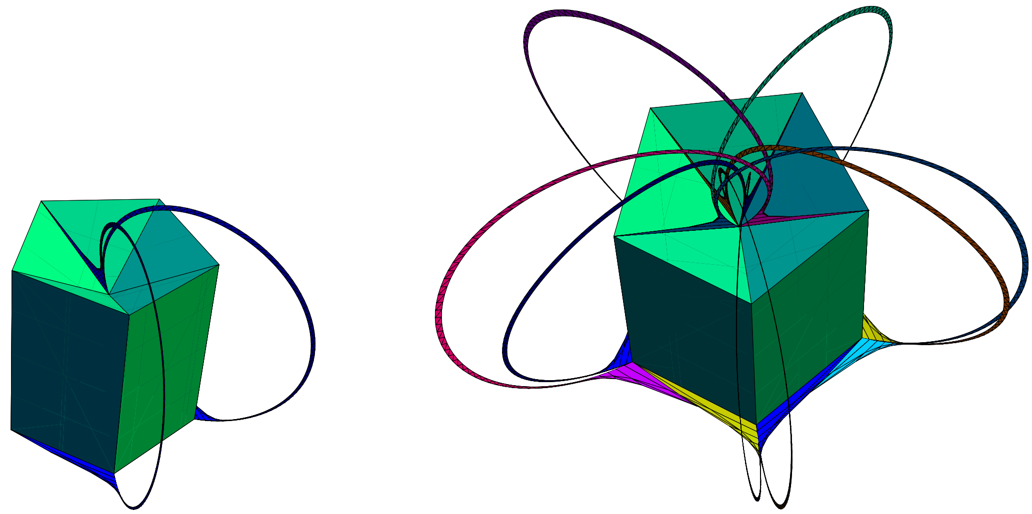

FIGURE 3. Left: Attaching one belt in the construction of Theorem 1.1. Right: The complete irrational complex with all four belts attached.

Thus in every geometric realization Y of X, the points corresponding to $\mathbf{p}_1, \ldots, \mathbf{p}_9$ form a realization Λ_P of $\mathcal{L}_P$, which is irrational by Lemma 1. Thus Y must have an irrational vertex coordinate. Otherwise, the supporting planes of Y would be the defined by equations with rational coefficients, whence the points of Λ_P would be the solutions of systems of linear equations with rational coefficients, hence rational, a contradiction. □

4. AN UNREALIZABLE POLYHEDRAL COMPLEX

Proof of Theorem 1.2. Let A be a 3-simplex in $\mathbb{R}^3$, with vertices labeled $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$. Let $e_{i,j} = \operatorname{conv}(\mathbf{x}_i, \mathbf{x}_j)$, $i \neq j$, denote the edges of A. Attach a topological belt B consisting of 2 triangular prisms to the two edges $\mathbf{x}_1\mathbf{x}_2$ and $\mathbf{x}_3\mathbf{x}_4$ as shown in Figure 4. Call the resulting topological polyhedral configuration X.

Suppose that X has a geometric realization Y, with vertices $\mathbf{y}_i$ corresponding to $\mathbf{x}_i$. Since the edges $e_{1,2}$ and $e_{3,4}$ both belong to the belt B, the corresponding edges of Y must be concurrent by Lemma 3.3. But then the vertices $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ are all coplanar. Hence they do not determine a 3-simplex, a contradiction. □

Proof of Theorem 1.3. Let X denote the polyhedral complex of Theorem 1.2. As shown in Figure 4, the vertices $\mathbf{x}_1, \mathbf{x}_3, \mathbf{a}_1$ and the edges between them determine a topological 2-simplex, call it Δ_1. Similarly, the vertices $\mathbf{x}_2, \mathbf{x}_4, \mathbf{a}_2$ and the edges between them determine a topological 2-simplex Δ_2. That is, Δ_1 and Δ_2 are topological 2-polyhedral complexes. We define $S = X \cup \{\Delta_1, \Delta_2\}$.

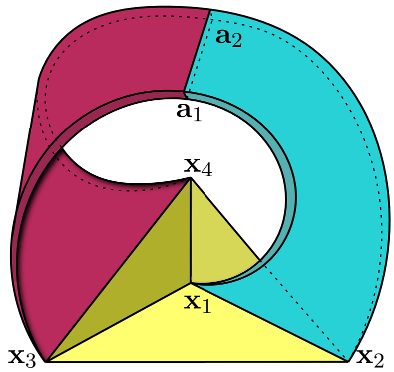

FIGURE 4. The unrealizable topological polyhedral complex of Theorem 1.2.

Note that S is not a 3-polyhedral complex by our definition, because it contains facets Δ_1 and Δ_2 which are not contained in any cell of S. However, we may create a polyhedral complex from S as follows. Let R denote the bounded component of the complement of S (i.e. R is the region surrounded by S). Let $c \in R$, and for each facet F in the boundary of R, add to S the topological cone with apex c and base F. In other words, cone from the point c. Let X' denote the resulting 3-polyhedral complex.

Then clearly X is a subcomplex of X', and $|X'| \sim B^3$. Furthermore, X' has exactly 9 vertices, 24 edges, 25 facets, and 9 cells. The cells of X' are comprised of 5 tetrahedra, 2 triangular prisms, and 2 tetragonal pyramids. $\square$

Remark 4.1 It is worth noting that the unrealizable complex of Theorem 1.2 is minimal, in the sense that any topological 3-polyhedral complex consisting of two polyhedra is geometrically realizable. To see this, let X be a topological 3-polyhedral complex consisting of two 3-polyhedra Q_1 and Q_2. If Q_1 and Q_2 share less than a 2-face, the result is immediate. So suppose that Q_1 and Q_2 share a 2-face F. Let P_1 and P_2 be polytopes isomorphic to Q_1 and Q_2, respectively. Let F_1 and F_2 denote the facets of P_1 and P_2, respectively, that correspond to F. Barnette and Grünbaum [BG] proved that the shape of one facet of a 3-polytope may be arbitrarily prescribed. Therefore we may choose P_1 and P_2 so that F_1 and F_2 are congruent. Apply an affine transformation that identifies F_1 with F_2. The result is a geometric 3-polyhedral complex isomorphic to X.

5. Universality theorems

Let us first note that the first irrational polytope result of Perles was later extended by Mnëv to a general universality theorem [Mn], and then further extended to all 4-polytopes [R]. Similarly, Brehm's result gives a universality theorem for self-intersecting 2-surfaces in $\mathbb{R}^3$ (see Subsection 7.2).

In what follows, we extend Theorem 1.1 to a similar universality result. Using belts, we can in fact mimic the constructions of Theorem 1.1 for *any* point and line configuration. In particular, for a point and line configuration $\mathcal{L}$, we can construct a geometric 3-polyhedral complex $X(\mathcal{L})$ such that every realization of $X(\mathcal{L})$ generates a realization of $\mathcal{L}$. The universality theorem for point and line configurations (see e.g. [P]) then implies, in particular, that

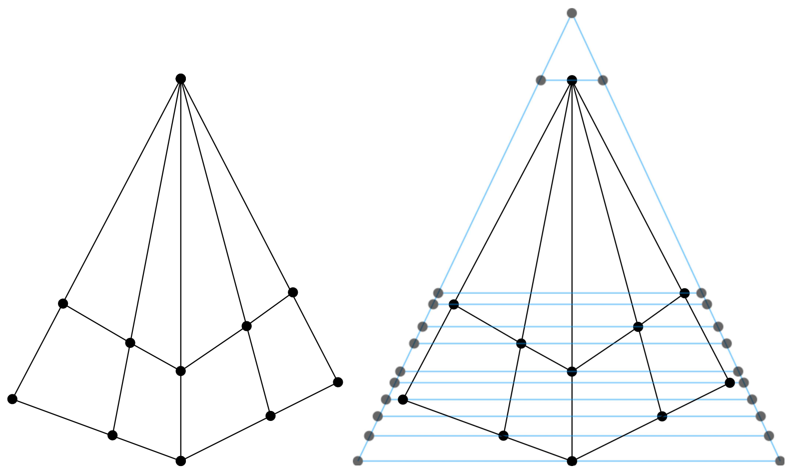

Figure 5. Left: A point and line configuration $\mathcal{L}$ that may have non-planar realizations in $\mathbb{RP}^3$. Right: The resulting planar configuration $\overline{\mathcal{L}}$ as constructed in Lemma 5.1.

for any proper subfield K of the algebraic closure of $\mathbb{Q}$, there is a geometric 3-polyhedral complex that cannot be realized with all vertex coordinates in K.

Technically, the universality theorem for point and line configurations assumes that the realizations of a configuration are restricted to the projective plane $\mathbb{RP}^2$. If we allow realizations in $\mathbb{RP}^d$ for $d > 2$, some realizations may not lie entirely in a single plane. A point and line configuration $\mathcal{L}$ is said to be *planar* if every realization of $\mathcal{L}$ in $\mathbb{RP}^d$ lies in a (projective) 2-plane. If a point and line configuration is not planar, there is a straightforward way to extend it to a planar configuration, which we describe in the following lemma.

Lemma 5.1 *Let $\mathcal{L} = ([n], E)$ be a point and line configuration. There is a point and line configuration $\overline{\mathcal{L}} = ([3n+1], \overline{E})$ such that $E \subseteq \overline{E}$, and the points of $\overline{\mathcal{L}}$ are coplanar in any realization of $\overline{\mathcal{L}}$ in $\mathbb{RP}^d$. Furthermore, a planar realization of $\overline{\mathcal{L}}$ contains a planar realization of $\mathcal{L}$.*

Proof. Let $\mathcal{L} = ([n], E)$. For each point $i \in [n]$, we add two points $w_i = n+2i-1, w_i' = n+2i$ and the line $e_i = \{i, w_i, w_i'\}$. Then we add a new point $a = 3n+1$, together with two lines $l = \{a, w_1, \ldots, w_n\}$ and $l' = \{a, w_1', \ldots, w_n'\}$. See Figure 5. Let $\overline{\mathcal{L}} = ([3n+1], \overline{E})$ denote the resulting point and line configuration. Clearly, in every realization of $\overline{\mathcal{L}}$ in $\mathbb{RP}^d$, each point $\mathbf{p}$ must lie in the plane determined by the two intersecting lines L and L' corresponding to l and l'.

Finally, let $\overline{\Lambda}$ denote a realization of $\overline{\mathcal{L}}$. Since $X \subseteq \overline{X}$ and $E \subseteq \overline{E}$, there is a subset $\Lambda \subseteq \overline{\Lambda}$ and a map $f : X \to \Lambda$ such that f is a bijection, and a collection of points $\{f(x_i)\}_{i \in I}$ is collinear if $I \subseteq E$. To see that a collection of points $\{f(x_i)\}_{i \in I}$ is collinear only if $I \subseteq E$, note that each abstract line $e_i = i, w_i, w_i'$ contains only one point of $\mathcal{L}$, namely i. Furthermore, the two lines L and L' contain no points of $\mathcal{L}$. So the lines added to E to form $\overline{E}$ do not enforce any new concurrencies among the points of $\mathcal{L}$. $\square$

We call the configuration $\overline{\mathcal{L}}$, constructed in Lemma 5.1, the *planar extension* of $\mathcal{L}$.

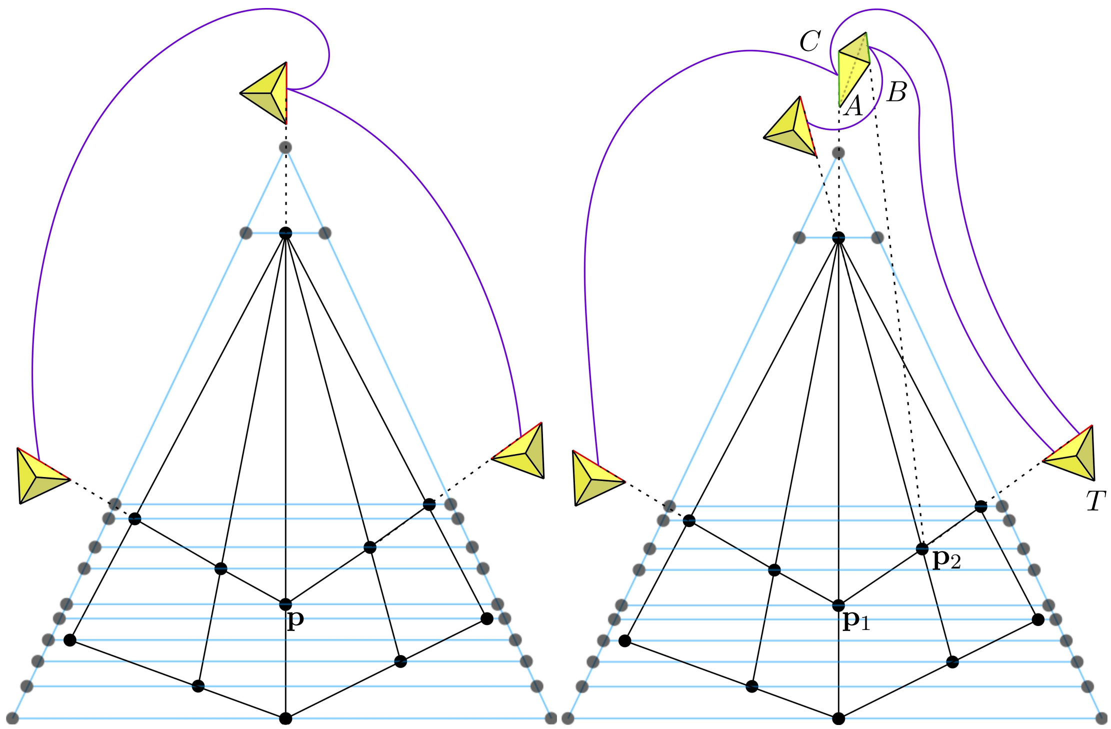

FIGURE 6. (Belts are shown schematically in purple) Left: A belt $B_{\mathbf{p}}$ which ensures that the three lines spanned by the red edges are concurrent in any realization of X. Right: Two belts B and C, attached to the tetrahedron A along the green edges, which ensure that the three lines spanned by the red edges are not concurrent in any realization of X (so that $\mathbf{p}_1 \neq \mathbf{p}_2$). Note that B and C share the red edge of the tetrahedron T, but they do not share a 2-face.

Theorem 5.2 (Weak Universality Theorem) *Let $\mathcal{L}$ be a point and line configuration, and let K be a proper subfield of the algebraic closure of $\mathbb{Q}$. There exists a geometric 3-polyhedral complex $X(\mathcal{L})$ such that if $X(\mathcal{L})$ has a realization over K then $\mathcal{L}$ has a planar realization over K. Moreover, the complex $X(\mathcal{L})$ may be constructed using only triangular prisms.*

Proof. We provide a sketch of the construction. By Lemma 5.1, we may assume that $\mathcal{L}$ is a planar configuration (if $\mathcal{L}$ is not planar, replace it with its planar extension). Let Λ denote a realization of $\mathcal{L}$ in $\mathbb{RP}^3$. For each line ℓ of Λ, place a tetrahedron with a marked edge e, such that $\ell = \mathrm{aff}(e)$.

For each point $\mathbf{p} \in \Lambda$, let $L_{\mathbf{p}}$ denote the set of lines of Λ containing $\mathbf{p}$. For each such set $L_{\mathbf{p}}$, add a belt $B_{\mathbf{p}}$ such that for each line $\ell_i \in L_{\mathbf{p}}$, the edge e_i is identified with a lateral edge of $B_{\mathbf{p}}$. See Figure 6 (left).

Finally, for each collection of 3 lines ℓ_i, ℓ_j, ℓ_k which are not concurrent in the realization Λ, place a tetrahedron A_{ijk} with vertices labeled $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$. Add a belt B_{ijk} such that each of the 3 edges e_i, e_j, and $\mathbf{x}_1\mathbf{x}_2$ is identified with a lateral edge of B_{ijk}. Add another belt C_{ijk} such that each of the 3 edges e_j, e_k, and $\mathbf{x}_3\mathbf{x}_4$ is attached along a lateral edge of C_{ijk}. See Figure 6 (right). Call the resulting geometric 3-polyhedral complex $X(\mathcal{L})$.

Let X' be a geometric realization of $X(\mathcal{L})$. The belts $B_{\mathbf{p}}$ ensure that the concurrencies present among the edges e_i of $X(\mathcal{L})$ hold among the corresponding edges e_i' of X'. The points of intersection of the edges e_i' of X' constitute a set of points Λ'. To show that Λ' is a realization of $\mathcal{L}$, it remains to show that no further concurrencies hold among the edges e_i'. That is, we wish to show that Λ' is not a *degenerate* realization of $\mathcal{L}$. But this is ensured by the tetrahedra A_{ijk} and the corresponding pairs of belts B_{ijk} and C_{ijk}.

To see this, let ℓ_i, ℓ_j, ℓ_k be distinct lines of Λ which are not concurrent, and let e_i, e_j, e_k denote the corresponding edges of $X(\mathcal{L})$. Then $X(\mathcal{L})$ contains the tetrahedron A_{ijk} and belts B_{ijk}, C_{ijk} as described above. Let $A'_{ijk}, B'_{ijk}, C'_{ijk}$ denote the corresponding prisms and belts of X', where the vertices of A'_{ijk} are labeled $\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3', \mathbf{x}_4'$. The belt B'_{ijk} ensures that the edges e_i', e_j' and $\mathbf{x}_1'\mathbf{x}_2'$ are concurrent, and the belt C'_{ijk} ensures that the edges e_j', e_k' and $\mathbf{x}_3'\mathbf{x}_4'$ are concurrent. Suppose that the edges e_i', e_j', e_k' of X' are concurrent. From the concurrencies forced by the belts, this implies that the edges $\mathbf{x}_1'\mathbf{x}_2'$ and $\mathbf{x}_3'\mathbf{x}_4'$ are concurrent. Thus the vertices $\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_2', \mathbf{x}_3'$ are coplanar, so they do not determine a tetrahedron, a contradiction.

Now let K be a proper subfield of the algebraic closure of $\mathbb{Q}$, and suppose that $X(\mathcal{L})$ is realizable over K. Let X' be a realization of $X(\mathcal{L})$ having all vertex coordinates in K, and let Λ be the realization of $\mathcal{L}$ generated by X'. Then the affine hulls of the faces of X' are defined by linear equations with coefficients in K. The points of Λ are the intersection of these affine hulls, hence they are solutions of a system of linear equations with coefficients in K. Thus the points of Λ have all coordinates in K. That is, $\mathcal{L}$ is realizable over K.

To prove the last claim of the theorem, note that each tetrahedron used in the above construction may be replaced in with a triangular prism. In fact, the tetrahedra whose marked edges generate the lines of Λ may be removed, as the edges of the attached belts suffice to define these lines. The tetrahedra A_{ijk} may be replaced with triangular prisms in the obvious way, by attaching the corresponding belts along two skew edges of the triangular prism, in the same way in which they were attached along two skew edges of the tetrahedron. $\square$

Theorem 5.2 is a *weak* universality theorem, in the sense that it does not imply that the *realization spaces* of $X(\mathcal{L})$ and $\mathcal{L}$ are *stably equivalent*. We would now like to investigate whether it is possible to obtain this latter type of result. We will find that by modifying our previous definitions slightly, we can in fact obtain a stronger and more general universality theorem. To this end, we adopt the definition of stable equivalence given in [R], and we define the realization spaces of $X(\mathcal{L})$ and $\mathcal{L}$ as follows.

For a point and line configuration $\mathcal{L} = ([n], E)$, we define the *(Euclidean) realization space* of $\mathcal{L}$ (in $\mathbb{R}^3$) to be the set

$$\mathcal{R}(\mathcal{L}) = \{(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{3n} \mid \Lambda = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \text{ is a realization of } \mathcal{L}\}.$$

In particular, we only allow realizations Λ in $\mathbb{R}^3$, rather than $\mathbb{RP}^3$ (that is, we do not allow points at infinity). This will be important for our final universality result. Notice that the coordinates of the realization space come with a particular order, induced by the natural order on $[n]$. That is, for each $i \in [n]$, if $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathcal{R}(\mathcal{L})$ then $\mathbf{p}_i$ must be the point corresponding to i. For a geometric 3-polyhedral complex X with N vertices, then the *realization space* of X is the set

$$\mathcal{R}(X) = \{(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N) \in \mathbb{R}^{3N} \mid \boldsymbol{v}_1, \ldots, \boldsymbol{v}_N \text{ are the vertices of a realization } X' \text{ of } X\}.$$

Consider the natural map $f : \mathcal{R}(X(\mathcal{L})) \to \mathcal{R}(\mathcal{L})$ that assigns to each realization X' of $X(\mathcal{L})$ the realization of $\mathcal{L}$ generated by X'. The following informal argument shows that f will not be a stable equivalence in general. Suppose that $X(\mathcal{L})$ is constructed so that its belts consist of a very large number of prisms. Let Λ be a realization of $\mathcal{L}$, and consider the fiber $\mathcal{A} = f^{-1}(\Lambda)$. Since the belts of $X(\mathcal{L})$ consist of a large number of prisms, for a given pair of belts B_1 and B_2 of $X(\mathcal{L})$, we may construct a realization $X' \in \mathcal{A}$ in which the corresponding belts are knotted, and a realization $Y' \in \mathcal{A}$ in which they are not knotted. Since we forbid the possibility that the belts B_1 and B_2 may intersect one another arbitrarily, the knot in X' is non-trivial. That is, there is no continuous path from X' to Y' in $\mathcal{R}(X(\mathcal{L})) \subset \mathbb{R}^{3N}$. Therefore $\mathcal{A}$ is not path-connected. But all fibers of a stable equivalence must be path-connected.

By modifying our definition of polyhedral complex slightly, we can eliminate the problem encountered in the previous paragraph. The idea is to preserve the face identifications in the complex, but allow polytopes to self-intersect (so in particular we will allow belts to intersect one another arbitrarily).

We define a *geometric d-polyhedral arrangement* $\mathcal{X} = (X, A)$ to consist of a set $X = \bigcup_{i=1}^n X_i$, where each X_i is a (face lattice of a) polytope, together with a set $A \subset X$ of *distinguished common faces*, such that any face $F \in A$ belongs to at least two polytopes, and any two polytopes have at most one common face that is contained in A. That is, if $F \in A$ and $F \in X_i \cap X_j$, then $G \notin A$ for all other $G \in X_i \cap X_j$.

A geometric polyhedral arrangement $\mathcal{Y} = (Y, B)$ is a *realization* of $\mathcal{X} = (X, A)$ if there is a bijection $f : X \to Y$ such that $X_i \simeq f(X_i)$, and the face poset isomorphisms $g_i : X_i \to f(X_i)$ satisfy

$$F \in A \text{ if and only if } g_i(F) \in B.$$

The realization space $\mathcal{R}(\mathcal{X})$ of a polyhedral arrangement $\mathcal{X}$ is defined in the obvious way.

Note that any two polytopes in $\mathcal{X}$ may *intersect* in more than a common face of both, but they can only have one common face *distinguished* by membership in A. That is, only the common face $F \in A$ is required to be a common face of both polytopes in every realization, although the intersection of the polytopes may consist of much more.

Given a geometric d-polyhedral complex $X = \bigcup_{i=1}^n X_i$, we may construct a corresponding geometric d-polyhedral arrangement $\mathcal{X}$ by taking

$$A = \{F \in X \mid F \in X_i \cap X_j \text{ for some } i \neq j\}$$

and $\mathcal{X} = (X, A)$. The only difference between $\mathcal{X}$ and X is that in realizations of $\mathcal{X}$, we allow the polytopes to self-intersect arbitrarily. However, the intersections corresponding to the faces in A are required to hold in all realizations of $\mathcal{X}$. For this reason, Theorem 5.2 holds if we replace “polyhedral complex” with “polyhedral arrangement”, and the proof is identical. With this understanding, we may prove the desired universality results.

Theorem 5.3 *Let $\mathcal{L}$ be a point and line configuration. Then there is a polyhedral arrangement $\mathcal{X}(\mathcal{L})$ such that $\mathcal{R}(\mathcal{X}(\mathcal{L}))$ is stably equivalent to $\mathcal{R}(\mathcal{L})$.*

Proof. Let $\mathcal{L}$ be a point and line configuration, with realization $\Lambda \subset \mathbb{R}^3$. Let $Z(\mathcal{L})$ denote the corresponding geometric 3-polyhedral complex constructed from Λ as in Theorem 5.2. Let $\mathcal{Z}(\mathcal{L})$ be the polyhedral arrangement corresponding to $Z(\mathcal{L})$. We begin by constructing a polyhedral arrangement $\mathcal{X} = \mathcal{X}(\mathcal{L})$ by adding additional polytopes to $\mathcal{Z}(\mathcal{L})$. The purpose of adding these polytopes is simply to force the realizations of $\mathcal{L}$ generated by realizations of $\mathcal{X}$ to lie in $\mathbb{R}^3$ (rather than $\mathbb{RP}^3$), hence in $\mathcal{R}(\mathcal{L})$.

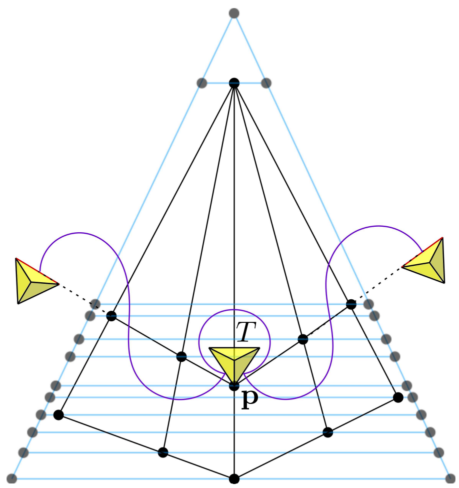

FIGURE 7. A belt, shown schematically in purple, attached to the four indicated edges of the tetrahedra. This belt forces the point $\mathbf{p}$ to be a vertex of the tetrahedron T in every realization of $\mathcal{X}$.

Let $\mathbf{p}_i \in \Lambda$, and place a tetrahedron (or triangular prism) T_i such that $\mathbf{p}_i$ is a vertex of T_i. Let e_1 and e_2 denote two of the edges of T_i containing $\mathbf{p}_i$. Let B denote a belt of $\mathcal{Z}(\mathcal{L})$ whose lateral edges are concurrent at $\mathbf{p}_i$. Let e_3 and e_4 denote two of the lateral edges of B. Now construct a new belt C_i that is attached along four of its lateral edges to the four edges e_1, e_2, e_3, e_4. See Figure 7. Adding T_i and C_i to $\mathcal{Z}(\mathcal{L})$ for each $i \in [n]$ yields the polyhedral arrangement $\mathcal{X} = \mathcal{X}(\mathcal{L})$.

Consider a realization $\mathcal{X}'$ of $\mathcal{X}$, and let Λ' be the realization of Λ generated by $\mathcal{X}'$. For a point $\mathbf{p}' \in \Lambda'$, let e_1', e_2', e_3', e_4' be the edges corresponding to e_1, e_2, e_3, e_4 in the above construction. Since they are the lateral edges of a belt, these edges will be concurrent, at $\mathbf{p}'$. Since the point of concurrency of e_1' and e_2' is a vertex of $\mathcal{X}'$, we have that $\mathbf{p}'$ is a vertex of $\mathcal{X}'$. Thus, for any realization $\mathcal{X}'$ of $\mathcal{X}$, the realization Λ' of $\mathcal{L}$ generated by $\mathcal{X}'$ consists entirely of vertices of $\mathcal{X}'$. In particular, Λ' does not contain points at infinity.

We will now define a map $F : \mathcal{R}(\mathcal{X}) \to \mathcal{R}(\mathcal{L})$ such that F assigns to each realization $\mathcal{X}'$ of $\mathcal{X}$ the corresponding realization of $\mathcal{L}$ generated by $\mathcal{X}'$. We show that F is a stable equivalence, by showing that it is the composition of stable projections and rational homeomorphisms.

Let n denote the number of points of $\mathcal{L}$ and N the number of vertices of $\mathcal{X}$. Then we have $\mathcal{R}(\mathcal{L}) \subset \mathbb{R}^{3n}$, and $\mathcal{R}(\mathcal{X}) \subset \mathbb{R}^{3N}$. From the construction of $X(\mathcal{L})$ in Theorem 5.2, we see that each point i of $\mathcal{L}$ corresponds to a belt B_i of $\mathcal{X}$. In particular, we choose B_i to be one of the belts such that the lateral edges of B_i are concurrent at the point $\mathbf{p}_i \in \Lambda$.

For each belt B_i, fix an ordering of the its vertices $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \ldots, \boldsymbol{v}_{is_i}$, such that the first four vertices are the (cyclically ordered) vertices of a lateral facet f_i of B_i, and such that $\boldsymbol{v}_{i1}\boldsymbol{v}_{i2}$ and $\boldsymbol{v}_{i3}\boldsymbol{v}_{i4}$ are the two lateral edges of f_i. Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r$ denote the remaining vertices of $\mathcal{X}$. Let $f_1 : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ be a map that permutes the coordinates, in such a way that

$$f_1(\mathbf{x}) = (\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{14}, \mathbf{x}_{21}, \mathbf{x}_{22}, \mathbf{x}_{23}, \mathbf{x}_{24}, \ldots, \mathbf{x}_{n1}, \mathbf{x}_{n2}, \mathbf{x}_{n3}, \mathbf{x}_{n4} \mid \mathbf{x}_{15}, \ldots, \mathbf{x}_{ns_n}, \mathbf{y}_1, \ldots, \mathbf{y}_r),$$

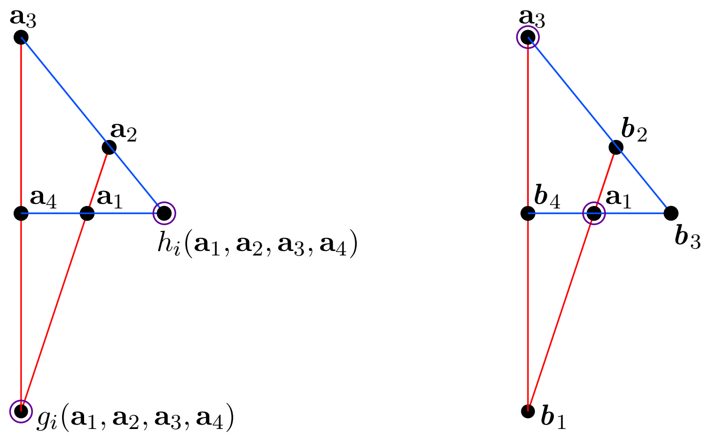

FIGURE 8. The map s_i (left) and its inverse (right). Encircled points indicate points obtained as the intersection of a red and blue line.

where each $\mathbf{x}_{ij}, \mathbf{y}_i \in \mathbb{R}^3$, and $\mathcal{X}$ is obtained by letting $\mathbf{x}_{ij}$ and $\mathbf{y}_i$ take on the values $\boldsymbol{v}_{ij}$ and $\boldsymbol{w}_i$, respectively. That is, each quadruple $\mathbf{x}_{i1}, \mathbf{x}_{i2}, \mathbf{x}_{i3}, \mathbf{x}_{i4}$ appears first in the ordering of the coordinates determined by f_1. We may choose any one of the many such maps f_1, since we do not care how the remaining coordinates (to the right of the bar) are permuted. Let $f_2 : \mathbb{R}^{3N} \to \mathbb{R}^{3(4n)}$ denote the standard projection onto the first $4n$ triples of coordinates.

Consider a realization $\mathcal{X}'$ of $\mathcal{X}$, and let $\Lambda = \{\mathbf{p}_1, \dots, \mathbf{p}_n\} \subseteq \mathbb{R}^3$ be the realization of $\mathcal{L}$ generated by $\mathcal{X}'$. Note that each point $\mathbf{p}_i$ is the point of intersection of the lateral edges of the belt B_i' of $\mathcal{X}'$ corresponding to the belt B_i of $\mathcal{X}$. Let $\boldsymbol{v}_{ij}'$ and $\boldsymbol{w}_i'$ denote the corresponding vertices of $\mathcal{X}'$

Note that $\boldsymbol{v}_{i2}'\boldsymbol{v}_{i3}'$ and $\boldsymbol{v}_{i1}'\boldsymbol{v}_{i4}'$ are the non-lateral edges of f_i. Let $\ell_1 = \operatorname{aff}(\boldsymbol{v}_{i1}', \boldsymbol{v}_{i2}')$, $\ell_2 = \operatorname{aff}(\boldsymbol{v}_{i3}', \boldsymbol{v}_{i4}')$, $\ell_3 = \operatorname{aff}(\boldsymbol{v}_{i2}', \boldsymbol{v}_{i3}')$, $\ell_4 = \operatorname{aff}(\boldsymbol{v}_{i1}', \boldsymbol{v}_{i4}')$ denote the lines spanned by these edges. By definition of B_i, the lines ℓ_1 and ℓ_2 intersect at $\mathbf{p}_i$. Thus the coordinates of $\mathbf{p}_i$ can be solved for in terms of those of the $\boldsymbol{v}_{ij}$. That is, $\mathbf{p}_i = g_i(\boldsymbol{v}_{i1}', \boldsymbol{v}_{i2}', \boldsymbol{v}_{i3}', \boldsymbol{v}_{i4}')$ for some rational function g_i with coefficients in $\mathbb{Q}$. Similarly, the lines ℓ_3 and ℓ_4 intersect in a point $\boldsymbol{q}_i$, so $\boldsymbol{q}_i = h_i(\boldsymbol{v}_{i1}', \boldsymbol{v}_{i2}', \boldsymbol{v}_{i3}', \boldsymbol{v}_{i4}')$ for some rational function h_i with coefficients in $\mathbb{Q}$.

Let $A \subset \mathbb{R}^{3(4)}$ denote the space of coplanar 4-tuples $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, $\mathbf{x}_j \in \mathbb{R}^3$, such that no three are collinear. For each i, we define a map $s_i : A \to A$ by

$$s_i(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \mathbf{x}_{i3}, \mathbf{x}_{i4}) = (g_i(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \mathbf{x}_{i3}, \mathbf{x}_{i4}), \mathbf{x}_{i2}, h_i(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \mathbf{x}_{i3}, \mathbf{x}_4), \mathbf{x}_{i4}).$$

From the definitions of g_i and h_i, we see that given a point in $\boldsymbol{b} = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4) \in A$, we may reconstruct the unique point $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \in A$ for which $s_i(\mathbf{a}) = \boldsymbol{b}$. Namely, $\mathbf{a}_2 = \boldsymbol{b}_2$, $\mathbf{a}_4 = \boldsymbol{b}_4$, $\mathbf{a}_1$ is the intersection of the lines $\operatorname{aff}(\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $\operatorname{aff}(\boldsymbol{b}_3, \boldsymbol{b}_4)$, and $\mathbf{a}_3$ is the intersection of the lines $\operatorname{aff}(\boldsymbol{b}_1, \boldsymbol{b}_4)$ and $\operatorname{aff}(\boldsymbol{b}_2, \boldsymbol{b}_3)$. See Figure 8. Thus s_i is a bijection, and clearly s_i and s_i^{-1} are continuous. Hence s_i is a homeomorphism. Since g_i and h_i are rational functions with coefficients in $\mathbb{Q}$, s_i is a rational homeomorphism.

It follows that the map $f_3 : A^n \to A^n$ defined by

$$f_3(\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{14}, \dots, \mathbf{x}_{n1}, \mathbf{x}_{n2}, \mathbf{x}_{n3}, \mathbf{x}_{n4}) = (s_1(\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{14}), \dots, s_n(\mathbf{x}_{n1}, \mathbf{x}_{n2}, \mathbf{x}_{n3}, \mathbf{x}_{n4}))$$

Is a rational homeomorphism. Let $f_4 : \mathbb{R}^{3(4n)} \to \mathbb{R}^{3(4n)}$ be the coordinate permutation defined by

$$f_4(\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{14}, \dots, \mathbf{x}_{n1}, \mathbf{x}_{n2}, \mathbf{x}_{n3}, \mathbf{x}_{n4}) = (\mathbf{x}_{11}, \dots, \mathbf{x}_{n1}, \mathbf{x}_{12}, \dots, \mathbf{x}_{n2}, \mathbf{x}_{13}, \dots, \mathbf{x}_{n3}, \mathbf{x}_{14}, \dots, \mathbf{x}_{n4}).$$

Let $f_5 : \mathbb{R}^{3(4n)} \to \mathbb{R}^{3n}$ denote the standard projection onto the first n triples of coordinates. Then note that

$$f_5 \circ f_4 \circ f_3(\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{14}, \ldots, \mathbf{x}_{n1}, \mathbf{x}_{n2}, \mathbf{x}_{n3}, \mathbf{x}_{n4}) = (g_1(\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{14}), \ldots, g_n(\mathbf{x}_{n1}, \mathbf{x}_{n2}, \mathbf{x}_{n3}, \mathbf{x}_{n4})),$$

so in particular

$$f_5 \circ f_4 \circ f_3(\boldsymbol{v}'_{11}, \boldsymbol{v}'_{12}, \boldsymbol{v}'_{13}, \boldsymbol{v}'_{14}, \ldots, \boldsymbol{v}'_{n1}, \boldsymbol{v}'_{n2}, \boldsymbol{v}'_{n3}, \boldsymbol{v}'_{n4}) = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n).$$

Let f'_1 denote the restriction of f_1 to $\mathcal{R}(\mathcal{X})$, and recursively, for $i > 1$ let f'_i denote the restriction of f_i to the range of f_{i-1}. We define $F = f'_5 \circ f'_4 \circ f'_3 \circ f'_2 \circ f'_1$. Then from the definition of the functions f'_i we see that $F : \mathcal{R}(\mathcal{X}) \to \mathcal{R}(\mathcal{L})$ and F is a surjection. We demonstrated above that f_3 is a rational homeomorphism, and the functions f_1 and f_4 are simply permutations of the coordinates, hence rational homeomorphisms. It follows that f'_1, f'_3, f'_4 are rational homeomorphisms.

Furthermore, note that the functions f_2 and f_4 are standard coordinate projections. Since realizations of $\mathcal{X}$ allow for self-intersection of polytopes, the only constraints on $\mathcal{R}(\mathcal{X})$ are those which require each vertex to be contained in the appropriate polytopes. Thus each fiber of f'_2 is the intersection of half spaces (in particular, no unions are taken over half spaces), hence each fiber is a convex polyhedron. Similarly for f'_4. It follows that f'_2 and f'_4 are stable projections. Thus F is a stable equivalence. $\square$

While the above result shows that the realization space of a polyhedral arrangement is stably equivalent to the underlying point and line configuration, we would like a stronger result, which states that realization spaces of polyhedral arrangements can be stably equivalent to arbitrary *semialgebraic* sets. For this, we will need to add an additional structure to our point and line configurations, to obtain an *oriented matroid*. We provide an equivalent definition of an oriented matroid, which will prove convenient for our purposes.

A line $L \subset \mathbb{R}^d$ has two possible orientations, each of which induces a linear order on the points $\mathbf{x} \in L$. Let $\mathcal{L} = (X, E)$ be a *planar* point and line configuration, such that every two abstract lines $e_1, e_2 \in E$ share a point of X (that is, $e_1 \cap e_2 \neq \emptyset$). Let $\Lambda \subset \mathbb{R}^d$ be a realization of $\mathcal{L}$. For each $e \in E$, let L be the line of Λ realizing e, and choose an orientation γ of L. Write $e = \{i_1, \ldots, i_k\}$, where $(\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \cdots \mathbf{p}_{i_k})$ is the order of the $\mathbf{p}_j \in L$ induced by γ. We define the *oriented line* e' to be the ordered tuple $(i_1, \ldots, i_k)$. Let E' denote the resulting set of oriented lines e'. Then $\mathcal{M} = (X, E')$ is an *oriented matroid*. One can readily check that this definition is equivalent to those given elsewhere (see e.g. [R]). In particular, the fact that we require every two lines of $\mathcal{M}$ to intersect in a point of $\mathcal{M}$ means that all realizations of $\mathcal{M}$ (which will agree on the order of the points) will agree on the set of half planes in which a given point lies.

If $\mathcal{M} = (X, E')$ is an oriented matroid, let E be the set obtained by replacing each tuple $(i_1, \ldots, i_k) \in E'$ with the set $\{i_1, \ldots, i_k\}$. Then we say that $\mathcal{L}(\mathcal{M}) = (X, E)$ is the point and line configuration corresponding to $\mathcal{M}$. A *realization* of $\mathcal{M}$ is a set $\Lambda = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^d$ such that Λ is a realization of $\mathcal{L}(\mathcal{M})$, and such that if L is a line of Λ corresponding to $e' = (i_1, \ldots, i_k)$, then there is an orientation γ of L such that $(\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \cdots \mathbf{p}_{i_k})$ is the order of the $\mathbf{p}_j \in L$ induced by γ. In other words, a realization of $\mathcal{M}$ is a realization of the underlying point and line configuration in which the points of each line have a prescribed order, up to reversing the orientation of the line.

Theorem 5.4 *Let $\mathcal{M}$ be an oriented matroid. Then there is a polyhedral arrangement $\mathcal{Y}(\mathcal{M})$ such that $\mathcal{R}(\mathcal{Y}(\mathcal{M}))$ is stably equivalent to $\mathcal{R}(\mathcal{M})$.*

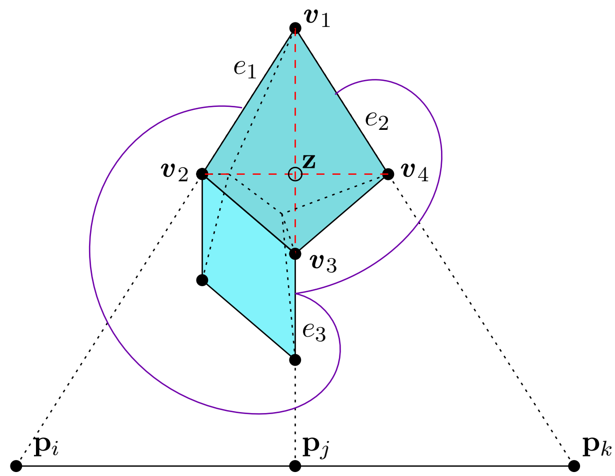

FIGURE 9. The arrangement I_{ijk}, with the two prisms T_1 and T_2 shown in blue and the belt H_{ijk} shown schematically in purple. The diagonals of the facet f of T_1 are shown in red.

Proof. Let $\mathcal{M}$ be an oriented matroid, and let $\Lambda \subset \mathbb{R}^3$ be a realization of $\mathcal{L}(\mathcal{M})$. Let $\mathcal{X}(\mathcal{L}(\mathcal{M}))$ be the polyhedral arrangement constructed from Λ as in Theorem 5.3. We write $\mathcal{X} = \mathcal{X}(\mathcal{L}(\mathcal{M}))$. We introduce a new polyhedral gadget which, when added to $\mathcal{X}$, will yield a polyhedral arrangement $\mathcal{Y} = \mathcal{Y}(\mathcal{M})$ such that every realization of $\mathcal{Y}$ generates a realization of $\mathcal{M}$. That is, every realization of $\mathcal{Y}$ generates a realization of $\mathcal{L}(\mathcal{M})$ in which the points occur on each line in the order prescribed by $\mathcal{M}$. The proof of stable equivalence is then identical to the proof of Theorem 5.3.

Let $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ denote three points of Λ which appear consecutively on a line L of Λ. We construct a belt G_{ijk} consisting of two triangular prisms, call them T_1 and T_2, as follows. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ denote the vertices of a lateral facet f of T_1 labeled cyclically, where $e_1 = \boldsymbol{v}_1\boldsymbol{v}_4$ and $e_2 = \boldsymbol{v}_2\boldsymbol{v}_3$ are the lateral edges of f. We may choose the vertices $\mathbf{x}_i$ so that if $\ell_1 = \operatorname{aff}(\boldsymbol{v}_1, \boldsymbol{v}_2)$, $\ell_2 = \operatorname{aff}(\boldsymbol{v}_1, \boldsymbol{v}_3)$, and $\ell_3 = \operatorname{aff}(\boldsymbol{v}_1, \boldsymbol{v}_4)$, then $\mathbf{p}_i \in \ell_1$, $\mathbf{p}_j \in \ell_2$, and $\mathbf{p}_k \in \ell_3$. Note that ℓ_2 is the line spanned by the diagonal $\boldsymbol{v}_1\boldsymbol{v}_3$ of the facet f. Let $\mathbf{z}$ denote the point of intersection of the diagonals $\boldsymbol{v}_1\boldsymbol{v}_3$ and $\boldsymbol{v}_2\boldsymbol{v}_4$. We attach T_2 to T_1 along the unique lateral facet g of T_1 that contains the edge e_2 and does not contain e_1. Let e_3 denote the non-lateral edge of T_2 containing the vertex $\boldsymbol{v}_3$, and not contained in g. We may choose the vertices of T_2 such that $e_3 \subset \ell_2$.

Attach a belt H_{ijk} to G_{ijk} along the edges e_1, e_2, e_3 of G_{ijk}. Call the resulting arrangement I_{ijk} (see Figure 9). Then in every realization of I_{ijk}, the edges e_1, e_2, e_3 will be concurrent. Thus the vertex $\boldsymbol{v}_1$ will be contained in $\operatorname{aff}(e_3)$. Since $\boldsymbol{v}_3 \in e_3$ by construction, this implies that $\ell_2 = \operatorname{aff}(e_3)$. That is, the diagonal $\boldsymbol{v}_1\boldsymbol{v}_3$ will be collinear with the line spanned by e_3. If B_a denotes a belt of $\mathcal{X}$ with all lateral edges concurrent at $\mathbf{p}_a$, then attach a lateral edge of the belt B_i to the edge e_1 of I_{ijk}, a lateral edge of B_k to e_2, and a lateral edge of B_j to e_3. Let $\mathcal{Y}$ denote the arrangement obtained by adding I_{ijk} to $\mathcal{X}$ for each such consecutive collinear triple of points $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$, and performing the belt attachments just described.

Now suppose $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ are three consecutive collinear points of Λ, contained in a line L of Λ. Let $\mathcal{Y}'$ be a realization of $\mathcal{Y}$, and for each point, edge, line, or facet a of $\mathcal{Y}$, let a' denote the corresponding object of $\mathcal{Y}'$. Then $\mathbf{z}'$ must lie between $\boldsymbol{v}_2'$ and $\boldsymbol{v}_4'$ on the line $\operatorname{aff}(\boldsymbol{v}_2', \boldsymbol{v}_4')$, since $\mathbf{z}'$ is the intersection of the diagonals $\boldsymbol{v}_1'\boldsymbol{v}_3'$ and $\boldsymbol{v}_2'\boldsymbol{v}_4'$ of f', and f' is convex. From the attachment of the belts B_i, B_j, B_k to I_{ijk}, we see that $\mathbf{p}_i \in \ell'_1$, $\mathbf{p}_j \in \operatorname{aff}(e'_3) = \ell'_2$, and $\mathbf{p}_k \in \ell'_3$. Thus the point $\mathbf{p}'_j$ must lie between the points $\mathbf{p}'_i$ and $\mathbf{p}'_k$ on the line L' containing them.

It follows that in every realization $\mathcal{Y}'$ of $\mathcal{Y}$, the realization Λ' of $\mathcal{L}(\mathcal{M})$ generated by $\mathcal{Y}'$ has the property that all points occur along each line in the order prescribed by $\mathcal{M}$. That is, Λ' is a realization of $\mathcal{M}$. $\square$

From Theorem 5.4, together with the universality theorem for oriented matroids (see [R]), we obtain the following universality theorem for polyhedral arrangements.

Corollary 5.5 (Strong Universality Theorem) *Let $V \subseteq \mathbb{R}^m$ be a basic primary semialgebraic set, defined over $\mathbb{Z}$. Then there is a polyhedral arrangement $\mathcal{X}$ such that $\mathcal{R}(\mathcal{X})$ is stably equivalent to V.*

6. Geometrically realizing a class of polyhedral complexes

In this section we prove two positive results. The first result is Theorem 1.4, which tells us that we may geometrically realize a certain class of simplicial complexes in arbitrary dimension. The second is an analogous result for general polyhedral complexes, and holds only for $d \leq 3$. Before we present the statement of these theorems and their proofs we will need to introduce some relevant definitions. Some of these definitions (such as *star*, *link*, and vertex decomposable) are standard, while others (such as *strongly vertex decomposable* and *vertex truncatable*) are not.

6.1. **Strongly vertex decomposable simplicial complexes.** Let X be a polyhedral complex and let F be a face of X. The *star* of F in X is the set $\operatorname{st}_X(F) = \{A \in X \mid F \subseteq A\}$. Let $\operatorname{cst}_X(F) = \{A \in X \mid A \subseteq B \in \operatorname{st}_X(F)\}$ denote the *closed star* of F. Note that $\operatorname{cst}_X(F)$ is a polyhedral complex, while $\operatorname{st}_X(F)$ may not be (since it may not be closed under taking subfaces). If $F \in X$, let $X \smallsetminus F = X - \operatorname{st}_X(F)$ denote the *deletion* of F from X, which is clearly a polyhedral complex. The *link* of F in X is the polyhedral complex $\operatorname{lk}_X(F) = \operatorname{cst}_X(F) \smallsetminus F$.

Let X be a topological d-polyhedral complex such that $|X| \sim B^d$, and let v be a boundary vertex of X. We say that v is *boundary minimal* if v is contained in exactly d boundary facets, each of which is a $(d-1)$-simplex. We call v a *shedding vertex* of X if $|X \smallsetminus v| \sim B^d$, and a *strong shedding vertex* if in addition v is boundary minimal. We say that X is *strongly vertex decomposable* if either X is a single polyhedron, or recursively, X has a strong shedding vertex v such that $X \smallsetminus v$ is strongly vertex decomposable. A boundary vertex w of X is a *solitary vertex* if there is exactly one polyhedron $P_w \in \mathcal{P}_X$ such that $w \in P_w$, and a *strong solitary vertex* if in addition w it is boundary minimal.

If we restrict ourselves to *simplicial* complexes, then we find that in any dimension d, a strongly vertex decomposable simplicial d-ball is always geometrically realizable. This is the statement of Theorem 1.4, and the proof is straightforward.

Proof of Theorem 1.4. Let X denote a d-simplicial complex, such that $|X| \sim B^d$ and X is strongly vertex decomposable and. We proceed by induction on n, the number of vertices of X. If $n = d+1$, then X consists of a single d-simplex, which is obviously realizable. If $n > d+1$, let v be a strong shedding vertex of X, and let $X' = X \smallsetminus v$. Then by definition X' is a strongly vertex decomposable d-ball. Since X' has one fewer vertex than X, by the induction hypothesis X' has a geometric realization Y'. Since v is a strong shedding vertex, it is adjacent to exactly d boundary vertices of X, call them $w_1, \ldots w_d$. Then $\operatorname{lk}_{\partial X}(v)$ (the link of v in the complex ∂X) has exactly d faces of maximal dimension, call them $F_1, \dots, F_d$, each of which is a $(d-2)$-face of X. Each face F_i is contained in exactly one facet τ_i of X'. Let $H_i = \operatorname{aff}(\tau_i)$ and let $H' = \operatorname{aff}(w_1, \dots, w_d)$.

Since d hyperplanes in $\mathbb{RP}^d$ always intersect in a point, we may let x denote the point of intersection of the hyperplanes H_i. If x is a point at infinity, or x lies on the same side of the hyperplane H' as $|Y'|$, then apply a projective transformation so that x is a finite point and x and $|Y'|$ lie on different sides of H'. Now let u be a point contained in $\operatorname{conv}(Y' \cup v)$ such that u is very close to v. Add straight line segments between u and all vertices of $|Y'|$ that correspond to neighbors of v in X. By taking u arbitrarily close to v, we may ensure that these added line segments intersect $|Y'|$ only in the desired vertices. These line segments, together with u and its neighbors, determine a collection of k-simplices for $2 \le k \le d$. Adding these simplices to Y' yields a geometric d-simplicial complex Y such that $X \sim Y$. □

6.2. **Vertex truncatable polyhedral complexes.** Now we consider the general case that X is a d-polyhedral complex. We will need to develop some more involved definitions in order to prove a result analogous to Theorem 1.4.

Suppose X_1 and X_2 are two d-polyhedra that share exactly one facet Q. Then let $X_1 \#_Q X_2$ denote the polyhedron obtained from $X_1 \cup X_2$ by replacing the two cells $|X_1|$ and $|X_2|$ by the single cell $|X_1| \cup |X_2|$ and removing the facet Q.

We now define a construction called *subdivision by facet*. Let X be a polyhedral complex, let v be a boundary vertex of X, and let $P \in \mathcal{P}_X(v)$. Let $\tau(v,P) \subseteq |P|$ denote a (topological) $(d-1)$-ball such that $\tau(v,P) \cap |\mathrm{lk}_{\partial P}(v)| = \partial\tau(v,P)$. In particular, the vertices of X contained in $\tau(v,P)$ are exactly the neighbors of v in ∂X. We may construct a new polyhedral complex $X \oplus \tau(v,P)$, called the *subdivision of X at v and P*, as follows. If X already contains a facet $F \subseteq P$ such that $F \cap |\mathrm{lk}_{\partial P}(v)| = \partial F$, then take $X \oplus \tau(v,P) = X$. If X contains no such facet, then $X \oplus \tau(v,P)$ is obtained by adding the facet $\tau(v,P)$ and replacing $P \in X$ with two new cells σ_1, σ_2 such that $\sigma_1 \cup \sigma_2 = P$ and $\sigma_1 \cap \sigma_2 = \tau(v,P)$. Note that subdivision by facet has no effect on simplicial complexes.

For a boundary vertex v of X we define the *full subdivision of X at v* by

$$X^*(v) = X \bigoplus_{P \in \mathcal{P}_X(v)} \tau(v,P).$$

That is, $X^*(v)$ is obtained from X by repeatedly doing subdivision by facet, in effect subdividing X at v and P for each $P \in \mathcal{P}_X(v)$ (see Fig. 10).

We say that X is *vertex truncatable* if $|X| \sim B^d$ and at least one of the following holds:

(a) X is a d-simplex.

(b) X has a strong shedding vertex v such that $X^*(v) \setminus v$ is vertex truncatable.

(c) X has a strong solitary vertex v such that $X^*(v) \setminus v$ is vertex truncatable.

Note that if $|X| \sim B^d$ and v is a strong shedding vertex of X, then v is clearly a strong shedding vertex of $X^*(v)$. Thus $|X^*(v) \setminus v| \sim B^d$. It follows that strongly vertex decomposable implies vertex truncatable.

Two k-faces $F_1, F_2 \in X$ are said to be *strongly adjacent* if $\dim(F_1 \cap F_2) = k-1$. If X_1 is a polyhedron of X and $F_1, F_2 \in X_1$ are strongly adjacent facets, then we define $\alpha(F_1, F_2)$ to be the interior angle of X_1 formed between F_1 and F_2. More precisely, $\alpha(F_1, F_2)$ is the angle between the normal vectors to the hyperplanes $\operatorname{aff}(F_1)$ and $\operatorname{aff}(F_2)$, with sign chosen

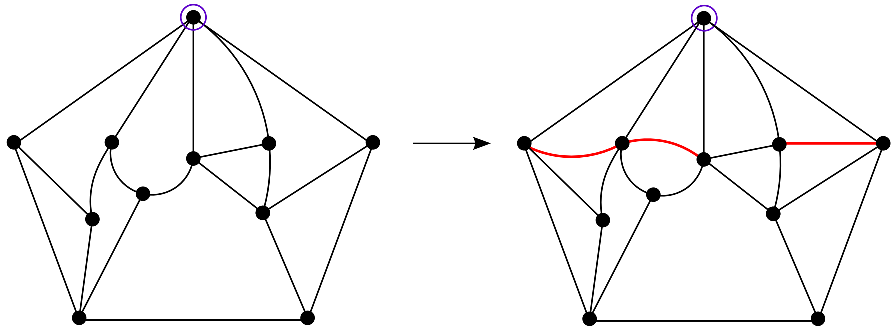

FIGURE 10. A 2-polyhedral complex with vertex v encircled, and the resulting complex $X^*(v)$ with the added facets $\tau(v, P)$ shown in red.

so that the angle is interior to X_1. If $d = 2$ then $\alpha(F_1, F_2)$ is a vertex angle, and if $d = 3$ then $\alpha(F_1, F_2)$ is a dihedral angle.

Given a d-polyhedron X in $\mathbb{R}^n$ and a facet $F \in X$, the *Schlegel set* of X with respect to F, which we denote by $\xi(X, F)$, is the closed convex set bounded by the hyperplanes $\mathrm{aff}(F)$ and $\mathrm{aff}(G_i)$ for each facet G_i strongly adjacent to F. That is, if $G_1, \ldots, G_k \in X$ denote the facets strongly adjacent to F, H_{G_i} denotes the closed halfspace bounded by $\mathrm{aff}(G_i)$ and meeting the interior of X, and H_F denotes the closed halfspace bounded by $\mathrm{aff}(F)$ and *not* meeting the interior of X, then $\xi(X, F) = H_F \cap \bigcap_{i=1}^k H_{G_i}$. A *Schlegel point* is a point $y \in \mathrm{int}(\xi(X, F))$.

If the facet F is a $(d-1)$-simplex, the hyperplanes $\mathrm{aff}(G_i)$ intersect in a (possibly infinite) point, which we call the *apex* of $\xi(X, F)$. If the apex x is a finite point, then $x \in \xi(X, F)$ if and only if x and X lie on opposite sides of the hyperplane $\mathrm{aff}(F)$. If x is a finite point and $x \in \xi(X, F)$, then $\xi(X, F)$ is just the cone with base F and apex x.

Theorem 6.1 *Let $d \leq 3$ and let X be a topological d-polyhedral complex in $\mathbb{R}^d$ such that $|X| \sim B^d$ and all interior facets $F \in X - \partial X$ are simplices. If X is vertex truncatable, then there is a geometric polyhedral complex Y in $\mathbb{R}^d$ such that $X \simeq Y$. Furthermore, we may choose Y to have all vertices rational.*

We prove in fact a stringer result, that such as embedding exists for *every* given embedding of ∂X. It is important to emphasize that this is a much stronger property, which does not extend to conditions of Theorem 1.4 (see Subsection **??**).

Proof. To prove the theorem, we strengthen it, which in turn strengthens our induction hypothesis. Namely, we claim that if such a Y exists, then furthermore:

(i) For any polytope Z such that $\partial Z \simeq \partial X$, we may choose Y such that $|Y| = |Z|$.

(ii) Let $\epsilon > 0$. Let $X_1, \ldots X_k \in \mathcal{P}_X$ such that $X_i \cap X_j \subseteq \partial X$ and $X_i \cap \partial X$ is a facet of X, call it F_i. Let $Y_i \in \mathcal{P}_Y$ be the polyhedron corresponding to X_i and let $G_i \in \partial Y$ be the face corresponding to $F_i \in \partial X$. Then we may choose Y such that $\alpha(H_j, G_i) < \epsilon$ for all $i = 1, \ldots, k$ and every face $H_j \in Y_i$ strongly adjacent to G_i.

If $d \leq 1$ the theorem is trivial, so assume $d \in \{2, 3\}$. Suppose that X is a vertex collapsible d-polyhedral complex with simplicial interior facets. We proceed by induction on the number n of vertices of X. In the base case X has $d+1$ vertices, and we may take Y to be any geometric d-simplex in $\mathbb{R}^d$. Now suppose $n > d+1$.

Since $\mathcal{X}$ is vertex truncatable, choose a boundary vertex w_0 satisfying either condition (b) or (c) of the definition. Let $X' = X^*(w_0) \smallsetminus w_0$. By definition, X' is vertex truncatable, so in particular $|X'| \sim B^d$. Furthermore, all interior facets of X' are clearly simplices.

Let Z be a polytope such that $\partial Z \simeq \partial X$. If $d = 3$, such a polytope Z exists by Steinitz's Theorem. If $d = 2$ then Z may be any strictly convex polygon with the same number of boundary vertices as X. Since w_0 is boundary minimal, it has d neighbors $w_1, \dots, w_d$ in ∂X. Let v_0 denote the vertex of Z corresponding to w_0. For each $i = 1, \dots, d$, the vertex w_i corresponds to a vertex v_i of Z, where v_i is a neighbor of v_0. The vertices v_i lie on a hyperplane $H = \operatorname{aff}(v_1, \dots, v_d)$. The hyperplane H splits Z into two polytopes Q_1, Q_2, each defined by taking all faces of Z lying on a given side of H, which includes the facet $T_Z = \operatorname{conv}(v_1, \dots, v_d)$ in both cases. One of these two polytopes, say Q_1, contains v_0. Then note that Q_1 is a d-simplex.

Clearly $|\operatorname{lk}_{X^*(w_0)}(w_0)|$ is homeomorphic to a $(d-1)$-ball. So if $d = 3$ then $\operatorname{lk}_{X^*(w_0)}(w_0)$ is isomorphic to the Schlegel diagram of some 3-polytope by Steinitz's theorem. Call this polytope A. If $d = 2$ then we may simply take A to be a convex polygon with one more edge than $\operatorname{lk}_{X^*(w_0)}(w_0)$. Let $u_1, \dots, u_d$ be the vertices of A corresponding to the vertices $v_1, \dots, v_d$ of $\operatorname{lk}_{X^*(w_0)}(w_0)$, and let $T_A = \operatorname{conv}(u_1, \dots, u_d)$. Note that T_A is a $(d-1)$-simplex, and a facet of A. Apply an affine transformation to $|A|$ so that $T_A = T_Z$. If A is not a simplex, apply a projective transformation to $|A|$ that fixes T_A and takes the apex x of $\xi(A, T_A)$ to a finite point on the side of the hyperplane $\operatorname{aff}(T_A)$ not containing $|A|$.

For a set $S \subseteq \mathbb{R}^d$, let $(S, 1) = \{(x, 1) \in \mathbb{R}^d \times \mathbb{R} \mid x \in S\}$. Form the $(d+1)$-cone C with base $(|A|, 1)$ and apex $a \in \mathbb{R}^{d+1}$, $a_{d+1} \neq 1$. Let F_C denote the (clearly simplicial) d-face of C containing a and $(T_A, 1)$. Let W denote the Schegel projection of C onto its facet F_C, with respect to a Schlegel point y. Then W is a geometric polyhedral complex. Clearly W contains a subcomplex B such that $B \simeq A$. Since $(T_A, 1) \subseteq F_C$, the $(d-1)$-face $(T_A, 1)$ is fixed by the Schlegel projection, so in fact $(T_A, 1) \in B$ is the face of B corresponding to $T_A \in A$. Apply an affine transformation to $|W|$ that maps $(T_A, 1)$ to $T_A = T_Z$ and maps a to v_0. This transforms $|B|$ accordingly. In particular, we now have $|W| = |Q_1|$. Thus $Z' = B \#_{T_Z} Q_2$ is a polytope such that $\partial Z' \simeq \partial X'$.

By the induction hypothesis and (i), since X' contains one fewer vertex than X, there is a geometric polyhedral complex Y' such that $X' \simeq Y'$ and $|Y'| = |Z'|$. Let $Y^* = Y' \cup W$. Then clearly $X^*(w_0) \simeq Y^*$. Removing the facets of Y^* corresponding to the facets $\tau(w_0, P)$ of $X^*(w_0)$, we obtain a polyhedral complex Y such that $X \simeq Y$ and $|Y| = |Z|$. This establishes (i), provided that each cell of Y is convex. We now show that this is the case.

Let $X'_1, \dots, X'_k \in \mathcal{P}_{X'}$ denote the d-polyhedra of X' such that $|X'_i| \subseteq |X|$ for some $X \in \mathcal{P}_X(w_0)$. We will let X_i denote the unique d-polyhedron of X such that $|X'_i| \subseteq |X_i|$. For all $i, j = 1, \dots, k$, since $w_0 \in X_i \cap X_j$ and X is a polyhedral complex, we must have $X'_i \cap X'_j \subseteq \partial X'$, for otherwise X_i and X_j would intersect in more than a unique common face. If w_0 is a strong solitary vertex of X, then $k = 1$. Thus the single polyhedron $Y_1 \in \mathcal{P}_Y$ corresponding to X_1 is clearly convex from the above construction.

Now suppose that w_0 is a strong shedding vertex of X. Note that any shedding vertex w_0 of X must satisfy $\operatorname{cst}_X(w_0) \cap \partial X = \operatorname{cst}_{\partial X}(w_0)$, for otherwise $|X \smallsetminus w_0|$ would not be homeomorphic to B^d. Thus we must have $X'_i \cap \partial X' = \tau(w_0, X_i)$. So $X'_1, \dots X'_k$ satisfy the hypotheses of (ii). Let $Y'_i \in \mathcal{P}_{Y'}$ be the polyhedron corresponding to X'_i and $G_i \in Y'$ the facet corresponding to $\tau(w_0, X_i)$. Let $Y_i \in \mathcal{P}_Y$ be the unique polyhedron such that $|Y'_i| \subseteq |Y_i|$. Since our induction hypothesis is enhanced by (ii), we may assume that the angles $\alpha(H_j, G_i)$ are arbitrarily small for each i and each face $H_j \in Y_i'$ strongly adjacent to G_i. Since the cells $|Y_i'|$ are convex by induction and form arbitrarily small angles with G_i, we may ensure that on removing G_i the resulting cell $|Y_i| \in Y$ is convex. Finally, if $\sigma \in Y$ is a cell not having any of the $|Y_i'|$ as a subset, then either σ is a d-simplex (hence convex) or $\sigma \in Y'$. In the latter case σ is convex because Y' is a geometric polyhedral complex.

Now we must show that (ii) holds. Let $X_1, \ldots, X_\ell \in \mathcal{P}_X$ be a collection of polytopes satisfying the hypotheses of (ii), and let $Y_1, \ldots Y_\ell \in \mathcal{P}_Y$ denote the corresponding polytopes of Y. If $Y_i \in \mathcal{P}_{Y'}$ then we obtain the conclusion of (ii) from the induction hypothesis. Now suppose $Y_i \notin \mathcal{P}_{Y'}$. Then $Y_i \in \mathcal{P}_W$. If the A in the above construction is a simplex, then $\ell = 1$. Let v be the vertex of A not contained in T_A. Then we may clearly choose v arbitrarily close to the facet $Y_i \cap \partial Y$. If A is not a simplex, then because the point x is finite and lies on the side of $\operatorname{aff}(T_A)$ opposite to that of A, $x \in \xi(C, F_C)$. In particular $\xi(C, F_C)$ is a cone with base F_C and apex $x \in \mathbb{R}^{d+1}$. By choosing the Schlegel point y in the above construction arbitrarily close to x, we may ensure that all such angles $\alpha(F_i, F_j)$ in the projection W are arbitrarily small.

Finally, that Y may be chosen to be rational is an immediate consequence of the methods of the proof. Broadly speaking, we obtained Y by first using Steinitz's theorem to produce polytopes Z and A, and then manipulating these polytopes using projective transformations. But from the proof of Steinitz's theorem, we may take both Z and A to be rational. By then using only rational affine and projective transformations in the above constructions, we may in fact obtain a *rational* geometric polyhedral complex Y such that $X \simeq Y$. □

It is straightforward to show that all 2-polyhedral complexes are vertex truncatable. Thus Theorem 6.1 implies that *all* 2-polyhedral complexes are geometrically realizable. Note that every 2-polyhedral complex is a 2-connected plane graph. However, the converse is not true. In fact, if a 2-connected plane graph is not a polyhedral complex (i.e. if the intersection of two faces is more than a unique common edge or vertex of both), then it clearly does not admit an embedding such that all faces are strictly convex.

Therefore we obtain necessary and sufficient conditions for a 2-connected plane graph G to admit an isotopic embedding G' such that all faces of G' are strictly convex. Specifically, a 2-connected plane graph G has a strictly convex isotopic embedding if and only if G is a 2-polyhedral complex. From this we recover Tutte's theorem [T1].

Finally, we note that the condition $d \leq 3$ in the statement of Theorem 6.1 plays a crucial role. Namely, it allows us to invoke Steinitz's theorem (for $d = 3$). For example, by Steinitz's theorem a simplicial 2-sphere is always isomorphic to the boundary of a 3-polytope. In higher dimensions the analogous statement is not true (see [GS]).

7. Further discussion

7.1. It is perhaps not obvious why Theorem 1.1 does not follow from existence of irrational 4-polytopes. Indeed, one can take a Schlegel diagram Q of an irrational 4-polytope P and conjecture that this is the desired irrational polyhedral complex. The logical fallacy here is that the implications go the other way. If the Schlegel diagram of P is irrational, then indeed P must be an irrational polytope. However, the converse is not true. There is no reason why all realizations of the Schegel diagram Q must have irrational coordinates, even if P is irrational. In fact, after computing degrees of freedom one should *expect* additional realizations of Q. Similarly, it is only in $\mathbb{R}^3$ that one can have (and does have) the Maxwell-Cremona theorem [R]; in $\mathbb{R}^4$ and higher dimensions it easily fails in full generality (cf. **??**).

On the other hand, it was noted by Richter-Gebert [**?**, §10] that the polytope operations used in the construction of irrational polytopes can be emulated in $\mathbb{R}^3$ by an analogous operations on the level of Schegel diagram.[1] Richter-Gebert briefly outlined both an irrational construction and a universality type theorem. If completed, the former construction would prove smaller than out 1278 polytope construction in Theorem 1.1, but each polytope is more complicated. The latter (universality result), is of a different nature from Theorem 5.5 as no intersections are allowed. As stated, it would imply only the first part of Theorem 5.2 (and only if the outside face is a tetrahedron), since we allow only triangular prisms as our polyhedra.

7.2. Let us here address a delicate issue of Brehm's universality theorem, as outlined in [Br, Z2]. Our Theorem 5.5 is clearly a variation on Brehm's announced result. In fact, the proof ideas do not seem very far from ours, even if different on a technical level. Unfortunately, the Brehm's complete proof has never appeared, and from our experience with universality type theorems these details are occasionally delicate and important. Thus, one can view our work as either an application of our tools to re-derive Brehm's results, or the first complete proof of a theorem of this type. Since in fact our construction has further properties is an unexpected bonus.

Let us mention that allowing intersections in our proof of Theorem 5.5 as the belts can get linked and knotted in a non-trivial way, a possibility we cannot account without further sub-triangulating the construction.

7.3. There is a rather simple reason why Tutte's theorems are delicate and unlikely to allow a direct extension to higher dimensions, even ignoring the topological obstructions as in the paper. Consider the first two graphs as in Figure 11 below. The smaller of the two has a non-strict convex realization, while the bigger does not. Tutte's result is "if and only if", and he explains that the difference between the two is a combinatorial rather than geometric or topological reason. Now of course, neither have a (strict) convex realization. This can be explained from the fact that this graph is not 3-connected and thus its *spring embedding* collapses. But in $\mathbb{R}^3$ if one replaces the middle square with an octahedron, the connectivity obstacle disappears.

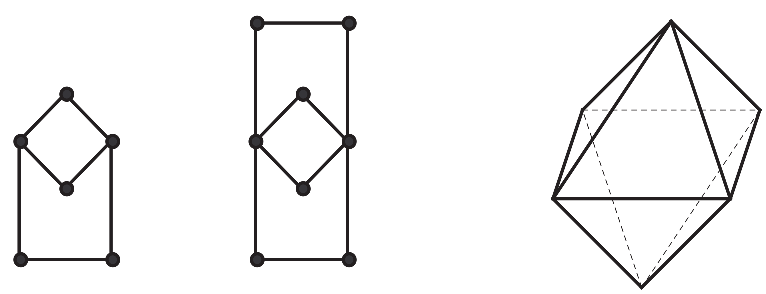

FIGURE 11. Examples and counterexamples.

[1]We learned about this work only after the results in this paper were completed.

7.4. The *vertex truncatability condition* in Theorem 1.4 is somewhat restrictive already in $\mathbb{R}^3$. For example, it cannot apply to any triangulation of the octahedron or the icosahedron since their boundaries have no vertices of degree three.

Recall now that the inductive assumption in the proof of Theorem 1.4 implied that the boundary can be prescribed in advance. Now the study of triangulations of the octahedron shows that vertex truncatability is critical for the result. Consider an octahedron $Q = (11'22'33')$ with topological triangulation $(122'3)$, $(122'3')$, $(1'233')$, $(1'2'33')$, and $(22'33')$. In this triangulation, the triangles $(122')$ and $(1'33')$ are *not linked* (topologically). But this is false for some realizations of Q. This means that Theorem 1.4, at least a stronger version where the boundary is prescribed, cannot possibly be extended to triangulations as simple as this one.

Interestingly, one can easily see why the "linked triangle obstacles" never appears in our situation. That is because the vertex truncatability condition forbids all *diagonals*, defined as inner edges connecting two boundary vertices. This follows easily by induction.

Finally, let us mention that we do not believe that Theorem 1.4 extends to $d \geq 4$. It would be interesting to find such a counterexample.

7.5. It is obvious that the triangulation produced in Theorem 1.4 can be made rational: simply perturb all the vertices. In particular, this explains why we must use non-simplicial polytopes in the proof of Theorem 1.1.

It is perhaps less obvious that all geometric realizations produced in Theorem 6.1 are rational. Although the resulting polyhedral complex must have simplicial *interior faces*, the boundary faces can be arbitrary. Here rationality is a corollary resulting from the nature of the proof: all steps, in particular all projective transformations can be done over $\mathbb{Q}$.

7.6. Recently, two new explicit examples of simplicial balls with further properties were announced in [BL]. They have 12 and 15 vertices, respectively. This can be contrasted with the 9 vertices of the topological polyhedral ball X' we construct in the proof of Theorem 1.3.

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References

[AB] K. Adiprasito and B. Benedetti, Metric Geometry and Collapsibility, `arXiv:1107.5789`.

[BG] D. W. Barnette and G. Grünbaum, Preassigning the shape of a face, *Pacific J. Math.* **32** (1970), 299–302.

[BL] B. Benedetti and F. Lutz, Non-evasiveness, collapsibility and explicit knotted triangulations, *Oberwolfach Report*, **8** (2011), 1–2.

[BFS] L. J. Billera, P. Filliman and B. Sturmfels, Constructions and complexity of secondary polytopes, *Adv. Math.* **83** (1990), 155–179.

[BP] L. Billera and J. Provan, Decompositions of simplicial complexes related to diameters of convex polyhedra, *Math. Op. Res.* **5** (1980), 576–594.

[Br] U. Brehm, *Realization spaces*, talk slides, 2007.

[Ca] S. S. Cairns, Triangulated manifolds which are not Brouwer manifolds, *Ann. Math.* **41** (1940), 792–795.

[Ch] D. R. J. Chillingworth, Collapsing three-dimensional convex polyhedra, *Proc. Camb. Phil. Soc.* **63** (1967), 353–357; Erratum in **88** (1980), 307–310.

[DRS] J. A. De Loera, J. Rambau and F. Santos, *Triangulations: Structures and Algorithms*, Springer, 2008.

[F] I. Fáry, On straight line representations of planar graphs, *Acta Univ. Szeged. Sect. Sci. Math.* **11** (1948), 229–233.

[FPP] H. de Fraysseix, J. Pach and R.Pollack, Small sets supporting Fary Embeddings of Planar Graphs, in *Proc. STOC 1988*, ACM, 426–433.

[G] B. Grünbaum, *Convex polytopes*, Springer, 1995.

[GS] B. Grünbaum and V.P. Sreedharan, An Enumeration of Simplicial 4-Polytopes with 8 vertices, *J. Comb. Theory* **2** (1967), 437–465.

[HZ] M. Hachimori and G. M. Ziegler, Decomposition of simplicial balls and spheres with knots consisting of few edges, *Math. Z.* **235** (2000), 159–171.

[K] E. R. van Kampen, Remark on the address of S. S. Cairns, in *Lectures in Topology*, 311–313, University of Michigan Press, Ann Arbor, MI, 1941.

[Mn] N. Mnëv, The universality theorems..., in *Lecture Notes in Math.* **1346**, Springer, Berlin, 1988, 527–543.

[MT] B. Mojar and C. Thomassen, *Graphs on surfaces*, JHU Press, 2001.

[P] I. Pak, *Lectures on discrete and polyhedral geometry*, monograph draft; available electronically at `http://www.math.ucla.edu/~pak/book.htm`.

[R] J. Richter-Gebert, *Realization spaces of polytopes*, *Lecture Notes in Math.*, Springer, 1996.

[RZ] J. Richter-Gebert and G. M. Ziegler, Realization spaces of 4-polytopes are universal, *Bull. Amer. Math. Soc.* **32** (1995), 403–412.

[T1] W. T. Tutte, Convex representations of graphs, *Proc. Lond. Math. Soc.* **10** (1960), 304–320.

[T2] W. T. Tutte, How to draw a graph, *Proc. Lond. Math. Soc.* **13** (1963), 743–768.

[Wi] N. Witte, A vertex decomposable 3-ball and 3-sphere with a knot consisting of 6 edges, *Electronic Geometry Model* No. 2001.05.003, available at `http://tinyurl.com/7bxdoxp`

[Wo] R. Woodroofe, Vertex decomposable graphs and obstructions to shellability, *Proc. Amer. Math. Soc.* **10** (2009), 3235–3234.

[Z1] G. M. Ziegler, *Lectures on polytopes*, Springer, New York, 1995.

[Z2] G. M. Ziegler, Non-rational configurations, polytopes, and surfaces, *Math. Intell.* **30** (2008), 36–42.

Appendix A. Belt construction in figures and numbers

It follows from Theorem 1.1 that the belts cannot be given by explicit (integer) coordinates. We give the explicit description of the belts by specifying the arcs on which the triangular facets of the prisms lie, as well as a function describing their lateral lengths (see Figure 12). Note that the prism lengths are made small except at the boundary to ensure that the belts do not intersect. Furthermore, we must ensure that the arcs bend sufficiently to avoid each other at the top of the core. To create the arcs we start with a family of circles, and then apply a parametrized family of rotations to stretch them. The Mathematica code describing the explicit details of the construction, and used to generate the complete irrational complex and the 3D graphics in this paper, can be found at `http://www.math.ucla.edu/~stedmanw/research/`.

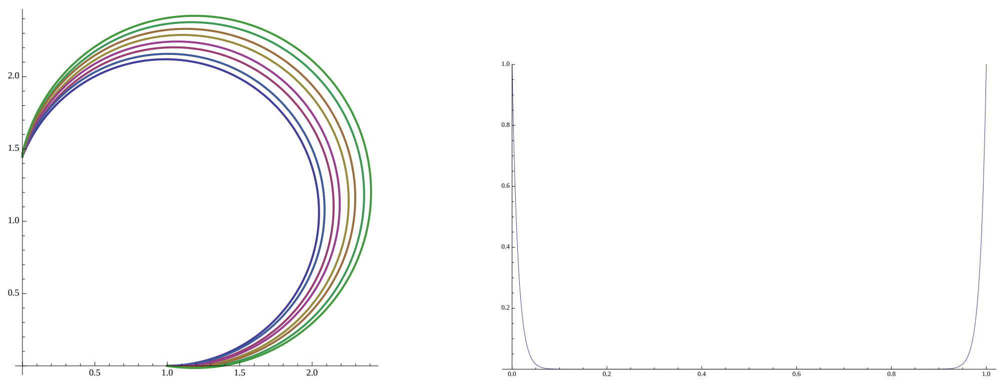

Figure 12. Left: Circles used in the first stage of the design of each belt. Right: The prism length function f_m for $m = 80$, described in the proof of Theorem 1.1.

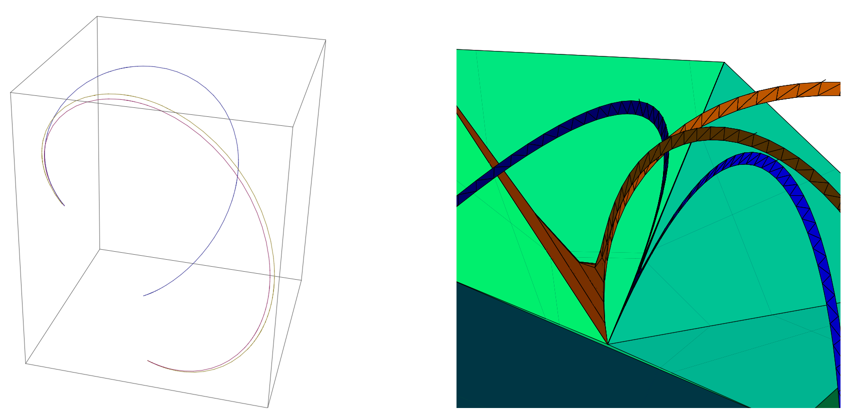

Figure 13. Left: The shape of one belt, obtained by stretching the circles of Fig. 12. Right: Belts do not intersect at the top of the core.

In each belt, our construction uses 318 triangular prisms, exactly $2(80 - 1) + 1 = 159$ prisms per semi-belt. The core consists of 5 triangular prisms and 1 pentagonal pyramid. The complete irrational complex thus consists of a total of $4 \cdot 318 + 5 = 1277$ triangular prisms and 1 pentagonal pyramid, as in the theorem.

Since the belts come close to intersecting near the boundary of the core, some checking is necessary. In Figure 13 we show how the belts near-miss each other due to their shape. We conclude with a rotated view of the irrational polyhedral complex.

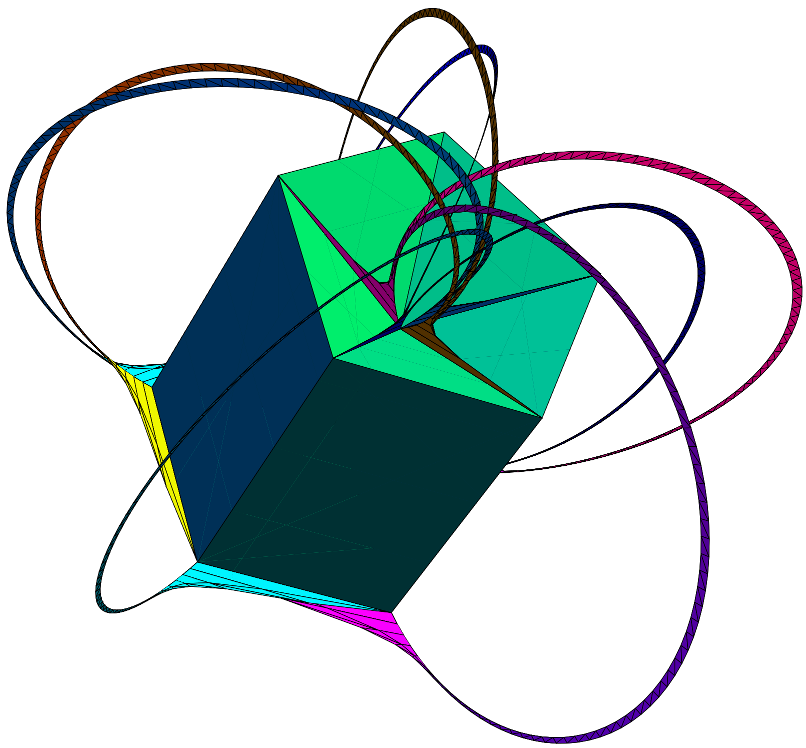

FIGURE 14. A rotated view of the irrational complex.

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