Tableaux bijections

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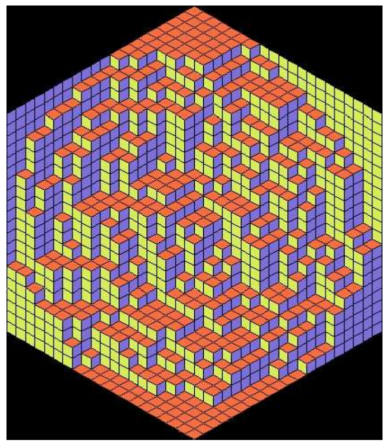


Standard Young tableaux

Let λ be a partition, $[\lambda]$ its Young diagram, $|\lambda| = n$.
A *standard Young tableau* of shape λ is an integer function
$A : [\lambda] \to \{1, \ldots, n\}$, which increases $\downarrow$ and $\to$.
$\mathrm{SYT}(\lambda)$ is a set of standard Young tableaux of shape λ.

1	2	4	5	12
3	6	8	13	
7	9	11	14	
10				

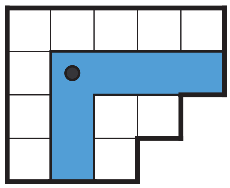
$A \in \mathrm{SYT}(5441)$

Observation:

$$|\mathrm{SYT}(\lambda)| = \sum_{\text{corner } s \in \lambda} |\mathrm{SYT}(\lambda - s)| .$$

Hook-length formula

For $(i,j) \in [\lambda]$, $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$ are the *hook lengths.*



$h_{22} = 6$

8	7	6	4	2
7	6	5	3	1
5	4	3	1	
3	2	1		

Theorem [Frame–Robinson–Thrall, 1954]
For every partition λ, such that $|\lambda| = n$:

$$|\mathrm{SYT}(\lambda)| \;=\; \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}}\,.$$

Example: $\lambda = (3,2)$, $n = 5$.

4	3	1
2	1	

$|\mathrm{SYT}(32)| = \frac{5!}{432} = 5$

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

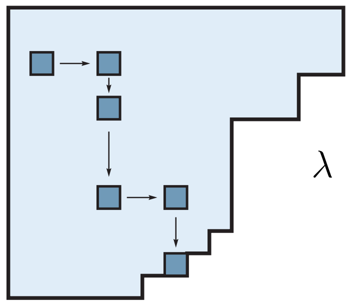
1	3	4
2	5	

1	3	5
2	4	

Note: FRT proved the theorem using the Frobenius determinant formula for $\dim(\pi_\lambda) = |\mathrm{SYT}(\lambda)|$. Now over a dozen different proofs of HLF is known, as well as a variety of generalizations.

Hook walk

Algorithm (*hook walk*)
Choose a uniform random square $x \in [\lambda]$.
Move x to a uniform random square y in the x-hook in $[\lambda]$.
Repeat until a corner is reached.



Theorem [Greene–Nijenhuis–Wilf, 1979]
The probability $p_\lambda(s)$ *of reaching a corner* $s \in [\lambda]$ *is equal to*

$$p_\lambda(s) \;=\; \frac{|\mathrm{SYT}(\lambda - s)|}{|\mathrm{SYT}(\lambda)|}\,.$$

Corollary: *One can efficiently sample from* $\mathrm{SYT}(\lambda)$.

Note: GNW algorithm is a tool in a simple proof of the HLF. The correctness is verified by induction. It is now generalized to q-walk (Kerov), (q,t)-walk (Garsia-Haiman), continuous process (Kerov), etc.

2-dim bubble sorting

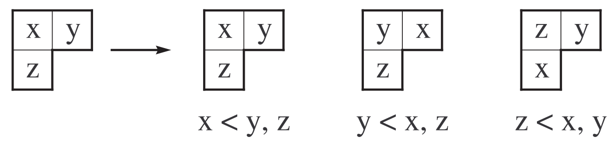
Algorithm (*NPS bubble sorting*)
Input: $B \in S_{[\lambda]}$, a permutation of squares in $[\lambda]$
For all $x \in [\lambda]$, from rightmost to leftmost column,
from the bottom to top square in the column:
Do: bubble-insert $x \downarrow$ and $\rightarrow$.
Output: $A \in \mathrm{SYT}(\lambda)$.

Theorem [Novelli–P.–Stoyanovsky, 1997]
The resulting standard Young tableau A is uniform in $\mathrm{SYT}(\lambda)$.

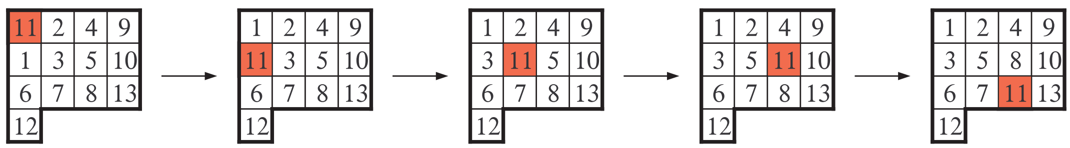
Thus, in particular, $|\mathrm{SYT}(\lambda)|$ divides $n!$.
This gives another way of sampling.

Note: PS first announced this in 1992. They extended this map to a full bijection: $|S_n| \leftrightarrow |\mathrm{SYT}(\lambda)| \times \prod_x h_x$. Now this bijection is extended and modified a number of times to work for other types of tableaux, trees, etc.

Explanation:

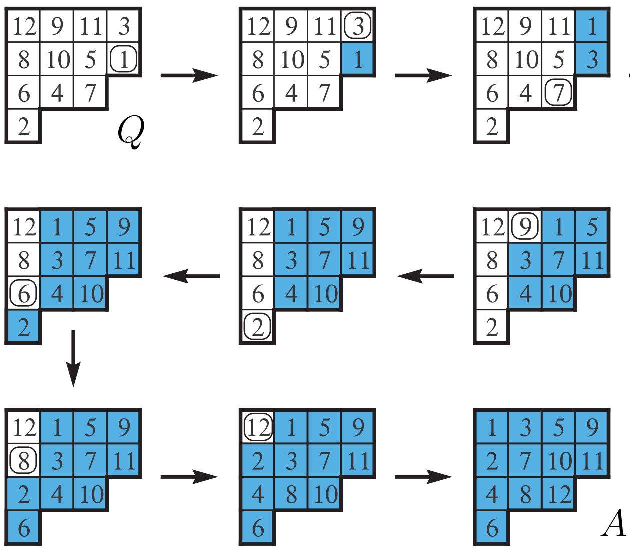


Bubble-insert step.



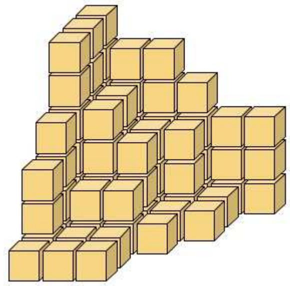
Example of a bubble-insertion of an element in a permutation tableau.

An example:



An example of the NPS bubble-sorting map: $Q \to A$, where $Q \in S_{12}$, $A \in \mathrm{SYT}(\lambda)$, and $\lambda = (4, 4, 3, 1)$.

Solid partitions



Theorem [MacMahon, 1912]
Let $p_3(n)$ be the number of solid partitions of n. Then

$$1 + \sum_{n=1}^{\infty} p_3(n)t^n = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^i}.$$

Note: MacMahon conjectured a generalization to higher dimensions, which was later proved incorrect (1967). The reason why this works for 3-dim solid partition is based on symmetric functions and representation theory of S_n and $\mathrm{GL}(n, \mathbb{C})$.

Theorem [MacMahon, 1912; Stanley, 1971; Macdonald, 1979]
Let $B(\ell, m, n)$ be the number of solid partitions which fit box $\ell \times m \times n$. Then:

$$B(\ell, m, n) \,=\, \prod_{i=1}^{\ell} \prod_{j=1}^{m} \prod_{k=1}^{n} \frac{i+j+k-1}{i+j+k-2}\,.$$

There are several common generalizations of these two formulas, which all now have bijective proofs. Below we present an intermediate generalization which gives a complete proof of MacMahon's theorem and an efficient Bolztmann sampling of all solid partitions.

Reverse plane partitions

A *reverse plane partition* of shape λ is a function $A : [\lambda] \to \{0, 1, 2, \ldots\}$ which non-decreases $\downarrow$ and $\to$.
Denote by $\mathrm{RPP}(\lambda)$ their set, $|A| = \sum_{x\in[\lambda]} A(x)$ the *size* of a rpp.

$$\begin{array}{|c|c|c|c|c|}
\hline
0 & 0 & 0 & 3 & 6 \\
\hline
0 & 0 & 2 & 3 \\
\cline{1-4}
1 & 2 & 2 & 4 \\
\cline{1-4}
4 \\
\cline{1-1}
\end{array}
\qquad A \in \mathrm{RPP}(5441), \ \ |A| = 27$$

Theorem [Stanley, 1971]
For every partition λ, we have:

$$1 + \sum_{A\in \mathrm{RPP}(\lambda)}^{\infty} t^{|A|} \, = \, \prod_{x\in[\lambda]}^{\infty} \frac{1}{1 - t^{h_x}} \, .$$

Observation: Stanley's formula implies MacMahon's formula:
set $[\lambda] = [N \times N]$ and let $N \to \infty$.

RPP archaeology

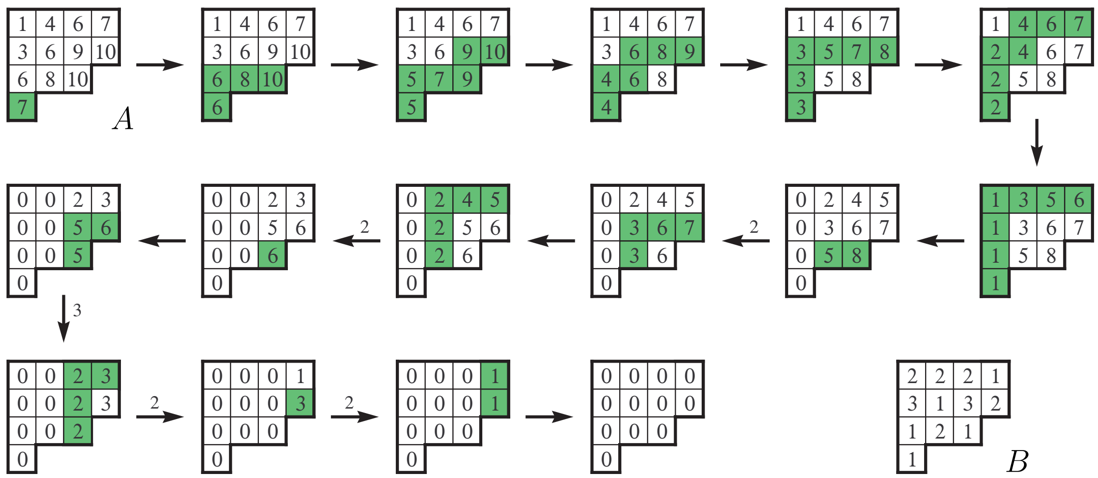
Algorithm (*Hillman–Grassl bijection*)
Input: $A \in \mathrm{RPP}(\lambda)$. Set *recording tableau* B to be zero.
For all $x \in [\lambda]$, from leftmost to rightmost column,
from the bottom to top square in the column:
Do: remove 1's along the most lower-right ribbon.
Add 1 to a square in the in a position
corresponding to the ribbon
Output: recording tableau $B : [\lambda] \to \{0, 1, 2, \ldots\}$.

Theorem [Hillman–Grassl, 1976]
The above map $A \to B$ is a bijection which satisfies

$$|A| = \sum_{(ij) \in [\lambda]} B(i,j) \cdot h_{ij}$$

Observation: Stanley's formula immediately follows from the theorem.

Example:



Note: Extensions of this bijection and variations on the theme were obtained by Gansner (1981), Krattenthaler (1995), P. (2001), and others.

Jeu de taquin

Burnside identity: $n! = \sum_{\lambda\,:\,|\lambda|=n} |\mathrm{SYT}(\lambda)|^2$

Example: $n = 3$, $|\mathrm{SYT}(111)| = |\mathrm{SYT}(3)| = 1$, $|\mathrm{SYT}(21)| = 2$

$$1^1 + 2^2 + 1^2 = 6$$

Jeu de taquin: = French name for the *Fifteen puzzle*

= name for *slide rules* to prove Burnside identity

= equivalent to the Robinson–Schensted–Knuth correspondence

= which is itself variation on patience sorting

The construction:

Jeu de Taquin Algorithm [Schützenberger, 1977]

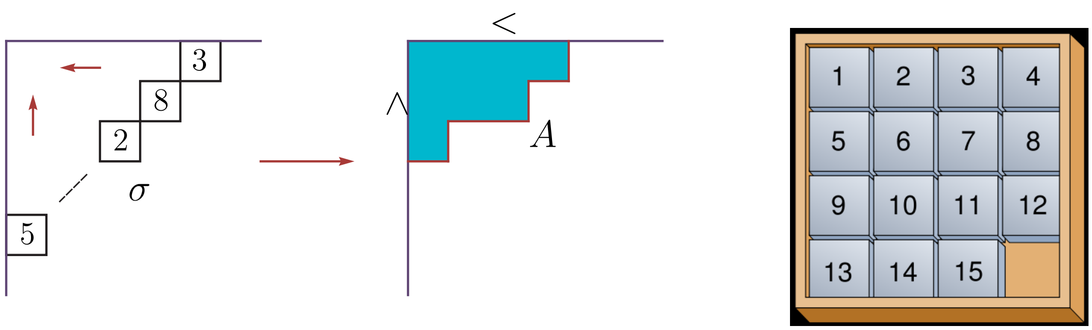
Input: permutation $\sigma \in S_n$. Arrange $\sigma \in S_n$ in boxes diagonally.

Slide rules: Push the smaller of the boxes up and left when possible.

Stop when a standard Young tableau A is obtained.

Do the same for inverse permutation: $\sigma^{-1} \to B$.

Output: (A, B).



Theorem [Schützenberger, 1977; Thomas, 1980]

Jeu de taquin algorithm is a well-defined bijection.

Littlewood–Richardson rule:

Fix partitions λ, μ.
Take all possible $A \in \mathrm{SYT}(\lambda)$ and $B \in \mathrm{SYT}(\mu)$ arranged diagonally.
Apply jeu de taquin algorithm to $A \circ B$.
The number of times $C \in \mathrm{SYT}(\nu)$ appears is $c^{\nu}_{\lambda,\mu}$.

Theorem [Zelevinsky, 1981; Kerov, 1984]
The integers $c^{\nu}_{\lambda,\mu}$ are well defined (i.e. independent on the choice of C), and are equal to the Littlewood–Richardson coefficients.

$S_\lambda \otimes S_\lambda = \sum_\nu c^{\nu}_{\lambda,\mu} S_\nu$, where S_λ is an irreducible $\mathrm{GL}(n)$ representation.

Note: The many hidden symmetries of $c^{\nu}_{\lambda,\mu}$ are extremely important. Knutson–Tao (1999) found a new YT bijection which eventually led to the proof of the *saturation conjecture*.