Chapter Six

Combinatorics

Combinatorial analysis—or, as it is coming to be called, combinatorial theory—is both the oldest and one of the least developed branches of mathematics. The reason for this apparent paradox will become clear toward the end of the present account.

The vast and ill-defined field of applied mathematics is rapidly coming to be divided into two clear-cut branches with little overlap. The first covers the varied offspring of what in the past century was called "analytical mechanics" or "rational mechanics," and includes such time-honored and distinguished endeavors as the mechanics of continua, the theory of elasticity, and geometric optics, as well as some modern offshoots such as plasmas, supersonic flow, and so on. This field is rapidly being transformed by the use of high-speed computers.

The second branch centers on what may be called "discrete phenomena" in both natural science and mathematics. The word "combinatorial," first used by the German philosopher and scientist G. W. Leibniz in a classic treatise, has been in general use since the seventeenth century. Combinatorial problems are found nowadays in increasing numbers in every branch of science, even in those where mathematics is rarely used. It is now becoming clear that, once the life sciences develop to the stage at which a mathematical apparatus becomes indispensable, their main support will come from combinatorial theory. This is already apparent in those branches of biology where the wealth of experimental data is gradually allowing the construction of successful theories, such as molecular biology and genetics. Physics itself, which has been the source of so much mathematical research, is now faced, in statistical mechanics and such fields as elementary particles, with difficult problems that will not be surmounted until entirely new theories of a combinatorial nature are developed to understand the discontinuous structure of the molecular and subatomic worlds.

To these stimuli we must again add the impact of high-speed computing. Here combinatorial theories are needed as an essential guide to the actual practice of computing. Furthermore, much interest in combinatorial problems has been stimulated by the possibility of testing on computers heretofore inaccessible hypotheses.

These symptoms alone should be sufficient to forecast an intensification of work in combinatorial theory. Another indication, perhaps a more important one, is the impulse from within mathematics toward the investigation of things combinatorial.

The earliest glimmers of mathematical understanding in civilized man were combinatorial. The most backward civilization, whenever it let fantasy roam as far as the world of numbers and geometric figures, would promptly come up with binomial coefficients, magic squares, or some rudimentary classification of solid polyhedra. Why then, given such ancient history, is combinatorial theory just now beginning to stir itself into a self-sustaining science? The reasons lie, we believe, in two very unusual circumstances.

The first is that combinatorial theory has been the mother of several of the more active branches of today's mathematics, which have become independent sometimes at the cost of a drastic narrowing of the range of problems to which they can be applied. The typical—and perhaps the most successful—case of this is algebraic topology (formerly known as combinatorial topology), which, from a status of little more than recreational mathematics in the nineteenth century, was raised to an independent geometric discipline by the French mathematician

Henri Poincaré, who displayed the amazing possibilities of topological reasoning in a series of memoirs written in the latter part of his life. Poincaré's message was taken up by several mathematicians, among whom were outstanding Americans such as Alexander, Lefschetz, Veblen, and Whitney. Homotopy theory, the central part of contemporary topology, stands today, together with quantum mechanics and relativity theory, as one of the great achievements in pure thought in this century, and the first that bears a peculiarly American imprint. The combinatorial problems that topology originally set out to solve are still largely unsolved. Nevertheless, algebraic topology has been unexpectedly successful in solving an impressive array of long-standing problems ranging over all mathematics. And its applications to physics have great promise.

What we have written of topology could be repeated about a number of other areas in mathematics. This brings us to the second reason why combinatorial theory has been aloof from the rest of mathematics (and that sometimes has pushed it closer to physics or theoretical chemistry). This is the extraordinary wealth of unsolved combinatorial problems, often of the utmost importance in applied science, going hand-in-hand with the extreme difficulty found in creating standard methods or theories leading to their solution. Yet relatively few men chose to work in combinatorial mathematics compared with the numbers active in any of the other branches of mathematics that have held the stage in recent years. One is reminded of a penetrating remark by the Spanish philosopher José Ortega y Gasset, who, in commenting upon the extraordinary achievements of physics, added that the adoption of advanced and accomplished techniques made possible "the use of idiots" in doing successful research work. While many scientists of today would probably shy away from such an extreme statement, it is nevertheless undeniable that research in one of the better developed branches of mathematics was often easier, especially for the beginner, than original work in a field like combinatorial theory, where sheer courage and a strong dose of talent of a very special kind are indispensable.

Thus, combinatorial theory has been slowed in its theoretical development by the very success of the few men who have solved some of the outstanding combinatorial problems of their day, for, just as the man of action feels little need to philosophize, so the successful problem-solver in mathematics feels little need for designing theories that would unify, and thereby enable the less talented worker to solve, problems of comparable and similar difficulty. But the sheer number and the rapidly increasing complexity of combinatorial problems have made this situation no longer tolerable. It is doubtful that one man alone could solve any of the major combinatorial problems of our day.

Challenging Problems

Fortunately, most combinatorial problems can be stated in everyday language. To give an idea of the present state of the field, we have selected a few of the many problems that are now being actively worked upon. Each of the problems has applications to physics, to theoretical chemistry, or to some of the more "businesslike" branches of discrete applied mathematics such as programming, scheduling, network theory, or mathematical economics.

1. *The Ising Problem*

A rectangular ($m \times n$)-grid is made up of unit squares, each colored either red or blue. How many different color patterns are there if the number of boundary edges between the red squares and the blue squares is prescribed?

This frivolous-sounding question happens to be equivalent to one of the problems most often worked upon in the field of statistical mechanics. The issue at stake is big: It is the explanation of the macroscopic behavior of matter on the basis of known facts at the molecular or atomic levels. The Ising problem, of which the above statement is one of many equivalent versions, is the simplest model that exhibits the macroscopic behavior expected from certain natural assumptions at the microscopic level.

A complete and rigorous solution of the problem was not achieved until recently, although the main ideas were initiated many years before. The three-dimensional analog of the Ising problem remains unsolved in spite of many attacks.

2. *Percolation Theory*

Consider an orchard of regularly arranged fruit trees. An infection is introduced on a few trees and spreads from one tree to an adjacent one with probability p. How many trees will be infected? Will the infection assume epidemic proportions and run through the whole orchard, leaving only isolated pockets of healthy trees? How far apart should the trees be spaced to ensure that p is so small that any outbreak is confined locally?

Consider a crystalline alloy of magnetic and nonmagnetic ions in proportions p to q. Adjacent magnetic ions interact, and so clusters of different sizes have different magnetic susceptibilities. If the magnetic ions are sufficiently concentrated, infinite clusters can form, and at a low enough temperature long-range ferromagnetic order can spread through the whole crystal. Below a certain density of magnetic ions, no such ordering can take place. What alloys of the two ions can serve as permanent magnets?

It takes a while to see that these two problems are instances of one and the same problem, which was brilliantly solved by Michael Fisher, a British physicist now at Cornell University. Fisher translated the problem into the language of the theory of graphs and developed a beautiful theory at the borderline between combinatorial theory and probability. This theory has now found application to a host of other problems. One of the main results of percolation theory is the existence of a critical probability p_c in every infinite graph G (satisfying certain conditions which we omit) that governs the formation of infinite clusters G. If the probability p of spread of the "epidemic" from a vertex of G to one of its nearest neighbors is smaller than the critical probability p_c, no infinite clusters will form, whereas if $p > p_c$, infinite clusters will form. Rules for computing the critical probability p_c were developed by Fisher from ingenious combinatorial arguments.

3. *The Number of Necklaces, and Polya's Problem*

Necklaces of n beads are to be made out of an infinite supply of beads in k different colors. How many distinctly different necklaces can be made?

This problem was solved quite a while ago, so much so that the priority is in dispute. Letting the number of different necklaces be $c(n,k)$, the formula is

$$c(n,k) = \frac{1}{n} \sum_{d|n} \phi(d) k^{n/d}.$$

Here, ϕ is a numerical function used in number theory, first introduced by Euler. Again, the problem as stated sounds rather frivolous and seems to be far removed from application. And yet, this formula can be used to solve a difficult problem in the theory of Lie algebras, which in turn has a deep effect on contemporary physics.

The problem of counting necklaces displays the typical difficulty of enumeration problems, which include a sizable number of combinatorial problems. This difficulty can be described as follows. A finite or infinite set S of objects is given, and to each object an integer n is attached—in the case of necklaces, the number of beads—in such a way that there are at most a finite number a_n of elements of S attached to each n. Furthermore, an equivalence relation is given on the set S—in this case, two necklaces are to be considered equivalent, or "the same," if they differ only by a rotation around their centers. The problem is to determine the number of equivalence classes, knowing only the integers a_n and as few combinatorial data as possible about the set S.

This problem was solved by the Hungarian-born mathematician George Polya (now at Stanford) in a famous memoir published in 1936. Polya gave an explicit formula for the solution, which has since been applied to the most disparate problems of enumeration in mathematics, physics, and chemistry (where, for example, the formula gives the number of isomers of a given molecule).

Polya's formula went a long way toward solving a great many problems of enumeration, and is being applied almost daily to count more and more complicated sets of objects. It is nevertheless easy to give examples of important enumeration problems that have defied all efforts to this day, for instance the one described in the next paragraph.

4. *Nonself-intersecting Random Walk*

The problem is to give some formula for the number R_n of random walks of n steps that never cross the same vertex twice. A random walk on a flat rectangular grid consists of a sequence of steps one unit in length, taken at random either in the x- or the y-direction, with equal probability in each of the four directions. Very little is known about this problem, although physicists have amassed a sizable amount of numerical data. It is likely that this problem will be at least partly solved in the next few years, if interest in it stays alive.

5. *The Traveling Salesman Problem*

Following R. Gomory, who has done some of the deepest work on the subject, the problem can be described as follows. "A traveling salesman is interested in only one thing, money. He sets out to pass through a number of points, usually called cities, and then returns to his starting point. When he goes from the ith city to the jth city, he incurs a cost c_{ij}. His problem is to find that tour of all the points (cities) that minimizes the total cost."

This problem clearly illustrates the influence of computing in combinatorial theory. It is obvious that a solution exists, because there is only a finite number of possibilities. What is interesting, however, is to determine the minimum number $S(n)$ of steps, depending on the number n of cities, required to find the solution. (A "step" is defined as the most elementary operation a computer can perform.) If the number $S(n)$ grows too fast (for example if $S(n) = n!$) as the integer n increases, the problem can be considered unsolvable since no computer will be able to handle the solution for any but small values of n. By extremely ingenious arguments, it has been shown that $S(n) \leqq cn^2 2^n$, where c is constant, but it has not yet been shown that this is the best one can do.

Attempts to solve the traveling salesman problem and related problems of discrete minimization have led to a revival and a great development of the theory of polyhedra in spaces of n dimensions, which lay practically untouched—except for isolated results—since Archimedes. Recent work has created a field of unsuspected beauty and power, which is far from being exhausted. Strangely, the combinatorial study of polyhedra turns out to have a close connection with topology, which is not yet understood. It is related also to the theory behind linear programming and similar methods widely used in business and economics.

The idea we have sketched, of considering a problem $S(n)$ depending on an integer n as unsolvable if $S(n)$ grows too fast, occurs in much the same way in an entirely different context, namely, number theory. Current work on Hilbert's tenth problem (solving Diophantine equations in integers) relies on the same principle and uses similar techniques.

6. *The Coloring Problem*

This is one of the oldest combinatorial problems and one of the most difficult. It is significant because of the work done on it and the unexpected applications of this work to other problems. The statement of the problem is deceptively simple: Can every planar map (every region is bounded by a polygon with straight sides) be colored with at most four colors, so that no two adjacent regions are assigned the same color?

It is true that recently a computer program has been discovered by Haken and Apel that verifies this conjecture. Nevertheless, mathematicians have not given up hope for a solution that can be followed logically. Thanks to the initiative of H. Whitney of the Institute for Advanced Study and largely to the work of W. T. Tutte (English-Canadian) a new and highly indirect approach to the coloring problem is being developed, called "combinatorial geometry" (or the theory of matroids). This is the first theory of a general character that has been completely successful in understanding a variety of combinatorial problems. The theory is a generalization of Kirchhoff's laws of circuit theory in a completely unforeseen — and untopological — direction. The basic notion is a closure relation with the MacLane-Steinitz exchange property. The exchange property is a closure relation, $A \rightarrow \overline{A}$, defined on all subsets A of a set S such that, if x and y are elements of S and $x \in \overline{A \cup y}$ but $x \notin \overline{A}$, then $y \in \overline{A \cup x}$. In general, one does not have $\overline{A \cup B} = \overline{A} \cup \overline{B}$, so that the resulting structure, called a combinatorial geometry, is not a topological space. The theory bears curious analogies with both point-set topology and linear algebra and lies a little deeper than either of them.

The most striking advance in the coloring problem is a theorem due to Whitney. To state it, we require the notion of "planar graph," which is a collection of points in the plane, called vertices, and nonoverlapping straight-line segments, called edges, each of them joining a pair of vertices. Every planar graph is the boundary of a *map* dividing the plane into *regions*. Whitney makes the following assumptions about the planar graph and the associated map: (a) Exactly three boundary edges meet at each vertex; (b) no pair of regions, taken together with any boundary edges separating them, forms a multiply connected region; (c) no three regions, taken together with any boundary edges separating them, form a multiply connected region. Under these assumptions, Whitney concludes that it is possible to draw a closed curve that passes through each region of the map once and only once. Whitney's theorem has found many applications since it was discovered.

7. *The Pigeonhole Principle and Ramsey's Theorem*

We cannot conclude this brief list of combinatorial problems without giving a typical example of combinatorial argument. We have chosen a little-known theorem of great beauty, whose short proof we shall give in its entirety. The lay reader who can follow the proof on first reading will have good reason to consider himself combinatorially inclined.

THEOREM: Given a sequence of $(n^2 + 1)$ distinct integers, it is possible to find a sequence of $(n + 1)$ entries which is either increasing or decreasing.

Before embarking upon the proof, let us see some examples. For $n = 1$, we have $n^2 + 1 = 2$ and $n + 1 = 2$; the conclusion is trivial since a sequence of two integers is always either increasing or decreasing. Let $n = 2$, so that $n^2 + 1 = 5$ and $n + 1 = 3$, and say the integers are 1,2,3,4,5. The theorem states that no matter how these integers are arranged, it is possible to pick out a string of at least three (not necessarily consecutive) integers that are either increasing or decreasing, for example,

1 2 3 4 5.

The subsequence 1 2 3 will do (it is increasing). Acutally, in this case every subsequence of three elements is increasing. Another example is

3 5 4 2 1.

Here all increasing subsequences, such as 3 4 and 3 5, have at most two integers. There is, however, a wealth of decreasing subsequences of three (or more) integers such as 5 4 2, 5 2 1.

One last example is

5 1 3 4 2.

Here there is one increasing subsequence with three integers, namely 1 3 4, and there are two decreasing subsequences with three integers, namely 5 3 2 and 5 4 2; hence, the statement of the theorem is again confirmed.

Proceeding in this way, we could eventually verify the statement for all permutations of five integers. There are altogether $5! = 120$ possibilities. For $n = 3$, we have to take $n^2 + 1 = 10$ integers, and the amount of work to be done to verify the conjecture case by case is overwhelming since the possibilities total $10! = 3{,}628{,}800$. We begin to see that an argument of an altogether different kind is needed if we are to establish the conclusion for all positive integers n.

The proof goes as follows. Let the sequence of integers (in the given order) be

$$a_1,\ a_2,\ a_3\ ,\ldots,\ a_{n^2+1} \tag{1}$$

We are to find a subsequence of Sequence 1, which we shall label

$$a_{i_1},\ a_{i_2},\ \ldots,\ a_{i_{n+1}},$$

where the entries are taken in the same order as Sequence 1 but with one of the following two properties: either

$$a_{i_1} \leqq a_{i_2} \leqq \cdots \leqq a_{i_{n+1}}, \tag{2}$$

or

$$a_{i_1} \geqq a_{i_2} \geqq \cdots \geqq a_{i_{n+1}}, \tag{3}$$

The argument is based on a *reductio ad absurdum.* Suppose that there is no subsequence of the type of Sequence 2, that is, no increasing subsequence of $(n + 1)$ or more entries. Our argument will then lead to the conclusion that, under this assumption, there must be a sequence of the type of Sequence 3, that is, a decreasing sequence with $(n + 1)$ entries.

Choose an arbitrary entry a_i of Sequence 1, and consider all increasing subsequences of Sequence 1 whose first element is a_i. Among these, there will be one with a maximum number of entries. Say this number is l (= length). Under our additional hypothesis, the number l can be 1, 2, 3,..., or n, *but not* $n + 1$ or any larger integer.

We have, therefore, associated to each entry a_i of Sequence 4 an integer l between 1 and n; for example, $l = 1$ if all subsequences of two or more integers starting with a_i are decreasing. We come now to the crucial part of the argument. Let $F(l)$ be the number of entries of Sequence 1 with which we have associated the integer l, by the procedure just described. Then

$$F(1) + F(2) + F(3) + \cdots + F(n) = n^2 + 1. \tag{4}$$

Identity 4 is just another way of saying that with each one of the $(n^2 + 1)$ entries, a_i of Sequence 1 we have associated a number between 1 and l. We claim that *at least one* of the summands on the left-hand side of Identity 4 must be an integer greater than or equal to $n + 1$. For if this were not so, then we should have

$$F(1) \leqq n, \ F(2) \leqq n, \ldots, F(n) \leqq n.$$

Adding all these n inequalities, we should have

$$F(1) + F(2) + \cdots + F(n) \leqq \underbrace{n + n + \cdots + n}_{n \text{ times}} = n^2,$$

and this contradicts Identity 4, since $n^2 < n^2 + 1$. Therefore, one of the summands on the left-hand side of Identity 4 must be at least $n + 1$. Say this is the lth summand:

$$F(l) \geqq n + 1.$$

We now go back to Sequence 1 and see what this conclusion means. We have found $(n + 1)$ entries of Sequence 1, call them (in the given order)

$$a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}} \tag{5}$$

with the property that each one of these entries is the beginning entry of an increasing subsequence of l entries of Sequence 1 but is not the beginning entry of any longer subsequence of Sequence 1.

From this we can immediately conclude that Sequence 5 is a *decreasing* sequence. Let us prove, for example, that $a_{i_1} > a_{i_2}$. If this were not true, then we should have $a_{i_1} < a_{i_2}$. The entry a_{i_2} is the beginning entry of an increasing subsequence of Sequence 1 containing exactly l entries. It would follow that a_{i_1} would be the beginning entry of a sequence of $(l + 1)$ entries, namely a_{i_1} itself followed by the sequence of l entries starting with a_{i_2}. But this contradicts our choice of a_{i_1}. We conclude that $a_{i_1} > a_{i_2}$. In the same way, we can show that $a_{i_2} > a_{i_3}$, etc., and complete the proof that Sequence 5 is decreasing and, with it, the proof of the theorem.

Looking over the preceding proof, we see that the crucial step can be restated as follows: If a set of $(n^2 + 1)$ objects is partitioned into n or fewer blocks, at least one block shall contain $(n + 1)$ or more objects or, more generally, if a set of n objects is partitioned into k blocks and $n > k$, at least one block shall contain two or more objects. This statement, generally known as the "pigeonhole" principle, has rendered good service to mathematics. Although the statement of the pigeonhole principle is evident, nevertheless the applications of it are often startling. The reason for this is that the principle asserts that an object having a certain property exists, without giving us a means for finding such an object; however, the mere existence of such an object allows us to draw concrete conclusions, as in the theorem just proved.

Some time ago, the British mathematician and philosopher F. P. Ramsey obtained a deep generalization of the pigeonhole principle, which we shall now state in one of its forms. Let S be an infinite set, and let $P_l(S)$ be the family of all finite subsets of S containing l elements. Partition $P_l(S)$ into k blocks, say $B_1, B_2, \ldots, B_k$; in other words, every l-element subset of S is assigned to one and only one of the blocks B_i for $1 \leq i \leq k$. Then there exists an infinite subset $R \subset S$ with the property that $P_l(R)$ is contained in one block, say $P_l(R) \subset B_i$ for some i, where $1 \leq i \leq k$; in other words, there exists an infinite subset R of S with the property that all subsets of R containing l elements are contained in one and the same of the B_i.

The Coming Explosion

It now seems that both physics and mathematics, as well as those life sciences that aspire to becoming mathematical, are conspiring to make further work in combinatorial theory a necessary condition for progress. For this and other reasons, some of which we have stated, the next few years will probably witness an explosion of combinatorial activity, and the mathematics of the discrete will come to occupy a position at least equal to that of the applied mathematics of continua, in university curricula as well as in the importance of research. Already in the past years, the amount of research in combinatorial theory has grown to the point that several specialized journals are being published. In the last few years, several textbooks and mongraphs in the subject have been published, and several more are now in print.

Before concluding this brief survey, we shall list the main subjects in which current work in combinatorial theory is being done. They are the following:

1. *Enumerative Analysis,* concerned largely with problems of efficient counting of (in general, infinite) sets of objects like chemical compounds, subatomic structures, simplicial complexes subject to various restrictions, finite algebraic structures, various probabilistic structures such as runs, queues, permutations with restricted position, and so on.

2. *Finite Geometries and Block Designs.* The work centers on the construction of finite projective planes and closely related structures, such as Hadamard matrices. The techniques used at present are largely borrowed from number theory. Thanks to modern computers, which allowed the testing of reasonable hypotheses, this subject has made great strides in recent years. It has significant applications to statistics and to coding theory.

3. *Applications to Logic.* The development of decision theory has forced logicians to make wide use of combinatorial methods.

4. *Statistical Mechanics.* This is one of the oldest and most active sources of combinatorial work. Some of the best work in combinatorial theory in the last twenty years has been done by physicists or applied mathematicians working in this field, for example in the Ising problem. Close connections with number theory, through the common medium of combinatorial theory, have been recently noticed, and it is very likely that the interaction of the two fields will produce striking results in the near future.

In conclusion, we should like to caution the reader who might gather the idea that combinatorial theory is limited to the study of finite sets. An infinite class of finite sets is no longer a finite set, and infinity has a way of getting into the most finite of considerations. Nowhere more than in combinatorial theory do we see the fallacy of Kronecker's well-known saying that "God created the integers; everything else is man-made." A more accurate description might be: "God created infinity, and man, unable to understand infinity, had to invent finite sets." In the ever-present interaction of finite and infinite lies the fascination of all things combinatorial.

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