Kronecker coefficients: combinatorics, complexity and beyond

Greta Panova

University of Pennsylvania

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The Kronecker coefficients: definitions

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

— the **Specht modules** $\mathbb{S}_\lambda$, indexed by partitions $\lambda \vdash n$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus ??}$$

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$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

Kronecker coefficients:

$g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$:

Tensor products of GL_N representations:

$$V_\mu \otimes V_\nu = \oplus_{\lambda \vdash |\mu|+|\nu|} V_\lambda^{\oplus c^\lambda_{\mu\nu}}$$

Characters of S_n

characters: $\text{char}\, \mathbb{S}_\lambda = \chi^\lambda : S_n \to \mathbb{C}$

$\chi^\lambda[\alpha]$ – value at permutation of cycle type $\alpha = (\alpha_1, \alpha_2, \ldots)$

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle$$

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Murnaghan–Nakayama rule:

$$\chi^\lambda[\alpha] = \sum_{T \,:\, \text{MN tableaux, shape } \lambda, \text{ content } \alpha} (-1)^{ht(T)}$$

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 & 4 & 4 \\ \hline 1 & 2 & 2 & 3 & 4 & 4 \\ \cline{1-6} 2 & 2 & 3 & 3 & 4 \\ \cline{1-5} \end{array}$$

$\lambda = (7, 6, 5)$, $\alpha = (4, 4, 5, 5)$,
$ht(T) = (2-1) + (2-1) + (3-1) + (3-1) = 6.$

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1	1	1	3	3	4	4
1	2	2	3	4	4	
2	2	3	3	4		

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A formula:

$$g(\lambda,\mu,\nu) = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda[w] \chi^\mu[w] \chi^\nu[w].$$

Corollary: $g(\lambda,\mu,\nu) = g(\mu,\lambda,\nu) = g(\nu,\mu,\lambda) = \cdots.$

The combinatorial problem

Problem (Murnaghan (1938), then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

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Motivation: Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\nu}$:
$\mathcal{O}_{\lambda,\mu,\nu} = \{$ LR tableaux of shape λ/μ, type $\nu\}$

$\lambda = (6, 4, 3), \mu = (3, 1), \nu = (4, 3, 2)$:

			1	1	1
	1	2	2		
2	3	3			

			1	1	1
	2	2	2		
1	3	3			

$\Rightarrow c^{\lambda}_{\mu\nu} = 2.$

Theorem (Murnaghan)

If $|\nu| + |\mu| = |\lambda|$ and $n > |\nu|$, then

$$g((n + |\mu|, \nu), (n + |\nu|, \mu), (n, \lambda)) = c^{\lambda}_{\mu\nu}.$$

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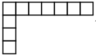
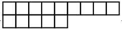
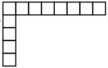
Results since then:

Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n-k, k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k, k)$, $\lambda = (n-r, r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n-k, 1^k)$ (), [*Blasiak, 2012*]

Complexity

Input: (binary) string of n bits, i.e. size($input$) $= n$.

Decision problems:

Is there an object, s.t.... ?

P $=$ solution can be found in time Poly(n)

NP $=$ solution can be *verified* in Poly(n) (polynomial witness)

NP **–Complete** $=$ in NP , and every NP problem can be reduced to it poly time; e.g.

existence of a Hamiltonian path,

Knapsack problem

Counting problems:

Compute $F(input) =?$

FP $=$ solution can be found in time Poly(n)

#P $=$ NP counting analogue; informally – $F(input)$ counts Exp-many objects, whose verification is in P .

e.g.:

number of Hamiltonian paths of given total cost;

computing the permanent of a matrix.

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The P vs NP Problem:
Is P = NP?
An approach [Mulmuley, Sohoni]: **Geometric Complexity Theory**

Kronecker complexity problems and GCT

Input: Partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, s.t. $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.

Size(Input)= $O(\ell \log N)$.

Positivity of Kronecker coefficients (KP):

Decide: whether $g(\lambda, \mu, \nu) > 0$

Kronecker coefficients (Kron):

Compute: $g(\lambda, \mu, \nu)$.

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Conjecture (Mulmuley)

KP *is in* P *.*

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Compute: $g(\lambda, \mu, \nu)$.

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When the input is in binary, KRON *is in* #P *.*

Problem (GCT)

Find estimates and bounds for $g(\lambda, \mu, \nu)$*, plethystic coefficients?*

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KP *is in* P . *(No positive combinatorial interpretation, not known: "*KP *is in* NP *"??)*

Kronecker coefficients (Kron):

Compute: $g(\lambda, \mu, \nu)$.

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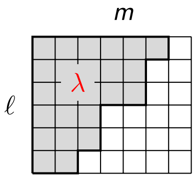
Find estimates and bounds for $g(\lambda, \mu, \nu)$, plethystic coefficients?

Partitions inside a rectangle

$$p_n(\ell, m) = \#\{\lambda \vdash n, \ell(\lambda) \le \ell, \lambda_1 \le m\}$$

$$\sum_{n \ge 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1 - q^{m+i}}{1 - q^i} = \binom{m+\ell}{m}_q$$



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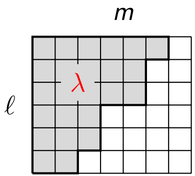

Theorem (Sylvester 1878, Cayley's conjecture 1856)

The sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ *is unimodal, i.e.*

$$p_0(\ell, m) \leq p_1(\ell, m) \leq \ldots \leq p_{\lfloor \ell m/2 \rfloor}(\ell, m) \geq \cdots \geq p_{\ell m}(\ell, m)$$

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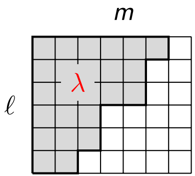

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"*I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.*"

J.J. Sylvester, 1878.

Partitions inside a rectangle

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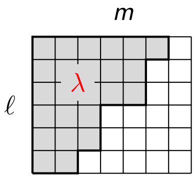

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Proofs:

Sylvester, 1878: *"by aid of a construction drawn from the resources of Imaginative Reason"* (Lie algebras, $\mathfrak{sl}_2$ representations)

Stanley, 1978: hard Lefschetz theorem (alg. geom.), gives Sperner property; 1982: Linear Algebra Paradigm.

Proctor, 1982: explicit linear operators.

O'Hara, 1990: constructive combinatorial proof.

Kronecker and partitions inside a rectangle

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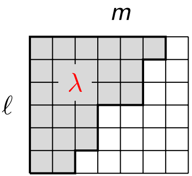

Theorem (Pak-P; Vallejo)

The Kronecker coefficient

$$g\left((\underbrace{m, \ldots, m}_{\ell}), (\underbrace{m, \ldots, m}_{\ell}), (m\ell - k, k) \right) = p_k(m, \ell) - p_{k-1}(m, \ell).$$

Corollary

[Sylvester's Theorem:] $p_0(m, \ell), p_1(m, \ell), \ldots, p_{m\ell}(m, \ell)$ *is unimodal.*

A combinatorial formula for $g(m^\ell, m^\ell, (m\ell - k, k))$

Partition tree $T(m,\ell,k)$: vertices labelled by (a,b,λ,j), $j \le ab$, $\lambda \vdash b$.
Leaves – $b = 1$, labels (a,i) with $0 \le i \le a$.
Root – (m,ℓ,λ,k) for some $\lambda \vdash \ell$.
Local conditions on vertices and children:
If label – (a,b,λ,j), with $\lambda = (1^{b_1},\ldots,n^{b_n})$, then $\le n$ children, labels – $(a_1,b_1,\lambda^1,j_1),\ldots,(a_n,b_n,\lambda^n,j_n)$. s.t.

- $a_i = i(a+2) - 2(\lambda'_1 + \cdots + \lambda'_i)$ for all $i = 1,\ldots,n$.
- $j_1 + \cdots + j_n = j - 2n(\lambda)$.

All leaves $(a_0,i_0),\ldots,(a_t,i_t)$, satisfy:

- For each $r < t$: $i_r \ge 2(i_{r+1} + \cdots + i_t) - (a_{r+1} + \cdots + a_t)$.

Theorem (Pak-P, 2015+)

The Kronecker coefficient for the partitions $(m^\ell),(m^\ell),(m\ell-k,k)$ is equal to the number of partition trees $T(m,\ell,k)$.

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Corollary (Pak-P,2015+)

Let r be fixed and $\lambda = \mu = (m^\ell)$. Then the partition trees $T(m, \ell, k)$ are of depth $O(\log \ell)$, width $O(\ell)$, and entries $O(m\ell)$.
Thus computing $g(\lambda, \mu, (m\ell - k, k))$ is in #P (input size is $O(\ell \log m)$).

Note: holds for $\lambda = (m^\ell, 1^r)$ and $\mu = (m + r, m^{\ell-1})$ for fixed r.

Character Lemma and Stanley's theorem

Lemma (Pak–P)

If $\mu = \mu'$, *then*

$$g(\lambda, \mu, \mu) \geq |\chi^\lambda [(2\mu_1 - 1, 2\mu_2 - 3, \ldots)]|.$$

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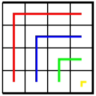
$\mu =$ 

$$\chi^\lambda = \chi^{(n^2-k)\circ(k)} - \chi^{(n^2-k+1)\circ(k-1)}$$

$$|\chi^\mu[(1, 3, 5, \ldots)]| = |b_k(n) - b_{k-1}(n)|,$$

$$\prod_{i=1}^{n} \left(1 + q^{2i-1}\right) =: \sum_{k=0}^{n^2} b_k(n)\, q^k$$

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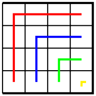
$\mu =$ 

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Theorem (Stanley, 1982)

The following polynomial in q is symmetric and unimodal

$$\binom{2n}{n}_q - \prod_{i=1}^{n} \left(1 + q^{2i-1}\right).$$

Effective bounds on $p_k(m,\ell) - p_{k-1}(m,\ell) = g(m^\ell, m^\ell, (m\ell - k, k))$

Theorem (Pak-P, 2014+)

For all $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$, *we have:*

$$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) - p_{k-1}(\ell, m) \; > \; A\, \frac{2^{\sqrt{s}}}{s^{9/4}}\,,$$

where $s = \min\{2k, \ell^2\}$, *and* $A = 0.00449$ *is an universal constant.*

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Proof:

1) For $m = \ell = n$ – Stanley's Theorem:
$p_k(n,n) - p_{k-1}(n,n) \geq b_k(n) - b_{k-1}(n)$, b_k asymptotics.

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2) For $m \neq \ell$:
Semigroup/monotonicity property:
If $g(\alpha,\beta,\gamma) > 0$, then $g(\lambda+\alpha, \mu+\beta, \nu+\gamma) \geq g(\lambda,\mu,\nu) \quad \forall \lambda,\mu,\nu$.

NP and #P from combinatorics

Theorem (Pak-P,2015+)

Let r be fixed and $\lambda = (m^\ell, 1^r)$ and $\mu = (m+r, m^{\ell-1})$. Then $g(\lambda, \mu, (m\ell + r - k, k))$ is equal to the number of certain partition trees with local conditions of depth $O(\log \ell)$, width $O(\ell)$, and entries $O(m\ell)$. Thus computing $g(\lambda, \mu, (m\ell + r - k, k))$ is in #P (input size is $O(\ell \log m)$).

Theorem (Pak-P, corollary of Blasiak's combinatorial interpretation)

When ν is a hook, $\mathrm{KP} \in \mathsf{NP}$ *and* $\textsc{Kron} \in \#\mathsf{P}$.

Complexity of KRON and KP

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu)$, $\ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)} .$$

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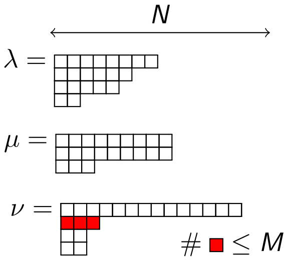
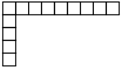


Corollary. *Suppose*

$$\log M, \ell = O\left(\frac{(\log \log N)^{1/3}}{(\log \log \log N)^{2/3}}\right).$$

Then there is a **polynomial time** *algorithm to compute $g(\lambda, \mu, \nu)$.*

Example: ℓ small and $\nu =$ .

Complexity of $\textsc{Kron}$ and KP

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

Corollary (Christandl-Doran-Walter)

When ℓ is fixed, the Kronecker coefficients can be computed in polynomial time, i.e. $\textsc{Kron} \in$ FP (this case: Mulmuley's conjecture ✓)

Theorem (Pak-P)

When the number of parts (ℓ) is fixed, there exists a **linear time** *algorithm to decide whether $g(\lambda, \mu, \nu) > 0$ (i.e. solve KP).*

Proofs I: the Reduction Lemma

Lemma (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ *and* $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. *Set* $s = n - \nu_1$. *Then:*

(i) *If* $|\lambda_i - \mu_i| > s$ *for some* i, *then* $g(\lambda, \mu, \nu) = 0$,

(ii) *If* $|\lambda_i - \mu_i| \leq s$ *for all* i, $1 \leq i \leq \ell$, *there* $\exists$ *an* $r \leq 2s\ell^2$, *s.t.*

$$g(\lambda, \mu, \nu) = g\big(\phi(\lambda), \phi(\mu), \phi(\nu)\big)$$

for certain explicitly defined partitions $\phi(\lambda), \phi(\mu), \phi(\nu) \vdash r$.

Proofs I: the Reduction Lemma

Lemma (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. Set $s = n - \nu_1$. Then:

(i) *If $|\lambda_i - \mu_i| > s$ for some i, then $g(\lambda, \mu, \nu) = 0$,*

(ii) *If $|\lambda_i - \mu_i| \leq s$ for all i, $1 \leq i \leq \ell$, there $\exists$ an $r \leq 2s\ell^2$, s.t.*

$$g(\lambda, \mu, \nu) \;=\; g\big(\phi(\lambda), \phi(\mu), \phi(\nu)\big)$$

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Corollary ([1])

For any m and partition $\alpha \vdash m$, we have that

$$g(\lambda + n\alpha, \mu + n\alpha, \nu + (nm))$$

is bounded and increasing as a function of $n \in \mathbb{N}$, i.e. **stable**.

[1] [Pak-P], indep in [Vallejo], [Stembridge], [Manivel]

Proofs II: Explicit bounds on KRON complexity

Lemma: [2]

$$g(\alpha,\beta,\gamma) = \sum_{\sigma^1,\sigma^2,\sigma^3\in S_\ell} \mathrm{sgn}(\sigma^1\sigma^2\sigma^3)\, C(\alpha+1-\sigma^1,\beta+1-\sigma^2,\gamma+1-\sigma^3)\,,$$

where $C(u,v,w)$ is the number of $\ell\times\ell\times\ell$ contingency arrays $[A_{i,j,k}]$:

$$\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_j, \quad \sum_{i,j} A_{i,j,k} = w_k$$

Lemma

Let $\alpha,\beta,\gamma\vdash n$ be partitions of the same size, such that $\alpha_1,\beta_1,\gamma_1\le m$ and $\ell(\alpha),\ell(\beta),\ell(\gamma)\le\ell$. Then $g(\alpha,\beta,\gamma)$ can be computed in time $(\ell\log m)^{O(\ell^3\log\ell)}$.

[2] In [Christandl-Doran-Walter],[Pak-P]

The theorem

Reduction Lemma:

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Lemma: Complexity of Kron

Let $\alpha, \beta, \gamma \vdash n$ be partitions of the same size, s.t. $\alpha_1, \beta_1, \gamma_1 \leq m$ and $\ell(\alpha), \ell(\beta), \ell(\gamma) \leq \ell$. Then $g(\alpha, \beta, \gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$.

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Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

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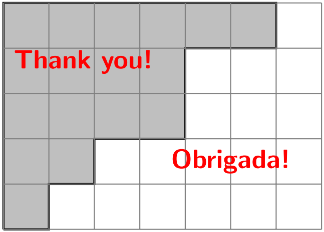
The analogous question for characters

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, where $0 \leq \lambda_i, \mu_i \leq N$, and $|\lambda| = |\mu|$.

Decide: whether $\chi^\lambda[\mu] = 0$

Proposition (Pak-P)

This problem is NP *–hard.*


Thank you!
Obrigada!