AN ELEMENTARY PROOF OF A q–BINOMIAL IDENTITY

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In the previous paper [Z] D. Zeilberger asks for an elementary, non–combinatorial proof of the identity (KOH). We shall give such a proof.

We shall use the notation of [M]. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition, let $|\lambda| = \sum \lambda_i$ denote the weight of λ, and λ' the conjugate partition. Then for each $i \geq 1$,

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$$

is the number of parts of λ equal to i. If $\mu = (\mu_1, \mu_2, \dots)$ is another partition, $\lambda + \mu$ denotes the partition $(\lambda_1 + \mu_1, \lambda_2, \mu_2, \dots)$ and $\lambda \cup \mu$ the partition whose parts are those of λ and μ, arranged in descending order. We have then $(\lambda \cup \mu)' = \lambda' + \mu'$.

Furthermore, for any partition λ we define

$$n(\lambda) = \sum (i-1)\lambda_i = \sum \binom{\lambda'_i}{2}$$

and

$$b_\lambda(q) = \prod_{i \geq 1} \varphi_{m_i(\lambda)}(q),$$

where q is an indeterminate, $\varphi_m(q) = (1-q)(1-q^2)\dots(1-q^m)$ if $m \geq 1$, and $\varphi_0(q) = 1$. Then the identity (KOH) in [Z] may be written in the form

$$\text{(KOH)} \qquad G(N,k) = \sum_{|\lambda|=k} q^{2n(\lambda)} F(N,\lambda)$$

where

$$F(N,\lambda) = \prod_{i \geq 1} G((N+2)i - 2\sum_{j=1}^{i} \lambda'_j, m_i(\lambda))$$

and $G(N,k)$ is the gaussian polynomial (or q–binomial coefficient)

$$G(N,k) = \begin{bmatrix} N+k \\ k \end{bmatrix} = \frac{(1-q^{N+1})\cdots(1-q^{N+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

The identity (KOH) is a consequence of the following two identities:

$$\text{(A)} \qquad G(N,k) = \sum_{r=0}^{k} q^{Nr}/\varphi_r(q^{-1})\varphi_{k-r}(q),$$

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$$\text{(B)} \qquad \sum_{|\lambda|=k} q^{2n(\lambda)} b_\lambda(q)^{-1} = \varphi_k(q)^{-1}.$$

We shall first give the proof of (KOH), and then comment on (A) and (B).

From (A) and the definition of $F(N,\lambda)$ we have

$$\text{(1)} \qquad F(N,\lambda) = \prod_{i\geq 1} \sum_{r_i=0}^{m_i(\lambda)} q^{\left((N+2)i-2\sum_1^i \lambda'_j\right)r_i} / \varphi_{r_i}(q^{-1})\varphi_{m_i(\lambda)-r_i}(q).$$

Let μ be the partition $(1^{r_1}2^{r_2}\ldots)$ and ν the partition $(1^{m_1(\lambda)-r_1}2^{m_2(\lambda)-r_2}\ldots)$, so that $\lambda = \mu \cup \nu$ (and therefore also $\lambda' = \mu' + \nu'$). Then we can rewrite (1) as

$$\text{(2)} \qquad q^{2n(\lambda)} F(N,\lambda) = \sum_{\mu,\nu} q^{a(\mu,\nu)} / b_\mu(q^{-1}) b_\nu(q)$$

where the sum is over (μ,ν) such that $\mu \cup \nu = \lambda$, and

$$\begin{aligned} a(\mu,\nu) &= 2n(\lambda) + \sum_{i\geq 1}((N+2)i - 2\sum_1^i \lambda'_j)(\mu'_i - \mu'_{i+1}) \\ &= \sum(\lambda'^2_i - \lambda'_i) + (N+2)|\mu| - 2\sum \lambda'_i\mu'_i \\ &= N|\mu| - 2n(\mu) + 2n(\nu) \end{aligned}$$

(since $\lambda'_i = \mu'_i + \nu'_i$). It follows that

$$\begin{aligned} \sum_{|\lambda|=k} q^{2n(\lambda)} F(N,\lambda) &= \sum_{\substack{\mu,\nu \\ |\mu|+|\nu|=k}} q^{N|\mu|-2n(\mu)+2n(\nu)} / b_\mu(q^{-1}) b_\nu(q) \\ &= \sum_{r=0}^k q^{Nr} \sum_{|\mu|=r} \frac{q^{-2n(\mu)}}{b_\mu(q^{-1})} \sum_{|\nu|=k-r} \frac{q^{2n(\nu)}}{b_\nu(q)} \\ &= \sum_{r=0}^k q^{Nr} / \varphi_r(q^{-1}) \varphi_{k-r}(q) \qquad \text{by (B)} \\ &= G(N,k) \qquad \text{by (A)} \end{aligned}$$

and the proof is complete.

REMARKS. The identity (A) is just the q-binomial theorem, applied to the numerator of $G(N,k)$. The identity (B) (which is also the limiting case $N \to \infty$ of (KOH)) is due to Philip Hall [H], who gave a simple and elegant combinatorial proof by counting partitions. Another proof of (B) is implicit in [M], Chapter III, by combining §2, Ex. 1 with §4, Ex. 1. More generally, these two exercises imply the identity

$$(B_n) \qquad \begin{bmatrix} n+k-1 \\ k \end{bmatrix} = \sum_{|\lambda|=k} q^{2n(\lambda)} \varphi_n(q) / \varphi_{n-\lambda'_1}(q) \varphi_{\lambda'_1-\lambda'_2}(q) \cdots,$$

of which (B) is the limiting case as $n \to \infty$. But George Andrews has remarked that (B_n) can be proved much more simply; since

$$\varphi_n(q)/\varphi_{n-\lambda'_1}(q)\varphi_{\lambda'_1-\lambda'_2}(q)\cdots = \begin{bmatrix} n \\ \lambda'_1 \end{bmatrix} \begin{bmatrix} \lambda'_1 \\ \lambda'_2 \end{bmatrix} \cdots,$$

(B_n) follows by iteration from the q–Vandermonde identity

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix} = \sum_{r \geq 0} q^{r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} k-1 \\ k-r \end{bmatrix}$$

which in turn follows from the q–binomial theorem, by picking out the coefficient of t^k on either side of

$$\prod_{i=0}^{n+k-2} (1+q^i t) = \prod_{i=0}^{k-2} (1+q^i t) \prod_{j=0}^{n-1} (1+q^{j+k-1} t).$$

Thus both (A) and (B), and therefore also (KOH), are consequences of the q–binomial theorem alone.

REFERENCES

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[Z] D. Zeilberger, *A one-line high school algebra proof of the unimodality of Gaussian polynomials* $\begin{bmatrix} n \\ k \end{bmatrix}$ *for* $k \leq 20$, in *Dennis Stanton (ed.), q-Series and Partitions, IMA Volumes in Mathematics and its Applications*, Springer-Verlag, New York (1989).