Dear Chern,

it was a great pleasure to receive your postcard from Nankai written jointly with Michael.

You ask about Catalan numbers. The n-th Catalan number $C_n$ is given by

$$C_n = \frac{(2n)!}{n!(n+1)!}.$$  

Thus for $n = 0, 1, 2, 3, \ldots$ we have

$$C_n : 1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

There is the characteristic function

$$(1) \sum_{n=0}^{\infty} C_n x^n = \frac{1}{2x} \left(1 - \sqrt{1 - 4x}\right)$$
Let $X_n$ be the manifold of all lines of the complex projective space $\mathbb{P}^{n+1}$.

$$\dim X_n = 2n$$

Over $X_n$ we have the tautological $\mathbb{C}^2$-vector bundle obtained by using that $X_n$ equals the Grassmannian of 2-dim. complex linear subspaces of $\mathbb{C}^{n+2}$. Using compact groups

$$X_n = U(n+2)/(U(2) \times U(n))$$

The Chern classes $c_1, c_2$ of the (dual) tautological bundle are according to one of your definitions dual to certain subvarieties of $X_n$ (of complex codimension $1/2$)

$c_1$ : Variety of all lines intersecting a fixed $\mathbb{P}^{n-1} \subset \mathbb{P}^{n+1}$

$c_2$ : $X_{n-1} \subset X_n$

Schubert (Math. Annalen 1885) already determines $c_1^{2n}[X_n]$. It is the number of lines intersecting all of $2n$ given projective subspaces of codimension 2 in $\mathbb{P}^{n+1}$ in general position.
We have

\[(2) \quad c_1^{2n} \left[ X_n \right] = C_n \]

and can determine all Chern numbers

\[(3) \quad c_1^{2r} c_2^{2s} \left[ X_n \right] = C_r \quad \text{for } 2r + 2s = 2n.\]

In particular the matrix of intersection

(for the signature) is a matrix of Catalan numbers which has determinant 1 and is equivalent over \( \mathbb{Z} \) to the standard diagonal matrix (all 1's in the diagonal).

Of course (3) does not give the Chern numbers of the tangent bundle of \( X_n \). But these, in principle, can be expressed by using (3).

The formulas of A. Borel and myself express the Chern classes of the tangent bundle of \( X_n \) in terms of \( c_1, c_2 \). For example, \( (n+2)c_2 \) is the first Chern class of the tangent bundle of \( X_n \).

We can embed

\[(4) \quad X_n \subset \mathbb{P} \left( \binom{n+2}{2} - 1 \right) \]

by the Plücker coordinates. Then \( c_1 \) is dual to the hyperplane section \( H_0 \). By (2) the Catalan number \( C_n \) is the degree.
of the embedding (4).


For example consider the variety of all lines in $X_n$ which intersect a given $P_{n-2} \subset P_{n+1}$. This variety has codimension 2.

It follows from Schubert that it is dual to

$$c_1^2 - c_2$$

Therefore the numbers

$$C_2(n) = (c_1^2 - c_2)^n [X_n] =$$

are interesting.

They occur in Schubert. $C_2(n)$ is the number of lines intersecting all of $n$ given projective subspaces of codimension 3 in $P_{n+1}$ in general position.

By (2) and (3) and (5)

$$C_2(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_k^n$$

For $n = 0, 1, 2, 3, \ldots$ we have

$$C_2(n) = 1, 0, 1, 4, 3, 6, 15, 36, 91, 232, 603, \ldots$$

Up to $n = 9$ these numbers are in Schubert.
I looked into Sloane's impressive list of integral sequences and found the sequence $C_2(n)$ under number M 2587. The given references show that $C_2(n)$ has several combinatorial interpretations (the Catalan numbers have dozens of combinatorial meanings, see the books by Stanley). I showed $C_2(n)$ to Don Zagier. He proved immediately that indeed $C_2(n)$ is M 2587 and

$$\sum_{n=0}^{\infty} C_2(n) x^n = \frac{1}{2x} \left( 1 - \sqrt{\frac{1 - 3x}{1 + x}} \right).$$

Formula (6) for M 2587 occurs in the literature. But I did not find anywhere that M 2587 are the Chern numbers

$$(c_1^2 - c_2)^n \left[ x^n \right].$$

Schubert calculus of lines is very amusing. I showed other things to Don Zagier and he developed a very interesting machinery. I could write many pages. But let me stop here. I wish you the best for your health. Inge just returned from hospital after a knee operation. Both of us send you our best wishes.

Fritz

Continued pages 6, 7
To Chern:

Apparently I am unable to stop. First let me mention that the Catalan numbers satisfy

\[ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \]

whereas the \( C_2(n) \) satisfy

\[ C_2(n+1) = \sum_{i=0}^{n} C_2(i) C_2(n-i) + (-1)^{n+1} \]

(Don Zagier).

Secondly let me mention the following fact which is proved using the relation between representation theory (Hermann Weyl) and my Riemann–Roch formulas observed by Borel and myself during our Princeton time 1952–54:

Consider the embedding (4) and let \( H \) be a hyperplane section of \( X_n \) dual to \( g \).

The Hilbert polynomial

\[ X(X_n, rH) = \dim H^0(X_n, rH) \]

for \( r > -(n+2) \)

(postulation formula) is given by

\[ -(n+2)H \] is the canonical divisor of \( X_n \)
(7) \( X(X_n, r+1) = \) 
\[
\frac{(r+1)(r+2) \cdots (r+n)(r+n+1)}{1 \cdot 2^2 \cdots n^2 \cdot (n+1)}
\]

It is a polynomial of degree 2n which vanishes for \( r = -1, -2, \ldots, -(n+1) \), so it must by the Kodaira vanishing theorem.

By Riemann-Roch the coefficient of \( r^{2n} \) (\( 2n = \dim C X_n \)) equals
\[
\frac{H^{2n} [X_n]}{(2n)!} = \frac{1}{(n+1)! \ n!}
\]

Hence
\[
H^{2n} [X_n] = \frac{(2n)!}{(n+1)! \ n!} = C_n
\]

Hence we obtain (2) by Riemann-Roch.

Once again, best wishes.

Fritz

* For \( r = 0 \) it has the value 1.