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Dear Chern,

it was a great pleasure to receive your postcard from Nankai written jointly with Michael.

You ask about Catalan numbers. The n-th Catalan number C_n is given by

$$C_n = \binom{2n}{n} \Big/ (n+1).$$

Thus for $n = 0, 1, 2, 3, \dots$ we have

$$C_n : 1, 1, 2, 5, 14, 42, 132, 429, \dots$$

There is the characteristic function

$$(1) \quad \sum_{n=0}^{\infty} C_n x^n = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right)$$

Let X_n be the manifold of all lines of the complex projective space P_{n+1}.

$$\dim_{\mathbb{C}} X_n = 2n$$

Over X_n we have the tautological $\mathbb{C}^2$-vectorbundle obtained by using that X_n equals the Grassmannian of 2-dim. complex linear subspaces of $\mathbb{C}^{n+2}$. Using compact groups

$$X_n = U(n+2)/(U(2) \times U(n)).$$

The Chern classes c_1, c_2 of the (dual) tautological bundle are according to one of your definitions dual to certain subvarieties of X_n (of complex codimension 1, 2)

c_1 : Variety of all lines intersecting a fixed $P_{n-1} \subset P_{n+1}$

c_2 : $X_{n-1} \subset X_n$

Schubert (Math. Annalen 1885) already determines $c_1^{2n}[X_n]$. It is the number of lines intersecting all of $2n$ given projective subspaces of codimension 2 in P_{n+1} in general position.

We have

(2) $$c_1^{2n}[X_n] = C_n$$

and can determine all Chern numbers

(3) $$c_1^{2r} c_2^{s}[X_n] = C_r \quad \text{for } 2r+2s=2n.$$

In particular the matrix of intersection (for the signature) is a matrix of Catalan numbers which has determinant 1 and is equivalent over $\mathbb{Z}$ to the standard diagonal matrix (all 1's in the diagonal).

Of course (3) does not give the Chern numbers of the tangent bundle of X_n. But these, in principal, can be expressed by using (3). The formulas of A. Borel and myself express the Chern classes of the tangent bundle of X_n in terms of c_1, c_2. For example, $(n+2)c_1$ is the first Chern class of the tangent bundle of X_n.

We can embed

(4) $$X_n \subset P_{\binom{n+2}{2}-1}$$

by the Plücker coordinates. Then c_1 is dual to the hyperplane section H. By (2) the Catalan number C_n is the degree

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of the embedding (4).

The Schubert paper contains much interesting material (Math. Ann. 1885).

For example consider the variety of all lines in X_n which intersect a given $P_{n-2} \subset P_{n+1}$. This variety has codimension 2. It follows from Schubert that it is dual to

$$e_1^2 - c_2$$

Therefore the numbers

$$(5) \qquad C_2(n) \underset{\text{Def}}{=} (c_1^2 - c_2)^n [X_n]$$

are interesting.

They occur in Schubert. <u>$C_2(n)$ is the number of lines intersecting all of n given projective subspaces of codimension 3 in P_{n+1} in general position</u>

By (2) and (3) and (5)

$$C_2(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_k$$

For $n = 0, 1, 2, 3, \ldots$ we have

$$C_2(n) = 1, 0, 1, 1, 3, 6, 15, 36, 91, 232, 603, \ldots$$

Up to $n = 9$ these numbers are in Schubert.

I looked into Sloane's impressive list of integral sequences and found the sequence $C_2(n)$ under number M 2587. The given references show that $C_2(n)$ has several combinatorial interpretations (the Catalan numbers have dozens of combinatorial meanings, see the books by Stanley), I showed $C_2(n)$ to Don Zagier, He proved immediately that indeed $C_2(n)$ is M 2587 and

$$(6) \qquad \sum_{n=0}^{\infty} C_2(n)\, x^n = \frac{1}{2x}\left(1 - \sqrt{\frac{1-3x}{1+x}}\right).$$

Formula (6) for M 2587 occurs in the literature. But I did not find anywhere that M 2587 are the Chern numbers

$$(c_1^2 - c_2)^n\,[X_n].$$

Schubert calculus of lines is very amusing. I showed other things to Don Zagier and he developed a very interesting machinery. I could write many pages. But let me stop here. I wish you the best for your health. Inge just returned from hospital after a knee operation. Both of us send you our best wishes.

Fritz

continued pages 6, 7

To Chern:

Apparently I am unable to stop. First let me mention that the Catalan numbers satisfy

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

whereas the $C_2(n)$ satisfy

$$C_2(n+1) = \sum_{i=0}^{n} C_2(i)\, C_2(n-i) + (-1)^{n+1}.$$

(Don Zagier).

Secondly let me mention the following fact which is proved using the relation between representation theory (Hermann Weyl) and my Riemann-Roch formulas observed by Borel and myself during our Princeton time 1952-54. Consider the embedding (4) and let H be a hyperplane section of X_n dual to c_1. The Hilbert polynomial

$$\chi(X_n, rH) = \dim H^0(X_n, rH)$$

for $r > -(n+2)$ (Kodaira vanishing)

("postulation" formula) is given by

$-(n+2)H$ is the canonical divisor of X_n

(7) $$\chi(X_n, rH) = \frac{(r+1)(r+2)^2 \cdots (r+n)^2 (r+n+1)}{1 \cdot 2^2 \cdots n^2 \cdot (n+1)}.$$

It is a polynomial of degree $2n$ which vanishes for $r = -1, -2, \dots, -(n+1)$, as it must by the Kodaira vanishing theorem.*
By Riemann-Roch the coefficient of r^{2n} ($2n = \dim_{\mathbb{C}} X_n$) equals

$$\frac{H^{2n}[X_n]}{(2n)!} = \frac{1}{(n+1)!\, n!} \quad (7)$$

Hence

$$H^{2n}[X_n] = \frac{(2n)!}{(n+1)!\, n!} = C_n$$

Hence we obtain (2) by Riemann-Roch

Once again

best wishes

Fritz

* For $r = 0$ it has the value 1.